Equivariant Periodic Cyclic Homology for Ample Groupoids

by

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Abstract

We develop an equivariant version of bivariant periodic cyclic homology for actions of Hausdorff ample groupoids, extending the classical bivariant theory of Cuntz and Quillen and its equivariant refinement for groups. For an ample groupoid $\mathcal{G}$, we construct a monoidal category of modules over its convolution algebra and study structural features of its objects, the $\mathcal{G}$-modules. In parallel, we present an equivalent comodule formulation and prove the equivalence between the module and comodule pictures. We introduce $\mathcal{G}$-algebras and give some important examples. After reviewing pro-categories, we define the equivariant X-complex, which is central to the construction of the bivariant equivariant periodic cyclic homology for $\mathcal{G}$-algebras. In analogy with the classical and group-equivariant settings, we establish homotopy invariance, stability, and excision for the resulting theory.

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Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

One of the guiding ideas in classical geometry is to study spaces through their algebras of functions. These algebras are typically commutative, since multiplication is defined pointwise. A cornerstone result in this framework is *Gelfand duality*, which establishes a duality between the category of commutative C^*-algebras and the category of locally compact Hausdorff spaces:

$$\{\text{Locally compact Hausdorff spaces}\} \underset{\operatorname{Spec}(\,\cdot\,)}{\overset{C_0(\,\cdot\,)}{\rightleftarrows}} \{\text{Commutative } C^*\text{-algebras}\}$$

The philosophy of *noncommutative geometry* extends this correspondence: instead of starting with a geometric space, one takes a noncommutative algebra and interprets it as the algebra of functions on a hypothetical noncommutative space. In this setting, algebraic invariants play the role of geometric and topological invariants.

Cyclic homology, and in particular *periodic* cyclic homology, was introduced by Connes as the noncommutative analogue of de Rham cohomology. This analogy is made precise in [Con85, Theorem 46]: if V is a compact smooth manifold, then for the Fréchet algebra $C^\infty(V)$ one has

$$HP_*\big(C^\infty(V)\big) \cong \bigoplus_{n\in\mathbb{Z}} H_{dR}^{*+2n}(V), \qquad * = 0, 1.$$

Periodic cyclic homology shares crucial features with K-theory: it is homotopy invariant and Morita invariant, and it pairs with K-theory via a Chern character.

A major breakthrough came with the bivariant framework of Cuntz and Quillen [CQ95a, CQ95b, CQ97]. Their approach centers on the X-complex. For a not necessarily unital or commutative algebra A, with unitarisation A^+, the noncommutative differential forms are

$$\Omega^n(A) = \begin{cases} A^+ \otimes A^{\otimes n} & \text{if } n > 0, \\ A & \text{if } n = 0, \end{cases}$$

and the X-complex is the $\mathbb{Z}_2$-graded complex defined by

$$X(A) : \Omega^0(A) \underset{\partial_1}{\overset{\partial_0}{\rightleftarrows}} \Omega^1(A)/\partial_1(\Omega^2(A)),$$

where

$$\partial_0(a) = da, \qquad \partial_1([a^0 da^1]) = a^0 a^1 - a^1 a^0.$$

Finally, the bivariant periodic cyclic homology of the algebras A and B is defined by the homology of the Hom-complex associated with the X-complexes of A and B respectively:

$$HP_*(A,B) = H_*(\mathrm{Hom}(X(\mathcal{T}A), X(\mathcal{T}B))).$$

Here $\mathcal{T}A$ is the *periodic* tensor algebra of A and represents a crucial ingredient in this definition. This approach provided the missing ingredient to establish the six-term exact sequences in periodic cyclic homology induced by an extension $0 \to K \to E \to Q \to 0$ of algebras, namely

$$\begin{array}{ccccc} HP_0(A,K) & \longrightarrow & HP_0(A,E) & \longrightarrow & HP_0(A,Q) \\ \uparrow & & & & \downarrow \\ HP_1(A,Q) & \longleftarrow & HP_1(A,E) & \longleftarrow & HP_1(A,K) \end{array}$$

and

$$\begin{array}{ccccc} HP_0(Q,A) & \longrightarrow & HP_0(E,A) & \longrightarrow & HP_0(K,A) \\ \uparrow & & & & \downarrow \\ HP_1(K,A) & \longleftarrow & HP_1(E,A) & \longleftarrow & HP_1(Q,A). \end{array}$$

The existence of these sequences is known as the excision problem and was at that time a longstanding problem, proved only in special cases. This further highlights the connection with bivariant K-theory.

These techniques were extended to algebras with group actions by Voigt, first for discrete groups [Voi03] and then for locally compact groups [Voi07]. In the equivariant theory one works in the monoidal category of G-modules, and constructions are adapted to respect the action. Given a G-algebra A, the equivariant noncommutative differential forms are

$$\Omega^n_G(A) := \mathcal{O}_G \otimes \Omega^n(A),$$

where $\mathcal{O}_G = C_c^\infty(G)$, with G acting diagonally and $\mathcal{O}_G$ carrying the adjoint action. A further ingredient in the definition of G-equivariant periodic cyclic homology is given by the G-algebra $\mathcal{K}_G$ associated to a certain bilinear pairing endowed with diagonal action. This algebra carries information about the action in the equivariant setting, while its classic cyclic homology contains no non-trivial information, being isomorphic to the homology of $\mathbb{C}$. A fundamental difference with the classical setting is that the equivariant X-complex $X_G(A)$ is typically a *paracomplex* rather than a chain complex, this means that the square of its differential need not vanish. This is resolved by working bivariantly from the beginning. Since the vanishing of the differential associated to the equivariant X-complex is controlled by a canonical map, the resulting Hom-complex is a genuine chain complex.

Motivated by the goal of extending equivariant periodic cyclic homology to broader algebraic settings, we now turn our attention to *groupoids*, which have emerged as central objects in several areas of mathematics, including operator theory, topology, and mathematical physics. Groupoids offer a remarkably flexible framework that generalizes many familiar structures, such as groups, topological spaces, and dynamical systems.

In particular, algebras associated with étale groupoids have attracted significant interest, see for example the work of Renault [Ren80], as they form a rich class of examples of noncommutative algebras, for instance, they arise naturally in topological dynamics and the classification of simple C^*-algebras as described in [Li20]. Among the key contributions in this direction are the works of Steinberg. In [Ste10], a connection is established between inverse semigroups and ample groupoids, including an isomorphism between their convolution algebras. While, in [Ste14], an equivalence between the category of nondegenerate modules over the convolution algebra of an ample groupoid and the category of sheaves of modules over the groupoid is proved. In this work we will consider topological groupoids, in particular locally compact, Hausdorff and ample groupoids $\mathcal{G}$.

A further motivation for this work comes from Matui's conjecture, see [Mat16, Conjecture 2.6]:

Conjecture (HK)**.** *Let $\mathcal{G}$ be an essentially principal minimal étale groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor set. Then we have*

$$\bigoplus_{n=0}^{\infty} H_{2n}(\mathcal{G}) \cong K_0\big(C_r^*(\mathcal{G})\big)$$

and

$$\bigoplus_{n=0}^{\infty} H_{2n+1}(\mathcal{G}) \cong K_1\big(C_r^*(\mathcal{G})\big).$$

The conjecture states a link between the K-theory of the reduced C^*-algebra of a certain class of ample groupoid and the groupoid homology as defined for étale groupoids by Crainic and Moerdijk in [CM00]. The conjecture was later shown not to hold in full generality by Scarparo in [Sca20]. At the same time, it is known to hold for several class of groupoids, see [Mat12], [FKPS19] and [BDGW23]. This shows how a deeper understanding of the relationship between groupoid homology and the K-theory of associated algebras remains an area of significant interest.

In this setting, to gain eventually insight into the K-theory of ample groupoids via a bivariant Chern character it is natural to investigate a generalisation of periodic cyclic homology to convolution algebras of groupoids, with particular emphasis on the class of ample groupoids. Our approach is inspired by the foundational techniques of both classical and equivariant periodic cyclic homology, adapted to the groupoid framework.

The primary goal of this work is to define an equivariant version of periodic cyclic homology for algebras arising from the convolution algebras of ample groupoids. Alongside this, we aim to develop a general framework suitable for such a generalisation. The construction of the core objects and tools of the theory will rely crucially on the structural properties of ample groupoids, which are collected in the first part of the thesis. Given an ample groupoid $\mathcal{G}$, we construct two categories: one is the category of $\mathcal{G}$-modules, given by essential modules over $\mathcal{D}(\mathcal{G})$ the convolution algebra of $\mathcal{G}$, the second is the category of $C_c^\infty(\mathcal{G})$-comodules given by essential modules over $C_c^\infty(\mathcal{G}^{(0)})$ and a certain isomorphism which encodes the information of the groupoid action. We then prove that these two categories are isomorphic. We introduce anti-Yetter-Drinfeld modules over $\mathcal{G}$ and the canonical automorphism T associated to such modules, which is crucial for turning equivariant differential forms into a paramixed complex and for defining the groupoid equivariant X-complex $X_\mathcal{G}(-)$.

Once we define $\mathcal{G}$-equivariant bivariant periodic cyclic homology $HP_*^\mathcal{G}$ for pro-$\mathcal{G}$-algebras, we investigate its fundamental properties.

The first important property is the homotopy invariance:
Theorem (A). *Let A and B be pro-$\mathcal{G}$-algebras and let $\Phi : A \to B[0,1]$ be a $\mathcal{G}$-equivariant homotopy. Then the elements $[\Phi_0]$ and $[\Phi_1]$ in $HP_0^\mathcal{G}(A,B)$ are equal.*

We discuss stability, considering a first result concerning a special case, we call *admissible* case, and finally the general case:
Theorem (B). *Let E be a $\mathcal{G}$-module equipped with a surjective $\mathcal{G}$-equivariant bilinear pairing. If $C_c^\infty(\mathcal{G}^{(0)})$ and E are projective as essential $C_c^\infty(\mathcal{G}^{(0)})$-modules then there exists an invertible element in*

$$HP_0^\mathcal{G}(A, A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$$

for any pro-$\mathcal{G}$-algebra A. It follows that we have natural isomorphisms

$$HP^{\mathcal{G}}_*(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E), B) \cong HP^{\mathcal{G}}_*(A, B) \cong HP^{\mathcal{G}}_*(A, B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$$

for all pro-$\mathcal{G}$-algebras A and B.

As a consequence of this theorem, we can simplify the computation of the periodic cyclic homology in the case of a proper groupoid, proving that:

Proposition (C). *Let $\mathcal{G}$ be a proper ample groupoid with $\mathcal{G}\backslash\mathcal{G}^{(0)}$ paracompact. Then we have a natural isomorphism*

$$HP^{\mathcal{G}}_*(A, B) \cong H_*\mathrm{Hom}_{A(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}A), X_{\mathcal{G}}(\mathcal{T}B))$$

for all $\mathcal{G}$-algebras A, B.

Finally, given an extension of pro-$\mathcal{G}$-algebras, which is admissible in the category of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules as in Definition 3.8, we prove the existence of a six-term exact sequence in both variables for the groupoid equivariant case:

Theorem (D). *Let A be a pro-$\mathcal{G}$-algebra and let $0 \to K \to E \to Q \to 0$ be an extension of pro-$\mathcal{G}$-algebras which is admissible as an extension of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules. Then there are two natural exact sequences*

$$\begin{array}{ccccc} HP^{\mathcal{G}}_0(A,K) & \longrightarrow & HP^{\mathcal{G}}_0(A,E) & \longrightarrow & HP^{\mathcal{G}}_0(A,Q) \\ \uparrow & & & & \downarrow \\ HP^{\mathcal{G}}_1(A,Q) & \longleftarrow & HP^{\mathcal{G}}_1(A,E) & \longleftarrow & HP^{\mathcal{G}}_1(A,K) \end{array}$$

and

$$\begin{array}{ccccc} HP^{\mathcal{G}}_0(Q,A) & \longrightarrow & HP^{\mathcal{G}}_0(E,A) & \longrightarrow & HP^{\mathcal{G}}_0(K,A) \\ \uparrow & & & & \downarrow \\ HP^{\mathcal{G}}_1(K,A) & \longleftarrow & HP^{\mathcal{G}}_1(E,A) & \longleftarrow & HP^{\mathcal{G}}_1(Q,A), \end{array}$$

where the horizontal maps in these diagrams are induced by the maps in the extension.

The work is organised as follows. Chapter 1 recalls the necessary preliminaries on groupoids and functions on totally disconnected spaces. Then it introduces the basics of the convolution algebras of an ample groupoid. In Chapter 2, we construct the category of modules over the convolution algebra of an ample groupoid. We study the main features of this category, focusing in particular on the construction of an internal tensor product, which turns this into a monoidal category. We also introduce the notion of Anti-Yetter–Drinfeld modules, which will play a central role. In Chapter 3, we start discussing about pro-categories, then we introduce the definition of equivariant differential forms for an ample groupoid and the equivariant X-complex. Finally we present the definition of bivariant equivariant periodic cyclic homology for ample groupoids, generalising the classical and equivariant theories. In Chapter 4, we investigate the key homological properties of this theory, such as homotopy invariance, stability, and excision.

Chapter 1

Preliminaries

In this first chapter, in order to make the work as self-contained as possible, we begin by reviewing some well-established definitions and results from the literature. In particular, we first collect some basic facts about topological groupoids. We then consider totally disconnected spaces, which will play a crucial role in our discussion. Finally, we introduce convolution algebras of ample groupoids and discuss some important features of proper groupoids.

§ 1.1 | Topological groupoids

This section provides essential definitions that will be used frequently throughout this thesis. The definition of a groupoid and its main properties form the starting point.

Groupoids first appeared about one hundred years ago, and a good historical survey can be found in [Bro87]. Since their introduction, groupoids have found applications in various fields, ranging from topology to operator algebras. A fundamental step in their development was the study of C^*-algebras associated with groupoids, initiated by Renault [Ren80], which remains a valuable source for basic definitions of topological groupoids. Several other good references are available for foundational concepts, such as [Pat99]. Moreover, an elementary treatment of finite groupoids and their representation theory can be found in [IR20].

Definition 1.1. *A* groupoid *is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication (or composition) map $m : \mathcal{G}^{(2)} \to \mathcal{G}$, $(\alpha, \beta) \mapsto \alpha\beta$ and an inversion map $i : \mathcal{G} \to \mathcal{G}$, $\alpha \mapsto \alpha^{-1}$ such that the following hold:*

(i) multiplication is associative: if $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$ for some $\alpha, \beta, \gamma \in \mathcal{G}$, then $(\alpha, \beta\gamma)$, $(\alpha\beta, \gamma) \in \mathcal{G}^{(2)}$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$;

(ii) inversion is involutive: for any $\alpha \in \mathcal{G}$, we have $(\alpha^{-1})^{-1} = \alpha$;

(iii) $(\alpha^{-1}, \alpha) \in \mathcal{G}^{(2)}$ *for any* $\alpha \in \mathcal{G}$, *and for all* $(\alpha, \beta) \in \mathcal{G}^{(2)}$ *we have* $\alpha^{-1}(\alpha\beta) = \beta$ *and* $(\alpha\beta)\beta^{-1} = \alpha$.

As the name suggests, this object is a generalisation of a group. However, as the previous definition shows, the composition is just partially defined. A consequence of this is that there are several partial units. The subset $\mathcal{G}^{(0)} := \{\alpha \in \mathcal{G} \mid \alpha = \alpha^{-1} = \alpha^2\}$ of $\mathcal{G}$ is the base space of the groupoid or its set of *units*. We also introduce the *source* map $s : \mathcal{G} \to \mathcal{G}^{(0)}$, $s(\alpha) = \alpha^{-1}\alpha$ and the *range* map $r : \mathcal{G} \to \mathcal{G}^{(0)}$, $r(\alpha) = \alpha\alpha^{-1}$. With this definition given, $\mathcal{G}^{(2)}$ can be expressed as the set $\{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$. The inclusion map $\mathcal{G}^{(0)} \to \mathcal{G}$ will often be denoted by u, and we will refer to it as the unit map.

We now turn to the notion of topological groupoids, which will play a central role throughout this thesis.

Definition 1.2. *A* topological *groupoid is a groupoid* $\mathcal{G}$ *endowed with a topology such that the multiplication map* $m : \mathcal{G}^{(2)} \to \mathcal{G}$ *is continuous with respect to the subspace topology on* $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ *and the inversion map* $i : \mathcal{G} \to \mathcal{G}$ *is continuous. Moreover, if* $\mathcal{G}$ *is locally compact and Hausdorff, we will say that it is a* locally compact Hausdorff *groupoid.*

Remark 1.3. *In a topological groupoid, the source and range maps* $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ *are automatically continuous when* $\mathcal{G}^{(0)}$ *has the subspace topology. Indeed, they have been defined as* $s(\alpha) = \alpha^{-1}\alpha$ *and* $r(\alpha) = \alpha\alpha^{-1}$ *using the groupoid operations, and thus inheriting continuity from the continuity of inversion and multiplication.*

Let us observe that a more category-theoretic approach is possible. A groupoid $\mathcal{G}$ is a small category in which all arrows are invertible. We denote by $\mathcal{G}^{(0)}$ the set of objects, by $\mathcal{G}$ the set of all morphisms, and by $\mathcal{G}^{(2)}$ the set of all composable pairs of morphisms. We identify $\mathcal{G}^{(0)}$ with the identity morphisms in $\mathcal{G}$ via the map $u : \mathcal{G}^{(0)} \to \mathcal{G}$, $x \mapsto \mathrm{id}_x$. In the topological setting, both $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, and the maps m, i, and u are continuous.

Remark 1.4. *The maps* $u : \mathcal{G}^{(0)} \to u(\mathcal{G}^{(0)})$ *and* $r : u(\mathcal{G}^{(0)}) \to \mathcal{G}^{(0)}$ *are inverse to each other, and both are continuous. Hence* $\mathcal{G}^{(0)}$ *is homeomorphic to* $u(\mathcal{G}^{(0)})$. *We may therefore identify the set of base points* $\mathcal{G}^{(0)}$ *of the category* $\mathcal{G}$ *with the subset*

$$\{\alpha \in \mathcal{G} \mid \alpha = \alpha^{-1} = \alpha^2\}$$

in Definition 1.1, endowed with the subspace topology. Accordingly, we will often move freely between the two, referring to the morphisms of the category as arrows *and to the objects as* points *in the base space.*

Moreover, for any $x, y \in \mathcal{G}^{(0)}$, we define

$$\mathcal{G}_x := s^{-1}(x), \quad \mathcal{G}^x := r^{-1}(x), \quad \text{and} \quad \mathcal{G}_x^y := s^{-1}(x) \cap r^{-1}(y)$$

as the sets of all arrows in $\mathcal{G}$ starting at x, ending at x, and starting at x and ending at y, respectively.

In the general treatment of non-Hausdorff groupoids, the only requirement is often that just the unit space $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ must be Hausdorff in the relative topology. In this case, we have the following result.

Lemma 1.5. *Let $\mathcal{G}$ be a locally compact groupoid with Hausdorff base space, then $\mathcal{G}^{(2)}$ is closed in $\mathcal{G} \times \mathcal{G}$ with the product topology.*

Proof. Consider the map $(s, r) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ and observe that $\mathcal{G}^{(2)} = (s, r)^{-1}(\Delta_{\mathcal{G}^{(0)}})$, where $\Delta_{\mathcal{G}^{(0)}}$ is the diagonal in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, which is closed in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ since $\mathcal{G}^{(0)}$ is Hausdorff by hypothesis. □

However, the following elementary lemma explains why dealing with a Hausdorff groupoid is useful.

Lemma 1.6. *Let $\mathcal{G}$ be a locally compact groupoid with Hausdorff base space, then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}^{(0)}$ is Hausdorff.*

Proof. Assume that $\mathcal{G}$ is Hausdorff and consider the map $(ur) \times \mathrm{id}_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$, which is continuous since it is the product of two continuous maps. Then we have $\mathcal{G}^{(0)} = (ur \times \mathrm{id}_{\mathcal{G}})^{-1}(\Delta_{\mathcal{G}})$, where $\Delta_{\mathcal{G}}$ is the diagonal in $\mathcal{G} \times \mathcal{G}$, which is closed in $\mathcal{G} \times \mathcal{G}$ since $\mathcal{G}$ is Hausdorff.

Conversely, to prove $\mathcal{G}$ being Hausdorff we will show the uniqueness of nets limit points. Assume that $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ and there exists a net $(\gamma_i)_{i \in I}$ converging simultaneously to α and β where $\alpha, \beta \in \mathcal{G}$. By continuity of the composition and inversion, we get that $\gamma_i \gamma_i^{-1}$ converges to $\alpha\beta^{-1}$. Since each $\gamma_i \gamma_i^{-1} = r(\gamma_i) \in \mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)}$ is closed, we have that $\alpha\beta^{-1} = r(\beta) \in \mathcal{G}^{(0)}$. From this we have $\alpha\beta^{-1}\beta = r(\beta)\beta$ and hence $\alpha = \beta$, which concludes the proof. □

From now on, all groupoids we consider, unless otherwise specified, will be *locally compact Hausdorff groupoids.* In particular, we are interested in a subclass of topological groupoids, namely the class of étale groupoids.

Definition 1.7. *Let X, Y be topological spaces. A function $f : X \to Y$ is called a local homeomorphism if, for every point $x \in X$, there exists an open neighbourhood U of x, such*

that the image $f(U)$ is open in Y and the restriction $f|_U : U \to f(U)$ is a homeomorphism.
Remark 1.8. *A local homeomorphism $f : X \to Y$ is automatically a continuous and open map between the topological spaces X and Y.*
Definition 1.9. *A topological groupoid $\mathcal{G}$ is called étale if the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.*
Remark 1.10. *If the range map is a local homeomorphism, it is immediate that the source map $s : \mathcal{G} \to \mathcal{G}^{(0)}$ is also a local homeomorphism since it can be written as the composition of the range map and the inverse map $i : \mathcal{G} \to \mathcal{G}$, which is a homeomorphism.*

In this setting, a central notion that makes étale groupoids distinctive is the definition of an open bisection.
Definition 1.11. *Let $\mathcal{G}$ be a topological groupoid. An open bisection of $\mathcal{G}$ is an open subset U of $\mathcal{G}$ such that the restriction of the source map $s|_U : U \to s(U)$ and the restriction of the range map $r|_U : U \to r(U)$ are homeomorphisms. Moreover, the set of all open bisections will be denoted by* $\mathrm{Bis}(\mathcal{G})$.

We now state and prove some properties of étale groupoids. Some of these results can be found in [Bö18].
Lemma 1.12. *Let $\mathcal{G}$ be a topological groupoid. If $\mathcal{G}$ is étale, then the following hold:*

(i) $\mathcal{G}^{(0)}$ *is open in* $\mathcal{G}$;

(ii) $\mathcal{G}^x$ *and* $\mathcal{G}_x$ *are discrete (in the subspace topology) for every* $x \in \mathcal{G}^{(0)}$;

(iii) If U and V are open subsets of $\mathcal{G}$, the set

$$UV := \{\alpha\beta \in \mathcal{G} \mid (\alpha, \beta) \in \mathcal{G}^{(2)} \cap (U \times V)\}$$

is open in $\mathcal{G}$.

Proof. To prove (i), let $x \in \mathcal{G}^{(0)}$, and let $A \subseteq \mathcal{G}$ be an open subset containing x, and $B \subseteq \mathcal{G}^{(0)}$ an open subset containing x, such that $r(A) = B$ and $r|_A : A \to B$ is a homeomorphism. Set $B' := A \cap \mathcal{G}^{(0)}$, which is non-empty (since it contains x) and open in $\mathcal{G}^{(0)}$ by construction. Consider $A' := r^{-1}(B') \cap A$, which is open in $\mathcal{G}$ and has the property that r is injective from A' to B'. To conclude, we check that $A' \subseteq \mathcal{G}^{(0)}$. If $a \in A'$, then $r(a) \in B'$ and $a \in A$. Thus a and $r(a)$ both belong to A and have the same image under the range map; by injectivity we conclude that $a = r(a)$, hence $a \in \mathcal{G}^{(0)}$.

We now prove (ii) only for $\mathcal{G}^x$, as the case of $\mathcal{G}_x$ is analogous. Let $\alpha \in \mathcal{G}^x$. Then there exists an open neighbourhood $U \subseteq \mathcal{G}$ such that $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is injective on U. It follows that $\mathcal{G}^x \cap U = \{\alpha\}$ is open in $\mathcal{G}^x$, so $\mathcal{G}^x$ is discrete.

Finally, for (iii), let $U,V \subseteq \mathcal{G}$ be open and $(\alpha,\beta) \in \mathcal{G}^{(2)} \cap (U \times V)$. Since $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism, there exists an open neighbourhood W of $\alpha\beta$ in $\mathcal{G}$ such that $r|_W$ is a homeomorphism onto its image. As $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is continuous, there exist open neighbourhoods $U', V' \subseteq \mathcal{G}$ of α and β, respectively, such that $U'V' \subseteq W$. By intersecting, we may assume $U' \subseteq U$, $V' \subseteq V$, and $U' \subseteq s^{-1}(r(V'))$. Then $r(U'V') = r(U')$ is open. Therefore,

$$U'V' = r^{-1}(r(U'V')) \cap W$$

is open and contained in UV, as required. □

Remark 1.13. *In many definitions of étale groupoids, one assumes that the range map $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism, meaning in particular that for any open subset $U \subseteq \mathcal{G}$, the image $r(U)$ is open in $\mathcal{G}$, not just in $\mathcal{G}^{(0)}$. However, the two definitions are equivalent. Indeed, if we assume that $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism we get that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$ (point (i) in Lemma 1.12), and this ensures that $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism as well. The other implication is trivial by definition of subspace topology.*

The class of étale groupoids has several good features. In particular, the set of open bisections is large enough to form a basis for the topology. More precisely, the following holds.

Lemma 1.14. *Let $\mathcal{G}$ be a topological groupoid. Then the following are equivalent:*

(i) $\mathcal{G}$ is étale;

(ii) The multiplication map $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is a local homeomorphism;

(iii) The collection $\mathrm{Bis}(\mathcal{G})$ of open bisections forms a basis for the topology of $\mathcal{G}$.

Proof. $(i) \Rightarrow (ii)$: Let $(\alpha,\beta) \in \mathcal{G}^{(2)}$, i.e. $s(\alpha) = r(\beta)$. Since $\mathcal{G}$ is étale, we can choose open bisections U_α and U_β containing α and β, respectively. Then define $V := (U_\alpha \times U_\beta) \cap \mathcal{G}^{(2)}$. This is open in $\mathcal{G}^{(2)}$ and contains (α,β). We claim that the restriction of the multiplication map $m|_V : V \to m(V)$ is a homeomorphism onto its image. Indeed, since U_α and U_β are bisections, the multiplication map is injective on V. If $\alpha\beta = \gamma\delta$ with $(\alpha,\beta), (\gamma,\delta) \in V$, then

$$s(\beta) = s(\alpha\beta) = s(\gamma\delta) = s(\delta),$$

and since $s|_{U_\beta}$ is a homeomorphism onto its image, this implies $\beta = \delta$. Similarly, since $r|_{U_\alpha}$ is injective, we also get $\alpha = \gamma$.

Moreover, since composition is continuous, and V is open in $\mathcal{G}^{(2)}$ (by Lemma 1.12, point (iii)), it follows that $m(V)$ is open in $\mathcal{G}$. Thus, $m|_V$ is a homeomorphism onto an open subset of $\mathcal{G}$, i.e., m is a local homeomorphism.

$(ii) \Rightarrow (iii)$: Suppose m is a local homeomorphism. We want to show that open bisections form a basis for the topology of $\mathcal{G}$.

Let $\gamma \in \mathcal{G}$. Since m is a local homeomorphism, there exists an open neighbourhood $W \subseteq \mathcal{G}^{(2)}$ of $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ such that $m(W)$ is open in $\mathcal{G}$ and $m|_W : W \to m(W)$ is a homeomorphism.

Take an open neighbourhood $U \subseteq \mathcal{G}$ of γ such that $\mathcal{G}^{(2)} \cap (U \times U^{-1}) \subseteq W$. Let $\alpha, \beta \in U$ such that $s(\alpha) = s(\beta)$, since the multiplication is injective on W then $\alpha^{-1}\alpha = s(\alpha) = s(\beta) = \beta^{-1}\beta$ implies that $\alpha = \beta$. Similarly, we can construct such a set for the range map. Without loss of generality, we can assume that both the source and the range maps are injective on U. Additionally, the injectivity of both source and range maps, combined with the fact that multiplication is open, implies that $s(U) = U^{-1}U$ is open, as is the case for the range map. Thus U becomes a bisection of $\mathcal{G}$. So, we have found an open neighbourhood $U \ni \gamma$ which is a bisection. Thus, open bisections containing γ form a neighbourhood basis at γ. Hence, taking the collection of these neighbourhood basis for all the elements in $\mathcal{G}$, we obtain a basis for the topology.

$(iii) \Rightarrow (i)$: Assume $\mathrm{Bis}(\mathcal{G})$ is a basis for the topology. It is sufficient to show that $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. Let $\gamma \in \mathcal{G}$. By assumption, there is an open bisection $U \subseteq \mathcal{G}$ containing γ. Then, by definition, $r(U)$ is open in $\mathcal{G}^{(0)}$ and the restriction $r|_U : U \to r(U)$ is a homeomorphism. Thus, r is a local homomorphism and $\mathcal{G}$ is étale. This concludes the proof. □

We now introduce a fundamental tool in the study of locally compact Hausdorff groupoids, analogous to the concept of Haar measure for locally compact groups.

Definition 1.15. *Let $\mathcal{G}$ be a topological groupoid. A (left) Haar system on $\mathcal{G}$ is a family $(\lambda^x)_{x \in \mathcal{G}^{(0)}}$ of positive regular Borel measures λ^x on $\mathcal{G}$ such that:*

(i) the support of λ^x is $\mathcal{G}^x$ for all $x \in \mathcal{G}^{(0)}$;

(ii) for every $f \in C_c(\mathcal{G})$ the function $\lambda(f) : \mathcal{G}^{(0)} \to \mathbb{C}$ given by

$$\lambda(f)(x) = \int_{\mathcal{G}^x} f(\beta) d\lambda^x(\beta)$$

is contained in $C_c(\mathcal{G}^{(0)})$;

(iii) we have

$$\int_{\mathcal{G}^{s(\alpha)}} f(\alpha\beta) d\lambda^{s(\alpha)}(\beta) = \int_{\mathcal{G}^{r(\alpha)}} f(\beta) d\lambda^{r(\alpha)}(\beta)$$

for all $f \in C_c(\mathcal{G})$ and $\alpha \in \mathcal{G}$.

A first consequence of having a Haar system has been outlined in [Ren80, Proposition 2.4].

Lemma 1.16. *Let $\mathcal{G}$ be a topological groupoid which admits a Haar system. Then the range and the source maps are open maps.*

The following result, which can be found in [Pat99, Proposition 2.2.5], shows a further feature of étale groupoids.

Lemma 1.17. *Let $\mathcal{G}$ be an étale groupoid. For each $x \in \mathcal{G}^{(0)}$ let λ^x be the counting measure on $\mathcal{G}^x$. Then $(\lambda^x)_{x \in \mathcal{G}^{(0)}}$ is a Haar system for $\mathcal{G}$. Then for any $f \in C_c(\mathcal{G})$ we have*

$$\lambda(f)(x) = \sum_{\beta \in \mathcal{G}^x} f(\beta).$$

A subclass of étale groupoids of particular interest is given by ample groupoids.

Definition 1.18. *A topological groupoid $\mathcal{G}$ is called ample if the set*

$$\mathrm{Bis_c}(\mathcal{G}) := \{U \subseteq \mathcal{G} \mid U \textit{ is a compact open bisection}\}$$

forms a basis for the topology of $\mathcal{G}$.

Remark 1.19. *If $\mathrm{Bis_c}(\mathcal{G})$ forms a basis of compact open bisections for the topology of $\mathcal{G}$, then $\mathcal{G}$ is étale. Indeed, for any $U \in \mathrm{Bis_c}(\mathcal{G})$ the restrictions $r|_U : U \to r(U)$ and $s|_U : U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Since such bisections form a basis, r and s are local homeomorphisms on $\mathcal{G}$, hence $\mathcal{G}$ is étale.*

We now show how to characterise ample groupoids in terms of their base space. So, we need a brief discussion about totally disconnected spaces.

Definition 1.20. *A topological space X is called totally disconnected if and only if the only non-empty connected components of X are the singletons.*

Examples 1.21. *The Cantor set, the topological space of the rational numbers $\mathbb{Q}$ and the topological space of irrationals $\mathbb{R} \setminus \mathbb{Q}$, all of them endowed with the subset topology of the usual topology on $\mathbb{R}$, are totally disconnected spaces. Moreover, while the Cantor set is compact, the latter two are not even locally compact.*

The following well-known result, see for instance [Wil04, Theorem 29.7], further characterises totally disconnected spaces.

Proposition 1.22. *Let X be a locally compact Hausdorff space. Then X is totally disconnected if and only if it has a basis consisting of compact open sets.*

Proof. First, suppose that X has a basis consisting of compact open sets. Let $x \in X$ and let C_x denote the connected component of x. Suppose, for the sake of contradiction, that there exists $y \in C_x$ with $x \neq y$. Then, since the topology has a basis of compact open sets and is Hausdorff, we can find a compact open set C that contains y but not x. Thus, it follows that $C_x = (C_x \setminus C) \cup (C_x \cap C)$, where both $C_x \setminus C$ and $C_x \cap C$ are non-empty, disjoint, and relatively open in C_x. This contradicts the connectedness of C_x, so we must have $C_x = \{x\}$ for all $x \in X$, and hence X is totally disconnected.

To prove the other implication, suppose that X is totally disconnected. Let $x \in X$ and let U be an open neighbourhood of x. Since X is locally compact, there exists an open neighbourhood V of x such that the closure $\overline{V} \subseteq U$ and $\overline{V}$ is compact. Since X is totally disconnected, for each $y \in \overline{V} \setminus V$ there exists a clopen subset V_y of $\overline{V}$ such that $x \in V_y$ and $y \notin V_y$. Then each V_y is closed in X, and $\{X \setminus V_y \mid y \in \overline{V} \setminus V\}$ is an open cover of $\overline{V} \setminus V$. Since $\overline{V} \setminus V$ is compact, there exists a finite set $F \subseteq \overline{V} \setminus V$ such that $\bigcup_{y \in F} (X \setminus V_y)$ is a cover of $\overline{V} \setminus V$. Let $W = \bigcap_{y \in F} V_y$ and observe that it contains x, it is clopen in $\overline{V}$ and disjoint from $\overline{V} \setminus V$, so $W \subseteq V \subseteq U$. Thus, W is closed in the closed set $\overline{V}$ and open in the open set V, hence W is clopen in X. Since every closed subset of a compact set is compact, we conclude that W is also compact. Therefore, X has a basis consisting of compact open sets. □

Finally, we provide a link to ample groupoids. The following characterises the ample groupoids as the étale groupoids with a totally disconnected base space. The proof of this result can be found in [Bö18] while a broad discussion about the topic can be found in [Exe10].

Proposition 1.23. *Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}$ is ample if and only if $\mathcal{G}^{(0)}$ is totally disconnected.*

Proof. If $\mathcal{G}$ is ample, then it has a basis of compact open bisections. So, $\mathcal{G}^{(0)}$ being open and closed in $\mathcal{G}$, it has a basis of compact open subsets. Thus, using Proposition 1.22, we get that $\mathcal{G}^{(0)}$ is totally disconnected.

Conversely, assume that $\mathcal{G}^{(0)}$ is totally disconnected. In the spirit of the proof of the Lemma 1.14, we need to show that given $\alpha \in A \subseteq \mathcal{G}$, with A open subset of $\mathcal{G}$, there exists a compact open bisection W of $\mathcal{G}$ such that $\alpha \in W \subseteq A$. Since $\mathcal{G}$ is étale, we start with an open bisection $U \subseteq A \subseteq \mathcal{G}$ containing α. Using that $\mathcal{G}$ is locally compact, we can find a compact subset V of $\mathcal{G}$ contained in U. Then for $r(\alpha)$ we can find an open and closed set $B \subseteq \mathcal{G}^{(0)}$ contained in the set $r(V)$ and containing $r(\alpha)$. Since $r|_V$ is a homeomorphism onto its image, the set $W = r|_V^{-1}(B) \subseteq V \subseteq U \subseteq A$ is the required compact open bisection containing α. □

§ 1.2 | Examples

In this section, we now illustrate these ideas with some relevant examples of groupoids, focusing on those that are ample.

Example 1.24 (Sets). *Any set X can be viewed as a groupoid in which the only arrows are the identity arrows* id_x *for $x \in X$. If we consider a locally compact and Hausdorff topological space, we obtain a topological groupoid, which is automatically étale since the source and range maps are the identity. Moreover, if that space X is totally disconnected, then X is an ample groupoid.*

Example 1.25 (Groups). *Any group Γ can be viewed as a groupoid with just one point, so that $\mathcal{G}^{(0)} = \{\star\}$, and the arrows given by the elements of the group. The source and range maps are trivial, while the inverse and multiplication functions are exactly those of the group.*

If the group is endowed with the discrete topology, then the groupoid is étale and even ample, since the unit space consists of a single point. It is helpful to remark that, in this case, the compact open bisections are given by singletons $\{g\}_{g\in\Gamma}$.

These two examples represent opposite ends of the spectrum: in the first, the focus lies entirely on the unit space, while in the second, the morphisms carry all the structure.

Example 1.26 (Disjoint union of groups). *Let I be an index set and, for each $i \in I$, let Γ_i be a group. Define a groupoid $\mathcal{G}$ by*

$$\mathcal{G}^{(0)} = \bigsqcup_{i\in I}\{e_i\} \cong I, \qquad \mathcal{G} = \bigsqcup_{i\in I}\Gamma_i,$$

where e_i is the identity of Γ_i. The range and source maps are $r(g) = s(g) = e_i$ for $g \in \Gamma_i$, and the multiplication is the group product within each component: if $g \in \Gamma_i$ and $h \in \Gamma_j$, then gh is defined if and only if $i = j$. In other words, $\mathcal{G}$ is the disjoint union of the groups Γ_i.

Example 1.27. *Let $\mathcal{G}$ be a groupoid, and define the isotropy subgroupoid $\mathcal{G}_{ad}$ by setting*

$$\mathcal{G}_{ad}^{(0)} = \mathcal{G}^{(0)}, \qquad \mathcal{G}_{ad} = \bigsqcup_{x\in\mathcal{G}^{(0)}} \mathcal{G}_x^x.$$

Equivalently, $\mathcal{G}_{ad}$ consists of all arrows $\gamma \in \mathcal{G}$ with $r(\gamma) = s(\gamma)$.

Example 1.28 (Equivalence relation). *Let X be a set and $R \subseteq X \times X$ an equivalence relation. Define a groupoid $\mathcal{G}$ by identifying the unit space with X via $\mathcal{G}^{(0)} = \{(x,x) \in R\} \subseteq X \times X$, and setting $\mathcal{G} = R$. The groupoid operations are induced by the properties of the equivalence relation. Since R is reflexive, $(x,x) \in R$ for all $x \in X$, so $\mathcal{G}^{(0)} \subseteq \mathcal{G}$. For* *any $(y,x) \in R$, define the range and source maps by*

$$r(y,x) = y, \qquad s(y,x) = x,$$

and the inverse by $i(y,x) = (x,y)$, which is well-defined because R is symmetric. Finally, the composition is given by

$$(z,y)(y,x) = (z,x)$$

whenever $(y,x),(z,y) \in R$, and this is well-defined because R is transitive.

So far, we have seen groupoids arising from spaces and groups; now we introduce an important source of examples that combines and generalises both constructions.

Example 1.29 (Transformation groupoid)**.** *Let X be a set and let Γ be a group acting on the left on X. Define the groupoid $\mathcal{G}$ with object set $\mathcal{G}^{(0)} = \{e\} \times X$ (where $e \in \Gamma$ is the identity) and arrow set $\mathcal{G} = \Gamma \times X$. We identify $\mathcal{G}^{(0)}$ with X. For $x \in X$ and $g,h \in \Gamma$, the source and range maps are*

$$s(g,x) = x, \qquad r(g,x) = g \cdot x,$$

the inverse is $i(g,x) = (g^{-1}, g \cdot x)$, and the composition is

$$(h, g \cdot x)(g,x) = (hg, x).$$

In the literature, this groupoid is often called the transformation groupoid *and is denoted by $\Gamma \ltimes X$.*

When Γ is discrete and X is locally compact Hausdorff, $\Gamma \ltimes X$ is étale, and every set of the form $\{g\} \times U$, with $g \in \Gamma$ and $U \subseteq X$ open, is an open bisection. Moreover, if X is totally disconnected, then $\Gamma \ltimes X$ is ample.

§ 1.3 | $\mathcal{G}$-spaces

Our next aim is to define groupoid actions on sets.

Definition 1.30 (Pullback)**.** *Let X, Y, and Z be sets, and let $f : X \to Z$ and $g : Y \to Z$ be maps. The (categorical)* pullback *of X and Y with respect to f and g is the set $X \times_{f,g} Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$. If X, Y, and Z are topological spaces and f, g are continuous maps, we equip $X \times_{f,g} Y$ with the subspace topology inherited from $X \times Y$. This construction is also often called the* fibre product*.*

Example 1.31. *Let $\mathcal{G}$ be a groupoid with source and range maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$. We often use the set $\mathcal{G} \times_{s,r} \mathcal{G} = \{(\alpha,\beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$, which coincides with the space of composable arrows $\mathcal{G}^{(2)}$. We may also consider the fibre product $\mathcal{G} \times_{r,r} \mathcal{G} = \{(\alpha,\beta) \in \mathcal{G} \times \mathcal{G} \mid r(\alpha) = r(\beta)\}$.*

Lemma 1.32. *Let X, Y, and Z be topological spaces with X, Y compact and Z Hausdorff, and let $f : X \to Z$ and $g : Y \to Z$ be continuous maps. Then the fibre product $X \times_{f,g} Y$ is compact.*

Proof. Since X and Y are compact, the product $X \times Y$ is compact. Consider the map

$$(f, g) : X \times Y \to Z \times Z, \qquad (x, y) \mapsto (f(x), g(y)),$$

which is continuous. Then we have

$$X \times_{f,g} Y = (f, g)^{-1}(\Delta_Z).$$

Since Δ_Z is closed in $Z \times Z$ (because Z is Hausdorff) and (f, g) is continuous, it follows that $X \times_{f,g} Y$ is closed in $X \times Y$, and hence compact as a closed subset of a compact space. □

Definition 1.33 (Groupoid action)**.** *Let $\mathcal{G}$ be a groupoid and X a set. A left action of $\mathcal{G}$ on X consists of:*

(i) an anchor map $\pi : X \to \mathcal{G}^{(0)}$;

(ii) a map $m : \mathcal{G} \times_{s,\pi} X \to X$, denoted $m(\alpha, x) = \alpha \cdot x$,

such that, for any $(\alpha, x) \in \mathcal{G} \times_{s,\pi} X$ and $\beta \in \mathcal{G}$ with $r(\alpha) = s(\beta)$, we have $(\beta, \alpha \cdot x) \in \mathcal{G} \times_{s,\pi} X$, $\beta \cdot (\alpha \cdot x) = (\beta\alpha) \cdot x$, and $u(\pi(x)) \cdot x = x$.

A set X equipped with a $\mathcal{G}$-action is called a $\mathcal{G}$-set. If $\mathcal{G}$ is a topological groupoid and X is a topological space, we further require π and m to be continuous, and we refer to X as a $\mathcal{G}$-space. If the anchor map π is a local homeomorphism, X is called an étale $\mathcal{G}$-space.

It is interesting to observe that a $\mathcal{G}$-space is related to the notion of local symmetries, whereas the group case corresponds to global symmetries.

Lemma 1.34. *Let $\mathcal{G}$ be an étale groupoid and let X be a $\mathcal{G}$-space with anchor map $\pi : X \to \mathcal{G}^{(0)}$. For every open bisection $U \subseteq \mathcal{G}$, the map*

$$\theta_U : \pi^{-1}\big(s(U)\big) \longrightarrow \pi^{-1}\big(r(U)\big), \qquad \theta_U(x) = (s|_U)^{-1}\big(\pi(x)\big) \cdot x,$$

is a homeomorphism with inverse $\theta_{U^{-1}}$.

Proof. Fix an open bisection $U \subseteq \mathcal{G}$. Since $s|_U : U \to s(U)$ and $r|_U : U \to r(U)$ are homeomorphisms, for each $x \in \pi^{-1}(s(U))$ there is a unique

$$\alpha_x := (s|_U)^{-1}\big(\pi(x)\big) \in U$$

with $s(\alpha_x) = \pi(x)$. Since the action map is continuous, we can define

$$\theta_U(x) := \alpha_x \cdot x \in \pi^{-1}(r(U)),$$

which is continuous because $x \mapsto \alpha_x$ is the composition $x \mapsto \pi(x)$ followed by $(s|_U)^{-1}$ (and $\pi(\alpha_x \cdot x) = r(\alpha_x)$).

For $y \in \pi^{-1}(r(U))$, define

$$\theta_{U^{-1}}(y) := (s|_{U^{-1}})^{-1}\big(\pi(y)\big) \cdot y.$$

If $y = \theta_U(x) = \alpha_x \cdot x$, then $\pi(y) = r(\alpha_x)$ and $(s|_{U^{-1}})^{-1}\big(\pi(y)\big) = \alpha_x^{-1}$, hence

$$\theta_{U^{-1}}(y) = \alpha_x^{-1} \cdot (\alpha_x \cdot x) = x.$$

The converse composition is analogous, so $\theta_{U^{-1}}$ is the inverse of θ_U. Therefore θ_U is a homeomorphism. □

Example 1.35. *Every groupoid $\mathcal{G}$ acts canonically on its unit space $\mathcal{G}^{(0)}$. The anchor map is the identity $\pi = \mathrm{id}_{\mathcal{G}^{(0)}}$, and for each $\alpha \in \mathcal{G}$ the action is defined by $\alpha \cdot s(\alpha) = r(\alpha)$.*

Example 1.36. *Every groupoid $\mathcal{G}$ acts on itself by composition of arrows. The anchor map is the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ and, for $(\alpha, \beta) \in \mathcal{G}^{(2)}$, the action is given by $\alpha \cdot \beta = \alpha\beta$.*

Example 1.37. *The groupoid $\mathcal{G}$ acts on its isotropy subgroupoid $\mathcal{G}_{ad}$ by conjugation. The anchor map is the restriction of r (equivalently, of s) to $\mathcal{G}_{ad} \to \mathcal{G}^{(0)}$ and, for $(\alpha, \gamma) \in \mathcal{G} \times_{s,r} \mathcal{G}_{ad}$, the action is given by $\alpha \cdot \gamma = \alpha\gamma\alpha^{-1}$.*

Example 1.38. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. The fibre product $\mathcal{G} \times_{s,r} \mathcal{G}$ is a $\mathcal{G}$-space, where the anchor map is $r\,\mathrm{pr}_1$ and, for $(\gamma, \alpha, \beta) \in \mathcal{G} \times_{s,r} \mathcal{G} \times_{s,r} \mathcal{G}$, the action is given by left multiplication on the first component: $\gamma \cdot (\alpha, \beta) = (\gamma\alpha, \beta)$. Similarly, $\mathcal{G} \times_{r,r} \mathcal{G}$ is a $\mathcal{G}$-space with anchor map $r\,\mathrm{pr}_1$ and action given by diagonal left multiplication: $\gamma \cdot (\alpha, \beta) = (\gamma\alpha, \gamma\beta)$.*

In analogy with group actions, we define the orbit space of a given $\mathcal{G}$-set.

Definition 1.39. *Let $\mathcal{G}$ be a groupoid and X a left $\mathcal{G}$-space with anchor map π. We define the space $\mathcal{G}\backslash X$ as the quotient $X/\sim$, where the equivalence relation is given by*

$$x \sim y \iff \exists\, \alpha \in \mathcal{G} \text{ with } \pi(x) = s(\alpha) \text{ and } y = \alpha \cdot x.$$

In the topological setting, when X is a $\mathcal{G}$-space, the orbit space $\mathcal{G}\backslash X$ is endowed with the quotient topology induced by the action. This topology need not be well-behaved: in particular, $\mathcal{G}\backslash X$ is not necessarily Hausdorff even if X is Hausdorff.

The following lemma outlines key features of the orbit space. A proof can be found in

[Tu04, Lemma 2.30].

Lemma 1.40. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. The range and source maps of $\mathcal{G}$ are open if and only if, for every $\mathcal{G}$-space X, the quotient map $X \to \mathcal{G}\backslash X$ is open. In that case, if X is locally compact, then $\mathcal{G}\backslash X$ is locally compact.*

§ 1.4 | Algebras and multipliers

Throughout this work, by an *algebra* we mean a (not necessarily unital) associative algebra over the complex numbers. We will mostly work with algebras A that are *essential*, in the sense that the multiplication map induces an isomorphism $A \otimes_A A \cong A$. Moreover, we shall focus on algebras with *nondegenerate* multiplication, meaning that $ab = 0$ for all $a \in A$ implies $b = 0$, and similarly, $ab = 0$ for all $b \in A$ implies $a = 0$. Note that every unital algebra satisfies both properties: it is essential and has nondegenerate multiplication.

The algebraic *multiplier algebra* $M(A)$ of an algebra A consists of all *two-sided multipliers* (L, R), as discussed for instance in [VD94, Appendix]. A two-sided multiplier is a pair where $L : A \to A$ is a right A-linear map (a left multiplier), and $R : A \to A$ is a left A-linear map (a right multiplier), such that for all $a, b \in A$ we have the compatibility condition

$$R(a)b = aL(b).$$

The vector space $M(A)$ forms a unital algebra under composition of maps, with unit given by the pair $(\mathrm{id}, \mathrm{id})$. If the multiplication of A is nondegenerate, we will write ab for $a_1(b)$ and ba for $a_2(b)$ when $a = (a_1, b_2) \in M(A)$. There is a canonical homomorphism $\iota : A \to M(A)$ defined by sending $a \in A$ to the multiplier (L_a, R_a), where $L_a(b) = ab$ and $R_a(b) = ba$. When A has nondegenerate multiplication, this map is injective, and we identify A with its image in $M(A)$.

A (left) A-module M is said to be *essential* if the canonical map $A \otimes_A M \to M$ induced by the module structure is an isomorphism. Note that in this case AM, the linear span of all elements $a \cdot m$ for $a \in A$ and $m \in M$, equals M. An algebra homomorphism $f : A \to M(B)$ is said to be *essential* if B is spanned by elements $f(a)b$ as a left A-module and by elements $bf(a)$ as a right A-module. When A is unital, these conditions reduce to the familiar notions: M is essential if and only if it is unital in the usual sense $(1 \cdot m = m)$, and f is essential if and only if it is a unital algebra homomorphism.

The following lemma, compare [VD94, Proposition A.5], ensures that an essential algebra homomorphism between algebras with nondegenerate multiplication extends uniquely to their multiplier algebras.

Lemma 1.41. *Let $f : A \to M(B)$ be an essential algebra homomorphism. If the mul-*

tiplication in B is nondegenerate there exists a unique unital algebra homomorphism $F : M(A) \to M(B)$ such that $F\iota = f$.

Proof. Suppose $F : M(A) \to M(B)$ is an extension of f. Then necessarily

$$F(c)(f(a)b) = f(ca)b$$

for all $c \in M(A)$, $a \in A$ and $b \in B$. Since the elements of the form $f(a)b$ span B by essentiality of f, this condition determines F uniquely.

We now define F on the spanning set by

$$F(c)(f(a)b) := f(ca)b$$

for $c \in M(A)$, $a \in A$, and $b \in B$ and extend linearly.

To ensure that $F(c)$ is well-defined on all of B, we must verify that this definition is independent of the representation of an element in B as a finite sum $\sum_i f(a_i)b_i$. That is, we must show that $\sum_i f(a_i)b_i = 0$ implies $\sum_i f(ca_i)b_i = 0$ for all $c \in M(A)$.

So, suppose $\sum_i f(a_i)b_i = 0$. For arbitrary $c \in M(A)$, $d \in A$, and $e \in B$, we compute

$$ef(d)\sum_i f(ca_i)b_i = e\sum_i f(dca_i)b_i = ef(dc)\sum_i f(a_i)b_i = 0.$$

Since the elements $ef(d)$ span B by essentiality of f, it follows that

$$\sum_i f(ca_i)b_i = 0.$$

This shows that the definition of $F(c)$ is well-defined on B, and therefore defines a linear map $F(c) : B \to B$.

Let $c_1, c_2 \in M(A)$. Then for all $a \in A$, $b \in B$, we compute

$$\begin{aligned} F(c_1c_2)(f(a)b) &= f(c_1c_2a)b \\ &= F(c_1)(f(c_2a)b) \\ &= F(c_1)F(c_2). \end{aligned}$$

Hence, $F(c_1c_2) = F(c_1)F(c_2)$, so F is multiplicative.

Finally, to check that F extends f, take $a \in A$. Then for all $a \in A$ and $b \in B$,

$$F(\iota(a))f(c)b = f(\iota(a)c)b = f(ac)b = f(a)f(c)b,$$

which shows $F(\iota(a)) = f(a)$. $\square$

We say that an algebra A has local units if for every finite set of elements $a_1, \dots, a_n$ of A there exists $e \in A$ such that $ea_i = a_i = a_i e$ for all i. The multiplication in such an algebra is clearly nondegenerate. We record the following well-known fact.

Lemma 1.42. *Let A be an algebra with local units. Then a left A-module M is essential if and only if $AM = M$. An analogous statement holds for right modules.*

Proof. We prove the statement for left A-modules.

Assume first that M is essential. Then any $m \in M$ lies in the image of the canonical map $\varphi : A \otimes_A M \to M$, so there exist finitely many $a_i \in A$ and $m_i \in M$ such that $m = \varphi(\sum_i a_i \otimes m_i) = \sum_i a_i \cdot m_i$.

Conversely, suppose $AM = M$, we need to prove that the canonical map $\varphi : A \otimes_A M \to M$ is an isomorphism. Let $\sum_i a_i \otimes m_i \in A \otimes_A M$ be in the kernel of the canonical map φ, i.e. $\sum_i a_i \cdot m_i = 0$. Since A has local units, there exists $e \in A$ such that $ea_i = a_i$ for all i. Then

$$\sum_i a_i \otimes m_i = \sum_i ea_i \otimes m_i = \sum_i e \otimes a_i \cdot m_i = \sum_i e \otimes 0 = 0,$$

so the kernel of φ is trivial, and hence φ is injective. Since we assume $AM = M$, for any $m \in M$ there exist finitely many $a_i \in A$ and $m_i \in M$ such that $m = \sum_i a_i \cdot m_i = \varphi(\sum_i a_i \otimes m_i)$, so it is also surjective, hence an isomorphism. $\square$

Remark 1.43. *Given an essential A-module M over an algebra with local units, we observe that any element $m \in M$ can be written as*

$$m = \sum_i a_i \cdot m_i = \sum_i ea_i \cdot m_i = \sum_i e \cdot (a_i \cdot m_i) = e \cdot m,$$

for finitely many $a_i \in A$, $m_i \in M$ and $e \in A$ such that $ea_i = a_i$ for all i.

Remark 1.44. *Observe that Lemma 1.42 implies, in particular, that an algebra with local units is essential.*

§ 1.5 | Convolution algebra of an ample groupoid

A fundamental step in this chapter is the construction of a function algebra associated with the ample groupoid $\mathcal{G}$. Specifically, we focus on the algebra of compactly supported, *locally constant* functions on $\mathcal{G}$.

§ 1.5.1 | Functions on a totally disconnected space

The primary motivation for this section is that ample groupoids are totally disconnected spaces, as we have seen previously, combining Definition 1.18 and Proposition 1.22.

Let us begin by recalling what we mean by a locally constant function.

Definition 1.45. *Let X be a topological space and Y a set. A function $f : X \to Y$ is said to be* locally constant *if for every $x \in X$ there exists an open neighbourhood $U_x \subseteq X$ of x such that $f(U_x) = \{f(x)\}$.*

Lemma 1.46. *Let X be a topological space and Y a set. A function $f : X \to Y$ is locally constant in the sense of Definition 1.45 if and only if f is continuous when Y is equipped with the discrete topology.*

Proof. Assume f is locally constant. Let $V \subseteq Y$ be any subset, which is automatically open in the discrete topology. For each $x \in f^{-1}(V)$, by local constancy there is an open set $U_x \ni x$ with $f(U_x) = \{f(x)\} \subseteq V$, hence $U_x \subseteq f^{-1}(V)$. Therefore

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

is open in X. Since this holds for every $V \subseteq Y$, f is continuous.

Conversely, assume f is continuous for the discrete topology on Y. Fix $x \in X$. Then the singleton $\{f(x)\}$ is open in Y, so

$$U_x := f^{-1}\big(\{f(x)\}\big)$$

is an open neighbourhood of x and $f(U_x) = \{f(x)\}$. Hence f is locally constant. □

In what follows, by a locally compact space we always mean a locally compact Hausdorff space.

Definition 1.47. *Let X be a locally compact space. Define $C_c^\infty(X)$ as the space of all locally constant functions $X \to \mathbb{C}$ with compact support.*

Remark 1.48. *For certain topological spaces, the previous definition may yield a trivial space of functions. For instance, when considering the real line $\mathbb{R}$ with the usual topology, the only locally constant function with compact support is the zero function. This is due to the fact that $\mathbb{R}$ has very few clopen subsets.*

More generally, the notion of locally constant functions is intimately connected to the abundance of clopen sets: the richer the collection of clopen sets in a space, the more non-trivial locally constant functions it admits.

Throughout the remainder of this section, we will restrict our attention to topological spaces that are both totally disconnected and locally compact, since these provide a natural setting in which the space of compactly supported, locally constant functions is rich and well-behaved.

A good description of these functions, in the totally disconnected case, can be given by using compact open subsets. This is made precise in the following lemma.

Lemma 1.49. *Let X be a totally disconnected locally compact space. Then every element $f \in C_c^\infty(X)$ can be written as a linear combination*

$$f = \sum_{k=1}^{n} c_k \, \chi_{U_k}$$

for a finite family of pairwise disjoint compact open subsets $U_k \subseteq X$ and coefficients $c_k \in \mathbb{C}$.

Proof. Since f is locally constant and $\operatorname{supp}(f)$ is compact, the image $f(X)$ is a finite subset of $\mathbb{C}$. If we denote by $c_1, \dots, c_n$ the nonzero elements of $f(X)$ and set $U_k = f^{-1}(c_k)$, then each $U_k \subseteq X$ is open and closed, being the preimage of a point in the discrete topology. Moreover, each U_k is compact since it is closed in $\operatorname{supp}(f)$, which is compact. The sets $U_1, \dots, U_n$ are pairwise disjoint, and we have $f = \sum_{k=1}^{n} c_k \, \chi_{U_k}$. □

Lemma 1.50. *The vector space $C_c^\infty(X)$ becomes naturally a commutative algebra with the pointwise multiplication. Moreover, it is an essential algebra with local units.*

Proof. Let $f, g \in C_c^\infty(X)$. Then fg is again locally constant with compact support. Indeed, for any $x \in X$, there exist open neighbourhoods $U_x, V_x \subseteq X$ such that f and g are respectively constant on them. Hence fg is constant on the open set $U_x \cap V_x$, so it is locally constant. The support of fg satisfies

$$\operatorname{supp}(fg) = \operatorname{supp}(f) \cap \operatorname{supp}(g),$$

which is compact as the intersection of two compact sets. Therefore $fg = gf \in C_c^\infty(X)$.

To show that $C_c^\infty(X)$ has local units, take $f_1, \dots, f_n \in C_c^\infty(X)$. By Lemma 1.49, we can write each f_i as a finite linear combination of characteristic functions of compact open sets. Let U be the finite union of all those compact open sets that appear in these decompositions; then U is compact open, and $e := \chi_U$ satisfies $ef_i = f_i = f_i e$ for all i. Hence $C_c^\infty(X)$ has local units. Finally, by Lemma 1.42, the algebra $C_c^\infty(X)$ is essential. □

We will write $C^\infty(X)$ for the algebra of all locally constant functions $f : X \to \mathbb{C}$.

Lemma 1.51. *Let X be a totally disconnected, locally compact Hausdorff space. The algebraic multiplier algebra $M(C_c^\infty(X))$ can be canonically identified with $C^\infty(X)$, the algebra of all locally constant functions on X.*

Proof. Let $m \in C^\infty(X)$ and $f, g \in C_c^\infty(X)$. Then, arguing as in the proof of Lemma 1.50, we have mf, $fm \in C_c^\infty(X)$. Define a pair (L_m, R_m) by $L_m(f) := mf$ and $R_m(f) := fm$. Moreover, we check that $L_m(fg) = L_m(f)g$, $R_m(fg) = fR_m(g)$, and $R_m(f)g = fL_m(g)$, so $(L_m, R_m) \in M(C_c^\infty(X))$.

Let $(L, R) \in M(C_c^\infty(X))$. Construct a function $m_{L,R} : X \to \mathbb{C}$ as follows: for any $x \in X$, choose a compact open subset $U \subseteq X$ containing x and set

$$m_{L,R}(x) := L(\chi_U)(x).$$

This does not depend on the choice of U nor on using L instead of R. Indeed, for a second compact open $V \ni x$, using commutativity in $C_c^\infty(X)$ and the right A-linearity of L we have

$$\chi_U L(\chi_V) = L(\chi_V)\chi_U = L(\chi_V \chi_U) = L(\chi_U)\chi_V = \chi_V L(\chi_U),$$

and by the multiplier identity we get

$$R(\chi_U)\chi_V = \chi_U L(\chi_V) = \chi_V L(\chi_U).$$

Restricting to $U \cap V$ yields

$$L(\chi_V)|_{U\cap V} = L(\chi_U)|_{U\cap V} = R(\chi_V)|_{U\cap V} = R(\chi_U)|_{U\cap V},$$

so $m_{L,R}$ is well-defined. Moreover, for each compact open U we have $m_{L,R}|_U = L(\chi_U)|_U \in C_c^\infty(X)$, hence $m_{L,R}$ is locally constant, i.e. $m_{L,R} \in C^\infty(X)$.

These two constructions are inverse to each other. Starting with $m \in C^\infty(X)$, for any $x \in X$ and compact open subset $U \ni x$, we get

$$m_{L_m,R_m}(x) = L_m(\chi_U)(x) = (m\chi_U)(x) = m(x).$$

We now show that $L = L_{m_{L,R}}$. Let $g \in C_c^\infty(X)$ and choose $V \subseteq X$ compact open with $g = \chi_V g$ (by Lemma 1.50). Then, for all $x \in X$,

$$L(g)(x) = L(\chi_V g)(x) = L(\chi_V)(x)\, g(x) = m_{L,R}(x)\, g(x) = L_{m_{L,R}}(g)(x).$$

Using the multiplier identity, the same argument shows $R = R_{m_{L,R}}$. This concludes the proof. ∎

The following definition will be used in several further discussions.

Definition 1.52. *Let X, Y be locally compact spaces. A continuous map $\varphi : X \to Y$ is proper if and only if $\varphi^{-1}(K)$ is compact for every compact subset $K \subseteq Y$.*

Lemma 1.53. *Let X and Y be totally disconnected locally compact spaces and let $\varphi : X \to Y$ be a continuous map. Then $\varphi^* : C_c^\infty(Y) \to C^\infty(X) = M(C_c^\infty(X))$, $\varphi^*(f) = f\varphi$ is a well-defined essential algebra homomorphism. If φ is proper then $\varphi^*(C_c^\infty(Y))$ is contained in $C_c^\infty(X)$.*

Proof. For $f \in C_c^\infty(Y)$ the function $\varphi^*(f) = f\varphi$ is locally constant since it is the composition of a continuous function and a locally constant function, hence a continuous function to $\mathbb{C}$ endowed with the discrete topology. It follows that $\varphi^* : C_c^\infty(Y) \to C^\infty(X) = M(C_c^\infty(X))$ is well-defined. Moreover, this map is clearly an algebra homomorphism.

We show that φ^* is an essential algebra homomorphism. Let $f \in C_c^\infty(X)$ and observe that $\varphi(\mathrm{supp}(f))$ is compact since $\mathrm{supp}(f)$ is compact and φ is continuous. We can cover $\varphi(\mathrm{supp}(f))$ by finitely many compact open subsets of Y, and if χ denotes the characteristic function of the union of these sets, then $f = \varphi^*(\chi)f$ is contained in $\varphi^*(C_c^\infty(Y))C_c^\infty(X)$.

Finally, assume that φ is proper. Let $g \in C_c^\infty(Y)$ and $K = \mathrm{supp}(g)$, which is compact open in Y. Then the preimage $\varphi^{-1}(K)$ is again compact open in X. If we write $e \in C_c^\infty(X)$ for the characteristic function of $\varphi^{-1}(K)$ then we get $\varphi^*(g) = \varphi^*(g)e = e\varphi^*(g)$ and since φ^* is essential, $\varphi^*(g)$ belongs to $C_c^\infty(X)$ as required. □

Proposition 1.54. *Let X and Y be totally disconnected locally compact spaces. Then the canonical linear map*

$$\gamma : C_c^\infty(X) \otimes C_c^\infty(Y) \to C_c^\infty(X \times Y),$$

given by $\gamma(f \otimes g)(x, y) = f(x)g(y)$, is an isomorphism.

Proof. Assume $F = \sum_i f_i \otimes g_i \in C_c^\infty(X) \otimes C_c^\infty(Y)$ satisfies $\gamma(F) = 0$. By Lemma 1.49 we can write each f_i as a linear combination of characteristic functions $\chi_{U_{ij}}$ for mutually disjoint compact open subsets $U_{ij} \subseteq X$, and similarly each g_i as a linear combination of characteristic functions $\chi_{V_{ik}}$ for mutually disjoint compact open subsets of Y. Upon taking intersections of these subsets, it follows that F can be written in the form $F = \sum_k c_k \chi_{U_k} \otimes \chi_{V_k}$, where $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ are mutually disjoint compact open subsets of X and Y, respectively. Without loss of generality, we may assume that these sets are all non-empty. For every index k pick $(x_k, y_k) \in U_k \times V_k$. Then the relation

$$0 = \gamma(F)(x_k, y_k) = c_k \chi_{U_k}(x_k)\chi_{V_k}(y_k) = c_k$$

gives $c_k = 0$. Hence $F = 0$, and it follows that γ is injective.

To show surjectivity, it suffices to verify that the characteristic function χ_W of an arbitrary compact open subset $W \subseteq X \times Y$ is contained in the image of γ. For this it is enough to write W as a disjoint union of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are compact open. In order to obtain such a decomposition of W, note first that since X and Y are totally disconnected and locally compact they both have a basis for their topology made up of compact open sets. In particular, for every point $w = (x, y) \in W$ we find compact open neighbourhoods $U_w \subseteq X$ of x and $V_w \subseteq Y$ of y such that the rectangle $R_w = U_w \times V_w$ is contained in W. Since W is compact we obtain a finite cover of W by rectangles $R_{w_1}, \dots, R_{w_n}$ for some $w_1, \dots, w_n \in W$. Upon taking intersections of the compact open sets U_{w_i} and V_{w_i} making up the rectangles R_{w_i}, we can refine this to a finite cover of W consisting of mutually disjoint compact open rectangles as required. □

Lemma 1.55. *Let X be a totally disconnected locally compact space and let $K \subseteq X$ be a closed subset. Then the canonical restriction map $C_c^\infty(X) \to C_c^\infty(K)$, mapping f to $f|_K$, is surjective.*

Proof. For any given $f \in C_c^\infty(K)$ we have to construct a function $F \in C_c^\infty(X)$ such that $F|_K = f$. Since every element of $C_c^\infty(K)$ is a linear combination of characteristic functions it suffices to consider the case that $f = \chi_U$ for some compact open set $U \subseteq K$. Observe that U is compact in a closed subset, so is again compact in X, and since it is open, there exists an open set $V \subseteq X$ such that $V \cap K = U$. Using that V is open and X is totally disconnected we can write V as a union of compact open subsets of X. Since we have $U \subseteq V$, these sets are in particular an open cover of the compact set U. This means that we can find finitely many compact open subsets $W_1, \dots, W_n \subseteq X$ such that $W_i \subseteq V$ for all i and the union W of the W_i satisfies $W \cap K = U$. It follows that the function $F = \chi_W$ has the desired properties. □

Let X, Y, Z be totally disconnected locally compact spaces and let $p : X \to Z$, $q : Y \to Z$ be continuous maps. The groups $C_c^\infty(X)$ and $C_c^\infty(Y)$ become essential $C_c^\infty(Z)$-modules via the pullback algebra homomorphisms p^*, q^* as seen in Lemma 1.53 and the pointwise multiplication.

Definition 1.56 (Balanced tensor product)**.** *Let X, Y, Z be totally disconnected locally compact spaces and let $p : X \to Z$, $q : Y \to Z$ be continuous maps. The* balanced tensor product *of $C_c^\infty(X)$ and $C_c^\infty(Y)$ over $C_c^\infty(Z)$ with respect to p, q is the quotient*

$$C_c^\infty(X) \overset{p,q}{\otimes} C_c^\infty(Y) := \big(C_c^\infty(X) \otimes C_c^\infty(Y)\big)\big/R,$$

where R is the linear subspace spanned by all elements of the form

$$f\,p^*(h) \otimes g \;-\; f \otimes q^*(h)\,g$$

for $f \in C_c^\infty(X)$, $g \in C_c^\infty(Y)$ and $h \in C_c^\infty(Z)$.

Example 1.57. *Let $\mathcal{G}$ be an ample groupoid. Since $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ are continuous maps between totally disconnected locally compact spaces, the pullbacks s^*, r^* endow $C_c^\infty(\mathcal{G})$ with essential $C_c^\infty(\mathcal{G}^{(0)})$-module structures. We will often consider the balanced tensor products induced by the source and range maps:*

$$C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G}) \quad \textit{and} \quad C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}).$$

Definition 1.58. *Let X, Z be totally disconnected locally compact spaces, let $p : X \to Z$ be a continuous maps and let M be an essential left $C_c^\infty(Z)$-module. We define*

$$C_c^\infty(X) \overset{p,\mathrm{id}}{\otimes} M := \big(C_c^\infty(X) \otimes M\big)/R$$

where R is the linear subspace spanned by all elements of the form

$$f\,p^*(h) \otimes m - f \otimes h \cdot m,$$

for $f \in C_c^\infty(X)$, $m \in M$, $h \in C_c^\infty(Z)$.

Proposition 1.59. *Let X, Y, Z be totally disconnected locally compact spaces and let $p : X \to Z, q : Y \to Z$ be continuous maps. Then the canonical $C_c^\infty(Z)$-linear map*

$$C_c^\infty(X) \overset{p,q}{\otimes} C_c^\infty(Y) \to C_c^\infty(X \times_{p,q} Y)$$

is an isomorphism.

Proof. It is straightforward to check that the composition of the canonical homomorphism $\gamma : C_c^\infty(X) \otimes C_c^\infty(Y) \to C_c^\infty(X \times Y)$ with the restriction homomorphism $C_c^\infty(X \times Y) \to C_c^\infty(X \times_{p,q} Y)$ factorises through $C_c^\infty(X) \overset{p,q}{\otimes} C_c^\infty(Y)$. We shall write $\gamma_{p,q}$ for the resulting $C_c^\infty(Z)$-linear map $C_c^\infty(X) \overset{p,q}{\otimes} C_c^\infty(Y) \to C_c^\infty(X \times_{p,q} Y)$. Due to Proposition 1.54 and Lemma 1.55 the map $\gamma_{p,q}$ is surjective, and it remains only to show that $\gamma_{p,q}$ is injective.

Assume that $F \in C_c^\infty(X) \overset{p,q}{\otimes} C_c^\infty(Y)$ satisfies $\gamma_{p,q}(F) = 0$. As explained in the proof of Proposition 1.54, we can represent F as a linear combination $F = \sum_k c_k \chi_{U_k} \otimes \chi_{V_k}$ where $U_1, \dots, U_n$ and $V_1, \dots, V_n$ are mutually disjoint compact open subsets of X and Y, respectively. If there exists an index k and points $x \in U_k, y \in V_k$ such that $p(x) = q(y)$ then $(x, y) \in X \times_{p,q} Y$ and $c_k = c_k \chi_{U_k}(x) \chi_{V_k}(y) = \gamma_{p,q}(F)(x, y) = 0$. Therefore we may assume without loss of generality that $p(U_k) \cap q(V_k) = \varnothing$ for all k. Using that Z is locally compact and hence regular we can then find compact open sets $E_k \subseteq Z$ such that $p(U_k) \subseteq E_k$ and $E_k \cap q(V_k) = \varnothing$ for all k. It follows that $e_k = \chi_{E_k} \in C_c^\infty(Z)$ satisfies $\chi_{U_k} \cdot e_k = \chi_{U_k}$ and $e_k \cdot \chi_{V_k} = 0$ for all k. Hence we conclude

$$F = \sum_k c_k \chi_{U_k} \otimes \chi_{V_k} = \sum_k c_k \chi_{U_k} \cdot e_k \otimes \chi_{V_k} - c_k \chi_{U_k} \otimes e_k \cdot \chi_{V_k} = 0$$

as required. □

§ 1.5.2 | Convolution algebra of an ample groupoid

Following the discussion about function algebras on totally disconnected spaces, we are now ready to introduce the main concept of this section, which serves as a cornerstone for the rest of this work.

Definition 1.60 (Convolution algebra)**.** *Let $\mathcal{G}$ be an ample groupoid. Define the vector space*

$$C_c^\infty(\mathcal{G}) := \{ f : \mathcal{G} \to \mathbb{C} \mid f \text{ is locally constant and compactly supported} \},$$

which becomes an algebra with the convolution defined for any $f, g \in C_c^\infty(\mathcal{G})$ and $\alpha \in \mathcal{G}$ as

$$(f * g)(\alpha) = \sum_{\beta \in \mathcal{G}^{r(\alpha)}} f(\beta) g(\beta^{-1}\alpha) = \sum_{\gamma \in \mathcal{G}_{s(\alpha)}} f(\alpha\gamma^{-1}) g(\gamma)$$

Since $C_c^\infty(\mathcal{G})$ is also an algebra with the pointwise multiplication to mark the difference when we refer to the convolution product, we denote this convolution algebra by $\mathcal{D}(\mathcal{G})$.

Remark 1.61. *In the definition of convolution, the sums are finite: for fixed $\alpha \in \mathcal{G}$, the sets $\operatorname{supp}(f) \cap \mathcal{G}^{r(\alpha)}$ and $\operatorname{supp}(g) \cap \mathcal{G}_{s(\alpha)}$ are finite since $\mathcal{G}$ is étale and f, g have compact support. Hence the convolution is well-defined. Moreover, the space $C_c^\infty(\mathcal{G})$ is closed under convolution. If $(f * g)(\alpha) \neq 0$, then $\alpha = \beta\gamma$ for some $\beta \in \operatorname{supp}(f)$ and $\gamma \in \operatorname{supp}(g)$ with $s(\beta) = r(\gamma)$, hence $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f)\operatorname{supp}(g)$. The latter being compact since the product map $\mathcal{G}^{(2)} \to \mathcal{G}$ is continuous. Local constancy of $f * g$ follows from the fact that f and g are locally constant. Finally, associativity and bilinearity of the convolution follow straightforwardly. Hence $C_c^\infty(\mathcal{G})$ is an associative algebra under convolution.*

In view of the discussion in [Wil07, Section 1.5.1], we endow the algebra $C_c(\mathcal{G})$ with the inductive limit topology. We write

$$C_c(\mathcal{G}) = \bigcup_{\substack{K \subseteq \mathcal{G} \\ \text{compact} \\ \text{open}}} C_K,$$

where $C_K := \{ f : \mathcal{G} \to \mathbb{C} \mid f \text{ continuous and } \operatorname{supp}(f) \subseteq K \}$, equipped with the uniform topology. This forms a directed system of topological vector spaces under inclusions

$K \subseteq L$ and extensions by zero $g_{KL} : C_K \to C_L$. We then define the inductive limit of the previous direct system of topological vector spaces as

$$\varinjlim C_K = \bigoplus_{\substack{K \subseteq \mathcal{G}\, compact \\ open}} C_K / D,$$

with the linear maps $t_K : C_K \to \varinjlim C_K$, where D is the vector space generated by $\{t_K(x) - (t_L g_{LK})(x) \mid x \in C_K \text{ and } K \subseteq L\}$, and endowed with the direct limit topology, that is the finest topology such that all the linear maps $t_K : C_K \to \varinjlim C_K$ are continuous.

The construction is summarised in the following diagram

$$\begin{array}{ccc} C_K & \xrightarrow{g_{KL}} & C_L \\ & \underset{t_K}{\searrow} & \downarrow{\scriptstyle t_L} \\ & & \varinjlim C_K. \end{array}$$

One can show that $(C_c(\mathcal{G}), \{s_K\})$, with $s_K : C_K \to C_c(\mathcal{G})$ being the extension by zero, is the inductive limit of this system.

We want to find a dense subalgebra of $C_c(\mathcal{G})$.

Proposition 1.62. *Let $\mathcal{G}$ be an ample groupoid. Then $\mathcal{D}(\mathcal{G})$ is a dense subalgebra of $C_c(\mathcal{G})$. Moreover,*

$$\mathcal{D}(\mathcal{G}) = \operatorname{span}\{\chi_U : \mathcal{G} \to \mathbb{C} \mid U \in \operatorname{Bis_c}(\mathcal{G})\}.$$

Proof. We first show that $\mathcal{D}(\mathcal{G}) = \operatorname{span}\{\chi_U : \mathcal{G} \to \mathbb{C} \mid U \in \operatorname{Bis_c}(\mathcal{G})\}$. The inclusion $\supseteq$ is obvious. For the reverse, let $f \in C_c^\infty(\mathcal{G})$. The same argument used in Lemma 1.49 shows that $f = \sum_i c_i \chi_{V_i}$, where $c_i \in \mathbb{C}$ and $V_i \subseteq \mathcal{G}$ are compact open. Since compact open bisections form a basis for the topology of $\mathcal{G}$, and each V_i is compact open, we can rewrite the sum using finitely many disjoint compact open bisections. Then we can write $f = \sum_j c_j \chi_{U_j}$, where $c_j \in \mathbb{C}$ and $U_j \in \operatorname{Bis_c}(\mathcal{G})$ for every index j.

Next we check closure under convolution using compact open bisections. If $U, V \in \operatorname{Bis_c}(\mathcal{G})$, then

$$\chi_U * \chi_V(t) = \sum_{\substack{(h,k) \in \mathcal{G}^{(2)} \\ hk = t}} \chi_U(h) \chi_V(k) = \chi_{UV}(t).$$

If $t \in UV$, there exists a unique pair (h, k) with $h \in U$, $k \in V$, and $hk = t$, so the sum is 1; if $t \notin UV$, the sum vanishes. The convolution is associative, as it is inherited from $C_c(\mathcal{G})$.

For the density, fix $f \in C_c(\mathcal{G})$ with support in a compact open set K and a real number $\epsilon > 0$. Since $\mathcal{G}$ is ample, K is totally disconnected, hence has a basis of clopen sets (Proposition 1.22). Cover $f(K)$ in $\mathbb{C}$ with finitely many open balls of radius less than

$\epsilon > 0$. Pulling back via f gives a finite open cover of K that can be refined to a finite clopen partition $\{Z_i\}$ of K. Define $\tilde{f}(x) := f(x_i)$ if $x \in Z_i$ for some $x_i \in Z_i$. Then $\tilde{f}$ is locally constant, has compact support, and $\|f - \tilde{f}\|_\infty < \epsilon$. □

We now gather some important structural results about $\mathcal{D}(\mathcal{G})$. Many of these can be found in [Ste10].

Lemma 1.63. *The algebra $\mathcal{D}(\mathcal{G})$ is unital if and only if $\mathcal{G}^{(0)}$ is compact.*

Proof. If $\mathcal{G}^{(0)}$ is compact, and it is also open by Lemma 1.12, then $\chi_{\mathcal{G}^{(0)}} \in \mathcal{D}(\mathcal{G})$. For $f \in C_c^\infty(\mathcal{G})$ and $\alpha \in \mathcal{G}$

$$(f * \chi_{\mathcal{G}^{(0)}})(\alpha) = \sum_{\substack{(\beta,\gamma)\in\mathcal{G}^{(2)} \\ \beta\gamma=\alpha}} f(\beta)\,\chi_{\mathcal{G}^{(0)}}(\gamma) = f(\alpha),$$

because $\chi_{\mathcal{G}^{(0)}}(\gamma) \neq 0$ only if $\gamma = s(\beta)$, hence $\beta = \alpha$. Similarly, we can prove that it is an identity on the left.

Conversely, if $\mathcal{D}(\mathcal{G})$ has a unit e, we show $e = \chi_{\mathcal{G}^{(0)}}$. If $\alpha \in \mathcal{G} \setminus \mathcal{G}^{(0)}$, then for a compact open set $U \subseteq \mathcal{G}^{(0)}$ with $s(\alpha) \in U$, we compute

$$0 = \chi_U(\alpha) = (e * \chi_U)(\alpha) = \sum_{\substack{(\beta,\gamma)\in\mathcal{G}^{(2)} \\ \beta\gamma=\alpha}} e(\beta)\,\chi_U(\gamma) = e(\alpha),$$

Similarly, for $\alpha \in \mathcal{G}^{(0)}$, $e(\alpha) = 1$. Hence $e = \chi_{\mathcal{G}^{(0)}}$ and $\mathcal{G}^{(0)}$ is compact. □

Lemma 1.64. *Let $\mathcal{G}$ be an ample groupoid. Then the extension-by-zero map $\phi : C_c^\infty(\mathcal{G}^{(0)}) \to \mathcal{D}(\mathcal{G})$ is a well-defined injective homomorphism of algebras.*

Proof. Since $\mathcal{G}^{(0)}$ is clopen in $\mathcal{G}$, from Lemmas 1.6 and 1.12, the extension $\phi(f)$ of any $f \in C_c^\infty(\mathcal{G}^{(0)})$ by zero is locally constant and compactly supported in $\mathcal{G}$. In fact, the support of $\phi(f)$ equals the support of f, and $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$. It is locally constant over $\mathcal{G}$ since, for every point $x \in \mathcal{G}^{(0)}$, there exists an open neighbourhood of x in $\mathcal{G}^{(0)}$ on which f is constant, and opens in $\mathcal{G}^{(0)}$ are open in $\mathcal{G}$ because $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. If $x \in \mathcal{G} \setminus \mathcal{G}^{(0)}$, then $\mathcal{G} \setminus \mathcal{G}^{(0)}$ is an open neighbourhood of x on which $\phi(f)$ vanishes. Moreover, convolution is preserved under extension because outside $\mathcal{G}^{(0)}$ all terms vanish and on $\mathcal{G}^{(0)}$ convolution reduces to pointwise multiplication. Hence ϕ is a well-defined homomorphism, which is injective since $\phi(f) = 0$ implies $f = 0$. □

Corollary 1.65. *The algebra $C_c^\infty(\mathcal{G}^{(0)})$, viewed as a subalgebra of $\mathcal{D}(\mathcal{G})$, is abelian.*

Proof. The result follows from the homomorphism $\phi : C_c^\infty(\mathcal{G}^{(0)}) \to \mathcal{D}(\mathcal{G})$ in Lemma 1.64; indeed, for any $f, g \in C_c^\infty(\mathcal{G}^{(0)})$,

$$\phi(f) * \phi(g) = \phi(fg) = \phi(gf) = \phi(g) * \phi(f).$$

Moreover, the convolution reduces to pointwise multiplication on $\mathcal{G}^{(0)}$. This concludes the proof. □

As we have already outlined, the algebras we consider are not always unital. Related to Lemma 1.50, we have the following.

Lemma 1.66. *The algebra $\mathcal{D}(\mathcal{G})$ has local units. Moreover, the local units can always be picked in $C_c^\infty(\mathcal{G}^{(0)})$, viewed as a subalgebra of $\mathcal{D}(\mathcal{G})$.*

Proof. Let $f_1, f_2, \dots, f_n \in \mathcal{D}(\mathcal{G})$ and consider the compact open set $U = \bigcup_{i=1}^n \operatorname{supp}(f_i)$. Then, construct the set $V = r(U) \cup s(U) \subseteq \mathcal{G}^{(0)}$, which is again compact open. Finally, let us consider the characteristic function $e = \chi_V \in \mathcal{D}(\mathcal{G})$, which is clearly an idempotent and such that $e * f_i = f_i = f_i * e$. □

§ 1.6 | Proper groupoids

In this section, we review the definition of proper groupoids and show the existence of particular cut-off functions for such groupoids.

Recall the definition of a proper map as given in Definition 1.52.

Definition 1.67. *Let X and Y be locally compact spaces, and let $\pi : X \to Y$ be a continuous map. A function $f : X \to \mathbb{C}$ is said to be* properly supported *if*

$$\pi|_{\operatorname{supp}(f)} : \operatorname{supp}(f) \to Y$$

is proper.

Properly supported functions, in the context of ample groupoids, behave nicely with respect to fibre integration. In particular, the following result holds.

Lemma 1.68. *Let $\mathcal{G}$ be an ample groupoid and let $f : \mathcal{G} \to \mathbb{C}$ be a locally constant function such that* $\operatorname{supp}(f) \cap r^{-1}(K)$ *is compact for all compact sets $K \subseteq \mathcal{G}^{(0)}$. Then the function $\lambda(f) : \mathcal{G}^{(0)} \to \mathbb{C}$ defined by*

$$\lambda(f)(x) = \sum_{\alpha \in \mathcal{G}^x} f(\alpha)$$

is locally constant.

Proof. Let $x \in \mathcal{G}^{(0)}$ and let V be a compact open neighbourhood of x. By assumption the set $\operatorname{supp}(f) \cap r^{-1}(V) \subseteq \mathcal{G}$ is compact. Hence $g = f\chi_{r^{-1}(V)}$ is contained in $C_c^\infty(\mathcal{G})$. Writing g as a linear combination of characteristic functions of compact open bisections of $\mathcal{G}$ it is straightforward to check that $\lambda(g)$ is locally constant. Since by construction the functions $\lambda(g)$ and $\lambda(f)$ agree on V it follows that $\lambda(f)$ is locally constant in a neighbourhood of x, and since x was arbitrary this yields the claim. □

Let us now introduce the definition of properness for groupoids. Recall that we write s, r for the source and range maps of $\mathcal{G}$, respectively.

Definition 1.69. *An étale groupoid $\mathcal{G}$ is called proper if $(s, r) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is a proper map.*

As mentioned in the section about $\mathcal{G}$-spaces, orbit spaces can misbehave in terms of Hausdorffness. A notable property of proper groupoids is given by the following. A proof can be found in [Bö18, Lemma 1.2.11].

Lemma 1.70. *Let $\mathcal{G}$ be a proper Hausdorff groupoid with open range and source maps. Then the quotient $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is Hausdorff.*

In particular, since étale groupoids always have the range and source map open, see Lemma 1.16, we can say that if $\mathcal{G}$ is a proper étale groupoid then the quotient space $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is Hausdorff.

Moreover, for ample groupoids we can say more.

Lemma 1.71. *Let $\mathcal{G}$ be an ample and proper groupoid. Then the quotient space $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is totally disconnected.*

Proof. By Lemma 1.40, the quotient map $q : \mathcal{G}^{(0)} \to \mathcal{G}\backslash\mathcal{G}^{(0)}$ is continuous and open. We aim to show that $\mathcal{G}\backslash\mathcal{G}^{(0)}$ has a basis of compact and open subsets, which by Proposition 1.22 implies that it is totally disconnected.

Let $[x] \in \mathcal{G}\backslash\mathcal{G}^{(0)}$ and let $A_{[x]}$ be an open neighbourhood of $[x]$. Then $q^{-1}(A_{[x]})$ is an open neighbourhood of a point $y \in \mathcal{G}^{(0)}$ with $q(y) = [x]$. Since $\mathcal{G}^{(0)}$ is totally disconnected and Hausdorff, there exists a compact open subset $C_y \subseteq \mathcal{G}^{(0)}$ such that $y \in C_y \subseteq q^{-1}(A_{[x]})$. Because q is continuous and open, the image $q(C_y)$ is an compact open subset of $\mathcal{G}\backslash\mathcal{G}^{(0)}$ containing $[x]$. Thus, $q(C_y) \subseteq A_{[x]}$ is a compact open neighbourhood of $[x]$. Thus we have constructed a neighbourhood basis at $[x]$ consisting of open and compact subsets. Collecting these neighbourhood basis for all classes in $\mathcal{G}\backslash\mathcal{G}^{(0)}$ gives a basis for the topology.

Finally, since $\mathcal{G}$ is proper, the quotient space is Hausdorff by Lemma 1.70, and hence $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is totally disconnected. □

Since $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is totally disconnected, we can consider the space of locally constant functions on it. Then we have the following.

Proposition 1.72. *Let $\mathcal{G}$ be an ample and proper groupoid. Then every essential $C_c^\infty(\mathcal{G}^{(0)})$-module becomes an essential $C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$-module in a canonical way.*

Proof. From Lemma 1.53, we have that the quotient map $q : \mathcal{G}^{(0)} \to \mathcal{G}\backslash\mathcal{G}^{(0)}$ induces an essential algebra homomorphism $q^* : C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)}) \to C^\infty(\mathcal{G}^{(0)})$. Moreover, since the quotient map is surjective, q^* is injective. Indeed let $f \in C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$ such that $q^*(f) = fq = 0$, then for every $y \in \mathcal{G}\backslash\mathcal{G}^{(0)}$, since q is surjective, there exists a $x \in \mathcal{G}^{(0)}$ such that $f(y) = f(q(x)) = 0$. Then the map q^* is an embedding of algebras.

Let M be an essential $C_c^\infty(\mathcal{G}^{(0)})$-module. We define, for any $f \in C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$, an action by

$$f \cdot m = q^*(f)e \cdot m,$$

where $e \in C_c^\infty(\mathcal{G}^{(0)})$ is such that $e \cdot m = m$, and $q^*(f) \in C^\infty(\mathcal{G}^{(0)}) = M(C_c^\infty(\mathcal{G}^{(0)}))$. This defines an essential $C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$-module structure. Indeed, since the algebra $C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$ has local units, we need to prove that $C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})M = M$. Since q^* is an essential algebra homomorphism, for any element $m \in M$, we have

$$m = e \cdot m = \sum q^*(f_i)g_i \cdot m = \sum q^*(f_i)e \cdot (g_i \cdot m),$$

where e, $g_i \in C_c^\infty(\mathcal{G}^{(0)})$ and $f_i \in C_c^\infty(\mathcal{G}\backslash\mathcal{G}^{(0)})$ for all i. This concludes the proof. □

Next, we review the concept of a cut-off function for étale groupoids, compare [Tu99, Definition 6.7].

Definition 1.73. *Let $\mathcal{G}$ be an étale groupoid. A cut-off function for $\mathcal{G}$ is a continuous function $c : \mathcal{G}^{(0)} \to [0, \infty)$ such that*

(i) for every $x \in \mathcal{G}^{(0)}$ we have $\sum_{\alpha\in\mathcal{G}^x} cs(\alpha) = 1$;

(ii) the map $r : \operatorname{supp}(cs) \to \mathcal{G}^{(0)}$ is proper.

The existence of a cut-off function for the groupoid is related to its properness, as shown in [Tu99, Proposition 6.10 and Proposition 6.11]. We now prove a variant of [Tu99, Proposition 6.11], which ensures the existence of a locally constant cut-off function.

Let us start by recalling the following definitions.

Definition 1.74. *A topological space X is said to be* paracompact *if it is Hausdorff and every open cover of X admits an open locally finite refinement that also covers X. That is, for every open cover $\{U_i\}_{i\in I}$ of X, there exists a second open cover $\{V_j\}_{j\in J}$ of X such* *that*

(i) for each $j \in J$, there exists $i \in I$ such that $V_j \subseteq U_i$;

(ii) for every $x \in X$, there exists a neighbourhood of x that intersects only finitely many V_j.

Definition 1.75. *A topological space X is said to be* σ-compact *if it can be written as a countable union of compact subsets.*

Proposition 1.76. *Let $\mathcal{G}$ be a proper ample groupoid with $\mathcal{G}\backslash\mathcal{G}^{(0)}$ paracompact. Then $\mathcal{G}$ admits a locally constant cut-off function. If $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is compact then $\mathcal{G}$ admits a locally constant cut-off function with compact support.*

Proof. The quotient $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is a totally disconnected locally compact Hausdorff space. By assumption it is also paracompact, and hence can be written as a disjoint union of a family of open σ-compact totally disconnected locally compact Hausdorff spaces, see [Bou66, Section 9.10, Theorem 5]. Every σ-compact totally disconnected locally compact space X, in turn, can be written as a disjoint union of a countable family of compact open subsets. Indeed, taking a finite cover made up of compact open subsets of each compact subset of X by Proposition 1.22, using σ-compactness we obtain a countable family $(U_i)_{i\in I}$ of compact open subsets that cover X. We can make this cover disjoint setting $V_1 := U_1$ and $V_n := U_n \setminus (\bigcup_{j=1} V_j)$ for $n \in \mathbb{N}$. Observe that the union and the difference of compact open subsets of X is compact open. As a consequence, there is a cover $(V_i)_{i\in I}$ of $\mathcal{G}\backslash\mathcal{G}^{(0)}$ consisting of mutually disjoint compact open subsets.

Since the quotient map $q : \mathcal{G}^{(0)} \to \mathcal{G}\backslash\mathcal{G}^{(0)}$ is open we can find $n_i \in \mathbb{N}$ and compact open subsets $U_{i,1} \ldots, U_{i,n_i}$ of $\mathcal{G}^{(0)}$ for each $i \in I$ such that $q(U_{i,j}) \subseteq V_i$ for all j and $q(U_{i,1}) \cup \cdots \cup q(U_{i,n_i}) = V_i$. Without loss of generality, we can arrange the sets $U_{i,j}$ to be mutually disjoint. We then define $d : \mathcal{G}^{(0)} \to [0, \infty)$ by

$$d = \sum_{i\in I} \sum_{j=1}^{n_i} \chi_{U_{i,j}}.$$

By construction, d is well-defined and locally constant. In fact, d is the characteristic function of the union of the sets $U_{i,j}$.

If $K \subseteq \mathcal{G}^{(0)}$ is compact then $q(K) \cap V_i$ is non-empty only for finitely many $i \in I$, and hence $\operatorname{supp}(d) \cap q^{-1}(q(K))$ is compact. As a consequence,

$$\operatorname{supp}(ds) \cap r^{-1}(K) = (s \times r)^{-1}(\operatorname{supp}(d) \times K) = (s \times r)^{-1}((\operatorname{supp}(d) \cap q^{-1}(q(K))) \times K)$$

is compact by properness of $\mathcal{G}$. According to Lemma 1.68 it follows that the function

$\lambda(ds)$ is locally constant.

Note that for every $x \in \mathcal{G}^{(0)}$ there exists an index $i \in I$ such that $q(x) \in V_i$. This implies that there exists some $1 \leq j \leq n_i$ and an element $\alpha \in \mathcal{G}^x$ such that $s(\alpha) \in U_{i,j}$, and we conclude that $\lambda(ds)(x) = \sum_{\alpha \in \mathcal{G}^x} d(s(\alpha)) > 0$. It is then straightforward to check that $c(x) = d(x)/\lambda(ds)(x)$ is a locally constant cut-off function for $\mathcal{G}$.

Finally, if $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is compact then the index set I in the above construction can be taken to be a singleton, and then both d and c have compact support. □

Chapter 2

The category of $\mathcal{G}$-modules

In this chapter, using notions presented previously, we introduce the category of essential modules over the convolution algebra of an ample groupoid $\mathcal{G}$. To give a different view of these objects, we introduce a second category, the category of $C_c^\infty(\mathcal{G})$-comodules, which we will prove to be equivalent to the first one. This alternative viewpoint will help us prove some of the main properties of these categories, such as being monoidal. Then, following a categorical approach, we define what a $\mathcal{G}$-algebra is in this context. These objects will recur from now on and will be the main target of investigation in this thesis. Finally, we introduce the category of $\mathcal{G}$-anti-Yetter–Drinfeld modules, which turns out to be the natural setting in which we will develop some of the homological tools in what follows.

§ 2.1 | $\mathcal{G}$-modules and $C_c^\infty(\mathcal{G})$-comodules

In the previous chapter, we defined the algebra of compactly supported locally constant functions over an ample groupoid $\mathcal{G}$. In this first part of the chapter we introduce two categories and show an isomorphism between them.

The category of $\mathcal{G}$-modules

We now define the category of modules over this algebra, study some of its properties, and give examples.

Definition 2.1. *Let* $\mathcal{G}$-Mod *be the category whose objects are essential left* $\mathcal{D}(\mathcal{G})$*-modules, and whose morphisms are* $\mathcal{D}(\mathcal{G})$*-linear maps. Objects in this category are called* $\mathcal{G}$*-modules, and morphisms are called* $\mathcal{G}$*-equivariant linear maps.*

To better familiarise ourselves with this category, we start by giving some examples arising naturally from the definition. The first trivial one is the following.

Example 2.2. *The algebra* $\mathcal{D}(\mathcal{G})$ *is a module over itself with the action given by convo-*

lution. Thus, for any $f, g \in \mathcal{D}(\mathcal{G})$, we have $f \cdot g = f * g$.

As pointed out in [BDGW23], a good source of examples of $\mathcal{G}$-modules are spaces with a topological action of $\mathcal{G}$.

Lemma 2.3. *Let $\mathcal{G}$ be an ample groupoid and let X be a locally compact, Hausdorff and étale $\mathcal{G}$-space with anchor map $\pi : X \to \mathcal{G}^{(0)}$. Then $C_c^\infty(X)$ is a $\mathcal{G}$-module and the action of $f \in \mathcal{D}(\mathcal{G})$ on $F \in C_c^\infty(X)$ is given by*

$$(f \cdot F)(x) = \sum_{\alpha \in \mathcal{G}^{\pi(x)}} f(\alpha) F(\alpha^{-1} \cdot x). \tag{2.1}$$

Proof. We begin by verifying that the action defined in (2.1) is well-defined. Since $\mathcal{G}$ is étale, the set $\mathcal{G}^{\pi(x)}$ is discrete (in the subspace topology) for each $x \in X$, and the support of $f \in \mathcal{D}(\mathcal{G})$ is compact. Hence, for each fixed $x \in X$, the sum involves only finitely many non-zero terms and is thus well-defined.

To prove that $f \cdot F \in C_c^\infty(X)$, we check that it is both compactly supported and locally constant. We may assume without loss of generality that $f = \chi_U$, where $U \in \mathrm{Bis_c}(\mathcal{G})$, since $\mathcal{D}(\mathcal{G})$ is spanned by such functions.

In this case, the action simplifies to

$$(\chi_U \cdot F)(x) = \begin{cases} F(\alpha^{-1} \cdot x) & \text{if } \exists \alpha \in U \text{ with } r(\alpha) = \pi(x), \\ 0 & \text{otherwise.} \end{cases}$$

Let $K = \mathrm{supp}(F)$, which is compact. Consider the subset

$$U \cdot K := \{\alpha \cdot x \in X \mid \alpha \in U, \, x \in K, \, s(\alpha) = \pi(x)\} \subseteq X.$$

This is the image under the action map $m : \mathcal{G} \times_{s,\pi} X \to X$ of the fibre product $U \times_{s,\pi} K$. Since both U and K are compact, and s, π are continuous maps, the fibre product is compact, see Lemma 1.32, and hence its image $U \cdot K$ is compact. Since $f \cdot F$ vanishes outside this set, we conclude that $\mathrm{supp}(f \cdot F)$ is compact.

Since F is locally constant, there exists an open neighbourhood $V \subseteq X$ of $\alpha^{-1} \cdot x$ on which F is constant. By shrinking V and U if necessary and using the same argument as in Lemma 1.34, we can conclude that the action map

$$m : U \times_{s,\pi} V \to X, \quad (\beta, z) \mapsto \beta \cdot z,$$

is a homeomorphism onto its image, which is precisely

$$U \cdot V = \{\beta \cdot z \in X \mid \beta \in U,\ z \in V,\ s(\beta) = \pi(z)\}.$$

Therefore, the set $U \cdot V$ is an open neighbourhood of x, and for every $y = \beta \cdot z \in U \cdot V$, we have the constant value

$$(f \cdot F)(y) = F(\beta^{-1} \cdot y) = F(z).$$

Thus, $f \cdot F$ is locally constant.

Finally, we show that the $\mathcal{D}(\mathcal{G})$-module structure is essential. Let $F \in C_c^\infty(X)$ with compact support $K = \operatorname{supp}(F)$. Since X is totally disconnected and locally compact, we can cover K with finitely many disjoint compact open subsets $\{U_i\}_i$ of X. For each i, let $V_i := \pi(U_i)$, which is compact open in $\mathcal{G}^{(0)}$ since π is étale. Then the function $\chi_{V_i} \in \mathcal{D}(\mathcal{G})$ satisfies

$$(\chi_{V_i} \cdot F)(x) = \chi_{V_i}(\pi(x))F(x) = \begin{cases} F(x) & \text{if } \pi(x) \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $F = \sum_i \chi_{V_i} \cdot F$, and this concludes the proof. □

Example 2.4. *In view of Lemma 2.3 we observe that $C_c^\infty(\mathcal{G}^{(0)})$ can be endowed with a $\mathcal{G}$-module structure on the left and the right. The actions are given by*

$$(f \cdot m)(x) = \sum_{\alpha \in \mathcal{G}^x} f(\alpha) m(s(\alpha))$$

and

$$(m \cdot f)(x) = \sum_{\alpha \in \mathcal{G}_x} m(r(\alpha)) f(\alpha)$$

for all $f \in \mathcal{D}(\mathcal{G})$, $m \in C_c^\infty(\mathcal{G}^{(0)})$ and $x \in \mathcal{G}^{(0)}$.

Lemma 2.5. *Let $\mathcal{G}$ be an ample groupoid. There exists a covariant functor from the category of locally compact, Hausdorff and étale $\mathcal{G}$-spaces to the category of $\mathcal{G}$-modules. This functor maps a $\mathcal{G}$-space X to the $\mathcal{D}(\mathcal{G})$-module $C_c^\infty(X)$, and a $\mathcal{G}$-equivariant and étale map $\phi : X \to Y$ to the $\mathcal{D}(\mathcal{G})$-linear map $\phi_* : C_c^\infty(X) \to C_c^\infty(Y)$ given by*

$$\phi_*(F)(y) = \begin{cases} \displaystyle\sum_{x \in \phi^{-1}(y)} F(x), & \text{if } \phi^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The $\mathcal{D}(\mathcal{G})$-module structure on $C_c^\infty(X)$ is described in Lemma 2.3. We now show that the assignment $X \mapsto C_c^\infty(X)$ and $\phi \mapsto \phi_*$ defines a covariant functor.

Let $A \subseteq X$ be a compact open subset and consider the characteristic function $\chi_A \in C_c^\infty(X)$.

Since ϕ is étale, each fibre $\phi^{-1}(y)$ is discrete (in the subspace topology) for $y \in Y$, and since χ_A has compact support, only finitely many points in $\phi^{-1}(y)$ contribute non-zero terms. Hence the sum defining $\phi_*(\chi_A)(y)$ is finite, and $\phi_*(\chi_A)$ is well-defined.

To show local constancy of $\phi_*(\chi_A)$, fix $y \in Y$, and for each $x_i \in \phi^{-1}(y) \cap A$ choose a compact open neighbourhood $U_i \subseteq X$ of x_i such that χ_A is constant on U_i and $\phi|_{U_i}$ is a homeomorphism onto an open subset of Y. Then $V := \bigcap_i \phi(U_i)$ is an open neighbourhood of y on which $\phi_*(\chi_A)$ is constant.

The support of $\phi_*(\chi_A)$ is contained in $\phi(A)$ because if $y \notin \phi(A)$, then $\phi^{-1}(y) \cap A = \varnothing$, so $\phi_*(\chi_A)(y) = 0$. Since ϕ is continuous and A is compact, $\phi(A)$ is compact. Thus $\operatorname{supp}(\phi_*(\chi_A))$ is a closed subset of a compact set and hence compact. It follows that $\phi_* : C_c^\infty(X) \to C_c^\infty(Y)$ is a well-defined linear map.

We now prove $\mathcal{D}(\mathcal{G})$-linearity. Let $F \in C_c^\infty(X)$ and $f \in \mathcal{D}(\mathcal{G})$. For $y \in Y$ we compute:

$$\phi_*(f \cdot F)(y) = \sum_{x \in \phi^{-1}(y)} (f \cdot F)(x) = \sum_{x \in \phi^{-1}(y)} \sum_{\alpha \in \mathcal{G}^{\pi(x)}} f(\alpha)\, F(\alpha^{-1} \cdot x).$$

Using the $\mathcal{G}$-equivariance of ϕ, we have $\phi(\alpha^{-1} \cdot x) = \alpha^{-1} \cdot y$, and setting $z = \alpha^{-1} \cdot x$ we get

$$\phi_*(f \cdot F)(y) = \sum_{\alpha \in \mathcal{G}^{\pi(y)}} f(\alpha) \sum_{z \in \phi^{-1}(\alpha^{-1} \cdot y)} F(z) = (f \cdot \phi_*(F))(y).$$

Hence, $\phi_*(f \cdot F) = f \cdot \phi_*(F)$ and ϕ_* is $\mathcal{D}(\mathcal{G})$-linear.

For functoriality, let $\phi : X \to Y$ and $\psi : Y \to Z$ be composable $\mathcal{G}$-equivariant and étale maps. For $F \in C_c^\infty(X)$ and $z \in Z$, we compute

$$(\psi_* \phi_*)(F)(z) = \sum_{y \in \psi^{-1}(z)} \phi_*(F)(y) = \sum_{y \in \psi^{-1}(z)} \sum_{x \in \phi^{-1}(y)} F(x) = \sum_{x \in \phi^{-1}(\psi^{-1}(z))} F(x).$$

Since $\phi^{-1}(\psi^{-1}(z)) = (\psi\phi)^{-1}(z)$ for every $z \in Z$, we can rewrite it as

$$(\psi_* \phi_*)(F)(z) = \sum_{x \in (\psi\phi)^{-1}(z)} F(x) = (\psi\phi)_*(F)(z).$$

Thus $(\psi\phi)_* = \psi_* \phi_*$. Moreover, if id_X is the identity on X, then

$$(\mathrm{id}_X)_*(F)(x) = \sum_{u \in \mathrm{id}_X^{-1}(x)} F(u) = F(x),$$

so $(\mathrm{id}_X)_* = \mathrm{id}_{C_c^\infty(X)}$. This completes the proof. □

The category of $C_c^\infty(\mathcal{G})$-comodules

Our first aim is to provide an alternative description of $\mathcal{G}$-modules, inspired by the discussion in [BHM18]. Consider the maps $d_0, d_1, d_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ given by

$$d_0(\alpha,\beta) = \beta, \quad d_1(\alpha,\beta) = \alpha\beta, \quad d_2(\alpha,\beta) = \alpha$$

for $(\alpha,\beta) \in \mathcal{G}^{(2)} = \mathcal{G} \times_{s,r} \mathcal{G}$. Each of these maps can be used to turn $C_c^\infty(\mathcal{G}^{(2)})$ into a $C_c^\infty(\mathcal{G})$-module by pulling back along d_i and using pointwise multiplication. Let $f \in C_c^\infty(\mathcal{G}^{(2)})$ and $g \in C_c^\infty(\mathcal{G})$ we define the action by

$$(f \cdot_i g)(\alpha,\beta) = f(\alpha,\beta) g(d_i(\alpha,\beta)),$$

for $(\alpha,\beta) \in \mathcal{G}^{(2)}$ and we write $C_c^\infty(\mathcal{G}^{(2)}, d_i)$ for the resulting $C_c^\infty(\mathcal{G})$-module for $i = 0,1,2$.

Hence, if P, Q are $C_c^\infty(\mathcal{G})$-modules and $T : P \to Q$ is a $C_c^\infty(\mathcal{G})$-linear map we get induced linear maps

$$\mathrm{id} \otimes T : C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} P \to C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} Q \tag{2.2}$$

for $i = 0,1,2$. We will denote these maps by $d_i^*(T)$ in the sequel, to keep track of the different module structures on $C_c^\infty(\mathcal{G}^{(2)})$ used in the construction. Consider the special case, as introduced in Definition 1.58, of $P = C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M$ and $Q = C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$ for a $C_c^\infty(\mathcal{G}^{(0)})$-module M, where both P and Q are viewed as $C_c^\infty(\mathcal{G})$-modules with the action by pointwise multiplication in the first tensor factor.

Lemma 2.6. *Using the notation introduced above, there is a canonical isomorphism between $C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M$ and $C_c^\infty(\mathcal{G}^{(2)}) \overset{rd_i,\mathrm{id}}{\otimes} M$. An analogous result holds when replacing the range map r with the source map s.*

Proof. Since $C_c^\infty(\mathcal{G}^{(2)}, d_i)$ is an essential right $C_c^\infty(\mathcal{G})$-module, there exists a canonical isomorphism

$$\varphi : C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(2)}, d_i)$$

defined by $\varphi(f \otimes g)(\alpha,\beta) = f(\alpha,\beta) g(d_i(\alpha,\beta))$.

We now consider the $C_c^\infty(\mathcal{G}^{(0)})$-module structure induced on $C_c^\infty(\mathcal{G}^{(2)})$ via the composition $rd_i : \mathcal{G}^{(2)} \to \mathcal{G}^{(0)}$ and the pointwise multiplication, so that for $h \in C_c^\infty(\mathcal{G}^{(0)})$ and $f \in C_c^\infty(\mathcal{G}^{(2)})$,

$$(f \cdot h)(\alpha,\beta) = f(\alpha,\beta)\, h(rd_i(\alpha,\beta)),$$

and the $C_c^\infty(\mathcal{G}^{(0)})$-module structure induced on $C_c^\infty(\mathcal{G})$ via the range map r and the pointwise multiplication.

Let us verify that φ is $C_c^\infty(\mathcal{G}^{(0)})$-linear with respect to these module structures. For all $f \in C_c^\infty(\mathcal{G}^{(2)})$, $g \in C_c^\infty(\mathcal{G})$ and $h \in C_c^\infty(\mathcal{G}^{(0)})$, we compute

$$\begin{aligned}(\varphi(f \otimes g) \cdot h)(\alpha, \beta) &= \varphi(f \otimes g)(\alpha, \beta)\, h(rd_i(\alpha, \beta)) \\ &= f(\alpha, \beta) g(d_i(\alpha, \beta))\, h(rd_i(\alpha, \beta)) \\ &= f(\alpha, \beta)(g \cdot h)(d_i(\alpha, \beta)) \\ &= \varphi(f \otimes (g \cdot h))(\alpha, \beta).\end{aligned}$$

Hence, φ is $C_c^\infty(\mathcal{G}^{(0)})$-linear.

Since φ is a bijective $C_c^\infty(\mathcal{G}^{(0)})$-linear map, we obtain an isomorphism of $C_c^\infty(\mathcal{G}^{(0)})$-modules. Tensoring with the identity map id_M over $C_c^\infty(\mathcal{G}^{(0)})$, we obtain the desired isomorphism.

The analogous result for the source map follows by symmetry, replacing r and rd_i with s and sd_i, respectively. □

In view of Lemma 2.6, writing the following compositions

$$v_0 = rd_1 = rd_2, \quad v_1 = rd_0 = sd_2, \quad v_2 = sd_0 = sd_1,$$

and recalling that $T : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$ here is an arbitrary $C_c^\infty(\mathcal{G})$-linear morphism, the maps introduced in (2.2),

$$d_i^*(T) : C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}^{(2)}, d_i) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$$

for $i = 0, 1, 2$ can be written as

$$\begin{aligned}d_0^*(T) &: C_c^\infty(\mathcal{G}^{(2)}) \overset{v_1,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,\mathrm{id}}{\otimes} M, \\ d_1^*(T) &: C_c^\infty(\mathcal{G}^{(2)}) \overset{v_0,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,\mathrm{id}}{\otimes} M, \\ d_2^*(T) &: C_c^\infty(\mathcal{G}^{(2)}) \overset{v_0,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}^{(2)}) \overset{v_1,\mathrm{id}}{\otimes} M.\end{aligned}$$

The following definition introduces the objects we will soon compare to $\mathcal{G}$-modules.

Definition 2.7. *A $C_c^\infty(\mathcal{G})$-comodule is an essential $C_c^\infty(\mathcal{G}^{(0)})$-module M together with a $C_c^\infty(\mathcal{G})$-linear isomorphism*

$$T_M : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$$

satisfying the coaction identity

$$d_0^*(T_M)d_2^*(T_M) = d_1^*(T_M).$$

A morphism of $C_c^\infty(\mathcal{G})$-comodules is a $C_c^\infty(\mathcal{G}^{(0)})$-linear map $f : M \to N$ such that $(\mathrm{id} \otimes f)T_M = T_N(\mathrm{id} \otimes f)$.

We will write $C_c^\infty(\mathcal{G})$-Comod for the category of $C_c^\infty(\mathcal{G})$-comodules.

In the second part of this section, we will prove that these new objects we have introduced are the same as $\mathcal{G}$-modules described at the beginning of the chapter.

From $\mathcal{G}$-modules to $C_c^\infty(\mathcal{G})$-comodules

Let us first explain how to pass from $\mathcal{G}$-modules to $C_c^\infty(\mathcal{G})$-comodules. Consider $M = C_c^\infty(\mathcal{G}) = \mathcal{D}(\mathcal{G})$ as a left module over itself.

Lemma 2.8. *Let $\mathcal{G}$ be an ample groupoid. Then there is a linear isomorphism $T : C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})$, given by*

$$T(f)(\alpha, \beta) = f(\alpha, \alpha\beta)$$

for $f \in C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) = C_c^\infty(\mathcal{G} \times_{r,r} \mathcal{G})$.

Proof. The spaces $\mathcal{G} \times_{s,r} \mathcal{G}$ and $\mathcal{G} \times_{r,r} \mathcal{G}$ are homeomorphic since the map

$$t : \mathcal{G} \times_{s,r} \mathcal{G} \to \mathcal{G} \times_{r,r} \mathcal{G}, \quad t(\alpha, \beta) = (\alpha, \alpha\beta)$$

is a continuous map whose inverse is given by

$$t^{-1} : \mathcal{G} \times_{r,r} \mathcal{G} \to \mathcal{G} \times_{s,r} \mathcal{G}, \quad t^{-1}(\alpha, \beta) = (\alpha, \alpha^{-1}\beta).$$

Since t is a homeomorphism, hence a proper map, using Lemma 1.53, we get the desired linear isomorphism

$$\begin{aligned} T : C_c^\infty(\mathcal{G} \times_{r,r} \mathcal{G}) &\to C_c^\infty(\mathcal{G} \times_{s,r} \mathcal{G}) \\ f &\mapsto T(f)(\alpha, \beta) = f(\alpha, \alpha\beta), \end{aligned}$$

induced by t. Finally, Proposition 1.59 concludes the proof. □

Moreover, the map T is left $C_c^\infty(\mathcal{G})$-linear, as explained in the following.

Lemma 2.9. *The map $T : C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})$ is a $C_c^\infty(\mathcal{G})$-linear map with respect to the pointwise multiplication action on the first tensor factor on both sides.*

Proof. Let $f \in C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})$ and $g \in C_c^\infty(\mathcal{G})$. Then we have

$$\begin{aligned} T(g \cdot f)(\alpha, \beta) &= (g \cdot f)(\alpha, \alpha\beta) \\ &= g(\alpha) f(\alpha, \alpha\beta) \\ &= g(\alpha) T(f)(\alpha, \beta) \\ &= (g \cdot T(f))(\alpha, \beta) \end{aligned}$$

Then we have $T(g \cdot f) = g \cdot T(f)$ and this concludes the proof. □

In the following, we will also refer to T as the *canonical* map of $M = C_c^\infty(\mathcal{G})$.

Let us show that the canonical map turns $C_c^\infty(\mathcal{G}) = \mathcal{D}(\mathcal{G})$ into a $C_c^\infty(\mathcal{G})$-comodule. To this end, note that using the definition of v_i for $i = 0, 1, 2$ we have the homeomorphisms

$$\begin{aligned} \mathcal{G}^{(2)} \times_{d_0,\pi} (\mathcal{G} \times_{s,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_2,r} \mathcal{G}, \\ \mathcal{G}^{(2)} \times_{d_0,\pi} (\mathcal{G} \times_{r,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_1,r} \mathcal{G}, \\ \mathcal{G}^{(2)} \times_{d_1,\pi} (\mathcal{G} \times_{s,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_2,r} \mathcal{G}, \\ \mathcal{G}^{(2)} \times_{d_1,\pi} (\mathcal{G} \times_{r,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_0,r} \mathcal{G}, \\ \mathcal{G}^{(2)} \times_{d_2,\pi} (\mathcal{G} \times_{s,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_1,r} \mathcal{G}, \\ \mathcal{G}^{(2)} \times_{d_2,\pi} (\mathcal{G} \times_{r,r} \mathcal{G}) &\cong \mathcal{G}^{(2)} \times_{v_0,r} \mathcal{G}, \end{aligned}$$

where π denotes the projection to the first copy of $\mathcal{G}$ in either case. Using these homeomorphisms, we can identify the maps induced by the map t, introduced in the proof of Lemma 2.8, on these fibre products as

$$\begin{aligned} &(\mathrm{id} \times_{d_0,\pi} t) : \mathcal{G}^{(2)} \times_{v_2,r} \mathcal{G} \to \mathcal{G}^{(2)} \times_{v_1,r} \mathcal{G}, \quad (\mathrm{id} \times_{d_0,\pi} t)(\alpha, \beta, \gamma) = (\alpha, \beta, \beta\gamma), \\ &(\mathrm{id} \times_{d_1,\pi} t) : \mathcal{G}^{(2)} \times_{v_2,r} \mathcal{G} \to \mathcal{G}^{(2)} \times_{v_0,r} \mathcal{G}, \quad (\mathrm{id} \times_{d_1,\pi} t)(\alpha, \beta, \gamma) = (\alpha, \beta, \alpha\beta\gamma), \\ &(\mathrm{id} \times_{d_2,\pi} t) : \mathcal{G}^{(2)} \times_{v_1,r} \mathcal{G} \to \mathcal{G}^{(2)} \times_{v_0,r} \mathcal{G}, \quad (\mathrm{id} \times_{d_2,\pi} t)(\alpha, \beta, \gamma) = (\alpha, \beta, \alpha\gamma). \end{aligned}$$

From this description it is immediate to check that $(\mathrm{id} \times_{d_2,\pi} t)(\mathrm{id} \times_{d_0,\pi} t) = (\mathrm{id} \times_{d_1,\pi} t)$. Using Lemma 1.53, as we have done in Lemma 2.8, and since $d_i^*(T)$ is the transpose of $\mathrm{id} \times_{d_i,\pi} t$ this yields the coaction identity $d_0^*(T) d_2^*(T) = d_1^*(T)$ for T.

Now let M be an arbitrary $\mathcal{G}$-module. We start with some preliminary results.

Lemma 2.10. *Let $\mathcal{G}$ be an ample groupoid. Then any $\mathcal{G}$-module M is naturally an essential $C_c^\infty(\mathcal{G}^{(0)})$-module via the restriction of the action along the inclusion $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$.*

Proof. Since M is a $\mathcal{G}$-module, it is by definition an essential $\mathcal{D}(\mathcal{G})$-module. The convolution algebra $\mathcal{D}(\mathcal{G})$ contains $C_c^\infty(\mathcal{G}^{(0)})$ as a subalgebra via the inclusion of units $\mathcal{G}^{(0)} \subseteq \mathcal{G}$

as shown in Lemma 1.64.

Then, we define the action of $C_c^\infty(\mathcal{G}^{(0)})$ on M by restricting the $\mathcal{D}(\mathcal{G})$-action

$$f \cdot m := \tilde{f} \cdot m,$$

for all $f \in C_c^\infty(\mathcal{G}^{(0)})$ and $m \in M$, where $\tilde{f}$ denotes the extension-by-zero of f. This defines a $C_c^\infty(\mathcal{G}^{(0)})$-module structure on M.

To show that this action is essential, we recall that $\mathcal{D}(\mathcal{G})$ has local units and these units can always be picked as elements of $C_c^\infty(\mathcal{G}^{(0)})$, see Lemma 1.66. In particular, in view of Remark 1.43, we have that for every given $m \in M$, there exists an $e \in C_c^\infty(\mathcal{G}^{(0)})$ such that $m = e \cdot m$.

Finally, Lemma 1.42 concludes the proof. □

Lemma 2.11. *The map* $T : C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})$ *is* $\mathcal{D}(\mathcal{G})$*-linear with respect to the right* $\mathcal{D}(\mathcal{G})$*-action on the second tensor factor on both sides.*

Proof. We equip both tensor products with the right $\mathcal{D}(\mathcal{G})$-action given by convolution on the second tensor factor. For $f \in C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})$ and $g \in \mathcal{D}(\mathcal{G})$, we compute

$$\begin{aligned} T(f * g)(\alpha, \beta) &= (f * g)(\alpha, \alpha\beta) \\ &= \sum_{\zeta \in \mathcal{G}^{r(\alpha)}} f(\alpha, \zeta)\, g(\zeta^{-1}\alpha\beta). \end{aligned}$$

Set $\zeta = \alpha\eta$, so that $\eta = \alpha^{-1}\zeta$. Then $\zeta \in \mathcal{G}^{r(\alpha)}$ implies $\eta \in \mathcal{G}^{r(\beta)}$, and the expression becomes

$$\begin{aligned} \sum_{\eta \in \mathcal{G}^{r(\beta)}} f(\alpha, \alpha\eta)\, g(\eta^{-1}\beta) &= \sum_{\eta \in \mathcal{G}^{r(\beta)}} T(f)(\alpha, \eta) g(\eta^{-1}\beta) \\ &= (T(f) * g)(\alpha, \beta). \end{aligned}$$

Hence, we have $T(f * g) = T(f) * g$, which proves the right $\mathcal{D}(\mathcal{G})$-linearity of T. □

Moreover, using the identification $\mathcal{D}(\mathcal{G}) \otimes_{\mathcal{D}(\mathcal{G})} M \cong M$ and Lemma 2.11, we obtain a $C_c^\infty(\mathcal{G})$-linear isomorphism $T_M : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$ as the unique map fitting into the commutative diagram

$$\begin{array}{ccc} (C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})} M & \xrightarrow{T \otimes \mathrm{id}} & (C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})} M \\ \downarrow \cong & & \downarrow \cong \\ C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M & \xrightarrow{T_M} & C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M. \end{array}$$

From the above construction, we obtain analogous commutative diagrams linking $d_i^*(T_M)$ and $d_i^*(T) \otimes \mathrm{id}$ for $i = 0, 1, 2$. Indeed, for example, taking $i = 0$, we get

$$\begin{array}{ccc} (C_c^\infty(\mathcal{G}^{(2)}) \overset{v_1,r}{\otimes} C_c^\infty(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})} M & \xrightarrow{d_0^*(T)\otimes \mathrm{id}} & (C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,r}{\otimes} C_c^\infty(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})} M \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ C_c^\infty(\mathcal{G}^{(2)}) \overset{v_1,\mathrm{id}}{\otimes} M & \xrightarrow{\quad d_0^*(T_M) \quad} & C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,\mathrm{id}}{\otimes} M, \end{array}$$

and hence the coaction identity $d_0^*(T_M)d_2^*(T_M) = d_1^*(T_M)$ holds. We will refer to T_M as the *canonical* map of M in the sequel.

In explicit calculations, the following result is useful.

Lemma 2.12. *Let M be a $\mathcal{G}$-module. For any compact open bisection U of $\mathcal{G}$ and $m \in M$ we have*

$$T_M(\chi_U \otimes m) = \chi_U \otimes \chi_{U^{-1}} \cdot m.$$

Proof. Using the essentiality of M and the existence of local units, we have

$$\begin{aligned} T_M(\chi_U \otimes m) &= (T \otimes \mathrm{id})(\chi_U \otimes e \otimes m) \\ &= T(\chi_U \otimes e) \otimes m, \end{aligned}$$

where $e \in C_c^\infty(\mathcal{G}^{(0)})$ has the property that $e \cdot m = m$ and without loss of generality, since we are working a balanced tensor product, we can assume that $e = \chi_{r(U)}$. Then, recalling the map T as defined in Lemma 2.8 we have

$$\begin{aligned} T(\chi_U \otimes e) \otimes m &= \chi_U \otimes \chi_{U^{-1}} \otimes m \\ &= \chi_U \otimes \chi_{U^{-1}} \cdot m, \end{aligned}$$

and this concludes the proof. □

These constructions lead to the following result.

Lemma 2.13. *Let M be a $\mathcal{G}$-module. Then M becomes a $C_c^\infty(\mathcal{G})$-comodule via the canonical map T_M defined above. This assignment defines a functor $A : \mathcal{G}$-Mod $\to C_c^\infty(\mathcal{G})$-Comod.*

Proof. The assignment at the level of objects is clear from the discussion so far.

We now verify functoriality. Let $f : M \to N$ be a morphism of $\mathcal{G}$-modules, that is, a $\mathcal{D}(\mathcal{G})$-linear map. We need to show that

$$(\mathrm{id} \otimes f)T_M = T_N(\mathrm{id} \otimes f), \tag{2.3}$$

for T_M and T_N the canonical map associated respectively with M and N. Using the identification $\mathcal{D}(\mathcal{G}) \otimes_{\mathcal{D}(\mathcal{G})} M \cong M$, the diagram in the construction of T_M, and $\mathcal{D}(\mathcal{G})$-linearity of f, we rewrite the compatibility condition (2.3) as

$$(\mathrm{id} \otimes \mathrm{id} \otimes f)(T \otimes \mathrm{id}_M) = (T \otimes \mathrm{id}_N)(\mathrm{id} \otimes \mathrm{id} \otimes f),$$

which clearly holds. Therefore, f is a morphism of $C_c^\infty(\mathcal{G})$-comodules. Let P be a $\mathcal{G}$-module and $g : N \to P$ be a $\mathcal{G}$-equivariant map. Compatibility with the composition of maps follows from $\mathrm{id} \otimes (gf) = (\mathrm{id} \otimes g)(\mathrm{id} \otimes f)$. Moreover, compatibility with the identity map $\mathrm{id}_M : M \to M$ holds trivially. Therefore, the assignment $M \mapsto (M, T_M)$ on objects and $f \mapsto f$ on morphisms defines a functor. □

From $C_c^\infty(\mathcal{G})$-comodules to $\mathcal{G}$-modules

The goal of this part is to construct a functor that goes in the opposite direction. Let us start by considering the integration map $\lambda : C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$ defined by

$$\lambda(f)(x) = \sum_{\alpha \in \mathcal{G}^x} f(\alpha).$$

Remark 2.14. *This map is clearly surjective, since the integration is the identity map when restricted to $C_c^\infty(\mathcal{G}^{(0)}) \subseteq C_c^\infty(\mathcal{G})$. Moreover, for any compact open bisection $U \subseteq \mathcal{G}$, we have $\lambda(\chi_U) = \chi_{r(U)}$.*

Lemma 2.15. *The integration map $\lambda : C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$ is $C_c^\infty(\mathcal{G}^{(0)})$-linear with respect to the action of $C_c^\infty(\mathcal{G}^{(0)})$ on $C_c^\infty(\mathcal{G})$ induced by the range map r and the pointwise multiplication action on $C_c^\infty(\mathcal{G}^{(0)})$.*

Proof. Let $f \in C_c^\infty(\mathcal{G})$ and $h \in C_c^\infty(\mathcal{G}^{(0)})$, then we have

$$\lambda(h \cdot f)(x) = \sum_{\alpha \in \mathcal{G}^x} h(r(\alpha))f(\alpha) = \sum_{\alpha \in \mathcal{G}^x} h(x)f(\alpha) = h(x)\lambda(f)(x) = (h \cdot \lambda(f))(x)$$

as required. □

Moreover, we have the following equivariant property.

Lemma 2.16. *The integration map $\lambda : C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$ is $\mathcal{G}$-equivariant with respect to the left multiplication action on $C_c^\infty(\mathcal{G}) = \mathcal{D}(\mathcal{G})$.*

Proof. Recall that $C_c^\infty(\mathcal{G}^{(0)})$ becomes a $\mathcal{G}$-module with action described in Example 2.4.

Let $f \in C_c^\infty(\mathcal{G})$ and let $U \subseteq \mathcal{G}$ be a compact open bisection. We compute

$$\begin{aligned}
\lambda(\chi_U * f)(x) &= \sum_{\beta \in \mathcal{G}^x} (\chi_U * f)(\beta) \\
&= \sum_{\alpha, \beta \in \mathcal{G}^x} \chi_U(\alpha) f(\alpha^{-1}\beta) \\
&= \sum_{\alpha \in \mathcal{G}^x} \chi_U(\alpha) \sum_{\gamma \in \mathcal{G}^{s(\alpha)}} f(\gamma) \\
&= \sum_{\alpha \in \mathcal{G}^x} \chi_U(\alpha) \lambda(f)(s(\alpha)) \\
&= \chi_U \cdot \lambda(f)(x),
\end{aligned}$$

and this yields the claim. □

Now assume that M is a $C_c^\infty(\mathcal{G})$-comodule with canonical map T_M and define the map $\mu_M : \mathcal{D}(\mathcal{G}) \otimes M \to M$ by the formula

$$\mu_M = (\lambda \otimes \mathrm{id}) T_M^{-1} q_M, \tag{2.4}$$

where $q_M : \mathcal{D}(\mathcal{G}) \otimes M = C_c^\infty(\mathcal{G}) \otimes M \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$ is the quotient map. Consider the diagram

$$\begin{array}{ccccc}
C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,\mathrm{id}}{\otimes} M & \xrightarrow{d_0^*(T_M^{-1})} & C_c^\infty(\mathcal{G}^{(2)}) \overset{v_1,\mathrm{id}}{\otimes} M & \xrightarrow{d_2^*(T_M^{-1})} & C_c^\infty(\mathcal{G}^{(2)}) \overset{v_0,\mathrm{id}}{\otimes} M \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M & \xrightarrow{\mathrm{id} \otimes T_M^{-1}} & C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M & \xrightarrow{(T_M^{-1})_{13}} & (C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})) \overset{v_0,\mathrm{id}}{\otimes} M \\
 & & \downarrow \mathrm{id} \otimes \lambda \otimes \mathrm{id} & & \downarrow \mathrm{id} \otimes \lambda \otimes \mathrm{id} \\
 & & C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M & \xrightarrow{T_M^{-1}} & C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \\
 & & \downarrow \lambda \otimes \mathrm{id} & & \downarrow \lambda \otimes \mathrm{id} \\
 & & M & \xrightarrow{=} & M.
\end{array}$$

Here $(T_M^{-1})_{13}$ is the map T_M^{-1} applied to the first and third tensor factors and the identity on the second. The two top squares are commutative by the definition of $d_0^*(T_M^{-1})$ and $d_2^*(T_M^{-1})$ as in Equation 2.2. The bottom right square commutes trivially since we are suppressing a factor ignored by the action. As a consequence, we obtain

$$(\lambda \otimes \mathrm{id})(\mathrm{id} \otimes \lambda \otimes \mathrm{id}) d_2^*(T_M^{-1}) d_0^*(T_M^{-1})(f \otimes g \otimes m) = \mu_M(\mathrm{id} \otimes \mu_M)(f \otimes g \otimes m)$$

for all $f, g \in C_c^\infty(\mathcal{G})$ and $m \in M$. Similarly, we have a commutative diagram

$$\begin{array}{ccccc}
C_c^\infty(\mathcal{G}^{(2)}) \overset{v_2,\mathrm{id}}{\otimes} M & \xrightarrow{d_1^*(T_M^{-1})} & C_c^\infty(\mathcal{G}^{(2)}) \overset{v_0,\mathrm{id}}{\otimes} M & & \\
\downarrow{\scriptstyle T^{-1}\otimes\mathrm{id}} & & \downarrow{\scriptstyle T^{-1}\otimes\mathrm{id}} & & \\
(C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})) \overset{s\pi,\mathrm{id}}{\otimes} M & \xrightarrow{\mathrm{id}\otimes T_M^{-1}} & (C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})) \overset{r\pi,\mathrm{id}}{\otimes} M & \xrightarrow{T\otimes\mathrm{id}} & (C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})) \overset{v_0,\mathrm{id}}{\otimes} M \\
\downarrow{\scriptstyle \lambda\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \lambda\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\lambda\otimes\mathrm{id}} \\
C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M & \xrightarrow{T_M^{-1}} & C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M & & C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \\
 & & \downarrow{\scriptstyle \lambda\otimes\mathrm{id}} & & \downarrow{\scriptstyle \lambda\otimes\mathrm{id}} \\
 & & M & \xrightarrow{=} & M,
\end{array}$$

where π is the projection onto the second factor. Observing that

$$(\lambda \otimes \mathrm{id})T^{-1}(f \otimes g) = f * g = \mu(f \otimes g)$$

is the convolution product, we obtain

$$\begin{aligned}
(\lambda \otimes \mathrm{id})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})d_1^*(T_M^{-1})(f \otimes g \otimes m) &= (\lambda \otimes \mathrm{id})T_M^{-1}(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(T^{-1} \otimes \mathrm{id})(f \otimes g \otimes m) \\
&= \mu_M(\mu \otimes \mathrm{id})(f \otimes g \otimes m),
\end{aligned}$$

for all $f, g \in C_c^\infty(\mathcal{G})$ and $m \in M$. Applying the coaction identity $d_2^*(T_M^{-1})d_0^*(T_M^{-1}) = d_1^*(T_M^{-1})$ we obtain

$$\mu_M(\mathrm{id} \otimes \mu_M) = \mu_M(\mu \otimes \mathrm{id}),$$

so we conclude that μ_M turns M into a left $\mathcal{D}(\mathcal{G})$-module. To conclude, we check that the module structure is essential. Since both λ and q_M are surjective and T_M is an isomorphism we see that $\mu_M(\mathcal{D}(\mathcal{G}) \otimes M) = M$. Then, using Lemma 1.42, it follows the claim.

This discussion leads to the following result.

Lemma 2.17. *Let M be a $C_c^\infty(\mathcal{G})$-comodule. Then M becomes a $\mathcal{G}$-module via the action μ_M defined above. This assignment defines a functor $B : C_c^\infty(\mathcal{G})\text{-Comod} \to \mathcal{G}\text{-Mod}$.*

Proof. The assignment at the level of objects is clear from the discussion above.

We now verify functoriality. Let M, N be $C_c^\infty(\mathcal{G})$-comodules, and $f : M \to N$ be a morphism between them. It is a $C_c^\infty(\mathcal{G}^{(0)})$-linear map satisfying the compatibility relation $(\mathrm{id} \otimes f)T_M = T_N(\mathrm{id} \otimes f)$. We need to show that f is $\mathcal{G}$-equivariant for the corresponding $\mathcal{G}$-module structures.

Observe that $f(\lambda \otimes \mathrm{id}) = (\lambda \otimes \mathrm{id})(\mathrm{id} \otimes f)$ because f is $C_c^\infty(\mathcal{G}^{(0)})$-linear and since both T_M and T_N are isomorphisms, the relation $(\mathrm{id} \otimes f)T_M^{-1} = T_N^{-1}(\mathrm{id} \otimes f)$ holds. So, using the definition of μ_M, see Equation 2.4, we have

$$\begin{aligned} f((\mu_M)(g\otimes m)) &= f(\lambda\otimes \mathrm{id})T_M^{-1}q_M(g\otimes m) \\ &= (\lambda\otimes \mathrm{id})(\mathrm{id}\otimes f)T_M^{-1}q_M(g\otimes m) \\ &= (\lambda\otimes \mathrm{id})T_N^{-1}(\mathrm{id}\otimes f)q_N(g\otimes m) \\ &= (\lambda\otimes \mathrm{id})T_N^{-1}q_N(g\otimes f(m)) \\ &= \mu_N(g\otimes f(m)), \end{aligned}$$

for any $g\otimes m \in \mathcal{D}(\mathcal{G})\otimes M$. Let P be a $C_c^\infty(\mathcal{G})$-module and $g : N \to P$ be a morphism between them. The compatibility with the composition of maps follows from the relation $\mathrm{id}\otimes(gf) = (\mathrm{id}\otimes g)(\mathrm{id}\otimes f)$. Moreover, the identity map $\mathrm{id}_M : M \to M$ of $C_c^\infty(\mathcal{G}^{(0)})$-comodules is trivially sent to the identity of $\mathcal{G}$-modules. Therefore, the assignment $(M, T_M) \mapsto (M, \mu_M)$ on objects and $f \mapsto f$ on morphisms defines a functor. □

Combining together Lemma 2.13 and Lemma 2.17 we are now ready to establish the correspondence between $\mathcal{G}$-modules and $C_c^\infty(\mathcal{G})$-comodules.

Proposition 2.18. *Let $\mathcal{G}$ be an ample groupoid. The constructions described above implement an isomorphism of categories between the category $\mathcal{G}$-Mod of $\mathcal{G}$-modules and the category $C_c^\infty(\mathcal{G})$-Comod of $C_c^\infty(\mathcal{G})$-comodules.*

Proof. We have already constructed functors $A : \mathcal{G}\text{-Mod} \to C_c^\infty(\mathcal{G})\text{-Comod}$ and $B : C_c^\infty(\mathcal{G})\text{-Comod} \to \mathcal{G}\text{-Mod}$, and it suffices to show that the compositions BA and AB equal the identity on $\mathcal{G}$-Mod and $C_c^\infty(\mathcal{G})$-Comod, respectively.

For the $\mathcal{G}$-module $\mathcal{D}(\mathcal{G})$, the $\mathcal{G}$-module $BA(\mathcal{D}(\mathcal{G}))$ is obtained by passing from the canonical map $T : C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} C_c^\infty(\mathcal{G})$ to the $\mathcal{G}$-module structure $\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{G})$ given by

$$(\lambda\otimes \mathrm{id})T^{-1}q(f\otimes g)(\alpha) = (\lambda\otimes \mathrm{id})T^{-1}(f\otimes g)(\alpha) = \sum_{\beta\in\mathcal{G}^{r(\alpha)}} f(\beta)g(\beta^{-1}\alpha)$$

for $f, g \in \mathcal{D}(\mathcal{G})$ and $\alpha\in\mathcal{G}$. This coincides with the left multiplication action of $\mathcal{D}(\mathcal{G})$ on itself. It follows that $BA(\mathcal{D}(\mathcal{G})) = \mathcal{D}(\mathcal{G})$ as $\mathcal{G}$-modules. For a general $\mathcal{G}$-module M, using the construction for $\mathcal{D}(\mathcal{G})$ and the canonical identification $\mathcal{D}(\mathcal{G})\otimes_{\mathcal{D}(\mathcal{G})} M \cong M$, we have that $A(M,\mu_M) = (M, T_{A(M)}) \cong (\mathcal{D}(\mathcal{G})\otimes_{\mathcal{D}(\mathcal{G})} M, T\otimes \mathrm{id})$ and the action map induced by $T_{A(M)}$ is then

$$\mu_{BA(M)} = (\lambda\otimes \mathrm{id}\otimes \mathrm{id})(T^{-1}\otimes \mathrm{id})(q\otimes \mathrm{id}).$$

Hence, we have that $BA(M) = M$ as $\mathcal{G}$-modules.

Conversely, let N be a $C_c^\infty(\mathcal{G})$-comodule with canonical map $T_N : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} N \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} N$. In order to describe the canonical map $T_{AB(N)}$ we start from the module structure $\mu_{B(N)} = (\lambda \otimes \mathrm{id})T_N^{-1} q_N$ on $B(N)$. Then, by construction,

$$\begin{aligned} T_{AB(N)}^{-1}(\chi_U \otimes n) &= \chi_U \otimes \mu_{B(N)}(\chi_U \otimes n) \\ &= \chi_U \otimes (\lambda \otimes \mathrm{id})T_N^{-1} q_N(\chi_U \otimes n) \\ &= \chi_U \otimes (\lambda \otimes \mathrm{id})T_N^{-1}(\chi_U \otimes n) \end{aligned}$$

for any compact open bisection $U \subseteq \mathcal{G}$ and $n \in N$. Let us write $T_N^{-1}(\chi_U \otimes n) = \sum_i \chi_{U_i} \otimes n_i$ for compact open bisections $U_i \subseteq \mathcal{G}$ and elements $n_i \in N$. Since T_N^{-1} is $C_c^\infty(\mathcal{G})$-linear we have

$$\begin{aligned} \sum_i \chi_{U_i} \otimes n_i &= T_N^{-1}(\chi_U \otimes n) \\ &= T_N^{-1}(\chi_U \chi_U \otimes n) \\ &= \chi_U \cdot T_N^{-1}(\chi_U \otimes n) \\ &= \chi_U \cdot (\sum_i \chi_{U_i} \otimes n_i) \\ &= \sum_i \chi_U \chi_{U_i} \otimes n_i \\ &= \sum_i \chi_{U \cap U_i} \otimes n_i, \end{aligned}$$

where recall that the action of $C_c^\infty(\mathcal{G})$ on itself is given by pointwise multiplication. Hence we can assume without loss of generality that $U_i \subseteq U$ for all i. With this observation done, we are ready to continue our calculation, and we have

$$\begin{aligned} T_{AB(N)}^{-1}(\chi_U \otimes n) &= \chi_U \otimes (\lambda \otimes \mathrm{id})(\sum_i \chi_{U_i} \otimes n_i) \\ &= \sum_i \chi_U \otimes \lambda(\chi_{U_i}) n_i \\ &= \sum_i \chi_U \cdot \lambda(\chi_{U_i}) \otimes n_i \\ &= \sum_i \chi_U \cdot \chi_{r(U_i)} \otimes n_i \\ &= \sum_i \chi_{U \cap U_i} \otimes n_i \\ &= T_N^{-1}(\chi_U \otimes n), \end{aligned}$$

using that the action of $C_c^\infty(\mathcal{G}^{(0)})$ on $C_c^\infty(\mathcal{G})$ in the tensor product is given by the range map in the penultimate step. Since U and n were arbitrary, we conclude $T_{AB(N)} = T_N$ as required. □

Proposition 2.18 can be interpreted as stating that a $\mathcal{G}$-module M is equivalently described by the data of an essential $C_c^\infty(\mathcal{G}^{(0)})$-module structure on M, together with a $C_c^\infty(\mathcal{G})$-linear map

$$T_M : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} M \longrightarrow C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} M$$

satisfying the coaction identity. This reformulation will be used throughout the sequel as a criterion to verify that certain objects are $\mathcal{G}$-modules, by constructing a suitable coaction map and checking the coaction identity.

§ 2.2 | Tensor products

The goal of this section is to show that the category $\mathcal{G}$-Mod admits a natural tensor product operation. More precisely, given two $\mathcal{G}$-modules M and N, their underlying $C_c^\infty(\mathcal{G}^{(0)})$-module structures, obtained by restricting the action along the inclusion $C_c^\infty(\mathcal{G}^{(0)}) \hookrightarrow C_c^\infty(\mathcal{G})$, are essential. We therefore consider the balanced tensor product $M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$, and we will construct a natural $\mathcal{G}$-module structure on it. We will then introduce the notion of monoidal category and prove that $\mathcal{G}$-Mod carries a canonical monoidal structure induced by this tensor product.

We shall describe this action using the comodule picture developed in the previous section. According to Proposition 2.18, it suffices to define a $C_c^\infty(\mathcal{G})$-linear isomorphism

$$T_{M\otimes N} : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} (M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N) \to C_c^\infty(\mathcal{G}) \overset{s,\mathrm{id}}{\otimes} (M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N),$$

satisfying the coaction identity. We start with the case $M = N = \mathcal{D}(\mathcal{G})$ and consider the homeomorphism

$$t_{\mathcal{G}\times\mathcal{G}} : \mathcal{G} \times_{s,r} (\mathcal{G} \times_{r,r} \mathcal{G}) \to \mathcal{G} \times_{r,r} (\mathcal{G} \times_{r,r} \mathcal{G}), \quad t_{\mathcal{G}\times\mathcal{G}}(\alpha, (\beta, \gamma)) = (\alpha, (\alpha\beta, \alpha\gamma)).$$

With the notation used in the proof of Lemma 2.13 we obtain natural identifications

$$\begin{aligned}
\mathcal{G}^{(2)} \times_{d_0,\pi} (\mathcal{G} \times_{s,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_2,r} (\mathcal{G} \times_{r,r} \mathcal{G}),\\
\mathcal{G}^{(2)} \times_{d_0,\pi} (\mathcal{G} \times_{r,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_1,r} (\mathcal{G} \times_{r,r} \mathcal{G}),\\
\mathcal{G}^{(2)} \times_{d_1,\pi} (\mathcal{G} \times_{s,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_2,r} (\mathcal{G} \times_{r,r} \mathcal{G}),\\
\mathcal{G}^{(2)} \times_{d_1,\pi} (\mathcal{G} \times_{r,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_0,r} (\mathcal{G} \times_{r,r} \mathcal{G}),\\
\mathcal{G}^{(2)} \times_{d_2,\pi} (\mathcal{G} \times_{s,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_1,r} (\mathcal{G} \times_{r,r} \mathcal{G}),\\
\mathcal{G}^{(2)} \times_{d_2,\pi} (\mathcal{G} \times_{r,r} (\mathcal{G} \times_{r,r} \mathcal{G})) &\cong \mathcal{G}^{(2)} \times_{v_0,r} (\mathcal{G} \times_{r,r} \mathcal{G}),
\end{aligned}$$

and one checks that the induced maps

$$\mathrm{id} \times_{d_0,\pi} t_{\mathcal{G}\times\mathcal{G}} : \mathcal{G}^{(2)} \times_{v_2,r} (\mathcal{G} \times_{r,r} \mathcal{G}) \to \mathcal{G}^{(2)} \times_{v_1,r} (\mathcal{G} \times_{r,r} \mathcal{G}),$$

$$\mathrm{id} \times_{d_1,\pi} t_{\mathcal{G}\times\mathcal{G}} : \mathcal{G}^{(2)} \times_{v_2,r} (\mathcal{G} \times_{r,r} \mathcal{G}) \to \mathcal{G}^{(2)} \times_{v_0,r} (\mathcal{G} \times_{r,r} \mathcal{G}),$$
$$\mathrm{id} \times_{d_2,\pi} t_{\mathcal{G}\times\mathcal{G}} : \mathcal{G}^{(2)} \times_{v_1,r} (\mathcal{G} \times_{r,r} \mathcal{G}) \to \mathcal{G}^{(2)} \times_{v_0,r} (\mathcal{G} \times_{r,r} \mathcal{G}),$$

are given by

$$(\mathrm{id} \times_{d_0,\pi} t_{\mathcal{G}\times\mathcal{G}})(\alpha, \beta, \gamma, \delta) = (\alpha, \beta, (\beta\gamma, \beta\delta)),$$
$$(\mathrm{id} \times_{d_1,\pi} t_{\mathcal{G}\times\mathcal{G}})(\alpha, \beta, \gamma, \delta) = (\alpha, \beta, (\alpha\beta\gamma, \alpha\beta\delta)),$$
$$(\mathrm{id} \times_{d_2,\pi} t_{\mathcal{G}\times\mathcal{G}})(\alpha, \beta, \gamma, \delta) = (\alpha, \beta, (\alpha\gamma, \alpha\delta)),$$

respectively.

From this description it is immediate to verify that

$$(\mathrm{id} \times_{d_2,\pi} t_{\mathcal{G}\times\mathcal{G}})(\mathrm{id} \times_{d_0,\pi} t_{\mathcal{G}\times\mathcal{G}}) = (\mathrm{id} \times_{d_1,\pi} t_{\mathcal{G}\times\mathcal{G}}).$$

It follows that the linear isomorphism

$$T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})} : C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} (C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})) \to C_c^\infty(\mathcal{G}) \overset{s,r}{\otimes} (C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G})),$$
$$T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}(f)(\alpha, \beta, \gamma) = f(\alpha, \alpha\beta, \alpha\gamma)$$

induced by the transpose of $t_{\mathcal{G}\times\mathcal{G}}$, as done in Lemma 2.8, satisfies

$$d_0^*(T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}) d_2^*(T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}) = d_1^*(T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}),$$

where $d_i^*(T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})})$ is the transpose of $\mathrm{id} \times_{d_i,\pi} t_{\mathcal{G}\times\mathcal{G}}$.

In analogy to Lemma 2.11, we get the following, which can be proved in the same way.

Lemma 2.19. *The map $T_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}$ is right $\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$-linear with respect to the $\mathcal{D}(\mathcal{G})$-action on the second and third tensor factors.*

Lemma 2.20. *Let M and N be $\mathcal{G}$-modules. Then there exists a linear isomorphism*

$$(\mathcal{D}(\mathcal{G}) \overset{r,r}{\otimes} \mathcal{D}(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})} (M \otimes N) \to M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N.$$

Proof. Recall the canonical isomorphism

$$(\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})} (M \otimes N) \to M \otimes N,$$

given by the essential $\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$-module structure on $M \otimes N$, that is,

$$(f \otimes g) \cdot (m \otimes n) := (f \cdot m) \otimes (g \cdot n),$$

for f, $g \in \mathcal{D}(\mathcal{G})$, $m \in M$ and $n \in N$. It induces a map at the level of the quotients

$$\Phi : (\mathcal{D}(\mathcal{G}) \overset{r,r}{\otimes} \mathcal{D}(\mathcal{G})) \otimes_{\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})} (M \otimes N) \to M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$$

by setting

$$\Phi\left((f \otimes g) \otimes (m \otimes n)\right) := (f \cdot m) \otimes (g \cdot n).$$

This map is well-defined. Indeed, for any $h \in C_c^\infty(\mathcal{G}^{(0)})$, we have

$$\begin{aligned} \Phi\left((h \cdot f \otimes g) \otimes (m \otimes n)\right) &= (h \cdot f \cdot m) \otimes g \cdot n \\ &= f \cdot m \otimes (h \cdot g \cdot n) \\ &= \Phi\left((f \otimes h \cdot g) \otimes (m \otimes n)\right). \end{aligned}$$

It remains to show that Φ is a bijection.

To prove surjectivity, take $m \otimes n \in M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$ and observe that it can be written as $f \cdot m \otimes g \cdot n$ with $f, g \in \mathcal{D}(\mathcal{G})$ and $m \in M$, $n \in N$, using the essentiality of the module structures of M and N and the existence of local units for $\mathcal{D}(\mathcal{G})$. Then we have $m \otimes n = \Phi((f \otimes g) \otimes (m \otimes n))$.

To prove injectivity, we observe that any element of the form $f \cdot h \otimes g \otimes m \otimes n - f \otimes h \cdot g \otimes m \otimes n$, which vanishes after passing to the quotient on the left-hand side, is sent to $f \cdot h \cdot m \otimes g \cdot n - f \cdot m \otimes h \cdot g \cdot n$, which is again zero after quotienting on the right-hand side. Conversely, every element of the form $h \cdot m \otimes n - m \otimes h \cdot n$ comes from an element $h \cdot f \otimes g \otimes m \otimes n - f \otimes h \cdot g \otimes m \otimes n$, where now f and g can be picked in $C_c^\infty(\mathcal{G}^{(0)})$ and are such that $f \cdot m = m$ and $g \cdot n = n$. This means that we are quotienting the same subspace on both sides of the original canonical isomorphism. This proves that Φ is an isomorphism and concludes the proof. □

Now let M, N be arbitrary $\mathcal{G}$-modules. Then using Lemma 2.20 and the left $\mathcal{D}(\mathcal{G})$-action on $\mathcal{D}(\mathcal{G}) \overset{r,r}{\otimes} \mathcal{D}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G})$ corresponding to the $C_c^\infty(\mathcal{G})$-comodule structure constructed above we obtain the desired $\mathcal{G}$-module structure on $M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$ through this identification.

For calculations it is useful to know how the tensor product $\mathcal{G}$-module structure looks in terms of compact open bisections.

Lemma 2.21. *Let M, N be $\mathcal{G}$-modules. Then the $\mathcal{D}(\mathcal{G})$-module structure on the tensor product $M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$ satisfies*

$$\chi_U \cdot (m \otimes n) = (\chi_U \cdot m) \otimes (\chi_U \cdot n),$$

for any compact open bisection U of $\mathcal{G}$.

Proof. In view of the construction of the tensor product action it suffices to consider the case $M = N = \mathcal{D}(\mathcal{G})$. Given compact open bisections U, V, W of $\mathcal{G}$ and $\alpha, \beta, \gamma \in \mathcal{G}$ such that $r(\alpha) = r(\beta) = r(\gamma)$ we compute

$$\begin{aligned} T^{-1}_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}(\chi_U \otimes \chi_V \otimes \chi_W)(\alpha,\beta,\gamma) &= (\chi_U \otimes \chi_V \otimes \chi_W)(\alpha, \alpha^{-1}\beta, \alpha^{-1}\gamma) \\ &= \chi_U(\alpha)\chi_V(\alpha^{-1}\beta)\chi_W(\alpha^{-1}\gamma). \end{aligned}$$

It follows that $T^{-1}_{\mathcal{D}(\mathcal{G})\otimes\mathcal{D}(\mathcal{G})}(\chi_U \otimes \chi_V \otimes \chi_W)$ is the characteristic function of the set $U \times UV \times UW$. Applying the integration map λ to the first tensor factor therefore gives

$$\chi_U \cdot (\chi_V \otimes \chi_W) = \chi_{UV} \otimes \chi_{UW} = (\chi_U \cdot \chi_V) \otimes (\chi_U \cdot \chi_W),$$

and since characteristic functions of compact open bisections span $\mathcal{D}(\mathcal{G})$ this yields the claim. □

Remark 2.22. *Observe that for an arbitrary function $f \in \mathcal{D}(\mathcal{G})$, we first use the decomposition given by Proposition 1.62, then the linearity of the action with Lemma 2.21.*

We will always view the tensor product $M \otimes_{C_c^\infty(\mathcal{G}^0)} N$ of $\mathcal{G}$-modules M, N as a $\mathcal{G}$-module with the action defined above.

Let us collect here some definitions about monoidal categories. For a classical reference, see [ML98, Chapter 7].

Definition 2.23. *A category $\mathcal{C}$ equipped with the following structures:*

(i) *a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;*

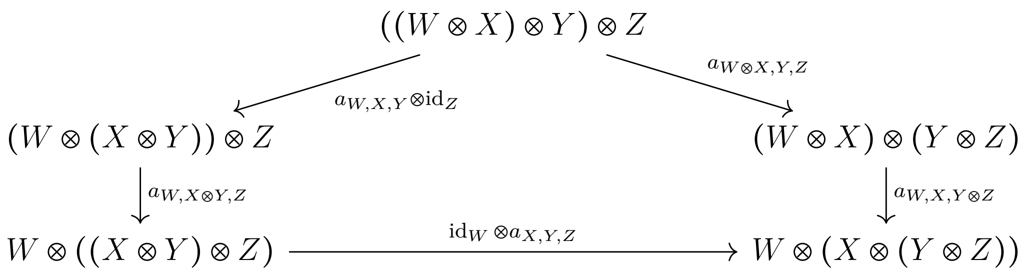
(ii) *an object $\mathbb{U} \in \mathrm{Ob}(\mathcal{C})$, called unit object;*

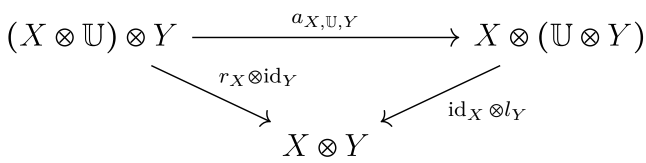
(iii) *a natural isomorphism $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, with components of the form $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, where $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$;*

(iv) *a natural isomorphism $l : \mathbb{U} \otimes - \to -$, with components of the form $l_X : \mathbb{U} \otimes X \to X$, where $X \in \mathrm{Ob}(\mathcal{C})$;*

(v) *a natural isomorphism $r : - \otimes \mathbb{U} \to -$, with components of the form $r_X : X \otimes \mathbb{U} \to X$, where $X \in \mathrm{Ob}(\mathcal{C})$,*

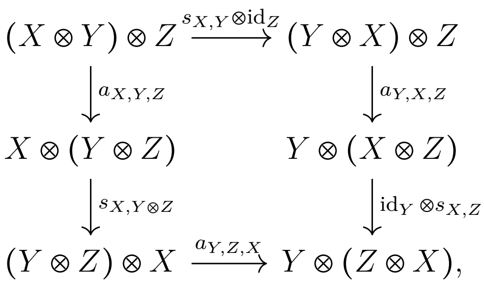
such that, for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagrams commute:

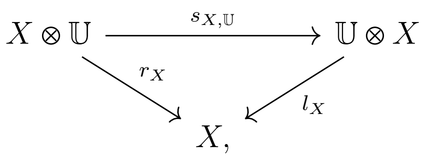


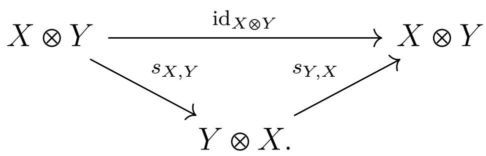


is called monoidal *category.*

Definition 2.24. *A symmetric monoidal category is a monoidal category* $(\mathcal{C}, \otimes, \mathbb{U})$ *together with a natural isomorphism* $s : - \otimes - \to - \otimes -$*, with components of the form* $s_{X,Y} : X \otimes Y \to Y \otimes X$*, where* $X, Y \in \mathrm{Ob}(\mathcal{C})$ *such that, for all* $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$*, the following diagrams commute:*







With these new definition in mind, we summarise our discussion so far as follows.

Proposition 2.25. *The category* $\mathcal{G}\text{-}\mathrm{Mod}$ *with the tensor product operation defined above is a symmetric monoidal category.*

Proof. Let M, N and P be $\mathcal{G}$-modules. Then we have a canonical isomorphism

$$(M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} P \cong M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N \otimes_{C_c^\infty(\mathcal{G}^{(0)})} P \cong M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} (N \otimes_{C_c^\infty(\mathcal{G}^{(0)})} P)$$

of $C_c^\infty(\mathcal{G}^{(0)})$-modules, and using Lemma 2.21, we see that the action on either side is given by

$$\chi_U \cdot (m \otimes n \otimes p) = (\chi_U \cdot m) \otimes (\chi_U \cdot n) \otimes (\chi_U \cdot p)$$

for $m \in M$, $n \in N$, $p \in P$ and U compact open bisection of $\mathcal{G}$. We conclude that the above isomorphism is $\mathcal{G}$-equivariant, thus giving the required associativity constraint.

The tensor unit is given by the $\mathcal{G}$-module $C_c^\infty(\mathcal{G}^{(0)})$, with the action $f \cdot h := \lambda(f)h$ for $f \in \mathcal{D}(\mathcal{G})$ and $h \in C_c^\infty(\mathcal{G}^{(0)})$, compare with Example 2.4.

Due to Lemma 2.21 it follows that the canonical identifications $M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \cong M \cong C_c^\infty(\mathcal{G}^{(0)}) \otimes_{C_c^\infty(\mathcal{G}^0)} M$ are $\mathcal{G}$-equivariant for every $\mathcal{G}$-module M.

Finally, since $C_c^\infty(\mathcal{G}^{(0)})$ is a commutative algebra, there exists an obvious isomorphism

$$M \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N \cong N \otimes_{C_c^\infty(\mathcal{G}^{(0)})} M$$

of $C_c^\infty(\mathcal{G}^{(0)})$-modules, and using Lemma 2.21 we see that the above isomorphism is $\mathcal{G}$-equivariant.

With these structures in place, the axioms for a symmetric monoidal category are readily verified. □

§ 2.3 | $\mathcal{G}$-algebras

Our main objects of study in this thesis are $\mathcal{G}$-algebras over an ample groupoid $\mathcal{G}$ in the following sense.

Definition 2.26. *A $\mathcal{G}$-algebra is a $\mathcal{G}$-module A together with a $\mathcal{G}$-equivariant linear map $m : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} A \to A$, written $m(a \otimes b) = ab$, such that $(ab)c = a(bc)$ for all $a, b, c \in A$.*

This can be phrased equivalently in a categorical way. Let us recall the following.

Given a monoidal category, we can introduce the definition of *algebra object* in the following way.

Definition 2.27. *Let $(\mathcal{C}, \otimes, \mathbb{U})$ be a monoidal category. A non-unital algebra object in $\mathcal{C}$ is an object $A \in \mathrm{Ob}(\mathcal{C})$ with a multiplication map $\mu : A \otimes A \to A$ such that the following diagram commutes:*

$$\begin{array}{ccccc} & & A & & \\ & \overset{\mu}{\nearrow} & & \overset{\mu}{\nwarrow} & \\ A \otimes A & & & & A \otimes A \\ \mu\otimes \mathrm{id}_A \uparrow & & & & \uparrow \mathrm{id}_A \otimes \mu \\ (A \otimes A) \otimes A & & \xrightarrow{a_{A,A,A}} & & A \otimes (A \otimes A), \end{array}$$

With this definition in mind, we equivalently define a $\mathcal{G}$-algebra as a non-unital algebra object in the monoidal category $\mathcal{G}\text{-}\operatorname{Mod}$.

By definition, if A, B are $\mathcal{G}$-algebras then a $\mathcal{G}$-equivariant algebra homomorphism $\phi : A \to B$ is a $\mathcal{G}$-equivariant linear map of the underlying $\mathcal{G}$-modules such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$.

We now show in detail some examples and constructions with $\mathcal{G}$-algebras that will be used in the following.

§ 2.3.1 | Algebras of functions

In Lemma 2.5 we have already proved that a good source of $\mathcal{G}$-modules comes from $\mathcal{G}$-spaces. In the following we prove that actions of $\mathcal{G}$ on totally disconnected locally compact spaces provide examples of commutative $\mathcal{G}$-algebras.

Proposition 2.28. *Let X be a totally disconnected locally compact $\mathcal{G}$-space. Then $C_c^\infty(X)$ with pointwise multiplication is a $\mathcal{G}$-algebra in a natural way.*

Proof. From Lemma 2.3 we have that $C_c^\infty(X)$ is an essential $\mathcal{G}$-module. Hence, we need to prove that pointwise multiplication is a $\mathcal{G}$-equivariant linear map. Indeed, using the action described in Lemma 2.5 and Lemma 2.21, for $f, g \in C_c^\infty(X)$ and U compact open bisection of $\mathcal{G}$, we compute

$$\begin{aligned} \chi_U \cdot m(f \otimes g)(x) &= \sum_{\alpha \in \mathcal{G}^{\pi(x)}} \chi_U(\alpha) m(f \otimes g)(\alpha^{-1} \cdot x) \\ &= \sum_{\alpha \in \mathcal{G}^{\pi(x)}} \chi_U(\alpha) f(\alpha^{-1} \cdot x) g(\alpha^{-1} \cdot x) \\ &= (\chi_U \cdot f)(x)(\chi_U \cdot g)(x) \\ &= m(\chi_U \cdot (f \otimes g))(x), \end{aligned}$$

and this concludes the proof. □

The construction in Proposition 2.28 is compatible with tensor products. More precisely, if X and Y are totally disconnected locally compact $\mathcal{G}$-spaces, with anchor maps π_1 and

π_2, respectively, then the canonical map

$$\phi : C_c^\infty(X) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(Y) \to C_c^\infty(X \times_{\pi_1,\pi_2} Y)$$

is an isomorphism of $\mathcal{G}$-algebras, compare Proposition 1.59. Indeed, using the action described in Lemma 2.5 and Lemma 2.21, $\mathcal{D}(\mathcal{G})$-linearity comes from the following calculation

$$\begin{aligned}\chi_U \cdot \phi(f \otimes g)(x,y) &= \sum_{\alpha \in \mathcal{G}^{\pi_1(x)}} \chi_U(\alpha)\phi(f \otimes g)(\alpha^{-1} \cdot x, \alpha^{-1} \cdot y) \\ &= \sum_{\alpha \in \mathcal{G}^{\pi_1(x)}} \chi_U(\alpha) f(\alpha^{-1} \cdot x) g(\alpha^{-1} \cdot y) \\ &= (\chi_U \cdot f)(x)(\chi_U \cdot g)(y) \\ &= \phi(\chi_U \cdot (f \otimes g))(x,y),\end{aligned}$$

where $f \in C_c^\infty(X)$, $g \in C_c^\infty(Y)$ and U is a compact open bisection of $\mathcal{G}$.

§ 2.3.2 | Algebras associated with pairings

Another class of examples of $\mathcal{G}$-algebras comes from $\mathcal{G}$-modules equipped with $\mathcal{G}$-equivariant pairings in the following sense.

Definition 2.29. *Let E be a $\mathcal{G}$-module. A $\mathcal{G}$-equivariant pairing on E is a $\mathcal{G}$-equivariant linear map $h : E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E \to C_c^\infty(\mathcal{G}^{(0)})$.*

We may equivalently view a $\mathcal{G}$-equivariant pairing as in Definition 2.29 as a $C_c^\infty(\mathcal{G}^{(0)})$-bilinear map $h : E \times E \to C_c^\infty(\mathcal{G}^{(0)})$ such that

$$h(\chi_U \cdot e, \chi_U \cdot f) = \chi_U \cdot h(e,f)$$

for all $e, f \in E$ and all compact open bisections $U \subseteq \mathcal{G}$. Given such a pairing, consider the tensor product

$$\mathcal{K}(E) = E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$$

as a $\mathcal{G}$-module with the diagonal action. Then $\mathcal{K}(E)$ becomes a $\mathcal{G}$-algebra with the multiplication defined by

$$(e_1 \otimes f_1)(e_2 \otimes f_2) = e_1 \otimes h(f_1, e_2) f_2 = e_1 h(f_1, e_2) \otimes f_2$$

for $e_1, e_2, f_1, f_2 \in E$. Note that the multiplication in $\mathcal{K}(E)$ depends on the pairing h, so it would be more accurate to write $\mathcal{K}(E, h)$ for the resulting $\mathcal{G}$-algebra. However, in the sequel the pairings we use will always be clear from the context.

The most important example of a $\mathcal{G}$-equivariant pairing is the following.

Example 2.30. *The* regular pairing $\lambda : \mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$ *is defined by*

$$\lambda(f \otimes g)(x) = \sum_{\alpha \in \mathcal{G}^x} f(\alpha) g(\alpha),$$

for $x \in \mathcal{G}^{(0)}$*. We will simply write* $\mathcal{K}_\mathcal{G} = \mathcal{K}(\mathcal{D}(\mathcal{G}))$ *for the associated* $\mathcal{G}$*-algebra.*

Lemma 2.31. *Let* E *and* F *be* $\mathcal{G}$*-modules equipped respectively with* $\mathcal{G}$*-equivariant pairings* h_E *and* h_F*. Then there is a natural* $\mathcal{G}$*-equivariant isomorphism of* $\mathcal{G}$*-algebras*

$$\mathcal{K}(E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F) \cong \mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(F).$$

Proof. By definition,

$$\mathcal{K}(E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F) = (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F).$$

Using associativity and symmetry of the tensor product, there is a canonical $\mathcal{G}$-equivariant isomorphism

$$\Phi : (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F) \to (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} (F \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F).$$

Let $h_{E\otimes F}$ be the induced $\mathcal{G}$-equivariant pairing on $E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} F$ defined by

$$h_{E\otimes F}(e_1 \otimes f_1, e_2 \otimes f_2) := h_E(e_1, e_2) h_F(f_1, f_2).$$

The canonical identification is multiplicative, indeed

$$\begin{aligned}\Phi((e_1 \otimes f_1) \otimes (e_2 \otimes f_2)(e_3 \otimes f_3) \otimes (e_4 \otimes f_4)) &= \Phi((e_1 \otimes f_1) \otimes h_E(e_2, e_3) h_F(f_2, f_3)(e_4 \otimes f_4)) \\ &= (e_1 \otimes h_E(e_2, e_3) e_4) \otimes (f_1 \otimes h_F(f_2, f_3) f_4) \\ &= ((e_1 \otimes e_2) \otimes (f_1 \otimes f_2))((e_3 \otimes e_4) \otimes (f_3 \otimes f_4)),\end{aligned}$$

for $e_1, e_2, e_3, e_4 \in E$ and $f_1, f_2, f_3, f_4 \in F$. Hence it defines a $\mathcal{G}$-equivariant isomorphism of $\mathcal{G}$-algebras. □

§ 2.3.3 | $C_c^\infty(X)$-algebras

We will now discuss a construction analogous to the notion of $C_0(X)$-algebras, where $C_0(X)$ denotes the space of continuous functions vanishing at infinity on a locally compact Hausdorff space X, in the C^*-algebra setting. See [Wil07, Appendix C] for more details.

Given an algebra A we write $ZM(A)$ for the centre of the multiplier algebra of A.

Definition 2.32. *Let* X *be a totally disconnected locally compact space. A* $C_c^\infty(X)$*-algebra is an algebra* A *together with an essential algebra homomorphism* $C_c^\infty(X) \to M(A)$ *which takes values in* $ZM(A)$*.*

Let us record the following observation, in analogy to the study of groupoid actions in the C^*-algebra setting.

Lemma 2.33. *Let $\mathcal{G}$ be an ample groupoid and let A be a $\mathcal{G}$-algebra. If the multiplication in A is nondegenerate, then A is canonically a $C_c^\infty(\mathcal{G}^{(0)})$-algebra.*

Proof. The essential $C_c^\infty(\mathcal{G}^{(0)})$-module structure on A determines an essential algebra homomorphism $\iota : C_c^\infty(\mathcal{G}^{(0)}) \to M(A)$ defined by $\iota(f)a = f \cdot a$ for all $a \in A$.

Let $m \in M(A)$ be arbitrary, we want to show that $\iota(f)m = m\iota(f)$.

Fix arbitrary $a, b \in A$. Then, using the definition of ι and the linearity of the multiplication in A, we get

$$am\iota(f)b = am(f \cdot b) = f \cdot (amb) = a(f \cdot mb) = a\iota(f)mb.$$

Hence, $am\iota(f)b = a\iota(f)mb$ for all $a, b \in A$. Since the multiplication in A is nondegenerate, this implies that $m\iota(f) = \iota(f)m$ in $M(A)$ and $\iota(f)$ belongs to the center $ZM(A)$. □

We also note that if the ample groupoid $\mathcal{G} = \mathcal{G}^{(0)} = X$ is obtained by viewing a totally disconnected locally compact space X as a groupoid then every $C_c^\infty(X)$-algebra is canonically a $\mathcal{G}$-algebra.

§ 2.3.4 | Unitarisation

Recall that at the beginning of this section we introduced $\mathcal{G}$-algebras as non-unital algebra objects in the category $\mathcal{G}$-Mod. In the main constructions of this work, it will be necessary to work with a suitable unitarisation of such algebras. We now turn to a discussion of this process.

Let us first specify what we mean by unital in this context.

Definition 2.34. *A unital $\mathcal{G}$-algebra object is a $\mathcal{G}$-algebra A in the sense of Definition 2.26 together with a $\mathcal{G}$-equivariant homomorphism $u : C_c^\infty(\mathcal{G}^{(0)}) \to A$ such that $u(f)a = au(f) = f \cdot a$ for $f \in C_c^\infty(\mathcal{G}^{(0)})$ and $a \in A$. A $\mathcal{G}$-equivariant algebra homomorphism between unital $\mathcal{G}$ algebra objects is called unital if it commutes with the unit maps in the obvious way.*

The first basic and immediate example is given by the following.

Example 2.35. *Let $A = C_c^\infty(\mathcal{G}^{(0)})$ with the canonical $\mathcal{G}$-action and $u = \mathrm{id}$.*

Remark 2.36. *The previous example shows already that a unital $\mathcal{G}$-algebra object does not need to have a unit element in general. For this reason, we speak of unital $\mathcal{G}$-algebra objects and not of unital $\mathcal{G}$-algebras.*

Definition 2.37. *Let A be a $\mathcal{G}$-algebra. The $\mathcal{G}$-unitarisation of A is defined as*

$$A^{+} = A \oplus C_c^{\infty}(\mathcal{G}^{(0)})$$

viewed as a $\mathcal{G}$-module with the given action on A and the canonical action on $C_c^{\infty}(\mathcal{G}^{(0)})$, and the multiplication given by

$$(a, f) \cdot (b, g) = (ab + g \cdot a + f \cdot b, fg)$$

for $a, b \in A$ and $f, g \in C_c^{\infty}(\mathcal{G}^{(0)})$. Here the dot product denotes the $C_c^{\infty}(\mathcal{G}^{(0)})$-action on A induced from its $\mathcal{G}$-module structure.

Let us write $\mathrm{Alg}_{\mathcal{G}}(A, B)$ for the set of all $\mathcal{G}$-equivariant algebra homomorphisms between $\mathcal{G}$-algebras A, B. If A, B are unital $\mathcal{G}$-algebra objects then we denote by $\mathrm{Alg}_{\mathcal{G}}^{u}(A, B)$ the set of all unital $\mathcal{G}$-equivariant algebra homomorphisms.

With this notation in place, let us show that the $\mathcal{G}$-unitarisation of a $\mathcal{G}$-algebra satisfies the following universal property.

Lemma 2.38. *Let A be a $\mathcal{G}$-algebra. Then A^{+} is a unital $\mathcal{G}$-algebra object, and there is a natural bijection*

$$\mathrm{Alg}_{\mathcal{G}}^{u}(A^{+}, B) \cong \mathrm{Alg}_{\mathcal{G}}(A, B)$$

for every unital $\mathcal{G}$-algebra object B.

Proof. The first claim is clear by construction. Indeed, the embedding $C_c^{\infty}(\mathcal{G}^{(0)}) \to A^{+}$ into the first summand is a $\mathcal{G}$-equivariant homomorphism which turns A^{+} into a unital $\mathcal{G}$-algebra object.

Suppose $\phi : A^{+} \to B$ is a unital $\mathcal{G}$-equivariant algebra homomorphism. Then its restriction to $A \subset A^{+}$ yields a $\mathcal{G}$-equivariant algebra homomorphism $\phi|_{A} : A \to B$. Conversely, given a $\mathcal{G}$-equivariant algebra homomorphism $\psi : A \to B$, we define a unital $\mathcal{G}$-equivariant algebra homomorphism $\psi^{+} : A^{+} \to B$ by

$$\psi^{+}(a, f) := \psi(a) + u(f),$$

where $u : C_c^{\infty}(\mathcal{G}^{(0)}) \to B$ denotes the unit map of the unital $\mathcal{G}$-algebra B.

One checks directly that ψ^{+} is an algebra homomorphism, is $\mathcal{G}$-equivariant, and extends ψ. These constructions are clearly inverse to one another, giving the claimed natural bijection. □

§ 2.3.5 | Crossed products

The algebraic *crossed product* $A \rtimes \mathcal{G}$ of a $\mathcal{G}$-algebra A can be defined analogously to the classical construction for discrete groups. We will give the definition and then prove a universal property, in analogy with the C^*-algebra setting.

Definition 2.39. *Let A be a $\mathcal{G}$-algebra. We define the algebraic crossed product $A \rtimes \mathcal{G} := A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G})$ as a vector space, with the left action of $C_c^\infty(\mathcal{G}^{(0)})$ on $\mathcal{D}(\mathcal{G})$ induced by the range map. The multiplication in $A \rtimes \mathcal{G}$ is determined by*

$$(a \otimes \chi_U)(b \otimes g) = a\chi_U \cdot b \otimes \chi_U * g$$

for $a, b \in A$, compact open bisections $U \subseteq \mathcal{G}$ and $g \in \mathcal{D}(\mathcal{G})$.

Lemma 2.40. *Let A be a $\mathcal{G}$-algebra. The algebraic crossed product $A \rtimes \mathcal{G}$ is an algebra.*

Proof. The only thing we need to check is that the multiplication is associative. Let $a, b, c \in A$ and let $U, V, W \subseteq \mathcal{G}$ be compact open bisections, we compute

$$\begin{aligned}((a \otimes \chi_U)(b \otimes \chi_V))(c \otimes \chi_W) &= (a(\chi_U \cdot b) \otimes \chi_U * \chi_V)(c \otimes \chi_W) \\ &= (a(\chi_U \cdot b))(\chi_{UV} \cdot c) \otimes (\chi_{UV} * \chi_W) \\ &= a((\chi_U \cdot b)(\chi_{UV} \cdot c)) \otimes (\chi_U * \chi_{VW}) \\ &= (a \otimes \chi_U)(b(\chi_V \cdot c) \otimes \chi_{VW}) \\ &= (a \otimes \chi_U)((b \otimes \chi_V)(c \otimes \chi_W)).\end{aligned}$$

□

Definition 2.41. *Let A be a $\mathcal{G}$-algebra. A covariant representation of $(A, \mathcal{G})$ on an algebra B is a pair of essential homomorphisms $\phi : A \to M(B)$ and $\pi : \mathcal{D}(\mathcal{G}) \to M(B)$ such that $\phi(f \cdot a)\pi(g) = \phi(a)\pi(f * g)$ for all $f \in C_c^\infty(\mathcal{G}^{(0)}), a \in A, g \in \mathcal{D}(\mathcal{G})$ and*

$$\phi(\chi_U \cdot a)\pi(\chi_U) = \pi(\chi_U)\phi(a)$$

for all compact open bisections $U \subseteq \mathcal{G}$ and $a \in A$.

The algebraic crossed product admits algebra homomorphisms $i_A : A \to M(A \rtimes \mathcal{G})$ and $i_\mathcal{G} : \mathcal{D}(\mathcal{G}) \to M(A \rtimes \mathcal{G})$ such that $i_A(a)i_\mathcal{G}(f) = a \otimes f$ for all $a \in A, f \in \mathcal{D}(\mathcal{G})$. Clearly the maps i_A and $i_\mathcal{G}$ define a covariant representation of $(A, \mathcal{G})$ on $A \rtimes \mathcal{G}$.

The following result states a universal property for the algebraic crossed products.

Proposition 2.42. *Let A be a $\mathcal{G}$-algebra. The algebraic crossed product $A \rtimes \mathcal{G}$ is universal for covariant representations of $(A, \mathcal{G})$, that is, for every algebra B and every covariant representation (ϕ, π) of $(A, \mathcal{G})$ on B there exists a unique essential algebra homomorphism*

$\psi : A \rtimes \mathcal{G} \to M(B)$ *such that* $\phi = \psi i_A$ *and* $\pi = \psi i_{\mathcal{G}}$.

Proof. We define $\psi(a \otimes f) = \phi(a)\pi(f)$. This gives a well-defined linear map $A \rtimes \mathcal{G} \to M(B)$ by the first part of the covariance condition. Using the second part of the covariance condition we calculate

$$\psi(a \otimes \chi_U)\psi(b \otimes \chi_V) = \phi(a)\pi(\chi_U)\phi(b)\pi(\chi_V) \\ = \phi(a)\phi(\chi_U \cdot b)\pi(\chi_U)\pi(\chi_V) = \psi((a \otimes \chi_U)(b \otimes \chi_V))$$

for $a, b \in A$ and all compact open bisections $U, V \subseteq \mathcal{G}$, and it follows that ψ is a homomorphism. It is straightforward to check that ψ is essential, satisfying $\phi = \psi i_A, \pi = \psi i_{\mathcal{G}}$, and since $A \rtimes \mathcal{G} = i_A(A) i_{\mathcal{G}}(\mathcal{D}(\mathcal{G}))$ it is uniquely determined. □

§ 2.4 | Anti-Yetter-Drinfeld modules

This section is devoted to the study of *anti-Yetter-Drinfeld modules* over an ample groupoid $\mathcal{G}$, a concept inspired by the theory developed in the setting of Hopf algebras and quantum groups [Voi08]. In our context, these objects will arise naturally when dealing with noncommutative equivariant differential forms.

Recall from Example 1.27 that for a given groupoid $\mathcal{G}$ its subset of loops

$$\mathcal{G}_{ad} = \{\alpha \in \mathcal{G} \mid r(\alpha) = s(\alpha)\}$$

is a subgroupoid of $\mathcal{G}$ with the same identities. If $\mathcal{G}$ is an ample groupoid, since $\mathcal{G}_{ad} = (s, r)^{-1}(\Delta)$, where $\Delta \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is the diagonal, we see that $\mathcal{G}_{ad}$ is a closed subset of $\mathcal{G}$, and thus a totally disconnected locally compact space with the subspace topology.

Due to Lemma 1.55, a function $f \in C_c^\infty(\mathcal{G}_{ad})$ can be represented as a linear combination of restrictions to $\mathcal{G}_{ad}$ of characteristic functions of compact open bisections of $\mathcal{G}$.

Remark 2.43. *However, even if such restriction is no longer a compact open bisection of* $\mathcal{G}$*, we will often refer to the characteristic function* χ_U *of a compact open bisection* $U \subseteq \mathcal{G}$ *as an element of* $C_c^\infty(\mathcal{G}_{ad})$ *without explicitly considering* $U \cap \mathcal{G}_{ad}$*.*

As we have already seen in Example 1.37, $\mathcal{G}_{ad}$ is a $\mathcal{G}$-space with the adjoint action

$$\alpha \cdot \beta = \alpha\beta\alpha^{-1}$$

for $\alpha \in \mathcal{G}, \beta \in \mathcal{G}_{ad}$, with anchor map $\pi = r = s : \mathcal{G}_{ad} \to \mathcal{G}^{(0)}$. According to Proposition 2.28 we therefore obtain a natural $\mathcal{G}$-algebra structure on $C_c^\infty(\mathcal{G}_{ad})$. We will write $\mathcal{O}_{\mathcal{G}}$ for this $\mathcal{G}$-algebra in the sequel.

Definition 2.44. *A $\mathcal{G}$-anti-Yetter-Drinfeld module is a $\mathcal{G}$-module M which is also an essential $\mathcal{O}_\mathcal{G}$-module such that the module action induces a $\mathcal{G}$-equivariant linear map $\mathcal{O}_\mathcal{G} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} M \to M$. A morphism of $\mathcal{G}$-anti-Yetter-Drinfeld modules is a $\mathcal{G}$-equivariant linear map which is also $\mathcal{O}_\mathcal{G}$-linear.*

A basic example of a $\mathcal{G}$-anti-Yetter-Drinfeld module is obtained by considering $M = \mathcal{O}_\mathcal{G} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$ for a $\mathcal{G}$-module E, with the diagonal action of $\mathcal{G}$ and the action of $\mathcal{O}_\mathcal{G}$ by pointwise multiplication on the first tensor factor.

One can view $\mathcal{G}$-anti-Yetter-Drinfeld modules equivalently as essential modules over the crossed product $A(\mathcal{G}) = \mathcal{O}_\mathcal{G} \rtimes \mathcal{G}$. This observation is a special case of Proposition 2.42, note that both $\mathcal{O}_\mathcal{G}$ and $\mathcal{D}(\mathcal{G})$ are subalgebras of the multiplier algebra $M(A(\mathcal{G}))$, and composition with the inclusion maps gives the asserted equivalence. We will frequently use this identification between $\mathcal{G}$-anti-Yetter-Drinfeld modules and $A(\mathcal{G})$-modules in the sequel.

Given a $\mathcal{G}$-anti-Yetter-Drinfeld module M our goal is to define a certain canonical automorphism $T = T_M : M \to M$. We start with $M = A(\mathcal{G}) = \mathcal{O}_\mathcal{G} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}) = C_c^\infty(\mathcal{G}_{ad} \times_{\pi,r} \mathcal{G})$, in which case we define T by the formula

$$T(f)(\alpha,\beta) = f(\alpha,\alpha\beta)$$

for $f \in C_c^\infty(\mathcal{G}_{ad} \times_{\pi,r} \mathcal{G})$, in a similar way as in the discussion of $C_c^\infty(\mathcal{G})$-comodules.

Lemma 2.45. *The map $T : A(\mathcal{G}) \to A(\mathcal{G})$ defined above is an isomorphism of $A(\mathcal{G})$-bimodules.*

Proof. It is clear that T is bijective with inverse given by $T^{-1}(f)(\alpha,\beta) = f(\alpha,\alpha^{-1}\beta)$. The left and right $\mathcal{O}_\mathcal{G}$-module structures on $A(\mathcal{G})$ are given by

$$(h \cdot f)(\alpha,\beta) = h(\alpha)f(\alpha,\beta), \qquad (f \cdot h)(\alpha,\beta) = h(\beta^{-1}\alpha\beta)f(\alpha,\beta)$$

for $h \in \mathcal{O}_\mathcal{G}$ and $f \in A(\mathcal{G})$. We thus obtain

$$\begin{aligned}(h \cdot T(f))(\alpha,\beta) &= h(\alpha)f(\alpha,\alpha\beta)\\ &= (h \cdot f)(\alpha,\alpha\beta)\\ &= T(h \cdot f)(\alpha,\beta)\end{aligned}$$

and

$$\begin{aligned}(T(f) \cdot h)(\alpha,\beta) &= h(\beta^{-1}\alpha\beta)f(\alpha,\alpha\beta)\\ &= (f \cdot h)(\alpha,\alpha\beta)\end{aligned}$$

$$= T(f \cdot h)(\alpha, \beta),$$

which shows that T is both left and right $\mathcal{O}_\mathcal{G}$-linear. For $g \in \mathcal{D}(\mathcal{G})$ we have

$$\begin{aligned}(g \cdot T(f))(\alpha, \beta) &= \sum_{\gamma \in \mathcal{G}^{r(\alpha)}} g(\gamma) f(\gamma^{-1}\alpha\gamma, \gamma^{-1}\alpha\gamma\gamma^{-1}\beta) \\ &= T(g \cdot f)(\alpha, \beta)\end{aligned}$$

and

$$\begin{aligned}(T(f) \cdot g)(\alpha, \beta) &= \sum_{\gamma \in \mathcal{G}_{s(\beta)}} f(\alpha, \alpha\beta\gamma^{-1}) g(\gamma) \\ &= T(f \cdot g)(\alpha, \beta),\end{aligned}$$

and it follows that T is left and right $\mathcal{D}(\mathcal{G})$-linear. Combining these observations yields the claim. □

In view of Lemma 2.45 we can define $T_M : M \to M$ for a $\mathcal{G}$-anti-Yetter-Drinfeld module M by

$$T_M = m_M (T \otimes \mathrm{id}) m_M^{-1}, \tag{2.5}$$

where $m_M : A(\mathcal{G}) \otimes_{A(\mathcal{G})} M \to M$ is the canonical isomorphism. This defines an automorphism of the $\mathcal{G}$-anti-Yetter-Drinfeld module M.

Lemma 2.46. *Let $\mathcal{G}$ be an ample groupoid and let $\phi : M \to N$ be a morphism of $\mathcal{G}$-anti-Yetter-Drinfeld modules. Then $T_N \phi = \phi T_M$.*

Proof. Using the canonical isomorphisms m_M, m_N and the relation $m_N(\mathrm{id} \otimes \phi) = \phi m_M$ we compute

$$\begin{aligned}T_N \phi &= m_N (T \otimes \mathrm{id}) m_N^{-1} \phi \\ &= m_N (T \otimes \mathrm{id})(\mathrm{id} \otimes \phi) m_M^{-1} \\ &= m_N (\mathrm{id} \otimes \phi)(T \otimes \mathrm{id}) m_M^{-1} \\ &= \phi m_M (T \otimes \mathrm{id}) m_M^{-1} \\ &= \phi T_M\end{aligned}$$

as required. □

In calculations, it is useful to have an explicit formula for the action of the canonical automorphism. We will only need this for $\mathcal{G}$-anti-Yetter-Drinfeld modules of the form $M = \mathcal{O}_\mathcal{G} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$ for a $\mathcal{G}$-module E. In this case, recalling (2.5) and following the spirit of the proof of Lemma 2.12, we get

$$T_M(\chi_U \otimes e) = \chi_U \otimes \chi_{U^{-1}} \cdot e \tag{2.6}$$

for any compact open bisection $U \subseteq \mathcal{G}$ and $e \in E$.

Example 2.47. *Let $E = C_c^\infty(\mathcal{G}^{(0)})$, then the canonical automorphism*

$$T : \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \to \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})$$

is the identity map. Indeed, recalling the Remark 2.43, the Equation 2.6 and the Example 1.35, for any open and compact bisection U of $\mathcal{G}$ we have $(U \cap \mathcal{G}_{ad}) \cdot s(U \cap \mathcal{G}_{ad}) = s(U \cap \mathcal{G}_{ad})$.

Chapter 3

Equivariant periodic cyclic homology

The main goal of this chapter will be the definition of bivariant equivariant periodic cyclic homology with respect to an ample groupoid $\mathcal{G}$. We start recalling the construction of the pro-category. In this context, we define paracomplexes and quasifree algebras. Finally, we state the main definition of this chapter and discuss some consequences.

§ 3.1 | Projective systems

One of the key aspects of the approach developed by Cuntz and Quillen [CQ97] to the study of periodic cyclic homology is the enlargement of the framework from algebras to projective systems of algebras. This generalisation proves convenient for discussing quasifreeness, even when the focus remains on algebras. One of the reasons to consider projective systems is that one of the main object we will soon define, the periodic tensor algebra, is a projective system. In this spirit, we will consider projective systems of $\mathcal{G}$-modules and $\mathcal{G}$-anti-Yetter-Drinfeld modules.

Most of the subsequent results and definitions are well-known and established in the literature. Our exposition follows the perspective developed in [CQ97], [Mey99] and [Voi03].

In the remaining part of this section $\mathcal{C}$ will denote an arbitrary additive category.

Definition 3.1. *A projective system over $\mathcal{C}$ consists of a covariant functor $F : I^{op} \to \mathcal{C}$, where I is a directed index set viewed as a small category.*

In more concrete terms, a projective system over $\mathcal{C}$ consists of a directed index set I, a collection of objects $(V_i)_{i\in I}$ in $\mathcal{C}$, and morphisms $p_{ij} : V_j \to V_i$ for all $j \geq i$, satisfying the compatibility conditions $p_{ij}p_{jk} = p_{ik}$ for all $k \geq j \geq i$ and $p_{ii} = \mathrm{id}_{V_i}$ for all $i \in I$.

Definition 3.2. *The* pro-category $\mathrm{pro}(\mathcal{C})$ *is the category whose objects are projective* *systems over $\mathcal{C}$, and whose morphism sets are defined by*

$$\mathrm{Mor}_{\mathrm{pro}(\mathcal{C})}(V, W) := \varprojlim_{j \in J} \varinjlim_{i \in I} \mathrm{Mor}_{\mathcal{C}}(V_i, W_j),$$

where $V = (V_i)_{i \in I}$ and $W = (W_j)_{j \in J}$, and the limits are taken in the category of abelian groups.

Unpacking the above definition, a morphism $\phi : V \to W$ can be described by the data of a function $j \mapsto i(j)$ from J to I, and a family of morphisms $\{\phi_j : V_{i(j)} \to W_j\}_{j \in J}$ in $\mathcal{C}$, such that the following compatibility condition holds: for any $j' \geq j$, there exists $i' \geq i(j), i(j')$ such that the diagram

$$\begin{array}{ccccc} V_{i'} & \longrightarrow & V_{i(j')} & \xrightarrow{\phi_{j'}} & W_{j'} \\ & \searrow & & & \downarrow \\ & & V_{i(j)} & \xrightarrow{\phi_j} & W_j \end{array}$$

commutes.

Moreover, two such families of morphisms $\{\phi_j : V_{i(j)} \to W_j\}_{j \in J}$ and $\{\phi'_j : V_{i'(j)} \to W_j\}_{j \in J}$ define the same morphism in $\mathrm{pro}(\mathcal{C})$ if there exists a function $j \mapsto i''(j)$ such that $i''(j) \geq i(j), i'(j)$ for all $j \in J$, and the following diagram

$$\begin{array}{ccccc} & & V_{i'(j)} & & \\ & \nearrow & & \searrow^{\phi'_j} & \\ V_{i''(j)} & & & & W_j \\ & \searrow & & \nearrow_{\phi_j} & \\ & & V_{i(j)} & & \end{array}$$

commutes for each j.

Definition 3.3. *A* constant pro-object *in* $\mathrm{pro}(\mathcal{C})$ *is a projective system indexed by a singleton set.*

Remark 3.4. *Any object of $\mathcal{C}$ can be viewed as a constant pro-object, and this gives rise to a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{pro}(\mathcal{C})$, which identifies $\mathcal{C}$ with the full subcategory of constant pro-objects inside* $\mathrm{pro}(\mathcal{C})$.

It will be useful to study pro-objects in comparison with constant pro-objects.

Remark 3.5. *From the description of morphisms in the pro-category, we observe that a morphism $\phi : V \to C$, where $V = (V_i)_{i \in I}$ is a pro-object and C is a constant pro-object associated to an object in $\mathcal{C}$, can be represented by a morphism $\phi_i : V_i \to C$ in $\mathcal{C}$, for some index $i \in I$.*

In the category $\mathrm{pro}(\mathcal{C})$, projective limits always exist. Given a projective system of pro-objects $(V_i)_{i\in I}$, where each V_i is itself a projective system $(V_{ij})_{j\in J_i}$ in $\mathcal{C}$, the projective limit of the system $(V_i)_i$ in $\mathrm{pro}(\mathcal{C})$ can be described as the pro-object associated to the double-indexed system $(V_{ij})_{(i,j)\in K}$, where $K := \{(i,j) \mid i \in I, j \in J_i\}$ with the ordering defined by declaring $(i,j) \le (i',j')$ if and only if $i \le i'$, and there exists a morphism $V_{i'j'} \to V_{ij}$ in $\mathcal{C}$ induced by the morphism $V_{i'} \to V_i$.

Moreover, the category $\mathrm{pro}(\mathcal{C})$ is canonically additive. In particular, we can form direct sums in $\mathrm{pro}(\mathcal{C})$. Let $V = (V_i)_{i\in I}$ and $W = (W_j)_{j\in J}$ be two pro-objects. Their direct sum is given by

$$V \oplus W := (V_i \oplus W_j)_{(i,j)\in I\times J},$$

where the index set $I \times J$ is ordered using the product ordering, that is, $(i,j) \le (i',j')$ if and only if $i \le i'$ and $j \le j'$. The structure maps in this system are defined component-wise as the direct sums of the corresponding structure maps of V and W.

If, in addition, the category $(\mathcal{C}, \otimes, \mathbb{U})$ is monoidal, then the pro-category $\mathrm{pro}(\mathcal{C})$ inherits a natural monoidal structure. Given two pro-objects $V = (V_i)_{i\in I}$ and $W = (W_j)_{j\in J}$, their tensor product is defined as the pro-object

$$V \otimes W := (V_i \otimes W_j)_{(i,j)\in I\times J},$$

where the index set $I \times J$ is ordered via the product ordering. The structure maps of the tensor product are given by the tensor products of the corresponding structure maps of V and W. The unit object in $\mathrm{pro}(\mathcal{C})$ is given by the constant pro-object $\mathbb{U}$.

Moreover, any morphism $\phi : V \otimes W \to C$ in $\mathrm{pro}(\mathcal{C})$, where C is a constant pro-object, can be written in the form $\phi = \psi(\phi_V \otimes \phi_W)$, where $\phi_V : V \to C_V$ and $\phi_W : W \to C_W$ are morphisms in $\mathrm{pro}(\mathcal{C})$ with constant targets and $\psi : C_V \otimes C_W \to C$ is a morphism between constant pro-objects.

With this tensor product, the category $\mathrm{pro}(\mathcal{C})$ becomes an additive monoidal category, and the embedding functor $\mathcal{C} \hookrightarrow \mathrm{pro}(\mathcal{C})$ is a monoidal functor. The existence of a tensor product in $\mathrm{pro}(\mathcal{C})$ allows us to define algebra objects within this setting.

Definition 3.6. *A* pro-algebra *is an algebra object in* $\mathrm{pro}(\mathcal{C})$. *A* pro-algebra homomorphism *is a homomorphism between pro-algebra objects.*

If we apply these general constructions to the category of $\mathcal{G}$-modules we obtain the category $\mathrm{pro}(\mathcal{G}\text{-Mod})$ of pro-$\mathcal{G}$-modules. A morphism in $\mathrm{pro}(\mathcal{G}\text{-Mod})$ will be called a $\mathcal{G}$-equivariant pro-linear map. Similarly, we have the category of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules.

According to Proposition 2.25, the category $\mathcal{G}$-Mod is additive monoidal. We then record the following definition.

Definition 3.7. *A* pro-$\mathcal{G}$-algebra *is an algebra object in* pro($\mathcal{G}$-Mod)*, in the same way as $\mathcal{G}$-algebras are algebra objects in* $\mathcal{G}$-Mod*. An algebra homomorphism $f : A \to B$ in* pro($\mathcal{G}$-Mod) *will simply be called a* $\mathcal{G}$-equivariant homomorphism.

Clearly, every $\mathcal{G}$-algebra is a pro-$\mathcal{G}$-algebra in a canonical way.

Occasionally we will encounter unital pro-$\mathcal{G}$-algebras. The $\mathcal{G}$-unitarisation A^+ of a pro-$\mathcal{G}$-algebra A is defined in the same way as for $\mathcal{G}$-algebras. Similarly, the construction of crossed products for $\mathcal{G}$-algebras carries over to pro-$\mathcal{G}$-algebras.

Let $\mathcal{C}$ be any additive category.

Definition 3.8. *Let K, E and Q be objects in* pro($\mathcal{C}$)*. An admissible extension is a diagram of the form*

$$0 \longrightarrow K \underset{\rho}{\overset{\iota}{\rightleftarrows}} E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0$$

in pro($\mathcal{C}$) *such that $\rho\iota = id_K$, $\pi\sigma = id_Q$ and $\iota\rho + \sigma\pi = id_E$.*

In other words, we require that E decomposes into a direct sum of K and Q in pro($\mathcal{C}$).

We will write

$$K \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} Q$$

or simply $0 \to K \to E \to Q \to 0$ for an admissible extension.

In the special case where $\mathcal{C}$ is $\mathcal{G}$-Mod, we will often use the following.

Definition 3.9. *Let K, E and Q be pro-$\mathcal{G}$-algebras. An admissible extension of pro-$\mathcal{G}$-algebras is an admissible extension $0 \to K \to E \to Q \to 0$ in* pro($\mathcal{G}$-Mod) *such that ι and π are $\mathcal{G}$-equivariant algebra homomorphism.*

Remark 3.10. *In the following, we will often describe morphisms between pro-objects by writing explicit formulas involving "elements" of the objects themselves. While such notation is not strictly rigorous, since pro-objects, being formal inverse systems, do not generally possess elements in the usual sense, this approach can be categorically justified.*

Indeed, any pro-object $V = (V_i)_{i\in I}$ naturally gives rise to a contravariant functor

$$h^V : \mathcal{C} \to \mathrm{Set}, \quad T \mapsto \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(T, V_i),$$

and the Yoneda Lemma asserts that an object is entirely determined by the hom-functor it represents. In this perspective, an "element" of V corresponds to a choice of T together *with a compatible family of morphisms $T \to V_i$.*

A morphism between two pro-objects $V = (V_i)$ and $W = (W_j)$ is then a natural transformation

$$\alpha : h^V \Longrightarrow h^W,$$

which consists of a family of maps

$$\alpha_T : \varprojlim_i \mathrm{Mor}_{\mathcal{C}}(T, V_i) \longrightarrow \varprojlim_j \mathrm{Mor}_{\mathcal{C}}(T, W_j),$$

natural in the object $T \in \mathcal{C}$. Thus, specifying how a morphism sends an "element" $x \in V$ to an element $f(x) \in W$ amounts to giving a natural transformation between functors, hence a well-defined morphism in $\mathrm{pro}(\mathcal{C})$*. This can be rephrased, saying that in many situations one can embed the category* $\mathrm{pro}(\mathcal{C})$ *in a concrete category of modules over a certain ring. This is known as the Freyd-Mitchell's embedding theorem, see [Wei94, Theorem 1.6.1] for reference.*

§ 3.2 | Paracomplexes

In this section, we introduce the notion of a paracomplex in a para-additive category. This concept will be crucial for the main definition of this chapter.

Definition 3.11. *A para-additive category is an additive category $\mathcal{C}$ together with a natural automorphism T of the identity functor* $\mathrm{id} : \mathcal{C} \to \mathcal{C}$*.*

More concretely, we are given a family of invertible morphisms $T_M : M \to M$ indexed by the objects M in the category $\mathcal{C}$ such that $\phi T_M = T_N \phi$ for all morphisms $\phi : M \to N$. In the sequel, we will simply write T instead of T_M if it is clear from the context.

Remark 3.12. *Any additive category is para-additive by setting* $T = \mathrm{id}$.

By Lemma 2.46, we know that the category of $\mathcal{G}$-anti-Yetter-Drinfeld modules is para-additive. This category, together with its pro-category, will serve as the main framework for our subsequent development.

Definition 3.13. *Let $\mathcal{C}$ be a para-additive category. A* paracomplex $C = C_0 \oplus C_1$ *in $\mathcal{C}$ consists of objects C_0 and C_1 in $\mathcal{C}$, together with morphisms $\partial_0 : C_0 \to C_1$ and $\partial_1 : C_1 \to C_0$ such that the differential*

$$\partial := \begin{pmatrix} 0 & \partial_1 \\ \partial_0 & 0 \end{pmatrix} : C \to C_1 \oplus C_0 \cong C$$

satisfies the relation

$$\partial^2 = \mathrm{id} - T,$$

where $T : C \to C$ is the automorphism associated with C.

A chain map $\phi : C \to D$ *between two paracomplexes is a morphism in $\mathcal{C}$ that preserves the $\mathbb{Z}_2$-grading and commutes with the differentials, that is, $\phi\partial = \partial\phi$.*

Remark 3.14. *Since the differential ∂ in the definition of a paracomplex, in general, is not a differential in the usual sense of homological algebra, in fact it does not square to zero but instead satisfies $\partial^2 = \mathrm{id} - T$, it does not make sense to speak of the homology of a paracomplex in the standard way. However, one can still define the notion of* homotopy equivalence *between paracomplexes, using the standard formulas for chain homotopies. That is, two paracomplexes C and D are said to be homotopy equivalent if there exist chain maps $f : C \to D$ and $g : D \to C$ such that gf and fg are homotopic to the identity on C and the identity on D, respectively.*

The paracomplexes we are interested in arise from paramixed complexes in the following sense.

Definition 3.15. *Let $\mathcal{C}$ be a para-additive category. A paramixed complex M in $\mathcal{C}$ is a sequence of objects M_n together with differentials b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and*

$$[b, B] = bB + Bb = \mathrm{id} - T.$$

This definition is crucial, as we will see, in the next section, that the equivariant version of noncommutative differential forms gives rise to a paracomplex. In that context, it is still possible to define Hochschild homology in the usual way, since the differential b, which corresponds to the Hochschild operator, satisfies the identity $b^2 = 0$.

Nevertheless, in this thesis, we are not concerned with Hochschild homology. Our focus will be entirely on periodic cyclic homology.

§ 3.3 | Equivariant differential forms

We will now define the space of equivariant noncommutative differential forms over a pro-$\mathcal{G}$-algebra. Throughout this section, we will review constructions that are already known in the literature. More details and motivations for these topics can be found in [CQ95a]. We will follow the notation used in [Voi03].

Definition 3.16. *Let A be a pro-$\mathcal{G}$-algebra. We define the space of noncommutative differential n-forms over A by the iterated tensor products over $C_c^\infty(\mathcal{G}^{(0)})$ given by*

$$\Omega^n_{\mathcal{G}^{(0)}}(A) = A^+ \otimes_{C_c^\infty(\mathcal{G}^{(0)})} A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})} n} \cong A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})} n+1} \oplus A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})} n}$$

for all $n > 0$, where A^+ denotes the $\mathcal{G}$-unitarisation of A as defined in Section 2.3.4 and we set $\Omega^0_{\mathcal{G}^{(0)}}(A) = A$.

Using the definition of $\mathcal{G}$-unitarisation and the essentiality of A, the space of noncommutative n-forms over A decomposes as

$$\Omega^n_{\mathcal{G}^{(0)}}(A) = (A \oplus C_c^\infty(\mathcal{G}^{(0)})) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})}n} \cong A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})}n+1} \oplus A^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})}n}.$$

Elements of $\Omega^n_{\mathcal{G}^{(0)}}(A)$ contained in the first summand of the above decomposition, namely, tensors of the form $a^0 \otimes a^1 \otimes \cdots \otimes a^n$ with $a^0, a^1, \ldots, a^n \in A$, will usually be written as $a^0 da^1 \cdots da^n$. Similarly, elements in the second summand will be denoted by $da^1 \cdots da^n$. If we want to treat both cases simultaneously, we shall write $\langle a^0 \rangle da^1 \cdots da^n$, following the notation used in [Mey99].

We always view $\Omega^n_{\mathcal{G}^{(0)}}(A)$ as a pro-$\mathcal{G}$-module with the diagonal action.

The pro-$\mathcal{G}$-module $\Omega^n_{\mathcal{G}^{(0)}}(A)$ becomes an A-A-bimodule object in $\text{pro}(\mathcal{G}\text{-Mod})$ with the left A-module structure given by

$$a \cdot (\langle a^0 \rangle da^1 \cdots da^n) = a \langle a^0 \rangle da^1 \cdots da^n,$$

and the right A-module determined by the Leibniz rule, that is,

$$(\langle a^0 \rangle da^1 \cdots da^n) \cdot a = \langle a^0 \rangle da^1 \cdots d(a^n a) + \sum_{j=1}^{n-1} (-1)^{n-j} \langle a^0 \rangle da^1 \cdots d(a^j a^{j+1}) \cdots da^n da$$
$$+ (-1)^n \langle a^0 \rangle a^1 da^2 \cdots da^n da,$$

for $a \in A$ and $\langle a^0 \rangle da^1 \cdots da^n \in \Omega^n_{\mathcal{G}^{(0)}}(A)$.

Remark 3.17. *The A-A-bimodule $\Omega^n_{\mathcal{G}^{(0)}}(A)$ can be identified with the n-fold tensor product of $\Omega^1_{\mathcal{G}^{(0)}}(A)$ over A in the category of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules.*

According to the Remark 3.17, one defines a map

$$\Omega^n_{\mathcal{G}^{(0)}}(A) \otimes \Omega^m_{\mathcal{G}^{(0)}}(A) \to \Omega^{n+m}_{\mathcal{G}^{(0)}}(A)$$

by considering the natural projection

$$\Omega^1_{\mathcal{G}^{(0)}}(A)^{\otimes_A n} \otimes \Omega^1_{\mathcal{G}^{(0)}}(A)^{\otimes_A m} \to \Omega^1_{\mathcal{G}^{(0)}}(A)^{\otimes_A n} \otimes_A \Omega^1_{\mathcal{G}^{(0)}}(A)^{\otimes_A m} = \Omega^1_{\mathcal{G}^{(0)}}(A)^{\otimes_A n+m}.$$

Definition 3.18. *We define $\Omega_{\mathcal{G}^{(0)}}(A)$ as the direct sum $\bigoplus_{n \geq 0} \Omega^n_{\mathcal{G}^{(0)}}(A)$.*

Remark 3.19. *A noncommutative differential form ω is called* homogeneous *if it belongs to $\Omega^n_{\mathcal{G}^{(0)}}(A)$ for some $n \in \mathbb{N}$.*

The maps

$$\Omega^n_{\mathcal{G}^{(0)}}(A) \otimes \Omega^m_{\mathcal{G}^{(0)}}(A) \to \Omega^{n+m}_{\mathcal{G}^{(0)}}(A)$$

assemble to give a multiplication over $\Omega_{\mathcal{G}^{(0)}}(A)$, which becomes a pro-$\mathcal{G}$-algebra in a natural way.

We also set the $C_c^\infty(\mathcal{G}^{(0)})$-linear operator $d : \Omega^n_{\mathcal{G}^{(0)}}(A) \to \Omega^{n+1}_{\mathcal{G}^{(0)}}(A)$ by

$$d(a^0da^1\cdots da^n) = da^0da^1\cdots da^n \qquad \text{and} \qquad d(da^1\cdots da^n) = 0,$$

for $a^0, a^1, \ldots, a^n \in A$.

Remark 3.20. *Observe that, by construction, one has that* $d^2 = 0$.

Next, we introduce the $\mathcal{G}$-equivariant version of noncommutative differential forms over a pro-$\mathcal{G}$-algebra A.

Definition 3.21. *We define*

$$\Omega^0_{\mathcal{G}}(A) \coloneqq \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} A \qquad \text{and} \qquad \Omega^n_{\mathcal{G}}(A) \coloneqq \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \Omega^n_{\mathcal{G}^{(0)}}(A)$$

for $n > 0$, *where we recall that* $\mathcal{O}_{\mathcal{G}}$ *is the* $\mathcal{G}$*-algebra of functions on* $\mathcal{G}_{ad}$ *with the adjoint action.*

This becomes a pro-$\mathcal{G}$-module with the diagonal action, and a pro-$\mathcal{O}_{\mathcal{G}}$-module with the multiplication action on the first tensor factor. These actions turn $\Omega^n_{\mathcal{G}}(A)$ into a pro-$\mathcal{G}$-anti-Yetter-Drinfeld module. We write $\Omega_{\mathcal{G}}(A)$ for the direct sum of all $\Omega^n_{\mathcal{G}}(A)$ for $n \geq 0$.

We need several operators on $\mathcal{G}$-equivariant differential forms. We start with the equivariant version of the differential operator d defined above.

Definition 3.22. *We define* $d_{\mathcal{G}} : \Omega^n_{\mathcal{G}}(A) \to \Omega^{n+1}_{\mathcal{G}}(A)$ *by*

$$d_{\mathcal{G}}(f \otimes \omega) = f \otimes d\omega,$$

where $f \in C_c^\infty(\mathcal{G}_{ad})$ *and* $\omega \in \Omega^n_{\mathcal{G}^{(0)}}(A)$.

Next we introduce an equivariant version of the Hochschild operator b.

Definition 3.23. *The equivariant Hochschild operator* $b_{\mathcal{G}} : \Omega^n_{\mathcal{G}}(A) \to \Omega^{n-1}_{\mathcal{G}}(A)$ *is defined by*

$$b_{\mathcal{G}}(f \otimes \omega da) = (-1)^{n-1}(f \otimes \omega a - (\mathrm{id} \otimes \mu)(T(f \otimes a) \otimes \omega))$$

for $n > 1$, *where* μ *denotes multiplication in* $\Omega_{\mathcal{G}^{(0)}}(A)$ *and* T *is the canonical map and by* $b_{\mathcal{G}} = 0$ *for* $n = 0$.

If $U \subseteq \mathcal{G}$ is a compact open bisection, then we can write this in the form

$$b_{\mathcal{G}}(\chi_U \otimes \omega da) = (-1)^{n-1}(\chi_U \otimes \omega a - \chi_U \otimes (\chi_{U^{-1}} \cdot a)\omega),$$

or, using the Leibniz rule, we can expand this explicitly as

$$\begin{aligned} b_{\mathcal{G}}(\chi_U \otimes \langle a^0\rangle da^1 \cdots da^n) = \chi_U \otimes \langle a^0\rangle a^1 da^2 \cdots da^n \\ + \sum_{j=1}^{n-1}(-1)^j \chi_U \otimes \langle a^0\rangle da^1 \cdots d(a^j a^{j+1}) \cdots da^n \\ + (-1)^n \chi_U \otimes (\chi_{U^{-1}} \cdot a^n)\langle a^0\rangle da^1 \cdots da^{n-1}, \end{aligned}$$

for $\langle a^0\rangle a^1 da^2 \cdots da^n \in \Omega^n_{\mathcal{G}^{(0)}}(A)$.

Lemma 3.24. *The operator $b_{\mathcal{G}}$ is a differential, that is, $b_{\mathcal{G}}^2 = 0$.*

Proof. Using the formulas developed above, we can calculate

$$\begin{aligned} b_{\mathcal{G}}^2(\chi_U \otimes \omega da^1 da^2) =& b_{\mathcal{G}}((-1)^{n+1}(\chi_U \otimes ((\omega da^1)a^2 - (\chi_{U^{-1}} \cdot a^2)\omega da^1))) \\ =& (-1)^{n+1} b_{\mathcal{G}}(\chi_U \otimes (\omega d(a^1a^2) - \omega a^1 da^2 - (\chi_{U^{-1}} \cdot a^2)\omega da^1)) \\ =& (-1)^{n+1} b_{\mathcal{G}}(\chi_U \otimes \omega d(a^1a^2) - \chi_U \otimes \omega a^1 da^2 - \chi_U \otimes (\chi_{U^{-1}} \cdot a^2)\omega da^1) \\ =& (-1)^{n+1}(-1)^n(\chi_U \otimes ((\omega a^1a^2 - (\chi_{U^{-1}} \cdot a^1a^2)\omega) \\ & - (\omega a^1a^2 - (\chi_{U^{-1}} \cdot a^2)\omega a^1) \\ & - ((\chi_{U^{-1}} \cdot a^2)\omega a^1 - (\chi_{U^{-1}} \cdot a^1)(\chi_{U^{-1}} \cdot a^2)\omega))) = 0, \end{aligned}$$

where $U \subseteq \mathcal{G}$ is a compact open bisection. □

Starting from $d_{\mathcal{G}}$ and $b_{\mathcal{G}}$ we define two more operators.

Definition 3.25. *Define the $\mathcal{G}$-equivariant Karoubi operator $\kappa_{\mathcal{G}}$ by*

$$\kappa_{\mathcal{G}} = \mathrm{id} - (b_{\mathcal{G}}d_{\mathcal{G}} + d_{\mathcal{G}}b_{\mathcal{G}}),$$

and the $\mathcal{G}$-equivariant Connes operator $B_{\mathcal{G}}$ by

$$B_{\mathcal{G}} = \sum_{j=0}^{n} \kappa_{\mathcal{G}}^j d_{\mathcal{G}}$$

on $\Omega^n_{\mathcal{G}^{(0)}}(A)$ for $n \geq 0$.

Lemma 3.26. *The operator $B_{\mathcal{G}}$ is a differential, that is, $B_{\mathcal{G}}^2 = 0$.*

Proof. Using $d_{\mathcal{G}}^2 = 0$ we see that $d_{\mathcal{G}}$ and $\kappa_{\mathcal{G}}$ commute. Indeed we compute

$$\begin{aligned} d_{\mathcal{G}}\kappa_{\mathcal{G}} &= d_{\mathcal{G}}(\mathrm{id} - (b_{\mathcal{G}}d_{\mathcal{G}} + d_{\mathcal{G}}b_{\mathcal{G}})) \\ &= d_{\mathcal{G}} - d_{\mathcal{G}}b_{\mathcal{G}}d_{\mathcal{G}} - d_{\mathcal{G}}^2 b_{\mathcal{G}} \\ &= d_{\mathcal{G}} - d_{\mathcal{G}}b_{\mathcal{G}}d_{\mathcal{G}} - b_{\mathcal{G}}d_{\mathcal{G}}^2 \\ &= (\mathrm{id} - (b_{\mathcal{G}}d_{\mathcal{G}} + d_{\mathcal{G}}b_{\mathcal{G}}))d_{\mathcal{G}} \\ &= \kappa_{\mathcal{G}}d_{\mathcal{G}}, \end{aligned}$$

and then we conclude that $B_{\mathcal{G}}^2 = 0$. □

We can write explicit formulas for $\kappa_{\mathcal{G}}$ and $B_{\mathcal{G}}$. For $n > 0$ and a compact open bisection $U \subseteq \mathcal{G}$ we obtain

$$\kappa_{\mathcal{G}}(\chi_U \otimes \omega da) = (-1)^{n-1}\chi_U \otimes (\chi_{U^{-1}} \cdot da)\omega,$$

and for $n = 0$ we get $\kappa_{\mathcal{G}}(\chi_U \otimes a) = \chi_U \otimes \chi_{U^{-1}} \cdot a$. For $B_{\mathcal{G}}$ one calculates

$$B_{\mathcal{G}}(\chi_U \otimes a^0 da^1 \cdots da^n) = \sum_{i=0}^{n} (-1)^{ni} \chi_U \otimes (\chi_{U^{-1}} \cdot (da^{n+1-i} \cdots da^n)) da^0 \cdots da^{n-i}.$$

Recalling the discussion of Section 2.4, we remark that the canonical operator T for $\Omega_{\mathcal{G}}(A)$ is given by

$$T(\chi_U \otimes \omega) = \chi_U \otimes \chi_{U^{-1}} \cdot \omega.$$

All the operators introduced above are morphisms of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules, and therefore commute with T by Lemma 2.46.

The following Lemma, which can be proved in a similar way as in the group case, see [Voi07, Lemma 7.2], collects some important properties of the operators defined so far.

Lemma 3.27. *The following identities hold on* $\Omega_{\mathcal{G}}^n(A)$*:*

(i) $\kappa_{\mathcal{G}}^{n+1} d_{\mathcal{G}} = T d_{\mathcal{G}}$*;*

(ii) $\kappa_{\mathcal{G}}^n = T + b_{\mathcal{G}}\kappa_{\mathcal{G}}^n d_{\mathcal{G}}$*;*

(iii) $b_{\mathcal{G}}\kappa_{\mathcal{G}}^n = b_{\mathcal{G}}T$*;*

(iv) $\kappa_{\mathcal{G}}^{n+1} = (\mathrm{id} - d_{\mathcal{G}}b_{\mathcal{G}})T$*;*

(v) $(\kappa_{\mathcal{G}}^{n+1} - T)(\kappa_{\mathcal{G}}^n - T) = 0$*;*

(vi) $B_{\mathcal{G}}b_{\mathcal{G}} + b_{\mathcal{G}}B_{\mathcal{G}} = \mathrm{id} - T$*.*

Proof. The first identity follows directly from the explicit formula for $\kappa_{\mathcal{G}}$. Using iteratively the explicit formula for $\kappa_{\mathcal{G}}$ again, we compute

$$\begin{aligned}\kappa_{\mathcal{G}}^n(\chi_U \otimes a^0 da^1 \cdots da^n) &= \chi_U \otimes \chi_{U^{-1}} \cdot (da^1 \cdots da^n) a^0 \\ &= \chi_U \otimes \chi_{U^{-1}} \cdot (a^0 da^1 \cdots da^n) + (-1)^n b_{\mathcal{G}}(\chi_U \otimes \chi_{U^{-1}} \cdot (da^1 \cdots da^n) da^0) \\ &= T(\chi_U \otimes a^0 da^1 \cdots da^n) + b_{\mathcal{G}} \kappa_{\mathcal{G}}^n d_{\mathcal{G}}(\chi_U \otimes a^0 da^1 \cdots da^n),\end{aligned}$$

and

$$\begin{aligned}\kappa_{\mathcal{G}}^n(\chi_U \otimes da^1 \cdots da^n) &= \chi_U \otimes \chi_{U^{-1}} \cdot (da^1 \cdots da^n) \\ &= T(\chi_U \otimes da^1 \cdots da^n),\end{aligned}$$

which prove (ii). To prove the third identity, apply $b_{\mathcal{G}}$ to both sides of (ii) and use that $b_{\mathcal{G}}^2 = 0$. To prove (iv), apply $\kappa_{\mathcal{G}}$ to both sides of (ii) and use (i) to get

$$\begin{aligned}\kappa_{\mathcal{G}}^{n+1} &= \kappa_{\mathcal{G}} T + \kappa_{\mathcal{G}} b_{\mathcal{G}} \kappa_{\mathcal{G}}^n d_{\mathcal{G}} \\ &= \kappa_{\mathcal{G}} T + b_{\mathcal{G}} \kappa_{\mathcal{G}}^{n+1} d_{\mathcal{G}} \\ &= \kappa_{\mathcal{G}} T + b_{\mathcal{G}} T d_{\mathcal{G}} \\ &= \kappa_{\mathcal{G}} T + b_{\mathcal{G}} d_{\mathcal{G}} T \\ &= (\mathrm{id} - d_{\mathcal{G}} b_{\mathcal{G}}) T,\end{aligned}$$

where we used the fact that $b_{\mathcal{G}}$ commutes with $\kappa_{\mathcal{G}}$ and $d_{\mathcal{G}}$ commutes with T. The identity (v) is a consequence of (ii) and (iv). Indeed, using both, we get

$$\begin{aligned}(\kappa_{\mathcal{G}}^{n+1} - T)(\kappa_{\mathcal{G}}^n - T) &= (T - d_{\mathcal{G}} b_{\mathcal{G}} T - T)(T + b_{\mathcal{G}} \kappa_{\mathcal{G}}^n d_{\mathcal{G}} - T) \\ &= -d_{\mathcal{G}} b_{\mathcal{G}} T b_{\mathcal{G}} \kappa_{\mathcal{G}}^n d_{\mathcal{G}} \\ &= -d_{\mathcal{G}} b_{\mathcal{G}}^2 T \kappa_{\mathcal{G}}^n d_{\mathcal{G}} \\ &= 0,\end{aligned}$$

since $b_{\mathcal{G}}^2 = 0$. Finally, to prove (vi), using the definition of $B_{\mathcal{G}}$, we directly compute

$$\begin{aligned}B_{\mathcal{G}} b_{\mathcal{G}} + b_{\mathcal{G}} B_{\mathcal{G}} &= \sum_{j=0}^{n-1} \kappa_{\mathcal{G}}^j d_{\mathcal{G}} b_{\mathcal{G}} + \sum_{j=0}^{n} \kappa_{\mathcal{G}}^j b_{\mathcal{G}} d_{\mathcal{G}} \\ &= \sum_{j=0}^{n-1} \kappa_{\mathcal{G}}^j (d_{\mathcal{G}} b_{\mathcal{G}} + b_{\mathcal{G}} d_{\mathcal{G}}) + \kappa_{\mathcal{G}}^n b_{\mathcal{G}} d_{\mathcal{G}} \\ &= \sum_{j=0}^{n-1} \kappa_{\mathcal{G}}^j (\mathrm{id} - \kappa_{\mathcal{G}}) + \kappa_{\mathcal{G}}^n b_{\mathcal{G}} d_{\mathcal{G}} \\ &= \mathrm{id} - \kappa_{\mathcal{G}}^n + \kappa_{\mathcal{G}}^n b_{\mathcal{G}} d_{\mathcal{G}} \\ &= \mathrm{id} - \kappa_{\mathcal{G}}^n (\mathrm{id} - b_{\mathcal{G}} d_{\mathcal{G}})\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{id} - \kappa_{\mathcal{G}}^{n}(\kappa_{\mathcal{G}} + d_{\mathcal{G}} b_{\mathcal{G}}) \\
&= \mathrm{id} - \kappa_{\mathcal{G}}^{n+1} - \kappa_{\mathcal{G}}^{n} d_{\mathcal{G}} b_{\mathcal{G}} \\
&= \mathrm{id} - T + d_{\mathcal{G}} b_{\mathcal{G}} T - T d_{\mathcal{G}} b_{\mathcal{G}} - b_{\mathcal{G}} \kappa_{\mathcal{G}}^{n} d_{\mathcal{G}}^{2} b_{\mathcal{G}} \\
&= \mathrm{id} - T,
\end{aligned}
$$

where we use (iv), (ii), and the fact that T commutes with $b_{\mathcal{G}}$ and $d_{\mathcal{G}}$. □

Observe that the final formula of Lemma 3.27, with Lemma 3.24 and Lemma 3.26, yields the following.

Proposition 3.28. *Let A be a pro-$\mathcal{G}$-algebra. The space $\Omega_{\mathcal{G}}(A)$ together with the operators $b_{\mathcal{G}}$ and $B_{\mathcal{G}}$ defines a paramixed complex in the category of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules.*

§ 3.4 | Quasifree pro-$\mathcal{G}$-algebras

In [CQ95a], one of the motivations to introduce and study quasifree algebras is that they are, in a broad sense, a noncommutative analogue of smooth algebras or manifolds. The link is given by the good behaviour of these algebras with nilpotent extensions. Such behaviour characterises smooth algebras in the commutative setting. Let us next discuss the main definitions and facts related to quasifreeness. For further background information in the non-equivariant case, we refer to [Mey99]. We are interested in quasifree pro-$\mathcal{G}$-algebras. In the following, we will review and adapt some of the definitions and results in [Voi03], [Voi07].

We endow the pro-$\mathcal{G}$-module $\Omega_{\mathcal{G}^{(0)}}(A)$ of differential forms over a pro-$\mathcal{G}$-algebra A with the Fedosov product, defined by

$$\omega \circ \eta := \omega\eta - (-1)^{m} d\omega d\eta$$

for forms $\omega \in \Omega^{m}_{\mathcal{G}^{(0)}}(A)$ and $\eta \in \Omega^{n}_{\mathcal{G}^{(0)}}(A)$.

Remark 3.29. *The Fedosov product preserves the forms of even degree.*

The second ingredient in Cuntz and Quillen's approach to periodic cyclic homology is the periodic tensor algebra of a pro-algebra.

Definition 3.30. *Let A be a pro-$\mathcal{G}$-algebra. The* periodic tensor algebra *$\mathcal{T}A$ of A is the pro-$\mathcal{G}$-algebra obtained as the projective limit of the projective system $(\mathcal{T}A/(\mathcal{J}A)^{n})_{n\in\mathbb{N}}$, where $\mathcal{T}A/(\mathcal{J}A)^{n} = A \oplus \Omega^{2}_{\mathcal{G}^{(0)}}(A) \oplus \cdots \oplus \Omega^{2n}_{\mathcal{G}^{(0)}}(A)$ and the structure maps are the canonical projections. Similarly one defines the pro-$\mathcal{G}$-algebra $\mathcal{J}A$ as the projective limit of the projective system $(\mathcal{J}A/(\mathcal{J}A)^{n})_{n\in\mathbb{N}}$, where $\mathcal{J}A/(\mathcal{J}A)^{n} = \Omega^{2}_{\mathcal{G}^{(0)}}(A) \oplus \cdots \oplus \Omega^{2n}_{\mathcal{G}^{(0)}}(A)$.*

The natural projection from $\mathcal{T}A$ to the first term of the projective system gives a $\mathcal{G}$-equivariant homomorphism $\tau_A : \mathcal{T}A \to A$. Moreover, for every $n \in \mathbb{N}$ we have natural inclusions $A \to A \oplus \Omega^2_{\mathcal{G}^{(0)}}(A) \oplus \cdots \oplus \Omega^{2n}_{\mathcal{G}^{(0)}}(A)$, and these maps assemble to a $\mathcal{G}$-equivariant pro-linear section σ_A for τ_A. Then we obtain an admissible extension

$$\mathcal{J}A \overset{\iota_A}{\rightarrowtail} \mathcal{T}A \overset{\tau_A}{\twoheadrightarrow} A$$

of pro-$\mathcal{G}$-algebras.

We discuss some properties of these two objects we have introduced. We will start with some preliminary definitions.

Definition 3.31. *Let N be a pro-$\mathcal{G}$-algebra, and let $m^n : N^{\otimes_{C_c^\infty(\mathcal{G}^{(0)})} n} \to N$ denote the n-fold iterated multiplication. We say that N is* k-nilpotent *for some $k \in \mathbb{N}$ if $m^k = 0$. If N is k-nilpotent for some $k \in \mathbb{N}$, we say that N is* nilpotent.

Moreover, we say that N is locally nilpotent *if for every $\mathcal{G}$-equivariant pro-linear map $f : N \to C$ with constant target C, there exists $n \in \mathbb{N}$ such that $fm^n = 0$.*

Definition 3.32. *An admissible extension*

$$K \rightarrowtail E \twoheadrightarrow Q$$

of pro-$\mathcal{G}$-algebras is called locally nilpotent *(respectively* k-nilpotent, nilpotent*) if the kernel K is locally nilpotent (respectively k-nilpotent, nilpotent) as a pro-$\mathcal{G}$-algebra.*

Lemma 3.33. *The pro-$\mathcal{G}$-algebra $\mathcal{J}A$ is locally nilpotent.*

Proof. Let $l : \mathcal{J}A \to C$ be a $\mathcal{G}$-equivariant pro-linear map with constant target. By Remark 3.5, there exists $n \in \mathbb{N}$ such that l factors through the quotient $\mathcal{J}A/(\mathcal{J}A)^n$. By definition of the Fedosov product, the algebra $\mathcal{J}A/(\mathcal{J}A)^n$ is n-nilpotent. It follows that $lm^n_{\mathcal{J}A} = 0$, as desired. □

Lemma 3.34. *Let N be a locally nilpotent pro-$\mathcal{G}$-algebra and let A be any pro-$\mathcal{G}$-algebra. Then the tensor product $A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$ is locally nilpotent as a pro-$\mathcal{G}$-algebra.*

Proof. Let $f : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N \to C$ be a $\mathcal{G}$-equivariant pro-linear map with constant target. By the description of tensor products in the pro-category, this map can be written in the form

$$f = g(f_A \otimes f_N),$$

where $f_A : A \to C_1$ and $f_N : N \to C_2$ are $\mathcal{G}$-equivariant pro-linear maps with constant targets, and $g : C_1 \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_2 \to C$ is a morphism of constant pro-$\mathcal{G}$-algebras.

Since N is locally nilpotent, there exists $n \in \mathbb{N}$ such that $f_N m_N^n = 0$. Since the multiplication is well-defined, we write

$$m^n_{A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N} = m_A^n \otimes m_N^n.$$

Then it follows that

$$f m^n_{A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N} = g\left(f_A m_A^n \otimes f_N m_N^n\right) = 0.$$

Hence $A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} N$ is locally nilpotent. □

Definition 3.35. *A $\mathcal{G}$-equivariant pro-linear map $l : A \to B$ between pro-$\mathcal{G}$-algebras is called a* $\mathcal{G}$-lonilcur *if its curvature, defined as*

$$\omega_l : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} A \to B, \quad \omega_l(a,b) = l(ab) - l(a)l(b),$$

is locally nilpotent. That is, for every $\mathcal{G}$-equivariant pro-linear map $f : B \to C$ with constant range C, there exists $n \in \mathbb{N}$ such that

$$f m_B^n \omega_l^{\otimes n} = 0.$$

Example 3.36. *Every $\mathcal{G}$-equivariant homomorphism $f : A \to B$ between pro-$\mathcal{G}$-algebras A and B is a $\mathcal{G}$-lonilcur. Indeed, being a homomorphism, one has $\omega_f(a_1, a_2) = 0$ for all $a_1, a_2 \in A$.*

Example 3.37. *The canonical map $\sigma_A : A \to \mathcal{T}A$ is a $\mathcal{G}$-lonilcur. Its curvature $\omega_{\sigma_A}(a,b) = \sigma_A(ab) - \sigma_A(a) \circ \sigma_A(b)$ takes values in $\mathcal{J}A$, which is locally nilpotent by Lemma 3.33. Hence, we conclude that σ_A is a lonilcur.*

The pro-$\mathcal{G}$-algebra $\mathcal{T}A$ together with the $\mathcal{G}$-equivariant pro-linear map $\sigma_A : A \to \mathcal{T}A$ satisfies the following universal property, which can be compared to the universal property satisfied by the usual tensor algebra.

Proposition 3.38. *Let A and B be pro-$\mathcal{G}$-algebras. For any $\mathcal{G}$-equivariant pro-linear map $l : A \to B$ which is a $\mathcal{G}$-lonilcur, there exists a unique $\mathcal{G}$-equivariant homomorphism $[\![l]\!] : \mathcal{T}A \to B$ of pro-$\mathcal{G}$-algebras such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\sigma_A} & \mathcal{T}A \\ & \searrow^{l} & \downarrow{\scriptstyle [\![l]\!]} \\ & & B \end{array}$$

commutes.

Proof. Define a $\mathcal{G}$-equivariant pro-linear map $\phi_l^k : \Omega^{2k}_{\mathcal{G}^{(0)}}(A) \to B$ by

$$\phi_l^k(\langle a^0\rangle da^1 \cdots da^{2k}) = l(\langle a^0\rangle)\omega_l(a^1, a^2)\cdots\omega_l(a^{2k-1}, a^{2k})$$

for all $k \geq 0$, where ω_l is the curvature of l, and l is extended naturally to a $\mathcal{G}$-equivariant pro-linear map $A^+ \to B^+$.

Let $f : B \to C$ be a $\mathcal{G}$-equivariant pro-linear map with constant range. Define $h : B^+ \otimes_{C_c^\infty(\mathcal{G}^{(0)})} B \to C$ by $h(b_0 \otimes b_1) \coloneqq f(b_0 b_1)$. We may write $h = g(f_1 \otimes f_2)$ with $f_1 : B^+ \to C$, $f_2 : B \to C$ and $g : C \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C \to C$.

Since l is a lonilcur, there exists $n \in \mathbb{N}$ such that $f_2 m_B^n \omega_l^{\otimes n} = 0$. Thus, for $k \geq n$,

$$f\phi_l^k = h\left(\phi_l^{k-n} \otimes m_B^n \omega_l^{\otimes n}\right) = g\left(f_1\phi_l^{k-n} \otimes f_2 m_B^n \omega_l^{\otimes n}\right) = 0.$$

Now write $B = (B_i)_i$ as a projective system. For each i, let $f_i : B \to B_i$ be the natural projection. By the above, there exists $n_i \in \mathbb{N}$ such that $f_i\phi_l^k = 0$ for all $k \geq n_i$. Define the map

$$[\![l]\!]_i \coloneqq f_i\left(\bigoplus_{j=0}^{n_i-1} \phi_l^j\right) : \bigoplus_{j=0}^{n_i-1} \Omega^{2j}_{\mathcal{G}^{(0)}}(A) \to B_i.$$

These maps $[\![l]\!]_i$ determine a morphism of projective systems $\left(\mathcal{T}A/(\mathcal{J}A)^n\right)_n \to (B_i)_i$, and hence define a $\mathcal{G}$-equivariant pro-linear map $[\![l]\!] : \mathcal{T}A \to B$.

It is straightforward to check that $[\![l]\!]$ is a homomorphism and satisfies $[\![l]\!]\sigma_A = l$. Moreover, the definition of the Fedosov product implies the uniqueness of such a homomorphism. □

The periodic tensor algebra plays a central role in the definition of quasifree pro-$\mathcal{G}$-algebras.

Definition 3.39. *A pro-$\mathcal{G}$-algebra R is called quasifree if there exists a $\mathcal{G}$-equivariant splitting homomorphism $R \to \mathcal{T}R$ for the canonical projection τ_R.*

Proposition 3.40. *Let A be any pro-$\mathcal{G}$-algebra. Then the periodic tensor algebra $\mathcal{T}A$ is quasifree.*

Proof. We prove the claim by constructing a $\mathcal{G}$-equivariant splitting homomorphism for the canonical projection $\tau_{\mathcal{T}A} : \mathcal{T}\mathcal{T}A \to \mathcal{T}A$.

To this end, we use the universal property of the periodic tensor algebra $\mathcal{T}A$. Consider the $\mathcal{G}$-equivariant pro-linear map $\sigma_A^2 \coloneqq \sigma_{\mathcal{T}A}\sigma_A : A \to \mathcal{T}\mathcal{T}A$.

We first show that σ_A^2 is a $\mathcal{G}$-lonilcur. Recall that the Fedosov product satisfies

$$\sigma_A(x) \circ \sigma_A(y) = \sigma_A(x)\sigma_A(y) - d\sigma_A(x)d\sigma_A(y),$$

where the multiplication $\sigma_A(x)\sigma_A(y)$ refers to the product in A, viewed as degree zero forms.

Thus, observing that

$$\sigma_A(x)\sigma_A(y) - d\sigma_A(x)d\sigma_A(y) = \sigma_A(xy) - d\sigma_A(x)d\sigma_A(y),$$

we compute the curvature of σ_A^2 as follows

$$\begin{aligned}\omega_{\sigma_A^2}(x,y) &= \sigma_A^2(xy) - \sigma_A^2(x) \circ \sigma_A^2(y)\\ &= \sigma_{\mathcal{T}A}(\sigma_A(xy)) - \sigma_{\mathcal{T}A}(\sigma_A(x) \circ \sigma_A(y)) + d\sigma_A^2(x)d\sigma_A^2(y)\\ &= \sigma_{\mathcal{T}A}(\omega_{\sigma_A}(x,y)) + d\sigma_A^2(x)d\sigma_A^2(y).\end{aligned}$$

Now consider the $\mathcal{G}$-equivariant pro-linear map $\sigma_A = \tau_{\mathcal{T}A}\sigma_A^2$. Since $\tau_{\mathcal{T}A}$ is a homomorphism, we have $\omega_{\sigma_A} = \tau_{\mathcal{T}A}\omega_{\sigma_A^2}$.

Let $l : \mathcal{T}\mathcal{T}A \to C$ be a $\mathcal{G}$-equivariant pro-linear map with constant target C. Composing with $\sigma_{\mathcal{T}A}$ gives $k := l\sigma_{\mathcal{T}A} : \mathcal{T}A \to C$. Since σ_A is a lonilcur, there exists $n \in \mathbb{N}$ such that

$$km^n_{\mathcal{T}A}\omega^{\otimes n}_{\sigma_A} = km^n_{\mathcal{T}A}\tau^{\otimes n}_{\mathcal{T}A}\omega^{\otimes n}_{\sigma_A^2} = \tau_{\mathcal{T}A}km^n_{\mathcal{T}\mathcal{T}A}\omega^{\otimes n}_{\sigma_A^2} = 0.$$

Moreover, since $\mathcal{T}\mathcal{T}A$ is constructed as a projective limit, the map l factors through some quotient $\mathcal{T}\mathcal{T}A/(\mathcal{J}(\mathcal{T}A))^m$ for some $m \in \mathbb{N}$. Therefore,

$$lm^{mn}_{\mathcal{T}\mathcal{T}A}\omega^{\otimes mn}_{\sigma_A^2} = 0,$$

showing that σ_A^2 is a lonilcur.

By the universal property of $\mathcal{T}A$, there exists a unique $\mathcal{G}$-equivariant homomorphism $[\![\sigma_A^2]\!] : \mathcal{T}A \to \mathcal{T}\mathcal{T}A$ such that $[\![\sigma_A^2]\!]\sigma_A = \sigma_A^2$. It follows that

$$\tau_{\mathcal{T}A}[\![\sigma_A^2]\!]\sigma_A = \tau_{\mathcal{T}A}\sigma_{\mathcal{T}A}\sigma_A = \sigma_A,$$

and by uniqueness in the universal property of $\mathcal{T}A$, we conclude that $\tau_{\mathcal{T}A}[\![\sigma_A^2]\!] = \mathrm{id}_{\mathcal{T}A}$.

Hence, $\mathcal{T}A$ admits a $\mathcal{G}$-equivariant splitting of $\tau_{\mathcal{T}A}$, which proves that $\mathcal{T}A$ is quasifree. □

We list a number of equivalent characterisations of the class of quasifree pro-$\mathcal{G}$-algebras.

Theorem 3.41. *Let R be a pro-$\mathcal{G}$-algebra. Then the following conditions are equivalent:*

(i) *R is quasifree;*

(ii) *There exists a $\mathcal{G}$-equivariant pro-linear map $\phi : R \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ satisfying*

$$\phi(xy) = \phi(x)y + x\phi(y) - dxdy$$

for all $x, y \in R$;

(iii) *There exists a $\mathcal{G}$-equivariant pro-linear map $\nabla : \Omega^1_{\mathcal{G}^{(0)}}(R) \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ satisfying*

$$\nabla(x\omega) = x\nabla(\omega), \qquad \nabla(\omega x) = \nabla(\omega)x - \omega dx$$

for all $x \in R$, $\omega \in \Omega^1_{\mathcal{G}^{(0)}}(R)$;

(iv) *The R-bimodule $\Omega^1_{\mathcal{G}^{(0)}}(R)$ is projective in the category* $\mathrm{pro}(\mathcal{G}\text{-}Mod)$.

Remark 3.42. *We observe that this result can be extended to a longer list of equivalent statements. We invite the reader to compare this with [CQ95b, Proposition 7.1], [Mey99, Definition and Lemma A.15] and, for the equivariant case, with [Voi07, Theorem 6.5]. We will give a sketch of the proof, since most of the details are a translation of the group equivariant case.*

Sketch of the proof of Theorem 3.41. $(i) \Leftrightarrow (ii)$: Assume R is quasifree. Then there exists a $\mathcal{G}$-equivariant splitting homomorphism $v : R \to \mathcal{T}R$ of the canonical projection $\tau_R : \mathcal{T}R \to R$. In particular, v factors through the quotient $\mathcal{T}R/(\mathcal{J}R)^2 \cong R \oplus \Omega^2_{\mathcal{G}^{(0)}}(R)$ equipped with the Fedosov product. Any section v must be of the form $v = \sigma_R + \phi$, for some $\mathcal{G}$-equivariant pro-linear map $\phi : R \to \Omega^2_{\mathcal{G}^{(0)}}(R)$. To make this map an algebra homomorphism, we must require

$$\begin{aligned}
0 =& (\sigma_R + \phi)(xy) - (\sigma_R + \phi)(x) \circ (\sigma_R + \phi)(y) \\
=& \sigma_R(xy) + \phi(xy) - \sigma_R(x)\sigma_R(y) + d\sigma_R(x)d\sigma_R(y) - \sigma_R(x)\phi(y) + d\sigma_R(x)d\phi(y) \\
& - \phi(x)\sigma_R(y) + d\phi(x)d\sigma_R(y) - \phi(x)\phi(y) + d\phi(x)d\phi(y) \\
=& \phi(xy) - x\phi(y) - \phi(x)y + dxdy,
\end{aligned}$$

where most of the elements vanish because they are higher differential forms. This yields the identity

$$\phi(xy) = \phi(x)y + x\phi(y) - dxdy.$$

Conversely, given such a map ϕ, the assignment $v = \sigma_R + \phi$ defines a $\mathcal{G}$-equivariant algebra homomorphism $v : R \to \mathcal{T}R/(\mathcal{J}R)^2$ that splits the projection. Composing with the canonical inclusion $\mathcal{T}R/(\mathcal{J}R)^2 \hookrightarrow \mathcal{T}R$ gives a splitting $R \to \mathcal{T}R$, showing that R is quasifree.

$(ii) \Leftrightarrow (iii)$: Suppose $\phi : R \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ is a $\mathcal{G}$-equivariant pro-linear map satisfying $\phi(xy) = \phi(x)y + x\phi(y) - dxdy$. Then define $\nabla : \Omega^1_{\mathcal{G}^{(0)}}(R) \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ on generators by

$$\nabla(xdy) := \phi(x)y - xdy,$$

and extend linearly. One checks that ∇ is well-defined and satisfies the relations

$$\nabla(x\omega) = x\nabla(\omega), \qquad \nabla(\omega x) = \nabla(\omega)x - \omega dx,$$

which define a bimodule map.

Conversely, if such a map ∇ exists, define $\phi : R \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ by $\phi(x) := \nabla(dx)$. Then the bimodule properties of ∇ imply:

$$\phi(xy) = \nabla(d(xy)) = \nabla(xdy + dxy) = x\nabla(dy) + \nabla(dx)y = x\phi(y) + \phi(x)y - dxdy.$$

$(ii) \Leftrightarrow (iv)$: Consider the short exact sequence of pro-$\mathcal{G}$-modules

$$0 \longrightarrow \Omega^2_{\mathcal{G}^{(0)}}(R) \xrightarrow{i} R^+ \otimes_{C_c^\infty(\mathcal{G}^{(0)})} R \otimes_{C_c^\infty(\mathcal{G}^{(0)})} R^+ \xrightarrow{p} \Omega^1_{\mathcal{G}^{(0)}}(R) \longrightarrow 0,$$

with R-R-bimodule homomorphisms defined by

$$i(\langle x\rangle dydz) = \langle x\rangle y \otimes z \otimes \chi_U - \langle x\rangle \otimes yz \otimes \chi_U + \langle x\rangle \otimes y \otimes z$$

and

$$p(\langle x\rangle \otimes y \otimes \langle z\rangle) = (\langle x\rangle dy)\langle z\rangle,$$

where $U \subseteq \mathcal{G}^{(0)}$ and its characteristic function is an identity for x, y and z.

Then $\Omega^1_{\mathcal{G}^{(0)}}(R)$ is a projective R-R-bimodule if and only if there exists an R-R-bimodule homomorphism $\rho : R^+ \otimes_{C_c^\infty(\mathcal{G}^{(0)})} R \otimes_{C_c^\infty(\mathcal{G}^{(0)})} R^+ \to \Omega^2_{\mathcal{G}^{(0)}}(R)$ such that $\rho i = \mathrm{id}$.

Moreover, such bimodule homomorphisms ρ correspond bijectively to $\mathcal{G}$-equivariant pro-linear maps $\phi : R \to \Omega^2_{\mathcal{G}^{(0)}}(R)$, via the assignment $\phi(x) := \rho(\chi_U \otimes x \otimes \chi_U)$, for $U \subseteq \mathcal{G}^{(0)}$ such that $\chi_U \cdot x = x$. This correspondence shows that the existence of a splitting ρ is equivalent to the existence of a map ϕ satisfying the identity

$$\phi(xy) = \phi(x)y + x\phi(y) - dxdy.$$

Thus, the projectivity of $\Omega^1_{\mathcal{G}^{(0)}}(R)$ is equivalent to the existence of such a $\mathcal{G}$-equivariant pro-linear map ϕ. □

Lemma 3.43. *The trivial $\mathcal{G}$-algebra $C_c^\infty(\mathcal{G}^{(0)})$ is quasifree.*

Proof. Let $f \in C_c^\infty(\mathcal{G}^{(0)})$ and define $\phi(f) = 2f d\chi_U d\chi_U - df d\chi_U$, where χ_U is the characteristic function of a compact open subset $U \subseteq \mathcal{G}^{(0)}$ such that $\chi_U f = f$. This does not depend on the choice of U, and one checks that ϕ satisfies condition (ii) in Theorem 3.41. □

We now study *universal* locally nilpotent extensions of pro-$\mathcal{G}$-algebras. These extensions play an important conceptual role in the theory of equivariant periodic cyclic homology.

Definition 3.44. *Let A be a pro-$\mathcal{G}$-algebra. A universal locally nilpotent extension of A is an admissible extension of pro-$\mathcal{G}$-algebras*

$$N \rightarrowtail R \twoheadrightarrow A$$

such that N is locally nilpotent and R is quasifree.

We first fix the notion of homotopy in our setting.

Definition 3.45. *Let A be a pro-$\mathcal{G}$-algebra. We denote by $A[0,1]$ the pro-$\mathcal{G}$-algebra $A \otimes C^\infty([0,1])$ of smooth functions on the unit interval with values in A, equipped with the $\mathcal{G}$-action on the first element.*

Definition 3.46. *Let A and B be pro-$\mathcal{G}$-algebras. A $\mathcal{G}$-*equivariant homotopy *between two $\mathcal{G}$-equivariant homomorphisms $f_0, f_1 : A \to B$ is a $\mathcal{G}$-equivariant homomorphism*

$$\Phi : A \to B[0,1]$$

such that for all $t \in [0,1]$, the evaluation map $\mathrm{ev}_t : B[0,1] \to B$ induces a $\mathcal{G}$-equivariant homomorphisms $\Phi_t := \mathrm{ev}_t \Phi : A \to B$ satisfying $\Phi_0 = f_0$ and $\Phi_1 = f_1$. We say that f_0 and f_1 are $\mathcal{G}$-equivariant homotopic *if such a map Φ exists.*

The terminology 'universal' is justified by the following universal property.

Proposition 3.47. *Let*

$$N \overset{\iota}{\rightarrowtail} R \overset{\pi}{\twoheadrightarrow} A$$

be a universal locally nilpotent extension of the pro-$\mathcal{G}$-algebra A. Let

$$K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$

be any other locally nilpotent extension such that there exists a $\mathcal{G}$-equivariant homomorphism $\phi : A \to Q$. Then there exist $\mathcal{G}$-equivariant homomorphisms $\xi : N \to K$ and

$\psi : R \to E$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\iota} & R & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & E & \xrightarrow{p} & Q & \longrightarrow & 0 \end{array}$$

Moreover, the homomorphisms ξ and ψ are unique up to $\mathcal{G}$-equivariant homotopy. Let (ξ_t, ψ_t, ϕ_t) for $t = 0, 1$ be $\mathcal{G}$-equivariant homomorphisms of extensions and let $\Phi : A \to Q[0,1]$ be a $\mathcal{G}$-equivariant homotopy connecting ϕ_0 and ϕ_1. Then Φ can be lifted to a $\mathcal{G}$-equivariant homotopy (Ξ, Ψ, Φ) between (ξ_0, ψ_0, ϕ_0) and (ξ_1, ψ_1, ϕ_1).

Proof. Since R is quasifree, let $v : R \to \mathcal{T}R$ be a $\mathcal{G}$-equivariant splitting of τ_R. Choose a $\mathcal{G}$-equivariant pro-linear section $s : Q \to E$. Then $s\phi\pi : R \to E$ is a $\mathcal{G}$-equivariant pro-linear map, and since $p(s\phi\pi) = \phi\pi$ is a homomorphism and the sequence is exact, the curvature of $s\phi\pi$ takes values in K. As K is locally nilpotent, $s\phi\pi$ is a $\mathcal{G}$-lonilcur. By the universal property of $\mathcal{T}R$, there exists a $\mathcal{G}$-equivariant homomorphism $[\![s\phi\pi]\!] : \mathcal{T}R \to E$ with $[\![s\phi\pi]\!]\sigma_R = s\phi\pi$. Define $\psi := [\![s\phi\pi]\!]v$, which satisfies $p\psi = \phi\pi$. Moreover, $\psi(N) \subseteq K$, so it restricts to a $\mathcal{G}$-equivariant homomorphism $\xi : N \to K$, yielding the desired morphism of extensions.

The assertion that ψ and ξ are uniquely defined up to homotopy follows by applying the first part of the proof to the homotopy $\Phi : A \to Q[0,1]$. □

We can now summarise what we have done in this section with the following result
Proposition 3.48. *Let A be a pro-$\mathcal{G}$-algebra. The extension $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$ is a universal locally nilpotent extension of A.*

§ 3.5 | The equivariant X-complex

A further ingredient in the definition of periodic cyclic homology introduced by Cuntz and Quillen [CQ95b] is the construction of the X-complex of a pro-algebra A. The X-complex is obtained by truncating the bicomplex used to define the periodic cyclic homology in the classical setting. For most pro-algebras A, the chain complex $X(A)$ can give less information than expected because it ignores all the higher levels. For quasi-free pro-algebras, however, there is no such additional information above degree 1, and the X-complex encodes all the relevant information. In this section, we will define the $\mathcal{G}$-equivariant version of the X-complex. The main sources for this are [Voi03] and [Voi07]. The definition in the groupoid case is similar to the group case. We will give the definition and state the main interesting features of this object.

We now return to the paramixed complex $\Omega_{\mathcal{G}}(A)$ of $\mathcal{G}$-equivariant differential forms over a pro-$\mathcal{G}$-algebra A.

Definition 3.49. *Let A be a pro-$\mathcal{G}$-algebra. We define the Hodge tower associated to $\Omega_{\mathcal{G}}(A)$ by defining the n-th level as*

$$\theta^n\Omega_{\mathcal{G}}(A) = \bigoplus_{j=0}^{n-1}\Omega^j_{\mathcal{G}}(A) \oplus \Omega^n_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^{n+1}_{\mathcal{G}}(A)).$$

The operators $d_{\mathcal{G}}$ and $b_{\mathcal{G}}$ descend to $\theta^n\Omega_{\mathcal{G}}(A)$ as follows

$$\begin{aligned} b_{\mathcal{G}} : \theta^n\Omega_{\mathcal{G}}(A) &\to \theta^n\Omega_{\mathcal{G}}(A) \\ (\omega^0, \omega^1, \dots, [\omega^n]) &\mapsto (b_{\mathcal{G}}(\omega^1), b_{\mathcal{G}}(\omega^2), \dots, b_{\mathcal{G}}(\omega^n), [0]), \end{aligned}$$

and

$$\begin{aligned} d_{\mathcal{G}} : \theta^n\Omega_{\mathcal{G}}(A) &\to \theta^n\Omega_{\mathcal{G}}(A) \\ (\omega^0, \omega^1, \dots, [\omega^n]) &\mapsto (0, d_{\mathcal{G}}(\omega^0), \dots, d_{\mathcal{G}}(\omega^{n-2}), [d_{\mathcal{G}}(\omega^{n-1})]), \end{aligned}$$

where $\omega^j \in \Omega^j_{\mathcal{G}}(A)$ for $j = 0, \dots, n-1$ and $[\omega^n] \in \Omega^n_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^{n+1}_{\mathcal{G}}(A))$. Observe that the first map is well-defined since $b^2_{\mathcal{G}} = 0$. Similarly, it is true for $\kappa_{\mathcal{G}}$ and $B_{\mathcal{G}}$. Using the natural grading into even and odd forms together with the last relation in Lemma 3.27, we see that $\theta^n\Omega_{\mathcal{G}}(A)$ together with the boundary operator $B_{\mathcal{G}} + b_{\mathcal{G}}$ becomes a pro-paracomplex of $\mathcal{G}$-anti-Yetter-Drinfeld modules.

For $m \geq n$ there exists a natural chain map $\theta^m\Omega_{\mathcal{G}}(A) \to \theta^n\Omega_{\mathcal{G}}(A)$ given by the obvious projection. By definition, the Hodge tower $\theta\Omega_{\mathcal{G}}(A)$ of A is the projective limit of the projective system $(\theta^n\Omega_{\mathcal{G}}(A))_{n\in\mathbb{N}}$ obtained in this way.

Definition 3.50. *Let A be a pro-$\mathcal{G}$-algebra. The equivariant X-complex $X_{\mathcal{G}}(A)$ of A is the pro-paracomplex $\theta^1\Omega_{\mathcal{G}}(A)$. Explicitly, we have*

$$X_{\mathcal{G}}(A) : \Omega^0_{\mathcal{G}}(A) \underset{b_{\mathcal{G}}}{\overset{\natural d_{\mathcal{G}}}{\rightleftarrows}} \Omega^1_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^2_{\mathcal{G}}(A))$$

where $\natural : \Omega^1_{\mathcal{G}}(A) \to \Omega^1_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^2_{\mathcal{G}}(A))$ denotes the canonical projection.

Remark 3.51. *An important difference with the classical setting is that the equivariant X-complex $X_{\mathcal{G}}(A)$ is typically not a chain complex but only a paracomplex. This is again a consequence of the relation (iv) in Lemma 3.27.*

A notable exception to the previous Remark is the case when $A = C^\infty_c(\mathcal{G}^{(0)})$ is the trivial $\mathcal{G}$-algebra.

Lemma 3.52. *The equivariant X-complex $X_{\mathcal{G}}(C^\infty_c(\mathcal{G}^{(0)}))$ of the trivial $\mathcal{G}$-algebra $C^\infty_c(\mathcal{G}^{(0)})$*

identifies canonically with

$$\mathcal{O}_{\mathcal{G}} \rightleftarrows 0$$

that is, it is equal to the trivial supercomplex $\mathcal{O}_{\mathcal{G}}[0]$.

Proof. By definition of the equivariant X-complex, the even part of $X_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}))$ is given by $\mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \cong \mathcal{O}_{\mathcal{G}}$.

Every element in the odd part of $X_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}))$ can be represented as a linear combination of terms of the form $\chi_U \otimes d\chi_V$ and $\chi_U \otimes \chi_V d\chi_V$ for compact open bisections $U \subseteq \mathcal{G}$ and compact open subsets $V \subseteq \mathcal{G}^{(0)}$. Moreover, the canonical map T associated with $\mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \cong \mathcal{O}_{\mathcal{G}}$ equals the identity, compare with the relation 2.6 and recall that we are considering the action of loop arrows. This implies that the Hochschild operator $b_{\mathcal{G}} : \Omega^2_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)})) \to \Omega^1_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}))$ satisfies

$$b_{\mathcal{G}}(\chi_U \otimes \langle \chi_V \rangle d\chi_V d\chi_V) = -\chi_U \otimes \langle \chi_V \rangle d(\chi_V)\chi_V + \chi_U \otimes \chi_V d\chi_V.$$

We therefore obtain that

$$\begin{aligned} \chi_U \otimes \chi_V d\chi_V &= \chi_U \otimes \chi_V d(\chi_V \chi_V) \\ &= \chi_U \otimes \chi_V d(\chi_V)\chi_V + \chi_U \otimes \chi_V d\chi_V \\ &= 2\chi_U \otimes \chi_V d\chi_V \end{aligned}$$

in $X_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}))$ since $b_{\mathcal{G}}(\chi_U \otimes \langle \chi_V \rangle d\chi_V d\chi_V)$ vanishes, and hence $\chi_U \otimes \chi_V d\chi_V = 0$.

Similarly,

$$\begin{aligned} \chi_U \otimes d\chi_V &= \chi_U \otimes d(\chi_V \chi_V) \\ &= \chi_U \otimes d(\chi_V)\chi_V + \chi_U \otimes \chi_V d\chi_V \\ &= 2\chi_U \otimes \chi_V d\chi_V = 0, \end{aligned}$$

and we conclude that the odd part of $X_{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}))$ vanishes as claimed. □

A central result regarding the equivariant X-complex is the following theorem, compare [Voi07, Theorem 8.6].

Theorem 3.53. *For any pro-$\mathcal{G}$-algebra A the equivariant X-complex $X_{\mathcal{G}}(\mathcal{T}A)$ and the Hodge tower $\theta\Omega_{\mathcal{G}}(A)$ are homotopy equivalent as pro-paracomplexes of $\mathcal{G}$-anti-Yetter-Drinfeld modules.*

The proof of Theorem 3.53 is a direct translation of the proof in the group equivariant case, building on the relations in Lemma 3.27.

§ 3.6 | Bivariant equivariant periodic cyclic homology

Now that we have all the ingredients, we can give the main definition of this chapter.

Definition 3.54. *Let $\mathcal{G}$ be an ample groupoid and let A and B be pro-$\mathcal{G}$-algebras. The* bivariant equivariant periodic cyclic homology *of A and B is*

$$HP_*^{\mathcal{G}}(A,B) = H_*(\mathrm{Hom}_{\mathcal{A}(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})), X_{\mathcal{G}}(\mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})))).$$

We pointed out earlier that the equivariant X-complex is not a chain complex in general. This marks a crucial difference with the ordinary approach and explains why we start directly with a bivariant approach. In fact, the Hom-complex in this definition is indeed an ordinary supercomplex, so that one can take its homology in the standard way. In order to explain this we write ∂_A and ∂_B for the differentials of the equivariant X-complexes in the source and the target, respectively. Recall that k-th element of the Hom-complex chain is given by

$$\Pi_{k=q-p}\mathrm{Hom}_{\mathcal{A}(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}))_p, X_{\mathcal{G}}(\mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}))_q),$$

where we observe that $X_{\mathcal{G}}(-)_{2n} = X_{\mathcal{G}}(-)_0$ and $X_{\mathcal{G}}(-)_{2n+1} = X_{\mathcal{G}}(-)_1$ for all $n \in \mathbb{Z}$ since the X-complex is a supercomplex. Moreover, the differential in the Hom-complex is given by

$$\partial(\phi) = \phi\partial_A - (-1)^{|\phi|}\partial_B\phi$$

for a homogeneous element ϕ, and we have

$$\begin{aligned}
\partial^2(\phi) &= \partial(\phi\,\partial_A - (-1)^{|\phi|}\partial_B\phi) \\
&= \phi\,\partial_A^2 - (-1)^{|\phi|-1}\partial_B\phi\,\partial_A - (-1)^{|\phi|}\partial_B\phi\,\partial_A + (-1)^{|\phi|-1}(-1)^{|\phi|}\partial_B^2\phi \\
&= \phi\,\partial_A^2 + (-1)^{|\phi|}(-1)^{|\phi|-1}\partial_B^2\,\phi \\
&= \phi(\mathrm{id} - T) - (\mathrm{id} - T)\phi \\
&= T\phi - \phi T.
\end{aligned}$$

Hence the commutation property showed in Lemma 2.46 gives the relation $\partial^2(\phi) = 0$.

It follows directly from the definition that $HP_*^{\mathcal{G}}$ is a bifunctor, contravariant in A and covariant in B. We define

$$HP_*^{\mathcal{G}}(B) := HP_*^{\mathcal{G}}(C_c^\infty(\mathcal{G}^{(0)}), B), \qquad HP^*_{\mathcal{G}}(A) := HP_*^{\mathcal{G}}(A, C_c^\infty(\mathcal{G}^{(0)}))$$

the $\mathcal{G}$-equivariant periodic cyclic homology of B, and the $\mathcal{G}$-equivariant periodic cyclic cohomology of A, respectively. Every $\mathcal{G}$-equivariant algebra homomorphism $f : A \to B$

induces naturally an element $[f] \in HP_0^{\mathcal{G}}(A,B)$. We have an associative product

$$HP_*^{\mathcal{G}}(A,B) \times HP_*^{\mathcal{G}}(B,C) \to HP_*^{\mathcal{G}}(A,C), \quad (x,y) \mapsto x \cdot y$$

induced by the composition, and this generalises the composition of $\mathcal{G}$-equivariant homomorphisms $f : A \to B$ and $g : B \to C$ in the sense that $[f] \cdot [g] = [g \circ f]$. In particular, we obtain a natural ring structure on $HP_*^{\mathcal{G}}(A,A)$ for every $\mathcal{G}$-algebra A with unit element given by $[\mathrm{id}]$.

Remark 3.55. *If the groupoid $\mathcal{G}$ is just a point, then we obviously obtain the constructions defined by Cuntz and Quillen. Moreover, if $\mathcal{G}^{(0)}$ is a singleton, or equivalently, if the groupoid $\mathcal{G}$ is a discrete group, then the above constructions reduce to the theory developed in [Voi03], [Voi07].*

§ 3.7 | Discrete groupoids

In this section, we describe in details how the calculation of $HP_*^{\mathcal{G}}$ can be reduced to the group equivariant case when the groupoid is discrete.

Start recording the following well-known result about the structure of discrete groupoids.

Lemma 3.56. *Any discrete groupoid $\mathcal{G}$ can be decomposed into the disjoint union of transitive groupoids.*

Proof. The groupoid $\mathcal{G}$ acts on its base space $\mathcal{G}^{(0)}$. We denote an orbit in the quotient space $\mathcal{G}\backslash\mathcal{G}^{(0)}$ by $[x]$, for a chosen representative x in the orbit. The restriction of $\mathcal{G}$ to $[x]$, denoted by $\mathcal{G}_{[x]}^{[x]}$ is a transitive subgroupoid of $\mathcal{G}$ and since the orbits are disjoint we can write

$$\mathcal{G} = \bigsqcup_{[x] \in \mathcal{G}\backslash\mathcal{G}^{(0)}} \mathcal{G}_{[x]}^{[x]}.$$

□

Let $x \in \mathcal{G}^{(0)}$ and write $\mathfrak{m}_x \subseteq C_c^\infty(\mathcal{G}^{(0)})$ for the maximal ideal of all functions vanishing at x. If A is a pro-$\mathcal{G}$-algebra then $A_x = A/\mathfrak{m}_x \cdot A$ is naturally a pro-$\mathcal{G}_x^x$-algebra.

Proposition 3.57. *Let $\mathcal{G}$ be a discrete groupoid and let A, B be pro-$\mathcal{G}$-algebras. Then we have a canonical isomorphism*

$$HP_*^{\mathcal{G}}(A,B) \cong \prod_{[x] \in \mathcal{G}\backslash\mathcal{G}^{(0)}} HP_*^{\mathcal{G}_x^x}(A_x, B_x),$$

where each x is an arbitrary representative of the orbit $[x] \in \mathcal{G}\backslash\mathcal{G}^{(0)}$.

Proof. Every discrete groupoid can be written as a disjoint union of transitive groupoids as seen in 3.56. This induces a direct product decomposition at level of the Hom-complexes. Therefore it suffices to consider the case that $\mathcal{G}$ is transitive.

In this case, given any $x \in \mathcal{G}^{(0)}$ one obtains an equivalence between the category of $\mathcal{G}$-anti-Yetter-Drinfeld modules and the category of $\mathcal{G}_x^x$-anti-Yetter-Drinfeld modules by sending a $\mathcal{G}$-anti-Yetter-Drinfeld module M to $\chi \cdot M$, where $\chi \in M(\mathcal{O}_{\mathcal{G}})) = C^\infty(\mathcal{G}_{ad})$ denotes the characteristic function of $\mathcal{G}_x^x$. Applying the extension of this functor to the corresponding pro-categories to the Hom-complex defining $HP_*^{\mathcal{G}}(A, B)$ yields the desired isomorphism. □

Chapter 4

Homological Properties

This chapter is devoted to the homological aspects of $\mathcal{G}$-equivariant periodic cyclic homology $HP_*^{\mathcal{G}}$. We will show that it shares many homological properties analogous to equivariant KK-theory, including excision, stability, and homotopy invariance. For further connections with equivariant KK-theory in a categorical framework, see [BP24].

§ 4.1 | Homotopy invariance

We first establish that $HP_*^{\mathcal{G}}$ is homotopy invariant with respect to $\mathcal{G}$-equivariant homotopies in both variables. The discussion about this topic, in the group case, can be found in [Voi07] and [Voi03]. In this section we will follow the same strategies and adapt the proofs to our situation.

Let A, B be pro-$\mathcal{G}$-algebras. Recall from the previous chapter the Definition 3.45 of $\mathcal{G}$-equivariant homotopy between $\mathcal{G}$-equivariant algebra homomorphisms $\phi_0, \phi_1 : A \to B$, that $\theta^n \Omega_{\mathcal{G}}(A)$ denotes the n-th level of the Hodge tower, and that $\theta^1 \Omega_{\mathcal{G}}(A) = X_{\mathcal{G}}(A)$ is the $\mathcal{G}$-equivariant X-complex. We have canonical projection maps $\xi_n : \theta^n \Omega_{\mathcal{G}}(A) \to \theta^{n-1} \Omega_{\mathcal{G}}(A)$ for all $n \geq 1$. The first step toward proving the main result of this section is to show that the map ξ_2 is a homotopy equivalence, provided that the pro-$\mathcal{G}$-algebra A is quasifree.

Lemma 4.1. *Let A be a quasifree pro-$\mathcal{G}$-algebra. Then the map $\xi_2 : \theta^2 \Omega_{\mathcal{G}}(A) \to X_{\mathcal{G}}(A)$ is a homotopy equivalence of pro-paracomplexes of $\mathcal{G}$-anti-Yetter-Drinfeld modules.*

Proof. The natural projection map ξ_2 is described by the following commutative diagram

$$\begin{array}{ccc} \Omega^0_{\mathcal{G}}(A) \oplus \Omega^2_{\mathcal{G}}(A)/b_{\mathcal{G}}\left(\Omega^3_{\mathcal{G}}(A)\right) & \overset{b_{\mathcal{G}}+B_{\mathcal{G}}}{\rightleftarrows} & \Omega^1_{\mathcal{G}}(A) \\ \downarrow \mathrm{pr}_1 & & \downarrow \natural \\ \Omega^0_{\mathcal{G}}(A) & \underset{b_{\mathcal{G}}}{\overset{\natural d}{\rightleftarrows}} & \Omega^1_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^2_{\mathcal{G}}(A)). \end{array}$$

Since A is quasifree, there exists by Theorem 3.41 a $\mathcal{G}$-equivariant pro-linear map $\nabla : \Omega^1_{\mathcal{G}^{(0)}}(A) \to \Omega^2_{\mathcal{G}^{(0)}}(A)$ such that

$$\nabla(a\omega) = a\nabla(\omega) \quad \text{and} \quad \nabla(\omega a) = \nabla(\omega)a - \omega da$$

for all $a \in A$ and $\omega \in \Omega^1_{\mathcal{G}^{(0)}}(A)$. We extend ∇ to forms of higher degree by setting

$$\nabla\left(\langle a^0\rangle da^1 \cdots da^n\right) = \nabla\left(\langle a^0\rangle da^1\right) da^2 \cdots da^n.$$

Then we have

$$\nabla(a\omega) = a\nabla(\omega), \quad \nabla(\omega\eta) = \nabla(\omega)\eta + (-1)^{|\omega|}\omega d\eta$$

for $a \in A$ and $\omega, \eta \in \Omega_{\mathcal{G}^{(0)}}(A)$. Moreover we set $\nabla(a) = 0$ for $a \in \Omega^0_{\mathcal{G}^{(0)}}(A) = A$.

One then obtains a map $\nabla_{\mathcal{G}} : \Omega^n_{\mathcal{G}}(A) \to \Omega^{n+1}_{\mathcal{G}}(A)$ of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules by setting

$$\nabla_{\mathcal{G}}(f \otimes \omega) = f \otimes \nabla(\omega).$$

We will use $\nabla_{\mathcal{G}}$ to construct an inverse of ξ_2 up to homotopy. Let $\omega \in \Omega^{n-1}_{\mathcal{G}^{(0)}}(A)$ with $n \geq 2$ and $a \in A$. Then an explicit computation gives

$$\begin{aligned}
[b_{\mathcal{G}}, \nabla_{\mathcal{G}}](\chi_U \otimes \omega da) &= b_{\mathcal{G}}\nabla_{\mathcal{G}}(\chi_U \otimes \omega da) + \nabla_{\mathcal{G}} b_{\mathcal{G}}(\chi_U \otimes \omega da) \\
&= b_{\mathcal{G}}(\chi_U \otimes \nabla(\omega)da) + \nabla_{\mathcal{G}}((-1)^{n-1}(\chi_U \otimes (\omega a - (\chi_{U^{-1}} \cdot a)\omega))) \\
&= (-1)^n(\chi_U \otimes (\nabla(\omega)a - (\chi_{U^{-1}} \cdot a)\nabla(\omega)) \\
&\quad + (-1)^{n-1}(\chi_U \otimes (\nabla(\omega a) - \nabla((\chi_{U^{-1}} \cdot a)\omega))) \\
&= (-1)^{n-1}(\chi_U \otimes ((\chi_{U^{-1}} \cdot a)\nabla(\omega) - \nabla(\omega)a + \nabla(\omega a) - \nabla((\chi_{U^{-1}} \cdot a)\omega)) \\
&= (-1)^{n-1}(\chi_U \otimes (\chi_{U^{-1}} \cdot a)\nabla(\omega) - \chi_U \otimes \nabla(\omega)a + \chi_U \otimes \nabla(\omega)a \\
&\quad + (-1)^{n-1}\chi_U \otimes \omega da - \chi_U \otimes (\chi_{U^{-1}} \cdot a)\nabla(\omega)) \\
&= \chi_U \otimes \omega da.
\end{aligned}$$

So this implies that $[b_{\mathcal{G}}, \nabla_{\mathcal{G}}] = \mathrm{id}$ on $\Omega^n_{\mathcal{G}}(A)$ for $n \geq 2$. Since $[b_{\mathcal{G}}, \nabla_{\mathcal{G}}]$ commutes with $b_{\mathcal{G}}$ this equality holds on $b_{\mathcal{G}}(\Omega^2_{\mathcal{G}}(A)) \subseteq \Omega^1_{\mathcal{G}}(A)$ as well. As a consequence, we obtain a well-defined map $\nu : X_{\mathcal{G}}(A) \to \theta^2\Omega_{\mathcal{G}}(A)$ by setting $\nu = \mathrm{id} - [\nabla_{\mathcal{G}}, B_{\mathcal{G}} + b_{\mathcal{G}}]$, noting that $[\nabla_{\mathcal{G}}, B_{\mathcal{G}}]$ increases the degree of differential forms by 2.

Using Lemma 2.46 with the fact that $\nabla_{\mathcal{G}}$ is a map of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules one checks that ν is a chain map with respect to $\partial = B_{\mathcal{G}} + b_{\mathcal{G}}$. Explicitly, we have

$$\begin{aligned}
\nu &= \mathrm{id} - \nabla_{\mathcal{G}} d && \text{on } \Omega^0_{\mathcal{G}}(A) \\
\nu &= \mathrm{id} - [\nabla_{\mathcal{G}}, b_{\mathcal{G}}] = \mathrm{id} - b_{\mathcal{G}}\nabla_{\mathcal{G}} && \text{on } \Omega^1_{\mathcal{G}}(A)/b_{\mathcal{G}}\left(\Omega^2_{\mathcal{G}}(A)\right),
\end{aligned}$$

and this implies $\xi_2 \nu = \mathrm{id}$. Moreover, by construction $\nu \xi_2 = \mathrm{id} - [\nabla_{\mathcal{G}}, B_{\mathcal{G}} + b_{\mathcal{G}}]$ is homotopic to the identity. □

Definition 4.2. *Let A, B be pro-$\mathcal{G}$-algebras and let $\Phi : A \to B[0,1]$ be a $\mathcal{G}$-equivariant homotopy. Recall that for $t \in [0,1]$ we write $\Phi_t := \mathrm{ev}_t \Phi$. We define the derivative of Φ as the $\mathcal{G}$-equivariant pro-linear map $\Phi' : A \to B[0,1]$ defined as $\Phi'_t(a) := \frac{\partial}{\partial t}\Phi_t(a)$, for $a \in A$.*

Then Φ' is a derivation with respect to Φ, that is,

$$\Phi'(ab) = \Phi'(a)\Phi(b) + \Phi(a)\Phi'(b)$$

for all $a, b \in A$.

In the same spirit, using the differential structure inherited by $C^\infty([0,1])$ we give the following.

Definition 4.3. *Let A, B be pro-$\mathcal{G}$-algebras and let $\Phi : A \to B[0,1]$ be a $\mathcal{G}$-equivariant homotopy with $\Phi' : A \to B[0,1]$ its derivative. We define $\eta : \Omega^n_{\mathcal{G}}(A) \to \Omega^{n-1}_{\mathcal{G}}(B)$ by*

$$\eta(f \otimes a^0 da^1 \cdots da^n) = \int_0^1 f \otimes \Phi_t(a^0)\Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^n) dt$$

for $n > 0$ and $\eta = 0$ on $\Omega^0_{\mathcal{G}}(A)$.

Using the fact that Φ' is a derivation with respect to Φ one computes

$$\begin{aligned}
\eta b_{\mathcal{G}}(\chi_U \otimes a^0 da^1 \cdots da^n) &= (-1)^n \eta(\chi_U \otimes a^0 da^1 \cdots da^{n-1} a^n - \chi_U \otimes (\chi_{U^{-1}} \cdot a^n) a^0 da^1 \cdots da^{n-1}) \\
&= \eta(\chi_U \otimes a^0 a^1 da^2 \cdots da^n + \sum_{j=1}^{n-1} (-1)^j a^0 da^1 \cdots d(a^j a^{j+1}) \cdots da^n \\
&\quad + (-1)^n \chi_U \otimes (\chi_{U^{-1}} \cdot a^n) a^0 da^1 \cdots da^{n-1}) \\
&= \int_0^1 (\chi_U \otimes \Phi_t(a^0 a^1) \Phi'_t(a^2) d\Phi_t(a^3) \cdots d\Phi_t(a^n) \\
&\quad - \chi_U \otimes \Phi_t(a^0) \Phi'_t(a^1 a^2) d\Phi_t(a^3) \cdots d\Phi_t(a^n) \\
&\quad + \sum_{j=2}^{n-1} (-1)^j \Phi_t(a^0) \Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^j a^{j+1}) \cdots d\Phi_t(a^n) \\
&\quad + (-1)^n \Phi_t((\chi_{U^{-1}} \cdot a^n) a^0) \Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^{n-1})) dt \\
&= -\int_0^1 (\chi_U \otimes \Phi_t(a^0) \Phi'_t(a^1) \Phi_t(a^2) d\Phi_t(a^3) \cdots d\Phi_t(a^n) \\
&\quad + \sum_{j=2}^{n-1} (-1)^{j-1} \Phi_t(a^0) \Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^j a^{j+1}) \cdots d\Phi_t(a^n) \\
&\quad + (-1)^{n-1} \Phi_t(\chi_{U^{-1}} \cdot a^n) \Phi_t(a^0) \Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^{n-1})) dt \\
&= -\int_0^1 b_{\mathcal{G}}(\chi_U \otimes \Phi_t(a^0) \Phi'_t(a^1) d\Phi_t(a^2) \cdots d\Phi_t(a^n)) dt
\end{aligned}$$

$$= -b_{\mathcal{G}}\eta(\chi_U \otimes a^0 da^1 \cdots da^n)$$

for any compact open bisection $U \subseteq \mathcal{G}$. We deduce that $\eta b_{\mathcal{G}} + b_{\mathcal{G}}\eta = 0$ on $\Omega^n_{\mathcal{G}}(A)$ for all $n \geq 0$. In particular, we have $\eta b_{\mathcal{G}}(\Omega^3_{\mathcal{G}}(A)) \subseteq b_{\mathcal{G}}(\Omega^2_{\mathcal{G}}(B))$, and hence we obtain a $\mathcal{G}$-equivariant pro-linear map $\eta : \theta^2\Omega_{\mathcal{G}}(A) \to X_{\mathcal{G}}(B)$.

Lemma 4.4. *Let $\Phi : A \to B[0,1]$ be a $\mathcal{G}$-equivariant homotopy between pro-$\mathcal{G}$-algebras A and B. Then we have*

$$X_{\mathcal{G}}(\Phi_1)\xi_2 - X_{\mathcal{G}}(\Phi_0)\xi_2 = \partial\eta + \eta\partial,$$

where $\eta : \theta^2\Omega_{\mathcal{G}}(A) \to X_{\mathcal{G}}(B)$ is the map introduced in Definition 4.3 and $\partial = B_{\mathcal{G}} + b_{\mathcal{G}}$. Hence the chain maps $X_{\mathcal{G}}(\Phi_t)\xi_2 : \theta^2\Omega_{\mathcal{G}}(A) \to X_{\mathcal{G}}(B)$ for $t = 0, 1$ are homotopic.

Proof. For $j = 0$ we have

$$\begin{aligned}
[\partial, \eta](f \otimes a) &= \eta(f \otimes da) \\
&= \int_0^1 f \otimes \Phi_t'(a)dt \\
&= f \otimes \Phi_1(a) - f \otimes \Phi_0(a).
\end{aligned}$$

For $j = 1$ we get

$$\begin{aligned}
[\partial, \eta](\chi_U \otimes a^0 da^1) &= d_{\mathcal{G}}\eta(\chi_U \otimes a^0 da^1) + \eta B_{\mathcal{G}}(\chi_U \otimes a^0 da^1) \\
&= \int_0^1 (\chi_U \otimes d(\Phi_t(a^0)\Phi_t'(a^1)) + \chi_U \otimes \Phi_t'(a^0)d\Phi_t(a^1) \\
&\quad - \chi_U \otimes \Phi_t'(\chi_{U^{-1}} \cdot a^1)d\Phi_t(a^0))dt \\
&= \int_0^1 (\chi_U \otimes d\Phi_t(a^0)\Phi_t'(a^1) + \chi_U \otimes \Phi_t(a^0)d\Phi_t'(a^1) \\
&\quad + \chi_U \otimes \Phi_t'(a^0)d\Phi_t(a^1) - \chi_U \otimes \Phi_t'(\chi_{U^{-1}} \cdot a^1)d\Phi_t(a^0))dt \\
&= \int_0^1 b_{\mathcal{G}}(\chi_U \otimes d\Phi_t(a^0)d\Phi_t'(a^1)) + \frac{\partial}{\partial t}(\chi_U \otimes \Phi_t(a^0)d\Phi_t(a^1))dt
\end{aligned}$$

for any compact open bisection $U \subseteq \mathcal{G}$. Since the first term vanishes in $X_{\mathcal{G}}(B)$ we conclude

$$[\partial, \eta](\chi_U \otimes a^0 da^1) = \chi_U \otimes \Phi_1(a^0)d\Phi_1(a^1) - \chi_U \otimes \Phi_0(a^0)d\Phi_0(a^1).$$

Finally, on $\Omega^2_{\mathcal{G}}(A)/b_{\mathcal{G}}(\Omega^3_{\mathcal{G}}(A))$ we have $\partial\eta + \eta\partial = \eta b_{\mathcal{G}} + b_{\mathcal{G}}\eta = 0$, with the last equality due to the calculation just before this Lemma. □

We are now ready to state and prove the following result.

Theorem 4.5 (Homotopy invariance)**.** *Let A and B be pro-$\mathcal{G}$-algebras and let $\Phi : A \to B[0,1]$ be a $\mathcal{G}$-equivariant homotopy. Then the elements $[\Phi_0]$ and $[\Phi_1]$ in $HP^{\mathcal{G}}_0(A, B)$*

are equal. More generally, if A is a quasifree pro-$\mathcal{G}$-algebra then the elements $[\Phi_0]$ *and* $[\Phi_1]$ *in* $H_0(\mathrm{Hom}_{A(\mathcal{G})}(X_{\mathcal{G}}(A), X_{\mathcal{G}}(B)))$ *are equal.*

Proof. The second part of the Theorem follows directly by combining Lemma 4.1 and Lemma 4.4.

In order to show that the first part of the Theorem can be viewed as a special case of the second, assume that $\Phi : A \to B[0,1]$ is a $\mathcal{G}$-equivariant homotopy. We tensor A and B with $\mathcal{K}_{\mathcal{G}}$ to obtain a $\mathcal{G}$-equivariant homotopy $\Phi \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \to (B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})[0,1]$. Passing to the periodic tensor algebras we obtain a $\mathcal{G}$-equivariant algebra homomorphism $\mathcal{T}(\Phi \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}) : \mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}) \to \mathcal{T}((B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})[0,1])$.

Consider the $\mathcal{G}$-equivariant pro-linear map

$$l : B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \otimes C^\infty([0,1]) \to \mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}) \otimes C^\infty([0,1])$$
$$l(b \otimes T \otimes f) = \sigma(b \otimes T) \otimes f,$$

where $\sigma : B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \to \mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})$ is the standard $\mathcal{G}$-equivariant pro-linear splitting. Then l is a lonilcur, and we get an associated $\mathcal{G}$-equivariant homomorphism

$$[\![l]\!] : \mathcal{T}((B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})[0,1]) \to \mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})[0,1]$$

by the universal property of the periodic tensor algebra from Proposition 3.38. Consider the $\mathcal{G}$-equivariant homotopy

$$\Psi = [\![l]\!]\mathcal{T}(\Phi \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}) : \mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}) \to \mathcal{T}(B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})[0,1]$$

and note that $\Psi_t = \mathcal{T}(\Phi_t \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})$ for all $t \in [0,1]$. Since $\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})$ is quasifree we are now in the setting of the second part of the Theorem, and this concludes the proof. □

We note that, as an application of the homotopy invariance, one can show that $X_{\mathcal{G}}(\mathcal{T}A)$ is homotopy equivalent to $X_{\mathcal{G}}(A)$ if A is a quasifree pro-$\mathcal{G}$-algebra.

Corollary 4.6. *Let* $0 \to N \to R \to A \to 0$ *be a universal locally nilpotent extension of the pro-$\mathcal{G}$-algebra A. Any morphism of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}A & \longrightarrow & \mathcal{T}A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \xi & & \downarrow \phi & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & N & \longrightarrow & R & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

induces a homotopy equivalence $X_{\mathcal{G}}(\phi) : X_{\mathcal{G}}(\mathcal{T}A) \to X_{\mathcal{G}}(R)$. *Moreover, the class of this* *homotopy equivalence in* $H_*(\mathrm{Hom}_{A(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}A), X_{\mathcal{G}}(R)))$ *does not depend on the choice of* ϕ*.*

Proof. By Proposition 3.48 we can deduce that $\phi : \mathcal{T}A \to R$ is a $\mathcal{G}$-equivariant homotopy equivalence of pro-$\mathcal{G}$-algebras. Then we can use Theorem 4.5 to get a homotopy equivalence $X_{\mathcal{G}}(\phi) : X_{\mathcal{G}}(\mathcal{T}A) \to X_{\mathcal{G}}(R)$. From the uniqueness of ϕ up to $\mathcal{G}$-equivariant homotopy, we immediately get the independence of the choice of ϕ. □

§ 4.2 | Stability

Next we show that $HP_*^{\mathcal{G}}$ is stable in both variables with respect to tensoring with the algebra $\mathcal{K}(E)$ associated to a $\mathcal{G}$-module E together with a $\mathcal{G}$-equivariant pairing as defined in Subsection 2.3.2.

Throughout this section, we denote by $h : E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E \to C_c^\infty(\mathcal{G}^{(0)})$ a given $\mathcal{G}$-equivariant pairing on E.

Definition 4.7. *Let E be a $\mathcal{G}$-module. The* twisted trace *map* $ttr : \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \to \mathcal{O}_{\mathcal{G}}$ *is defined by setting*

$$ttr(f \otimes e_1 \otimes e_2) = (\mathrm{id} \otimes h)(T(f \otimes e_2) \otimes e_1)$$

for $f \in \mathcal{O}_{\mathcal{G}}$ *and* $e_1, e_2 \in E$*.*

Explicitly, we have

$$ttr(\chi_U \otimes e_1 \otimes e_2) = \chi_U \otimes h(\chi_{U^{-1}} \cdot e_2 \otimes e_1) \in \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \cong \mathcal{O}_{\mathcal{G}}$$

for any compact open bisection $U \subseteq \mathcal{G}$.

Lemma 4.8. *The twisted trace map introduced above satisfies*

$$ttr(\chi_U \otimes L_0 L_1) = ttr(\chi_U \otimes (\chi_{U^{-1}} \cdot L_1) L_0)$$

for any compact open bisection $U \subseteq \mathcal{G}$ *and* $L_0, L_1 \in \mathcal{K}(E)$*.*

Proof. It suffices to prove the claim for $L_0 = e_1 \otimes e_2$ and $L_1 = e_3 \otimes e_4$ for any $e_1, e_2, e_3, e_4 \in E$. Recall that product in $\mathcal{K}(E)$ is given by

$$L_0 L_1 = e_1 \otimes h(e_2 \otimes e_3) e_4.$$

With these assumptions, we obtain

$$
\begin{aligned}
ttr(\chi_U \otimes L_0 L_1) &= ttr(\chi_U \otimes e_1 \otimes h(e_2 \otimes e_3)e_4) \\
&= \chi_U \otimes h(\chi_{U^{-1}} \cdot (h(e_2 \otimes e_3)e_4) \otimes e_1) \\
&= \chi_U \otimes \chi_{U^{-1}} \cdot h(e_2 \otimes e_3) h(\chi_{U^{-1}} \cdot e_4 \otimes e_1) \\
&= \chi_U \otimes h(e_2 \otimes e_3) h(\chi_{U^{-1}} \cdot e_4 \otimes e_1)
\end{aligned}
$$

and

$$
\begin{aligned}
ttr(\chi_U \otimes (\chi_{U^{-1}} \cdot L_1) L_0) &= ttr(\chi_U \otimes \chi_{U^{-1}} \cdot e_3 \otimes h(\chi_{U^{-1}} \cdot e_4 \otimes e_1) e_2) \\
&= \chi_U \otimes h(\chi_{U^{-1}} \cdot (h(\chi_{U^{-1}} \cdot e_4 \otimes e_1) e_2) \otimes \chi_{U^{-1}} \cdot e_3) \\
&= \chi_U \otimes \chi_{U^{-1}} \cdot h(h(\chi_{U^{-1}} \cdot e_4 \otimes e_1) e_2 \otimes e_3) \\
&= \chi_U \otimes h(e_2 \otimes e_3) h(\chi_{U^{-1}} \cdot e_4 \otimes e_1).
\end{aligned}
$$

Observe that we used the $\mathcal{G}$-equivariance of the pairing in the third equality and that $\chi_U \otimes f = \chi_U \otimes \chi_{U^{-1}} \cdot f$ for all $f \in C_c^\infty(\mathcal{G}^{(0)})$, or equivalently, that the canonical map T of $\mathcal{O}_\mathcal{G} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})$ equals the identity as shown in Example 2.47. □

§ 4.2.1 | Admissible pairings

Let us consider a particular class of such pairings.

Definition 4.9. *Let E be a $\mathcal{G}$-module. A $\mathcal{G}$-equivariant pairing h on E is said to be* admissible *if there exists a $\mathcal{G}$-equivariant linear embedding $C_c^\infty(\mathcal{G}^{(0)}) \hookrightarrow E$ such that the restriction of h to $C_c^\infty(\mathcal{G}^{(0)}) \subseteq E$ agrees with the canonical isomorphism $C_c^\infty(\mathcal{G}^{(0)}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \cong C_c^\infty(\mathcal{G}^{(0)})$.*

Definition 4.10. *Let E be a $\mathcal{G}$-module equipped with an admissible $\mathcal{G}$-equivariant pairing. Define the linear map*

$$\iota : C_c^\infty(\mathcal{G}^{(0)}) \cong C_c^\infty(\mathcal{G}^{(0)}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \hookrightarrow \mathcal{K}(E),$$

as the composition of the canonical isomorphism for $C_c^\infty(\mathcal{G}^{(0)})$ with the tensor product of the embedding by itself.

Remark 4.11. *Admissibility, that is the existence of such an embedding, ensures that the map ι defined above is a $\mathcal{G}$-equivariant algebra homomorphism. Indeed, if we denote by $\mathrm{e} : C_c^\infty(\mathcal{G}^{(0)}) \hookrightarrow E$ the embedding, then the claim follows by observing that both the canonical isomorphism for $C_c^\infty(\mathcal{G}^{(0)})$ and the map $\mathrm{e} \otimes \mathrm{e}$ are $\mathcal{G}$-equivariant algebra homomorphisms.*

More generally, we consider the following construction.

Definition 4.12. *Let E be a $\mathcal{G}$-module equipped with an admissible $\mathcal{G}$-equivariant bilinear* *pairing, and let A be a $\mathcal{G}$-algebra. We define the map*

$$\iota_A : A \cong A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)}) \longrightarrow A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)$$

as the tensor product of the identity on A with the map $\iota : C_c^\infty(\mathcal{G}^{(0)}) \to \mathcal{K}(E)$.

Remark 4.13. *The map ι_A defined in Definition 4.12 is a $\mathcal{G}$-equivariant algebra homomorphism.*

With these preparations in place, we can now state one of the main theorems of this section. This result will serve as a cornerstone for the general stability theorem that will be presented later in the section.

Theorem 4.14. *Let A be a pro-$\mathcal{G}$-algebra, and let E be a $\mathcal{G}$-module equipped with an admissible $\mathcal{G}$-equivariant bilinear pairing. Then the class*

$$[\iota_A] \in H_0\mathrm{Hom}_{A(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}A), X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))))$$

is invertible.

Proof. We have to find an inverse for $[\iota_A]$. First observe that the canonical $\mathcal{G}$-equivariant linear map $A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \to \mathcal{T}A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)$ is a lonilcur and hence induces a $\mathcal{G}$-equivariant homomorphism $\lambda_A : \mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)) \to \mathcal{T}A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)$, which concretely acts by

$$\lambda_A(f \otimes \langle a^0 \otimes L_0\rangle d(a^1 \otimes L_1)\dots d(a^{2n} \otimes L_{2n})) = f \otimes \langle a^0\rangle da^1 \dots a^{2n} \otimes \langle L_0\rangle L_1 \dots L_{2n},$$

for $f \in \mathcal{O}_{\mathcal{G}}$, $a^i \in A$ and $L_i \in \mathcal{K}(E)$.

Define $tr_A : X_{\mathcal{G}}(\mathcal{T}A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)) \to X_{\mathcal{G}}(\mathcal{T}A)$ by

$$tr_A(f \otimes x \otimes L) = ttr(f \otimes L) \otimes x$$

on $\Omega^0_{\mathcal{G}}(\mathcal{T}A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$ and

$$\begin{aligned} tr_A(f \otimes (x_0 \otimes L_0)d(x_1 \otimes L_1)) &= ttr(f \otimes L_0L_1) \otimes x_0dx_1 \\ tr_A(f \otimes d(x_1 \otimes L_1)) &= ttr(f \otimes L_1) \otimes dx_1, \end{aligned}$$

on $\Omega^1_{\mathcal{G}}(\mathcal{T}A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$ for $f \in \mathcal{O}_{\mathcal{G}}$, $x, x_0, x_1 \in \mathcal{T}A$ and $L, L_0, L_1 \in \mathcal{K}(E)$. Observe that we used the twisted trace $ttr : \mathcal{O}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \to \mathcal{O}_{\mathcal{G}}$ as in Definition 4.7. By construction, tr_A is a map of $\mathcal{G}$-anti-Yetter-Drinfeld modules. We have

$$\begin{aligned} tr_A d_{\mathcal{G}}(f \otimes x \otimes L) &= tr_A(f \otimes d(x \otimes L)) \\ &= ttr(f \otimes L) \otimes dx \end{aligned}$$

$$
\begin{aligned}
&= d_{\mathcal{G}}(ttr(f \otimes L) \otimes x) \\
&= d_{\mathcal{G}} tr_A(f \otimes x \otimes L),
\end{aligned}
$$

and for a compact open bisection $U \subseteq \mathcal{G}$ we calculate

$$
\begin{aligned}
b_{\mathcal{G}} tr_A(\chi_U \otimes (x_0 \otimes L_0) d(x_1 \otimes L_1)) &= b_{\mathcal{G}}(ttr(\chi_U \otimes L_0 L_1) \otimes x_0 dx_1) \\
&= ttr(\chi_U \otimes L_0 L_1) \otimes (x_0 x_1 - (\chi_{U^{-1}} \cdot x_1) x_0) \\
&= ttr(\chi_U \otimes L_0 L_1) \otimes x_0 x_1 - ttr(\chi_U \otimes (\chi_{U^{-1}} \cdot L_1) L_0) \otimes (\chi_{U^{-1}} \cdot x_1) x_0 \\
&= tr_A(\chi_U \otimes (x_0 x_1 \otimes L_0 L_1) - \chi_U \otimes (\chi_{U^{-1}} \cdot x_1) x_0 \otimes (\chi_{U^{-1}} \cdot L_1) L_0) \\
&= tr_A b_{\mathcal{G}}(\chi_U \otimes (x_0 \otimes L_0) d(x_1 \otimes L_1)),
\end{aligned}
$$

using the twisted trace property from Lemma 4.8. Similarly one checks

$$
b_{\mathcal{G}} tr_A(\chi_U \otimes d(x_1 \otimes L_1)) = tr_A b_{\mathcal{G}}(\chi_U \otimes d(x_1 \otimes L_1)).
$$

It follows that tr_A is a chain map of paracomplexes.

We define $\tau_A = tr_A X_{\mathcal{G}}(\lambda_A)$ and claim that $[\tau_A]$ is an inverse for $[\iota_A]$. Since ι_A is an $\mathcal{G}$-equivariant algebra homomorphism, we consider $\mathcal{T}\iota_A : \mathcal{T}A \to \mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$ and observe that $\lambda_A \mathcal{T}\iota_A = \iota_{\mathcal{T}A}$. Moreover, because $ttr(f \otimes \iota(e)) = f$ for $f \in \mathcal{O}_{\mathcal{G}}$ and $e \in C_c^\infty(\mathcal{G}^{(0)})$, we have that

$$
\begin{aligned}
\tau_A X_{\mathcal{G}}(\mathcal{T}\iota_A) &= tr_A X_{\mathcal{G}}(\lambda_A) X_{\mathcal{G}}(\mathcal{T}\iota_A) \\
&= tr_A X_{\mathcal{G}}(\lambda_A \mathcal{T}\iota_A) \\
&= tr_A X_{\mathcal{G}}(\iota_{\mathcal{T}A}) \\
&= \mathrm{id}_{X_{\mathcal{G}}(\mathcal{T}A)}
\end{aligned}
$$

Then it follows that $[\iota_A] \cdot [\tau_A] = \mathrm{id}$. It thus remains to show that $[\tau_A] \cdot [\iota_A] = \mathrm{id}$. Consider the $\mathcal{G}$-equivariant homomorphisms

$$
i_j : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \to A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)
$$

for $j = 0, 1$ given by

$$
i_0 = \mathrm{id} \otimes \iota, \quad i_1 = (\mathrm{id} \otimes \sigma) i_0,
$$

where we used the canonical identification

$$
A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \cong A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})
$$

in the definition of i_0, and the tensor flip automorphism σ of $\mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)$ given by $\sigma(L_1 \otimes L_2) = L_2 \otimes L_1$ in the definition of i_1.

Similarly as above, we calculate $[i_0] \cdot [\tau_{A\otimes\mathcal{K}(E)}] = \mathrm{id}$ and $[i_1] \cdot [\tau_{A\otimes\mathcal{K}(E)}] = [\tau_A] \cdot [\iota_A]$. Let us show that the maps i_0 and i_1 are $\mathcal{G}$-equivariant homotopic. To this end observe that $\mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \cong \mathcal{K}(E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E)$ as $\mathcal{G}$-algebras and denote by Σ the flip automorphism of $E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$ given by $\Sigma(e \otimes f) = f \otimes e$. For $t \in [0,1]$ we then obtain a $\mathcal{G}$-equivariant linear endomorphism Σ_t of $E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$ given by

$$\Sigma_t = \cos(\pi t/2)\,\mathrm{id} + \sin(\pi t/2)\Sigma,$$

and we note that Σ_t is invertible with inverse $\Sigma_t^{-1} = \cos(\pi t/2)\,\mathrm{id} - \sin(\pi t/2)\Sigma$. Since Σ is isometric with respect to the tensor product pairing on $E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E$, that is, it preserves the value of the pairing, the same holds for Σ_t. It follows that $\sigma_t = \Sigma_t \otimes \Sigma_t$ defines $\mathcal{G}$-equivariant algebra automorphism of

$$\mathcal{K}(E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E) = (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} (E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E).$$

The family $(\sigma_t)_{t\in[0,1]}$ depends smoothly on t, and by construction we have $\sigma_0 = \mathrm{id}$ and $\sigma_1 = \sigma$. Now define

$$h_t : A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \to A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E)$$

by $h_t = (\mathrm{id} \otimes \sigma_t) i_0$ for $t \in [0,1]$. Then each h_t is a $\mathcal{G}$-equivariant algebra homomorphism, and by construction $h_j = i_j$ for $j = 0,1$. Since the family $(h_t)_{t\in[0,1]}$ depends again smoothly on t we have thus constructed a $\mathcal{G}$-equivariant homotopy between i_0 and i_1. According to Theorem 4.5 we obtain $[i_0] = [i_1]$, and hence $[\tau_A] \cdot [\iota_A] = \mathrm{id}$ as required. □

§ 4.2.2 | A more general case

In order to discuss the implications of Theorem 4.14 for the stability properties of the functor $HP_*^{\mathcal{G}}$ in a more general setting, we need some preparation.

Lemma 4.15. *Let E, F be $\mathcal{G}$-modules, and suppose that $E \cong F$ as $C_c^\infty(\mathcal{G}^{(0)})$-modules. Then*

$$C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} E \cong C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} F$$

as $\mathcal{G}$-modules.

Proof. Let $\phi : E \to F$ be a $C_c^\infty(\mathcal{G}^{(0)})$-linear isomorphism. Consider the map

$$T_F^{-1}(\mathrm{id} \otimes \phi) T_E : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} E \to C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} F.$$

This is a $\mathcal{D}(\mathcal{G})$-linear isomorphism from $C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} E$ to $C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} F$ both sides endowed with the natural diagonal action. Indeed, for U and V compact open bisections of $\mathcal{G}$ and $e \in E$, we compute

$$\begin{aligned}\chi_V \cdot (T_F^{-1}(\mathrm{id} \otimes \phi) T_E)(\chi_U \otimes e) &= \chi_{VU} \otimes \chi_{VU} \cdot \phi(\chi_{U^{-1}} \cdot e) \\ &= \chi_{VU} \otimes \chi_{VU} \cdot \phi(\chi_{U^{-1}V^{-1}} * \chi_V \cdot e) \\ &= T_F^{-1}(\mathrm{id} \otimes \phi) T_E(\chi_V \cdot (\chi_U \otimes e)),\end{aligned}$$

where we have used the definition of the maps T_E and T_F as constructed in Section 2.1. □

For a transformation groupoid $\mathcal{G} = \Gamma \ltimes X$, associated to a discrete group Γ acting on a totally disconnected locally compact space X, we have $\mathcal{D}(\mathcal{G}) \cong \bigoplus_{\gamma \in \Gamma} C_c^\infty(\mathcal{G}^{(0)})$ as left $C_c^\infty(\mathcal{G}^{(0)})$-modules. Hence Lemma 4.15 shows that there is a $\mathcal{G}$-equivariant isomorphism

$$C_c^\infty(\mathcal{G}) \overset{r,r}{\otimes} C_c^\infty(\mathcal{G}) \cong \bigoplus_{\gamma \in \Gamma} C_c^\infty(\mathcal{G})$$

in this case. The same is true for ample groupoids $\mathcal{G}$ which can be covered by a family of disjoint global range sections.

However, not every ample groupoid admits such a covering. This fails already in the case of finite groupoids, as the example of a disjoint union of finite groups of different orders shows. In this case we cannot write the underlying $C_c^\infty(\mathcal{G}^{(0)})$-module of $\mathcal{D}(\mathcal{G})$ as a direct sum of copies of $C_c^\infty(\mathcal{G}^{(0)})$.

The following lemma allows one to circumvent this by passing to infinite direct sums of copies of $\mathcal{D}(\mathcal{G})$. In the sequel we write $V^{\oplus \kappa}$ for a direct sum of copies of V indexed by a set of cardinality κ.

Lemma 4.16. *Let E be a $\mathcal{G}$-module equipped with a $\mathcal{G}$-equivariant bilinear pairing $h : E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E \to C_c^\infty(\mathcal{G}^{(0)})$. If $C_c^\infty(\mathcal{G}^{(0)})$ and E are projective as essential $C_c^\infty(\mathcal{G}^{(0)})$-modules and the map h is surjective then there exists an isomorphism*

$$E^{\oplus \kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus \kappa}$$

of $C_c^\infty(\mathcal{G}^{(0)})$-modules for any infinite cardinal κ such that E admits a generating set of cardinality at most κ.

Proof. We fix I with $|I| = \kappa$ and a family $(e_i)_{i \in I}$ of elements of E which generate E as a $C_c^\infty(\mathcal{G}^{(0)})$-module. Then we obtain a surjection $C_c^\infty(\mathcal{G}^{(0)})^{\oplus \kappa} \to E$ of $C_c^\infty(\mathcal{G}^{(0)})$-modules by mapping $(f_i)_{i \in I}$ to $\sum_{i \in I} f_i \cdot e_i$. Since E is projective this surjection splits, so that E can be written as a direct summand of $C_c^\infty(\mathcal{G}^{(0)})^{\oplus \kappa}$. By our assumption that κ is infinite it follows that $E^{\oplus\kappa}$ is a direct summand of $(C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa})^{\oplus\kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$. Explicitly, let us choose a direct complement P, so that

$$P \oplus E^{\oplus\kappa} \cong E^{\oplus\kappa} \oplus P \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}.$$

By writing again $C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \cong (C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa})^{\oplus\kappa}$ we then get

$$E^{\oplus\kappa} \oplus C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \cong E^{\oplus\kappa} \oplus (P \oplus E^{\oplus\kappa})^{\oplus\kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}.$$

Similarly, using that the pairing h is surjective we obtain a surjection $E^{\oplus\kappa} \to C_c^\infty(\mathcal{G}^{(0)})$ of $C_c^\infty(\mathcal{G}^{(0)})$-modules mapping $(x_i)_{i\in I}$ to $\sum_{i\in I} h(x_i \otimes e_i)$. Since $C_c^\infty(\mathcal{G}^{(0)})$ is projective it follows that $C_c^\infty(\mathcal{G}^{(0)})$ is a direct summand in $E^{\oplus\kappa}$. In the same way as above we can then write $C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$ as a direct summand of $E^{\oplus\kappa}$, and construct an isomorphism

$$E^{\oplus\kappa} \oplus C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \cong E^{\oplus\kappa}.$$

Combining these considerations, we therefore obtain an isomorphism

$$E^{\oplus\kappa} \cong E^{\oplus\kappa} \oplus C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$$

of $C_c^\infty(\mathcal{G}^{(0)})$-modules as required. □

Let us make some comments on Lemma 4.16. Projectivity of $C_c^\infty(\mathcal{G}^{(0)})$, viewed as an essential module over itself, is a mild assumption which is satisfied whenever $\mathcal{G}^{(0)}$ is paracompact. This follows easily from the fact that one can write $\mathcal{G}^{(0)}$ as a disjoint union of compact open subsets in this case, compare the discussion in the proof of Proposition 1.76. If $\mathcal{G}$ is paracompact, admitting a covering by mutually disjoint compact open range sections indexed by a set of cardinality μ, then the same argument as in [BDGW23, Lemma 2.13] shows that $\mathcal{D}(\mathcal{G})$, viewed as $C_c^\infty(\mathcal{G}^{(0)})$-module, is projective and admits a generating set of cardinality μ. Since the standard pairing on $\mathcal{D}(\mathcal{G})$ is always surjective, Lemma 4.16 yields an isomorphism

$$\mathcal{D}(\mathcal{G})^{\oplus\kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$$

of left $C_c^\infty(\mathcal{G}^{(0)})$-modules for any infinite cardinal $\kappa \geq \mu$ in this case. If $\mathcal{G}$ is σ-compact there exists a countable such covering family, so that the countable direct sum of copies of $\mathcal{D}(\mathcal{G})$ is isomorphic to a countable direct sum of copies of $C_c^\infty(\mathcal{G}^{(0)})$.

We also note that if $\mathcal{G}^{(0)}$ is discrete then every $C_c^\infty(\mathcal{G}^{(0)})$-module is projective. In contrast, for a general totally disconnected base space $\mathcal{G}^{(0)}$, a module of the form $E = \mathbb{C}$ with the action of $C_c^\infty(\mathcal{G}^{(0)})$ given by point evaluation at some point $x \in \mathcal{G}^{(0)}$ will typically fail to be projective.

Theorem 4.17 (Stability)**.** *Let E be a $\mathcal{G}$-module equipped with a surjective $\mathcal{G}$-equivariant bilinear pairing. If $C_c^\infty(\mathcal{G}^{(0)})$ and E are projective as essential $C_c^\infty(\mathcal{G}^{(0)})$-modules then there exists an invertible element in*

$$HP_0^{\mathcal{G}}(A, A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$$

for any pro-$\mathcal{G}$-algebra A. It follows that we have natural isomorphisms

$$HP_*^{\mathcal{G}}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E), B) \cong HP_*^{\mathcal{G}}(A, B) \cong HP_*^{\mathcal{G}}(A, B \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))$$

for all pro-$\mathcal{G}$-algebras A and B.

Proof. According to Lemma 4.16 we obtain an isomorphism $E^{\oplus\kappa} \cong C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$ of $C_c^\infty(\mathcal{G}^{(0)})$-modules for some infinite cardinal κ. Now let us view $C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$ as a $\mathcal{G}$-module via the canonical action. Using Lemma 4.15 we obtain an isomorphism

$$\mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E^{\oplus\kappa} \cong \mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \cong \mathcal{D}(\mathcal{G})^{\oplus\kappa}$$

of $\mathcal{G}$-modules. The pairing $C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa} \to C_c^\infty(\mathcal{G}^{(0)})$ induced from $E^{\oplus\kappa}$ splits as a map of $C_c^\infty(\mathcal{G}^{(0)})$-modules because $C_c^\infty(\mathcal{G}^{(0)})$ is projective. Moreover, since we are considering the canonical $\mathcal{G}$-module structure on $C_c^\infty(\mathcal{G}^{(0)})$ the pairing splits as $\mathcal{G}$-equivariant linear map. Then we have that the pairing on the $\mathcal{G}$-module $C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}$ induced from $E^{\oplus\kappa}$ is admissible.

Hence, the proof of Theorem 4.14 gives homotopy equivalences

$$\begin{aligned}
X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})) &\simeq X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}))) \\
&\cong X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(\mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} C_c^\infty(\mathcal{G}^{(0)})^{\oplus\kappa}))) \\
&\cong X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(\mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E^{\oplus\kappa}))) \\
&\simeq X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(\mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} E))) \\
&\cong X_{\mathcal{G}}(\mathcal{T}(A \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}(E))).
\end{aligned}$$

This yields the assertion. □

§ 4.2.3 | Stability for proper groupoids

Let $\mathcal{G}$ be a proper ample groupoid such that $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is paracompact. According to Proposition 1.76 there exists a locally constant cut-off function c for $\mathcal{G}$. It follows that for any

$f \in C_c^\infty(\mathcal{G}^{(0)})$ the function $s^*(c)r^*(f) : \mathcal{G} \to \mathbb{C}$ given by

$$s^*(c)r^*(f)(\alpha) = c(s(\alpha))f(r(\alpha))$$

has compact support and is thus contained in $C_c^\infty(\mathcal{G})$.

Now let E, F be $\mathcal{G}$-modules and let $\phi : E \to F$ be a $C_c^\infty(\mathcal{G}^{(0)})$-linear map. As in the proof of Lemma 4.15 we obtain a $\mathcal{G}$-equivariant linear map $\phi^T = T_F^{-1}(\mathrm{id} \otimes \phi)T_E : C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} E \to C_c^\infty(\mathcal{G}) \overset{r,\mathrm{id}}{\otimes} F$. Recall moreover from Lemma 2.16 that the integration map $\lambda : C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$ is $\mathcal{G}$-equivariant with respect to the left multiplication action on $C_c^\infty(\mathcal{G}) = \mathcal{D}(\mathcal{G})$. Hence we obtain a linear map $\phi^\mathcal{G} : E \to F$ by defining

$$\phi^\mathcal{G}(f \otimes e) = (\lambda \otimes \mathrm{id})\phi^T(s^*(c)r^*(f) \otimes e),$$

using the canonical identification $X \cong C_c^\infty(\mathcal{G}^{(0)}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} X$ for $X = E, F$.

Lemma 4.18. *Let $\mathcal{G}$ be a proper ample groupoid with $\mathcal{G}\backslash\mathcal{G}^{(0)}$ paracompact and let E, F be $\mathcal{G}$-modules. If $\phi : E \to F$ is a $C_c^\infty(\mathcal{G}^{(0)})$-linear map then $\phi^\mathcal{G} : E \to F$ is a $\mathcal{G}$-equivariant linear map.*

Proof. For a compact open bisection $U \subseteq \mathcal{G}$ and a compact open set $V \subseteq \mathcal{G}^{(0)}$ we have $\chi_U \cdot \chi_V = \chi_{r(U \cap s^{-1}(V))} = \chi_{U \cdot V}$. Using this we calculate

$$\begin{aligned}
\lambda(\chi_U * (s^*(c)r^*(\chi_V)))(x) &= \chi_U \cdot \lambda(s^*(c)r^*(\chi_V))(x) \\
&= \sum_{\gamma \in \mathcal{G}^x} \chi_U(\gamma)\lambda(s^*(c)r^*(\chi_V))(\gamma^{-1} \cdot x) \\
&= \sum_{\gamma \in \mathcal{G}^x} \sum_{\beta \in \mathcal{G}^{\gamma^{-1} \cdot x}} \chi_U(\gamma)c(s(\beta))\chi_V(r(\beta)) \\
&= \sum_{\beta \in \mathcal{G}^x} \sum_{\gamma \in \mathcal{G}^x} \chi_U(\gamma)c(s(\gamma^{-1}\beta))\chi_V(r(\gamma^{-1}\beta)) \\
&= \sum_{\gamma \in \mathcal{G}^x} \chi_U(\gamma)\chi_V(r(\gamma^{-1})) \\
&= \chi_{r(U \cap s^{-1}(V))}(x) \\
&= \sum_{\beta \in \mathcal{G}^x} cs(\beta)\chi_{r(U \cap s^{-1}(V))}(x) \\
&= \lambda(s^*(c)r^*(\chi_{U \cdot V}))(x)
\end{aligned}$$

for all $x \in \mathcal{G}^{(0)}$.

Moreover, observing that ϕ^T is $C_c^\infty(\mathcal{G})$-linear on the left with respect to the first factor, for $f \in C_c^\infty(\mathcal{G})$, $e \in E$ and W a compact open bisection of $\mathcal{G}$ such that $f\chi_W = f$, we have

$$\begin{aligned}\phi^T(f \otimes e) &= \phi^T(f\chi_W \otimes e)\\ &= f \cdot \phi^T(\chi_W \otimes e)\\ &= f \cdot (\chi_W \otimes \chi_W \cdot \phi(\chi_{W^{-1}} \cdot e))\\ &= f\chi_W \otimes \chi_W \cdot \phi(\chi_{W^{-1}} \cdot e)\\ &= f \otimes \chi_W \cdot \phi(\chi_{W^{-1}} \cdot e).\end{aligned}$$

Using the previous computations and recalling that λ is $\mathcal{G}$-equivariant, for $e \in E$ we then compute

$$\begin{aligned}\phi^{\mathcal{G}}(\chi_U \cdot (\chi_V \otimes e)) &= \phi^{\mathcal{G}}(\chi_{U\cdot V} \otimes \chi_U \cdot e)\\ &= (\lambda \otimes \mathrm{id})\phi^T(s^*(c)r^*(\chi_{U\cdot V}) \otimes \chi_U \cdot e)\\ &= (\lambda \otimes \mathrm{id})\phi^T(\chi_U * (s^*(c)r^*(\chi_V)) \otimes \chi_U \cdot e)\\ &= \chi_U \cdot (\lambda \otimes \mathrm{id})\phi^T(s^*(c)r^*(\chi_V) \otimes e)\\ &= \chi_U \cdot \phi^{\mathcal{G}}(\chi_V \otimes e)\end{aligned}$$

as required. □

We remark that if the map ϕ in Lemma 4.18 is already $\mathcal{G}$-equivariant then $\phi^{\mathcal{G}} = \phi$. Indeed, analogously to what done in the proof of Lemma 4.18, choosing a compact open bisection W of $\mathcal{G}$ such that $\chi_W s^*(c)r^*(f) = s^*(c)r^*(f)$, we have

$$\begin{aligned}\phi^{\mathcal{G}}(f \otimes e) &= (\lambda \otimes \mathrm{id})\phi^T(s^*(c)r^*(f) \otimes e)\\ &= (\lambda \otimes \mathrm{id})(s^*(c)r^*(f) \otimes \chi_W \cdot \phi(\chi_{W^{-1}} \cdot e))\\ &= \lambda(s^*(c)r^*(f)) \otimes \phi(e)\\ &= f \otimes \phi(e)\end{aligned}$$

for all $f \in C_c^\infty(\mathcal{G}^{(0)})$ and $e \in E$. In a similar way we get $(\phi\psi)^{\mathcal{G}} = \phi^{\mathcal{G}}\psi$ and $(\theta\phi)^{\mathcal{G}} = \theta\phi^{\mathcal{G}}$ if ψ, θ are $\mathcal{G}$-equivariant linear maps.

Proposition 4.19. *Let $\mathcal{G}$ be a proper ample groupoid with $\mathcal{G}\backslash\mathcal{G}^{(0)}$ paracompact. Then we have a natural isomorphism*

$$HP_*^{\mathcal{G}}(A, B) \cong H_*\mathrm{Hom}_{A(\mathcal{G})}(X_{\mathcal{G}}(\mathcal{T}A), X_{\mathcal{G}}(\mathcal{T}B))$$

for all $\mathcal{G}$-algebras A, B.

Proof. The integration map defines a $\mathcal{G}$-equivariant surjection $\lambda : \mathcal{D}(\mathcal{G}) \to C_c^\infty(\mathcal{G}^{(0)})$, compare Lemma 2.16. Moreover, the extension of functions by zero induces a $C_c^\infty(\mathcal{G}^{(0)})$-linear inclusion map $\iota : C_c^\infty(\mathcal{G}^{(0)}) \to \mathcal{D}(\mathcal{G})$. Since

$$\lambda\iota(f)(x) = \sum_{\alpha \in \mathcal{G}^x} \iota(f)(\alpha) = f(x)$$

we see that $C_c^\infty(\mathcal{G}^{(0)})$ is a direct summand of the $C_c^\infty(\mathcal{G}^{(0)})$-module $\mathcal{D}(\mathcal{G})$. According to Lemma 4.18 it follows that $\iota^{\mathcal{G}}$ is a $\mathcal{G}$-equivariant splitting of λ, so the $\mathcal{G}$-module $C_c^\infty(\mathcal{G}^{(0)})$ is a direct summand of $\mathcal{D}(\mathcal{G})$ in the category of $\mathcal{G}$-modules as well. We conclude that the regular pairing on $\mathcal{D}(\mathcal{G})$ is admissible, so that the claim follows from Theorem 4.14. □

§ 4.3 | Excision

In the final part of this chapter, we discuss excision. This property was first established by Cuntz and Quillen in [CQ97], and represents one of the main achievements in their series of papers on bivariant periodic cyclic homology. Their result provided a further conceptual link between periodic cyclic homology and K-theory.

We show that equivariant periodic cyclic homology satisfies excision in both variables. An excellent source for the main ideas of these proofs is [Mey99], and our argument closely follows the proof in the group equivariant case presented in [Voi07]. For this reason, we shall be rather brief and only sketch the main strategy.

We consider an admissible extension

$$K \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} Q$$

of pro-$\mathcal{G}$-algebras, with a fixed $\mathcal{G}$-equivariant pro-linear splitting $\sigma : Q \to E$ for the quotient homomorphism $\pi : E \to Q$.

Let $X_{\mathcal{G}}(\mathcal{T}E : \mathcal{T}Q)$ be the kernel of the map $X_{\mathcal{G}}(\mathcal{T}\pi) : X_{\mathcal{G}}(\mathcal{T}E) \to X_{\mathcal{G}}(\mathcal{T}Q)$ induced by π. The splitting σ yields a direct sum decomposition $X_{\mathcal{G}}(\mathcal{T}E) = X_{\mathcal{G}}(\mathcal{T}E : \mathcal{T}Q) \oplus X_{\mathcal{G}}(\mathcal{T}Q)$ of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules. Moreover, since $X_{\mathcal{G}}(\mathcal{T}\pi)X_{\mathcal{G}}(\mathcal{T}\iota) = 0$ there is a natural map $\rho : X_{\mathcal{G}}(\mathcal{T}K) \to X_{\mathcal{G}}(\mathcal{T}E : \mathcal{T}Q)$ of paracomplexes of pro-$\mathcal{G}$-anti-Yetter-Drinfeld modules. At this point, if the map ρ is a homotopy equivalence of paracomplexes, then the long exact sequence in homology induced by the above extension will give the excision result.

With this introduction, we see that the key step in the proof of the excision theorem is the following result.

Theorem 4.20. *Let*

$$K \rightarrowtail^{\iota} E \overset{\pi}{\twoheadrightarrow} Q$$

an admissible extension of pro-$\mathcal{G}$-algebras, with a fixed $\mathcal{G}$-equivariant pro-linear splitting $\sigma : Q \to E$ for the quotient homomorphism $\pi : E \to Q$. Then the map $\rho : X_{\mathcal{G}}(\mathcal{T}K) \to X_{\mathcal{G}}(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.

As a consequence of Theorem 4.20 one obtains excision in both variables for $\mathcal{G}$-equivariant periodic cyclic homology.

Theorem 4.21 (Excision)**.** *Let A be a pro-$\mathcal{G}$-algebra and let $0 \to K \to E \to Q \to 0$ be an extension of pro-$\mathcal{G}$-algebras which is admissible as an extension of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules. Then there are two natural exact sequences*

$$\begin{array}{ccccc} HP_0^{\mathcal{G}}(A,K) & \longrightarrow & HP_0^{\mathcal{G}}(A,E) & \longrightarrow & HP_0^{\mathcal{G}}(A,Q) \\ \uparrow & & & & \downarrow \\ HP_1^{\mathcal{G}}(A,Q) & \longleftarrow & HP_1^{\mathcal{G}}(A,E) & \longleftarrow & HP_1^{\mathcal{G}}(A,K) \end{array}$$

and

$$\begin{array}{ccccc} HP_0^{\mathcal{G}}(Q,A) & \longrightarrow & HP_0^{\mathcal{G}}(E,A) & \longrightarrow & HP_0^{\mathcal{G}}(K,A) \\ \uparrow & & & & \downarrow \\ HP_1^{\mathcal{G}}(K,A) & \longleftarrow & HP_1^{\mathcal{G}}(E,A) & \longleftarrow & HP_1^{\mathcal{G}}(Q,A), \end{array}$$

where the horizontal maps in these diagrams are induced by the maps in the extension.

Remark 4.22. *In Theorem 4.21 we only require that the given extension is admissible as an extension of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules, or equivalently, that there exists a pro-$C_c^\infty(\mathcal{G}^{(0)})$-linear splitting for the quotient homomorphism $E \to Q$.*

Sketch of the proof of Theorem 4.21. We start considering the extension

$$K \rightarrowtail E \twoheadrightarrow Q,$$

and tensoring with $\mathcal{K}_{\mathcal{G}}$ gives the extension

$$K \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \rightarrowtail E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \twoheadrightarrow Q \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}$$

of pro-$\mathcal{G}$-algebras which is admissible as an extension of pro-$C_c^\infty(\mathcal{G}^{(0)})$-modules. Recalling that $\mathcal{K}_{\mathcal{G}} = \mathcal{D}(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G})$ and using twice the same argument as in the proof of Lemma 4.15, first on $Q \to E$ and then on $Q \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G}) \to E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{D}(\mathcal{G})$, we obtain a $\mathcal{G}$-equivariant pro-linear splitting for the quotient map $E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}} \to Q \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}$.

Hence, the hypotheses of Theorem 4.20 are satisfied and the claim follows by considering long exact sequences in homology in both variables induced by the short exact sequence

$$\ker(X_{\mathcal{G}}(\mathcal{T}(\pi \otimes \mathrm{id}_{\mathcal{K}_{\mathcal{G}}}))) \rightarrowtail X_{\mathcal{G}}(\mathcal{T}(E \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}})) \twoheadrightarrow X_{\mathcal{G}}(\mathcal{T}(Q \otimes_{C_c^\infty(\mathcal{G}^{(0)})} \mathcal{K}_{\mathcal{G}}))$$

of paracomplexes and the homotopy invariance of $HP_*^{\mathcal{G}}$. □

Future directions

The results obtained in this thesis provide a general framework for the study of equivariant bivariant periodic cyclic homology associated with groupoid actions. Having established the basic algebraic and homological machinery, a number of directions for further investigation naturally emerge.

A first important problem is to clarify the relationship between the theory developed here and equivariant KK-theory. In this work, we have already shown that the two theories share several common properties. Moreover, a further result in this direction has already been obtained, proving a homological analogue of the Green–Julg theorem for the equivariant K-theory of proper groupoids due to Tu; see [PV25, Theorem 6.1].

Secondly, the computation of $HP_*^{\mathcal{G}}$ for specific families of $\mathcal{G}$-algebras constitutes a natural direction for future research. Closely related to this problem is the comparison with other homological theories, as has been done for example in [Voi03, Chapter 5].

Thirdly, in view of the growing interest in non-Hausdorff groupoids, it would be natural to investigate possible extensions of the present theory to the more general non-Hausdorff setting.

Finally, at the end of Chapter 3, we stated Proposition 3.57, which compares $\mathcal{G}$-equivariant homology with group-equivariant homology in the special case of discrete groupoids. One could pursue this line of investigation further by proving analogous results under Morita equivalence of groupoids, thereby extending the discrete case.

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