Legendre transformations for special solutions of the WDVV equations

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Abstract

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations admit many interrelated classes of exact solutions, some of which are associated with the richer structure of a Frobenius manifold. This thesis explores a symmetry of the WDVV equations known as a Legendre transformation. Remarkably, Legendre transformations have previously been shown to connect rational and trigonometric solutions in limited examples.

Legendre transformations are generated by a choice of vector field, which must satisfy a geometric requirement called the Legendre field condition. In the first part of the thesis, we analyse the Legendre field condition in the setting of each of the two-dimensional Frobenius manifolds. We explicitly describe all homogeneous Legendre fields for these manifolds. We also consider so-called twisted Legendre fields for the Frobenius manifold associated with the Coxeter root system A_2. Twisted fields arise from almost duality for Frobenius manifolds, which provides a mapping between certain polynomial and rational solutions of the WDVV equations.

In the second part of the thesis, we investigate particular examples of Legendre transformations applied to multi-parameter families of rational solutions. The solutions we consider are associated with deformations of the A_n and B_n root systems; for special values of the parameters, these are almost dual to the corresponding polynomial Frobenius manifold. In all cases that we consider, Legendre transformations of the rational solutions produce trigonometric solutions which are also related to root systems. The B_n-type rational solutions are mapped to a sub-class of a family of BC_{n-1}-type trigonometric solutions already known in the literature.

Motivated by results in the type A case, we introduce a new multi-parameter family of A-type trigonometric solutions. We show that this family generalises and relates some previously known trigonometric solutions of the same form. We find that two different types of Legendre transformations applied to A_n-type rational solutions both produce sub-families of the new A_{n-1}-type trigonometric solutions.

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Author's Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

1.1 The Witten–Dijkgraaf–Verlinde–Verlinde equations

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations are a nonlinear system of third-order partial differential equations for a function $F = F(t^1, \dots, t^n)$. Since their origins in two-dimensional topological quantum field theory, they have appeared in many different mathematical settings such as quantum cohomology, singularity theory, and the study of Frobenius manifolds.

In the 1980s, the WDVV equations appeared for the first time as equations of associativity for an operator algebra in two-dimensional topological field theories [15, 61]. In these works, the physicists Witten, Dijkgraaf, Verlinde, and Verlinde constructed a family of algebras A_t depending on $t = (t^1, \dots, t^n)$ with a commutative, associative, and unital multiplication. This multiplication, which we denote as $\circ$, can be written as

$$e_i \circ e_j = \sum_{k=1}^{n} c_{ij}^k(t) e_k,$$

where $e_1, \dots, e_n$ form a basis in an n-dimensional space. The structure constants $c_{ij}^k(t)$ depend on a three-point function $c_{ijk}(t)$ and two-point function (or metric) $\eta_{ij}(t) = c_{1ij}(t)$, as follows:

$$c_{ij}^k(t) = \sum_m \eta^{km} c_{ijm}(t),$$

where $(\eta^{ij}) = \eta^{-1}$ is the matrix inverse of $(\eta_{ij}) = \eta$. The element e_1 is the multiplicative identity of A_t. Sometimes, the object η is required to have constant entries and the algebras A_t are identified with the tangent spaces of a manifold $\mathcal{M}$. In this case, η defines a flat metric on $\mathcal{M}$, as discussed below in Section 1.2.

The symmetry of the three-point function and that of its derivatives imply the existence of a potential function $F = F(t)$, called the free energy for the field theory, such that

$$c_{ijk}(t) = \frac{\partial^3 F(t)}{\partial t^i \partial t^j \partial t^k}.$$

The associativity of the multiplication $\circ$ is then equivalent to the system of equations

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i, \tag{1.1.1}$$

where F_i denotes the $n \times n$ matrix of third-order derivatives of F and $F_1 = \eta$. The $(r, s)^{th}$ entry of F_i is given by

$$(F_i)_{rs} = \frac{\partial^3 F}{\partial t^i \partial t^r \partial t^s}.$$

Equations (1.1.1) became known as the WDVV equations; however, we refer to (1.1.1) as the *ordinary* WDVV equations in contrast with a later formulation, given below and known as the generalised WDVV equations.

The ordinary WDVV equations have a distinguished direction, t^1. In the late 1990s, Marshakov, Mironov, and Morozov showed that similar systems of equations with no distinguished direction arise in Seiberg–Witten theory [46, 45]. The Seiberg–Witten prepotential F satisfies the equations

$$F_i Q^{-1} F_j = F_j Q^{-1} F_i, \tag{1.1.2}$$

for some linear combination $Q = \sum_{k=1}^{n} q^k F_k$ with $q^k = q^k\left(t^1, \ldots, t^n\right)$. We refer to (1.1.2) as the *generalised* WDVV equations. Note that the "metric" Q is not required to be constant. The ordinary WDVV equations can be recovered by setting $Q = \eta = F_1$; in fact, the equations (1.1.1) and (1.1.2) are equivalent for non-degenerate Q [45].

Solutions of the generalised WDVV equations also give a family of associative algebras A_t defined in the same way as for the ordinary WDVV equations, but with identity element $\sum_{k=1}^{n} q^k e_k$. The algebras A_t are examples of Frobenius algebras when Q is constant. A Frobenius algebra A can be described in a coordinate-free way as having an associative, symmetric, unital multiplication $\circ$ and non-degenerate inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle u \circ v, w \rangle = \langle u, v \circ w \rangle$$

for all $u, v, w \in A$. In the context of the (generalised) WDVV equations, the inner product is defined by the matrix Q so that $\langle e_i, e_j \rangle := Q_{ij}$. A Frobenius algebra uniquely defines a two-dimensional topological quantum field theory, as discussed by Kock [37].

A differential-geometric approach to the WDVV equations was developed as a field of study in its own right by Dubrovin, in his theory of Frobenius manifolds [18]. At the same time, the WDVV equations and their associated Frobenius algebras continued to be studied in supersymmetric gauge theories (see, for example, work by Martini and Gragert [47]). The WDVV equations and Frobenius manifolds are also connected to quantum cohomology, which can be thought of as a deformation of, for example, de Rham cohomology on a manifold. The cup product is replaced with a (big or small) quantum product, which can be defined using Gromov–Witten invariants. These are rational numbers which in some sense count curves intersecting certain submanifolds. Both the big and small quantum cohomology define families of Frobenius algebras; see, for example, the treatment by Cotti, Dubrovin, and Guzzetti in [13]. Kontsevich and Manin showed that the Gromov–Witten potential, a generating function for the Gromov–Witten invariants, satisfies the WDVV equations [38].

The WDVV equations admit many interconnected classes of exact solutions — some of which are associated with the richer structure of a Frobenius manifold — as well as a symmetry known as a Legendre transformation. We now introduce some important examples of solutions from Seiberg–Witten theory and quantum cohomology, before discussing Frobenius manifolds and symmetries of the WDVV equations.

1.1.1 Rational solutions

We consider the class of solutions $F = F_{\mathcal{A}}^{\text{rat}}$ where

$$F_{\mathcal{A}}^{\text{rat}}(x) = \sum_{\alpha \in \mathcal{A}} (\alpha, x)^2 \log{(\alpha, x)}, \tag{1.1.3}$$

$x \in V \cong \mathbb{C}^n$, and $\mathcal{A} \subset V$. Solutions of this form, which we call rational solutions, appear as (the perturbative parts of) Seiberg–Witten prepotentials in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories; see [46] for an early example. In particular, Martini and Gragert proved that $F_{\mathcal{A}}^{\text{rat}}$ satisfies the generalised WDVV equations when $\mathcal{A} = \mathcal{R}$ is the root system of a Weyl group [47]. Veselov derived a set of geometric conditions for a collection of vectors $\mathcal{A}$ which ensure that $F_{\mathcal{A}}^{\text{rat}}$ is a solution of the generalised WDVV equations [60]. These configurations are known as $\vee$-systems and include the Coxeter root systems [60]. Later work by Feigin and Veselov showed that certain projections and deformations of Coxeter root systems are also $\vee$-systems [26]. The corresponding multi-parameter deformations of the root systems of A_n and B_n, and their associated rational solutions, were first presented by Chalykh and Veselov in [11].

1.1.2 Trigonometric solutions

Another important class of solutions of the generalised WDVV equations have the form

$$F_{\mathcal{A}}^{\text{trig}}(x, y) = \sum_{\alpha \in \mathcal{A}} f\big((\alpha, x)\big) + P(x, y), \tag{1.1.4}$$

where P is a cubic polynomial in $x = (x^1, \ldots, x^n) \in \mathbb{C}^n$ and $y \in \mathbb{C}$. The function $f(z)$ is given by

$$f(z) = \frac{1}{6}z^3 - \frac{1}{4}\,\mathrm{Li}_3\left(e^{-2z}\right),$$

where $\mathrm{Li}_3(z)$ is the trilogarithm function. These solutions are called trigonometric solutions, in reference to the third-order derivative $f'''(z) = \coth z$. Trigonometric solutions arise in five-dimensional Seiberg–Witten theory, as originally shown in [45]. Solutions of the form $F_{\mathcal{A}}^{\mathrm{trig}}$ when $\mathcal{A} = \mathcal{R}$ is a crystallographic root system were constructed in this context by Hoevenaars and Martini [35, 36]. Of particular interest is the following A_{n-1}-type solution, which appears only in the preprint version of [35]:

$$F^{\mathrm{HM}}(x) = \sum_{1\le i<j\le n} f\left(x_i - x_j\right) + a\left(\sum_{i=1}^{n} x_i\right)^3 + b\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j^2\right) + c\sum_{i=1}^{n} x_i^3, \tag{1.1.5}$$

where the parameters $a, b, c \in \mathbb{C}$ are subject to some constraints. Bryan and Gholampour obtained Frobenius algebras corresponding to trigonometric solutions for the ADE-type Weyl groups, by studying the (equivariant) quantum cohomology of resolutions of the ADE singularities $\mathbb{C}^2/G$ where G is a finite subgroup of $SU(2)$ [10]. In the same paper, they produced Frobenius algebras for arbitrary reduced irreducible root systems, which also correspond to solutions of the form $F_{\mathcal{A}}^{\mathrm{trig}}$. Trigonometric solutions associated with multi-parameter deformations of classical root systems have been found by Pavlov [49]; Pavlov's approach uses reductions of Egorov hydrodynamic chains to produce explicit solutions for the related WDVV equations.

More recently, Shen obtained trigonometric solutions, and their corresponding Frobenius algebra structures, by investigating toric mirror arrangements associated with root systems [56]. A family of flat, torsion-free connections — originally defined in the context of Frobenius manifolds in [18] — is used to produce a multiplication on the tangent bundle of the complement of these mirror arrangements. The resulting solutions are of the form $F_{\mathcal{A}}^{\mathrm{trig}}$, where $\mathcal{A} = \mathcal{R}$ is the root system of a Weyl group W, and include arbitrary W-invariant multiplicity parameters. These results include solutions presented in [35, 36, 10] for special values of these multiplicity parameters. The A-type solution found by Shen is equivalent to (1.1.5).

In analogy to the rational case, geometric and algebraic conditions for a collection of vectors $\mathcal{A}$ such that $F_{\mathcal{A}}^{\mathrm{trig}}$ satisfies the WDVV equations have been developed. This work on trigonometric $\vee$-systems, begun by Feigin [24] and expanded on jointly with Alkadhem [2], provides an alternative construction of multi-parameter families of trigonometric solutions from (deformations of) root systems. Solutions of BCD-type previously found in [35, 36, 10, 49, 56] are generalised by an $(n+3)$-parameter family $F_{\mathcal{A}}^{\mathrm{trig}}$, where $\mathcal{A}$ is the non-reduced root system BC_n [2]. However, the A-type solutions obtained in [2] are not equivalent, for any choice of the parameters, to (1.1.5) with $c \neq 0$.

1.2 Frobenius manifolds

Solutions of the ordinary WDVV equations, with certain extra conditions, were interpreted geometrically by Dubrovin in the early 1990s [18]. We now provide a brief summary of this structure, ahead of a detailed description in the following chapter.

A Frobenius manifold $\mathcal{M}$ is a complex manifold with a flat metric η and a unital multiplication $\circ$ on $T_p\mathcal{M}$, such that each tangent space forms a commutative Frobenius algebra. A flat metric is such that its Riemann curvature tensor is zero everywhere; equivalently, there exists a local coordinate system —referred to here as flat coordinates—in which the entries of the metric are constant. The multiplicative identity e is required to be flat with respect to the Levi-Civita connection ∇ of η; that is, $\nabla_X e = 0$ for all vector fields X. Frobenius manifolds are also equipped with another distinguished vector field E, known as the Euler field, which is linear in the flat coordinates of η. Both the metric and the multiplication are required to be homogeneous with respect to the Lie derivative along E. Finally, a symmetric $(0,3)$-tensor c is defined by

$$c\,(u, v, w) = \eta\,(u \circ v, w)$$

for $u, v, w \in T_p\mathcal{M}$. The covariant derivative of c, denoted $\nabla c = \nabla_W c(X, Y, Z)$, is required to be totally symmetric in the vector fields W, X, Y, Z.

Dubrovin showed that the existence of a Frobenius manifold structure on some n-dimensional manifold $\mathcal{M}$ is equivalent to the existence of a solution $F(t)$ of the ordinary WDVV equations, defined on a chart of $\mathcal{M}$ [18]. The variables $t^1, \ldots, t^n$ are flat coordinates of the metric η. The two- and three-point functions of F can be reinterpreted as the components of η and c, so that

$$\eta_{ij}(p) = \eta\,(\partial_i, \partial_j)$$

and

$$c_{ijk}(p) = c\,(\partial_i, \partial_j, \partial_k)\,,$$

where $\partial_i = \dfrac{\partial}{\partial t^i} \in T_p\mathcal{M}$. A solution associated with a Frobenius manifold has the additional property of quasi-homogeneity:

$$\mathcal{L}_E F(t) = (3-d)F(t) + \text{quadratic terms.}$$

Here, E is the Euler field of the corresponding Frobenius manifold, and $d \in \mathbb{C}$ is known as the charge of the manifold. We refer to a quasi-homogeneous solution of the ordinary WDVV equations as a prepotential.

In this thesis, we will focus on semisimple Frobenius manifolds in concrete examples. A Frobenius manifold $\mathcal{M}$ is said to be semisimple if the Frobenius algebra on $T_p\mathcal{M}$ is semisimple for a generic point $p \in \mathcal{M}$; that is, $T_p\mathcal{M}$ contains no nilpotent elements. Semisimplicity provides Frobenius manifolds with additional structure and constrains the associated solutions of the WDVV equations, which has aided their classification (see, for example [18, Lecture 3]). A semisimple Frobenius manifold can be completely described by a function of one variable, called the Landau-Ginzburg superpotential [18].

Example 1.2.1 [18, 34]**.** In two dimensions, there exist six families of Frobenius manifolds, consisting of one trivial case — case (1.2.1) below — and five semisimple cases. Their prepotentials are as follows.

$$F\left(t^1,t^2\right) = \frac{1}{2}\left(t^1\right)^2 t^2; \tag{1.2.1}$$

$$F\left(t^1,t^2\right) = \frac{1}{2}\left(t^1\right)^2 t^2 + e^{\frac{2}{r}t^2}, \quad r \neq 0; \tag{1.2.2}$$

$$F\left(t^1,t^2\right) = \frac{1}{2}\left(t^1\right)^2 t^2 + \log t^2; \tag{1.2.3}$$

$$F\left(t^1,t^2\right) = \frac{1}{2}\left(t^1\right)^2 t^2 + \left(t^2\right)^2 \log t^2; \tag{1.2.4}$$

$$F\left(t^1,t^2\right) = \frac{1}{2}\left(t^1\right)^2 t^2 + \left(t^2\right)^k, \quad k \neq 0,1,2; \tag{1.2.5}$$

$$F\left(t^1,t^2\right) = \frac{1}{6}\left(t^1\right)^3 + \frac{1}{2}t^1\left(t^2\right)^2 + \frac{k}{6}\left(t^2\right)^3, \quad k \neq 0. \tag{1.2.6}$$

1.2.1 Orbit spaces of reflection groups

An important class of polynomial prepotentials is related to finite reflection groups, known also as finite Coxeter groups. The corresponding Frobenius manifolds originate in singularity theory, notably in work by Saito on universal unfoldings of functions with an isolated critical point. Saito [54] showed that the orbit space $\mathcal{M}_W$ of a complexified vector space $V \cong \mathbb{C}^n$ carrying the reflection representation of a finite Coxeter group W of rank n is endowed with a flat metric, now known as the Saito metric. The flat coordinates of the Saito metric are special W-invariant polynomials and were found by Saito, Yano, and Sekiguchi in most cases [55]. Dubrovin then found that $\mathcal{M}_W$ carries the structure of a semisimple Frobenius manifold, with a prepotential which is polynomial in the flat coordinates of the Saito metric η [19, 18]. He conjectured that all polynomial prepotentials, under some assumptions, correspond to the orbit space $\mathcal{M}_W$ for some finite Coxeter group W. This was later proved by Hertling [32].

For example, the prepotential associated with the Frobenius manifold on the orbit space $\mathbb{C}^2/A_2$, where A_2 is the symmetric group on 3 elements, can be written as

$$F_{A_2}(t^1,t^2) = \frac{1}{2}\left(t^1\right)^2 t^2 + \left(t^2\right)^4. \tag{1.2.7}$$

The group A_2 can also be written as $I_2(3)$, where the dihedral group $I_2(\kappa)$ is the group of symmetries of a regular κ-sided polygon. The prepotentials associated with $I_2(\kappa)$ are given by (1.2.5) with $k = \kappa + 1$. Closed form expressions for A_n and B_n prepotentials were found by Natanzon [48]. In general, there is no unifying formula for polynomial prepotentials.

It has since been shown that Frobenius manifold structures can also be defined on the orbit spaces of extended affine Weyl groups. Given a Weyl group of rank n, with a root system defined in a vector space V, the extended affine Weyl group W^k acts on $V \oplus \mathbb{R}$ where $k \in \{1, \ldots, n\}$ represents a choice of simple root. The Frobenius manifold on the orbit space of W^k was initially constructed, with some assumptions, by Dubrovin and Zhang in [22], and expanded on by Dubrovin, Strachan, Zhang, and Zuo in [21]. The prepotentials for these Frobenius manifolds are polynomial or rational functions of $t^1, \ldots, t^{n+1}, e^{t^{n+1}}$. The prepotential for the extended affine Weyl A_1 is given by (1.2.2) with $r = 2$.

1.2.2 Almost duality

In [54], Saito also introduced an inner product known as the intersection form (in the context of singularity theory). The intersection form defines a second flat metric, denoted g, which is related to the Saito metric η via the expression

$$\eta^{ij}(t) = \frac{\partial}{\partial t^1} g^{ij}(t).$$

The second flat metric can be defined for any Frobenius manifold $\mathcal{M}$ [18], along with a second multiplication $\star$ on $T_p\mathcal{M}$ such that

$$u \star v = E^{-1} \circ u \circ v.$$

Here, E^{-1} is the inverse of the Euler field as an operator with respect to $\circ$; the new multiplication $\star$ is therefore only defined on the complement of the discriminant locus

$$\Sigma := \{p \in \mathcal{M} \mid E \in T_p\mathcal{M} \text{ is not invertible}\}.$$

Dubrovin used the new metric g and multiplication $\star$ to define an almost dual Frobenius manifold structure in [20]. Given a Frobenius manifold $\mathcal{M}$, this construction on $\mathcal{M} \setminus \Sigma$ satisfies most but not all of the axioms of a Frobenius manifold. Namely, the unity field for the almost dual is not flat. A second solution of the WDVV equations is associated with the almost dual, known as the almost dual prepotential or $F^\star$. However, this is a solution of the generalised, rather than the ordinary, WDVV equations.

While keeping in mind that almost duality is defined for Frobenius manifolds only, it can be thought of as a symmetry of the WDVV equations; given a function F and metric $Q = \eta$ satisfying (1.1.2), almost duality produces a new solution $F^\star$ with metric $Q = g$.

In [20], the almost dual prepotential for the orbit space $\mathcal{M}_W$ was shown to have the form $F^\star(x) = F^{\text{rat}}_{\mathcal{A}}(x)$ for $F^{\text{rat}}_{\mathcal{A}}$ as in (1.1.3), where $\mathcal{A} = \mathcal{R}$ is a root system of W, and $x = (x^1, \ldots, x^n)$ are the flat coordinates of g.

Example 1.2.2 [20, Example 2]**.** Setting $W = A_2$, the almost dual prepotential of (1.2.7) can be written as

$$F^\star_{A_2}\left(x^1, x^2\right) = \left(x^1\right)^2 \log x^1 + \left(x^2\right)^2 \log x^2 + \left(x^1 - x^2\right)^2 \log\left(x^1 - x^2\right). \tag{1.2.8}$$

Trigonometric solutions of the form (1.1.4) associated with a root system $\mathcal{A} = W$ have appeared as the almost dual prepotentials of Frobenius manifolds defined on the orbit spaces of the extended affine Weyl group W^k [22, 21]. The almost dual prepotential for the extended affine Weyl group of type A was found by Riley and Strachan in [52], along with an example of a transformation, known as a (twisted) Legendre transformation, relating a rational A-type solution to a trigonometric A-type solution. The almost dual prepotentials for the BCD-type extended affine Weyl groups were found in [21]. Recently, the G_2-type almost dual prepotential was derived by Wright [62], using the corresponding Landau-Ginzburg superpotential found by Brini and van Gemst [8]. In the ADE case, Brini, Ma, and Strachan [9] have shown that the almost dual Frobenius manifolds for the extended affine Weyl groups are isomorphic to the Frobenius manifold structures considered in [10].

1.2.3 Other developments

Elliptic solutions of the WDVV equations have also been found, from the study of both Frobenius manifolds and supersymmetric gauge theories. Frobenius manifold structures on the orbit spaces of Jacobi groups, as constructed for types A and B by Bertola in [5], were shown to have elliptic almost dual prepotentials for types A and B by Riley and Strachan [53, 57]. Strachan conjectured that this is the case for arbitrary Weyl groups, and developed an elliptic version of $\vee$-systems [57]. Elliptic solutions have appeared as Seiberg–Witten prepotentials in six-dimensional Seiberg–Witten theory, in works by Braden, Marshakov, Mironov, and Morozov [44, 7].

The range of constructions associated with Frobenius manifolds has motivated the definition and study of weaker objects, which typically satisfy some but not all of the Frobenius manifold axioms. Hertling and Manin introduced the F-manifold, which is, roughly, a Frobenius manifold without a fixed flat metric η and without the assumption that the multiplicative identity is flat [33]. Manin then extended the notion of almost duality to F-manifolds with a certain flat structure, that is a torsionless flat connection which is compatible with the multiplication $\circ$ [42]. Integrable hierarchies of hydrodynamic types associated with F-manifolds were studied by Arsie and Lorenzoni, leading to the definition of bi-flat F-manifolds [4, 3]. Bi-flat structures are associated with the multiplication $\circ$ and the dual multiplication $\star$ defined via a version of almost duality.

Generalised Frobenius manifolds, which satisfy all the axioms of a Frobenius manifold except quasi-homogeneity and flatness of the identity field, have been defined by Liu, Qu, and Zhang [39]. A generalised Frobenius manifold locally defines a solution of the generalised WDVV equations.

1.3 Legendre transformations

Legendre transformations were first defined as a symmetry of Frobenius manifolds by Dubrovin [18]. They were later generalised to include symmetries of the generalised WDVV equations by Strachan and Stedman [59]. In this thesis, we use the definition given in [59], which we refer to simply as a Legendre transformation. We start with a solution $F(t)$ of the generalised WDVV equations, whose domain is (a chart of) an n-dimensional manifold $\mathcal{M}$ with flat metric η, corresponding Levi-Civita connection ∇, and multiplication $\circ$ defined on the tangent spaces. The multiplication can be described by structure constants c_{ij}^k such that

$$\partial_i \circ \partial_j = \sum_{k=1}^{n} c_{ij}^k \partial_k,$$

where $\partial_i = \frac{\partial}{\partial t^i} \in T_p\mathcal{M}$. A Legendre transformation $(F, t, \eta) \mapsto \left(\widehat{F}, \hat{t}, \widehat{\eta}\right)$ is defined by a choice of vector field $\delta = \sum_{i=1}^n \delta^i \partial_i \in T\mathcal{M}$ such that

$$X \circ \nabla_Y \delta = Y \circ \nabla_X \delta \quad \forall\, X, Y \in T\mathcal{M}. \tag{1.3.1}$$

Equation (1.3.1) is called the Legendre field condition. The new structures are defined by the relations,

$$\widehat{\eta}_{ij} = \eta_{ij}, \tag{1.3.2}$$

$$\frac{\partial \hat{t}^i}{\partial t^j} = \sum_{k=1}^{n} \delta^k c_{jk}^i, \tag{1.3.3}$$

$$\frac{\partial^2 \widehat{F}}{\partial \hat{t}^i \partial \hat{t}^j} = \frac{\partial^2 F}{\partial t^i \partial t^j}. \tag{1.3.4}$$

Then $\widehat{F}(\hat{t})$ solves the generalised WDVV equations (1.1.2) with metric $\widehat{\eta}$.

In this language, Dubrovin's earlier definition of a Legendre transformation corresponds to the requirement that a Legendre field δ is flat, i.e. $\nabla_X \delta = 0$. When the Legendre field is flat, one may choose new coordinates $\hat{t}$ defined by

$$\hat{t}^i = \sum_{k=1}^{n} \sum_{j=1}^{n} \eta^{ij} \delta^k \partial_j \partial_k \left(F(t) \right). \tag{1.3.5}$$

The right-hand side of this expression is straightforward to compute. However, to find the new solution $\widehat{F}$, one needs to be able to write the second-order derivatives of F in terms of $\hat{t}$ so that relation (1.3.4) can be integrated. This requires that (1.3.5) can be inverted explicitly; this step typically presents some difficulty and is not always possible. It is still unclear for which combinations of Legendre field δ and solution F it is possible to find $\widehat{F}$ explicitly, or what properties one should look for in a Legendre field to allow this.

In [52], a low-dimensional example was presented of a Legendre transformation mapping a trigonometric to a rational solution of the WDVV equations. Notably, these two solutions are respectively almost dual to two prepotentials, themselves connected by another Legendre transformation. We reproduce this example as follows.

Example 1.3.1 [52]**.** The Legendre transformation produced by the field $\delta = \partial_2$ transforms prepotential (1.2.2) with $r = 2$, which we denote F, into (1.2.4), which we denote $\widehat{F}$; see also [18, Example B.1]. The almost dual prepotential of F is given by the trigonometric solution

$$F^\star = \frac{1}{4} x^1 x^2 \left(x^1 + x^2 \right) - \frac{1}{12} \left(\left(x^1 \right)^3 + \left(x^2 \right)^3 \right) + \frac{1}{2} \left(\mathrm{Li}_3 \left(e^{x^1 - x^2} \right) + \mathrm{Li}_3 \left(e^{x^2 - x^1} \right) \right).$$

The almost dual prepotential of $\widehat{F}$ is given by the rational solution

$$\widehat{F}^\star = \frac{1}{4} \left(\hat{x}^1 \right)^2 \log \left(\hat{x}^1 \right)^2 + \frac{1}{4} \left(\hat{x}^2 \right)^2 \log \left(\hat{x}^2 \right)^2 - \left(\hat{x}^1 - \hat{x}^2 \right)^2 \log \left(\hat{x}^1 - \hat{x}^2 \right)^2.$$

The functions $F^\star$ and $\widehat{F}^\star$ are connected by a "twisted" Legendre transformation generated by the field $E \circ \delta$, where E is the Euler field associated with F.

Strachan and Stedman showed that the twisted field $E \circ \delta$ of a Legendre field δ for a Frobenius manifold $\mathcal{M}$ is always a Legendre field for the almost dual of $\mathcal{M}$ [59]. More recently, other properties of Frobenius manifolds related by Legendre transformations have been studied. In [63], Yang found that two semisimple Frobenius manifolds related by a (flat) Legendre transformation share the same monodromy data. Liu, Qu, and Zhang showed that certain integrable hierarchies associated with (generalised) Frobenius manifolds related by Legendre transformations are themselves related by linear transformations [40]. In this last work, the authors also showed that Legendre transformations produced by homogeneous Legendre fields map Frobenius manifolds to generalised Frobenius manifolds

[40, Proposition 2.7]; conversely, certain flat Legendre fields map generalised Frobenius manifolds to Frobenius manifolds.

The Legendre field condition (1.3.1) appears in an equivalent form in work by Lorenzoni, Pedroni, and Raimondo on integrable hierarchies associated with F-manifolds [41]. In this paper, (1.3.1) characterises the symmetries of these integrable hierarchies, which generalise the principal hierarchy defined for Frobenius manifolds in [18]. Solutions of (1.3.1) act as generators for commuting flows.

1.4 Present work and thesis structure

1.4.1 Main results

The focus of this thesis is two-fold. We explore the conditions in which a Legendre transformation can be defined for two-dimensional Frobenius manifolds. We then investigate specific examples of Legendre transformations which map rational solutions of the WDVV equations to trigonometric solutions. These mappings give rise to a new family of trigonometric solutions associated with (deformations of) a root system of the Weyl group A_n.

The two overarching questions that motivate this work are:

(I) When is a vector field a Legendre field?

(II) What happens to (rational) solutions of the WDVV equations under a Legendre transformation?

The results presented in this thesis can be thought of as investigating these two questions in restricted settings.

I The Legendre field condition for two-dimensional Frobenius manifolds

We begin by studying the Legendre field condition (1.3.1) for each of the two-dimensional Frobenius manifolds listed in Example 1.2.1 (Chapter 3). We describe all homogeneous Legendre fields for these manifolds.

Consider an arbitrary vector field $\delta = u\partial_1 + v\partial_2$, for some functions $u = u\,(t^1, t^2)$, $v = v\,(t^1, t^2)$. Then δ is a Legendre field for a two-dimensional Frobenius manifold $\mathcal{M}$ with multiplication $\circ$, metric η, and associated Levi-Civita connection ∇ if (1.3.1) is satisfied.

For prepotentials (1.2.1)–(1.2.5), we show that (1.3.1) is equivalent to the system of partial differential equations

$$\frac{\partial u}{\partial t^2} = c_{222}\frac{\partial v}{\partial t^1}, \tag{1.4.1}$$

$$\frac{\partial u}{\partial t^1} = \frac{\partial v}{\partial t^2}, \tag{1.4.2}$$

where $c_{222} = c\left(\partial_2, \partial_2, \partial_2\right)$ is dependent on the Frobenius manifold in question.

We obtain general solutions to (1.4.1), (1.4.2) for the trivial Frobenius manifold described by (1.2.1).

In cases (1.2.2)–(1.2.5), we restrict our attention to homogeneous Legendre fields; that is, we assume that δ additionally satisfies

$$\mathcal{L}_E\delta = \mu\delta, \tag{1.4.3}$$

where E is the Euler field of the Frobenius manifold and $\mu \in \mathbb{C}$ is the degree of homogeneity of δ. We find that (1.4.3) implies that the components of δ have the form

$$\begin{cases} u = \left(t^1\right)^{\mu+1} A(z), \\ v = \left(t^1\right)^{\mu+\alpha} B(z), \end{cases} \tag{1.4.4}$$

where $A(z)$, $B(z)$ are unspecified functions, and both $z = z\left(t^1, t^2\right)$ and $\alpha \in \mathbb{C}$ depend on the Frobenius manifold. Substituting the ansatzes in (1.4.4) into (1.4.1), (1.4.2) produces a pair of ordinary differential equations (ODEs) in $A(z), B(z)$ of the form

$$z(1-z)X''(z) + [c - (a+b+1)z]X'(z) - abX(z) = 0, \tag{1.4.5}$$

where $X = A, B$. The parameters $a, b, c \in \mathbb{C}$ depend on the choice of X, on the degree of homogeneity μ, and on the Frobenius manifold. The ODE (1.4.5) is a hypergeometric differential equation. We use the extensive literature on solutions of hypergeometric differential equations, and other methods, to describe all homogeneous Legendre fields for the Frobenius manifolds (1.2.2)–(1.2.5).

For the prepotential (1.2.6), we show that the potential of a Legendre field must satisfy the wave equation

$$\frac{\partial^2 h}{\partial\left(t^2\right)^2} = \frac{\partial^2 h}{\partial\left(t^1\right)^2}.$$

The potential $h = h(t^1, t^2)$ is a scalar function that defines a Legendre field δ via the relation $\delta = \text{grad} h = \sum_{i,j} \eta^{ij}\partial_j(h)\partial_i$. We find that a homogeneous Legendre field in this case must be of degree 0 or -1.

We also consider a specific example of twisted Legendre fields for the almost dual of the Frobenius manifold on $\mathcal{M}_{A_2} = \mathbb{C}^2/A_2$, that is the almost dual Frobenius manifold to

$F(t)$ as written in (1.2.5) with $k = 4$. As we have seen in Example 1.2.2, the almost dual prepotential is the A-type rational solution (1.2.8), given here in the coordinates x. We are interested in flat twisted fields, that is $E \circ \delta = \alpha \partial_{x^1} + \beta \partial_{x^2}$ for $\alpha, \beta \in \mathbb{C}$. We show that for some vector field δ to produce a flat twisted field, it must be homogeneous. Using the homogeneous Legendre fields for the Frobenius manifold (1.2.5), we describe all Legendre fields δ for $\mathcal{M}_{A_2}$ such that $E \circ \delta$ is flat.

II Legendre transformations connecting rational and trigonometric solutions

We apply Legendre transformations, for certain flat Legendre fields, to families of rational solutions associated with multi-parameter A- and B-type $\vee$-systems (respectively Chapters 5 and 6). The resulting solutions are trigonometric in all cases studied here. We relate these trigonometric solutions to existing solutions in the literature; in doing so, we generalise results from [35, 51, 56] by introducing a new family of $(n+3)$-parameter A_{n-1}-type trigonometric solutions of the WDVV equations (Chapter 4).

The following A- and B-type $\vee$-systems were found by Chalykh and Veselov [11]. Let $A_n(k)$ be the n-parameter collection of vectors

$$A_n(k) = \left\{ \sqrt{k_i} e_i | 1 \leq i \leq n \right\} \cup \left\{ \sqrt{k_i k_j} (e_i - e_j) | 1 \leq i < j \leq n \right\},$$

where $k = (k_1, \ldots, k_n) \in \mathbb{C}^n$ describes the deformation parameters, and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. The root system of type A_n (in a non-standard realisation) is recovered by setting all $k_i = 1$. Similarly, $B_n(k)$ is the $(n+1)$-parameter configuration

$$B_n(k) = \left\{ \sqrt{2k_i(k_i + k_0)} e_i | 1 \leq i \leq n \right\} \cup \left\{ \sqrt{k_i k_j} (e_i \pm e_j) | 1 \leq i < j \leq n \right\},$$

with deformation parameters $k = (k_0, k_1, \ldots, k_n) \in \mathbb{C}^{n+1}$. The root system of type B_n is recovered by setting $k_0 = -\frac{1}{2}$ and $k_i = 1 \ \forall \ 1 \leq i \leq n$.

A-type solutions:

The family of rational solutions associated with configuration $A_n(k)$ is given by

$$F^{\text{rat}}_{A_n(k)}(x) = \sum_{i=1}^{n} k_i \left(x^i\right)^2 \log\left(x^i\right) + \sum_{1 \leq i < j \leq n} k_i k_j \left(x^i - x^j\right)^2 \log\left(x^i - x^j\right), \tag{1.4.6}$$

for $x = (x^1, \ldots, x^n) \in \mathbb{C}^n$. In Chapter 5, we apply Legendre transformations generated by the fields $\delta_R = \partial_{x^\gamma}$, for arbitrary $\gamma \in \{1, \ldots, n\}$, and $\delta_W = \sum_{k=1}^n \partial_{x^k}$. In the Coxeter case $k_i = 1$, the field δ_R corresponds to a root vector while the field δ_W corresponds to the n^{th} fundamental weight of A_n.

We find that Legendre transformations via both δ_R and δ_W applied to $F^{\text{rat}}_{A_n(k)}$ produce trigonometric solutions of the form

$$F^{\text{trig}}_{A_{n-1}}(y) = \sum_{1\le i<j\le n} m_i m_j f\left(y_i - y_j\right) + a\left(\sum_{i=1}^{n} m_i y_i\right)^3 + b\left(\sum_{i=1}^{n} m_i y_i\right)\left(\sum_{j=1}^{n} m_j y_j^2\right) + c\sum_{i=1}^{n} m_i y_i^3. \quad (1.4.7)$$

The parameters $a, b, c \in \mathbb{C}$ and $m = (m_1, \dots, m_n) \in \mathbb{C}^n$ are all functions of the parameters k_i. In the Coxeter case, both types of Legendre transformation produce solutions which are equivalent to each other and to (1.4.7) with all $m_i = 1$ and special values of a, b, c.

Example 1.4.1. The almost dual A_2 prepotential $F = F^{\star}_{A_2}(x)$ as written in (1.2.8) is included in the family (1.4.6) when $k_i = 1$. Applying the Legendre transformation generated by ∂_{x^1} to F produces the solution

$$\widehat{F}(\hat{x}) = \frac{2}{3}\left(\hat{x}^1\right)^3 + \left(\hat{x}^1\right)^2\hat{x}^2 + 2\,\hat{x}^1\left(\hat{x}^2\right)^2 - \frac{1}{3}\left(\hat{x}^2\right)^3 - \frac{2}{9}\operatorname{Li}_3\left(e^{-3\hat{x}^2}\right),$$

where $\hat{x}$ are new coordinates defined by the transformation. The function $\widehat{F}$ can be written in the form (1.4.7) when $n = 2$ using the following relations (cf. Theorem 5.4.1):

$$\begin{aligned} y_1 &= -\frac{3}{2}\hat{x}^1, \quad y_2 = \frac{3}{2}\left(\hat{x}^2 - \hat{x}^1\right); \\ m_1 &= m_2 = 1; \\ a &= -\frac{1}{9}, \quad b = \frac{1}{2}, \quad c = -\frac{2}{3}. \end{aligned}$$

Hoevenaars and Martini showed that functions of the form (1.4.7) satisfy the WDVV equations when $m_i = 1\,\forall\, i \in \{1, \dots, n\}$, under some conditions on a, b, c [35] (note that this result appears explicitly only in the preprint version of [35]). The solutions found in [35] include the Legendre transformations via δ_R and δ_W of $F^{\text{rat}}_{A_n(k)}$ in the undeformed case where all $k_i = 1$. However, the Legendre transformations for arbitrary k are not in general equivalent to solutions in [35], or to other known A-type trigonometric solutions (for example, those found in [49, 51, 10, 2, 56]). This motivated the question of whether functions of the form (1.4.7) satisfy the WDVV equations for arbitrary m.

We therefore state and prove the following theorem in Chapter 4.

Theorem 1.4.2 (cf. Theorem 4.0.1)**.** *Let $F = F^{\text{trig}}_{A_{n-1}}$ denote the function given by (1.4.7), and let $M = \sum_{i=1}^{n} m_i$. Suppose that $bM + 3c \neq 0$, $aM^2 + bM + c \neq 0$, and $m_i \neq 0\,\forall\, i \in \{1, \dots, n\}$. Then $\eta = \sum_{k=1}^{n} F_k$ is a constant non-degenerate matrix. Furthermore, F solves the generalised WDVV equations with $Q = \eta$ if the following relation holds:*

$$4b^3 M + 6b^2 c - 108ac^2 + 3aM^2 + 3bM + 3c = 0. \quad (1.4.8)$$

Conversely, the WDVV equations (1.1.2) imply relation (1.4.8) when $n \geq 3$.

The trigonometric solutions of the form (1.4.7) connect several existing results in the literature. In particular, they draw directly on work by Hoevenaars and Martini, being multi-parameter deformations of (1.1.5). The A-type trigonometric solutions found by Riley and Shen are also included for special values of the parameters a, b, c, m, as shown in Section 4.3.

B-type solutions:
The family of rational solutions associated with the $\vee$-system $B_n(k)$ is

$$F^{\text{rat}}_{B_n(k)} = \sum_{i=1}^{n} 2k_i(k_0 + k_i)(x^i)^2 \log(x^i) + \sum_{1 \leq i < j \leq n} k_i k_j (x^i \pm x^j)^2 \log(x^i \pm x^j), \tag{1.4.9}$$

for $x = (x^1, \ldots, x^n) \in \mathbb{C}^n$ [11]. In Chapter 6, we apply Legendre transformations via the fields $\delta = \partial_{x^\gamma}$ for arbitrary $\gamma \in \{1, \ldots, n\}$.

We find that the resulting solutions belong to a large class of trigonometric solutions associated with the non-reduced root system BC_{n-1}, found by Alkadhem and Feigin [2] (see also [49]). In general, these solutions have the form

$$\begin{aligned} F^{\text{trig}}_{BC_{n-1}} = {} & \frac{1}{3}\xi_0^3 + h\xi_0 \sum_{i=1}^{n-1} m_i \xi_i^2 + \lambda r \sum_{i=1}^{n-1} m_i f(\xi_i) \\ & + \lambda \sum_{i=1}^{n-1} \left(s m_i + \frac{1}{2} q m_i (m_i - 1) \right) f(2\xi_i) + \lambda q \sum_{1 \leq j < k \leq n-1} m_j m_k f(i\xi_j \pm i\xi_k), \end{aligned}$$

with coordinates $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^n$, and independent parameters $q, r, s \in \mathbb{C}$, $m = (m_1, \ldots, m_{n-1}) \in \mathbb{C}^{n-1}$. The constants h, λ are functions of q, r, s, m_i.

The solutions obtained by Legendre transformations from $F^{\text{rat}}_{B_n(k)}$ form a sub-class of solutions of the form $F^{\text{trig}}_{BC_{n-1}}$.

1.4.1.3 Explanatory schematic diagrams

Included below are two schematics showing the relationships between different objects studied in this thesis. Figure 1.1 depicts the main classes of known solutions of the WDVV equations that are relevant to this work, mappings between different solutions, and some of the new results discussed above. In this diagram, horizontal arrows indicate that two solutions are related by almost duality for Frobenius manifolds. Vertical arrows (annotated with "LT") represent a Legendre transformation. Red dashed lines represent a new result presented in this thesis.

In Figure 1.2, red arrows represent Legendre transformations generated by different Legendre fields. Frobenius manifolds are depicted as a subset of generalised Frobenius manifolds, as defined by Liu, Qu, and Zhang [39].

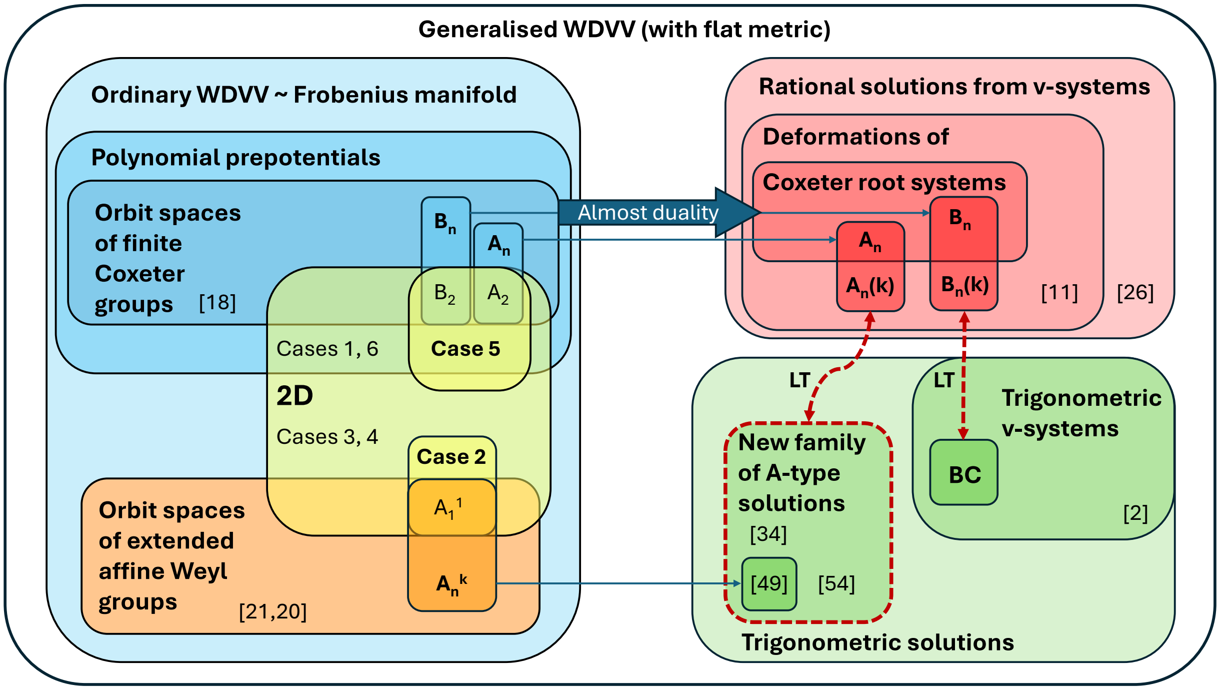

Figure 1.1: Relationships between different classes of solutions of the WDVV equations.

1.4.2 Structure of the thesis

Chapter 2 presents a self-contained overview of the theory of Frobenius manifolds and the WDVV equations, with an emphasis on the material most relevant to subsequent chapters. We introduce the notational conventions used throughout the thesis in Section 2.1.

Section 2.2 introduces the WDVV equations and associated differential-geometric properties. We begin with the generalised WDVV equations and build up to the more restricted ordinary WDVV equations, which appear in the context of Frobenius manifolds. This material is well-known but some results are reformulated to be applicable to the generalised WDVV equations. For example, we show that the generalised WDVV equations with constant Q are equivalent to the flatness of a certain deformed connection, first defined by Dubrovin for Frobenius manifolds in [18].

In Section 2.3, we provide definitions and examples of basic concepts related to Frobenius manifolds. This includes a subsection on the Frobenius manifolds defined on the orbit spaces of Coxeter groups, which correspond to polynomial prepotentials. The material in this section closely follows Dubrovin's works [18, 20].

Section 2.4 discusses symmetries of the WDVV equations; in particular, we focus on almost duality for Frobenius manifolds, inversion, and Legendre transformations. In Subsection 2.4.1, on almost duality, we present an important result from [20], which states that the almost dual of a polynomial prepotential is given by a rational solution. In Subsection

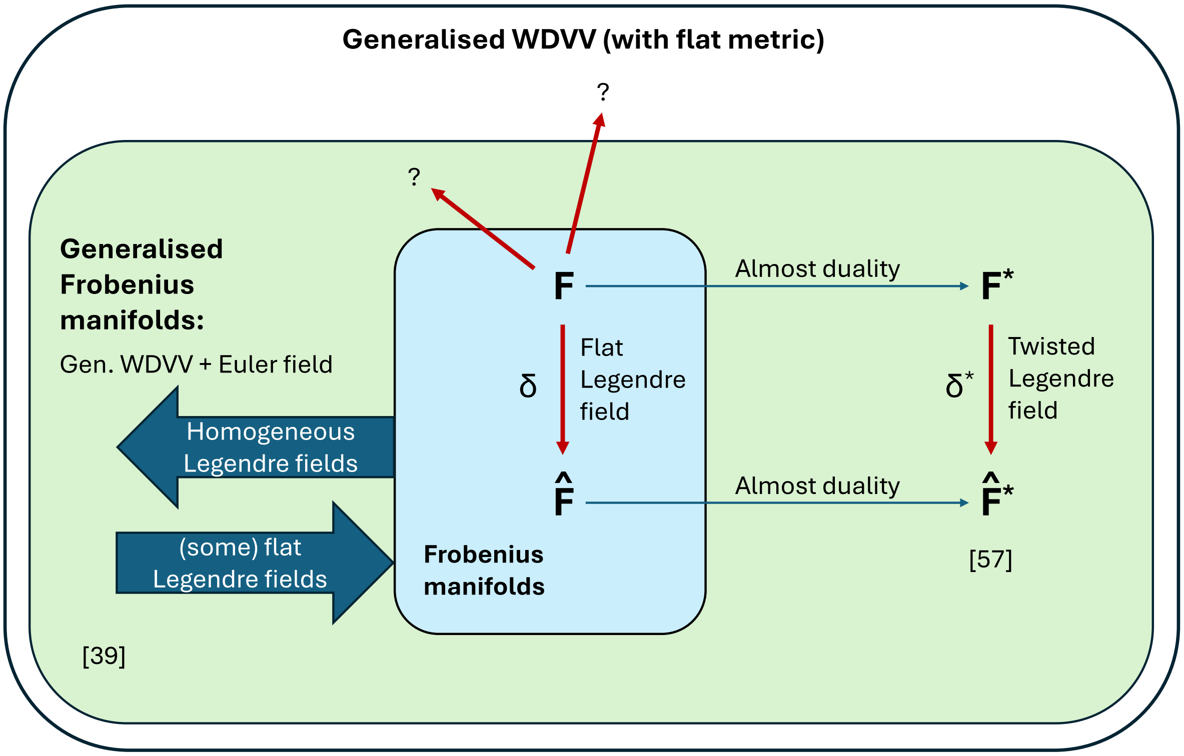

Figure 1.2: Transformations of Frobenius manifolds produced by Legendre fields with different properties.

2.4.2, on the inversion symmetry, we show that the two two-dimensional Frobenius manifolds given by prepotentials (1.2.3) and (1.2.4) are related under inversion. In Subsection 2.4.3, we define Legendre transformations for solutions of the WDVV equations following results in [59]. Dubrovin's definition of Legendre transformations is discussed as a special case. We also discuss Legendre transformations in combination with almost duality, looking at twisted Legendre fields for almost dual Frobenius manifolds as introduced in [59]. Note that Subsection 2.4.3 includes a few unpublished results. Propositions 2.4.26 and 2.4.27 deal with properties of homogeneous twisted Legendre fields. All Legendre fields are shown to have potential functions in Theorem 2.4.31, and the remaining results in § 2.4.3.3 explore relations between properties of Legendre fields and properties of their potentials.

In Section 2.5, we present some important known classes of rational and trigonometric solutions of the WDVV equations. Subsection 2.5.1 summarises the theory of (rational) $\vee$-systems, following material in [60], [27]. We see that certain root systems of Coxeter groups are examples of $\vee$-systems, and that the corresponding rational solutions are exactly the almost dual prepotentials for the polynomial Frobenius manifolds. Multi-parameter deformations of A- and B-type root systems, found in [11], are defined. In Subsection 2.5.2, we discuss trigonometric solutions. Trigonometric $\vee$-systems, as developed by Feigin and Alkadhem in [24], [2], and their corresponding solutions are introduced. We review other families of trigonometric solutions related to root systems: of particular relevance are results from [35, 51, 56].

In Chapter 3, we analyse the Legendre field condition in the context of each of the two-dimensional Frobenius manifolds in turn. Section 3.1 presents some general considerations valid for the Frobenius manifolds (1.2.1)–(1.2.5), followed by a brief overview of the hypergeometric differential equation and its solutions in Section 3.2. The results for each Frobenius manifold are presented in Subsections 3.3.1–3.3.6, labelled Cases 1–6, following the ordering used in Example 1.2.1.

The almost dual structure of the A_2 Frobenius manifold, with prepotential (1.2.7), is discussed in Section 3.4. Twisted fields for arbitrary vector fields are found in Subsection 3.4.1. In Subsection 3.4.2, we find the Legendre fields for the original Frobenius manifold which produce flat twisted fields for the almost dual.

Section 3.5 deals briefly with the Legendre field condition in canonical coordinates for the semisimple Frobenius manifolds (1.2.2)–(1.2.5). This section is the result of comments by Paolo Lorenzoni, whom I would like to thank for his input.

In Chapter 4, we introduce a new family of trigonometric solutions of the WDVV equations. Theorem 1.4.2 is proved in Section 4.1. In Section 4.2, we discuss the case when $bM + c = 0$. We show that the function $F^{\mathrm{trig}}_{A_{n-1}}$ still satisfies equations (1.1.2) when Q is not in general a linear combination of its third-order derivatives. The results of Sections 4.1 and 4.2 are published in [25]. In Section 4.3, we show that trigonometric solutions found in [51], [56] are equivalent to $F^{\mathrm{trig}}_{A_{n-1}}$ for special values of the parameters a, b, c.

In Chapter 5, we investigate Legendre transformations of A_n-type rational solutions. Section 5.1 discusses the geometrical meaning of the chosen Legendre fields for the root system case. In Section 5.2, we consider the Legendre transformation produced by a Legendre field of the form $\frac{\partial}{\partial x^\gamma}$, for arbitrary $\gamma \in \{1, \dots, n\}$, and find the resulting (trigonometric) solution. In Section 5.3, we carry out the same process for the Legendre field $\sum_{i=1}^n \frac{\partial}{\partial x^i}$. Both types of Legendre transformation produce trigonometric solutions; we show in Section 5.4 that the two resulting families of functions are sub-classes of the family $F^{\mathrm{trig}}_{A_{n-1}}$ introduced in Chapter 4. The results of Section 5.2 and some results from Section 5.4 were published in [25].

In Chapter 6, we explore Legendre transformations of the B_n-type rational solutions. In Section 6.1, we compute the result of applying a Legendre transformation produced by the field $\frac{\partial}{\partial x^\gamma}$, for arbitrary $\gamma \in \{1, \dots, n\}$, to $F^{\mathrm{rat}}_{B_n(k)}$. In Section 6.2, the resulting solutions are shown to form a subclass of the BC_{n-1}-type trigonometric solutions obtained in [2]. The results of this chapter are published in [25].

In Chapter 7, we comment on future research directions suggested by the problems explored in this thesis.

Chapter 2

Background

2.1 Notation and terminology

This section introduces the basic objects and notation used in the thesis. Familiarity with fundamental concepts from differential geometry is assumed. Note that although many concepts are introduced in the smooth setting, one can equivalently take all geometric data associated with a manifold to be holomorphic.

2.1.1 Differentiable manifolds

Let $\mathcal{M}$ be an n-dimensional smooth complex manifold, with a local coordinate system $u = (u^1, \ldots, u^n)$ defined on an open chart $\mathcal{U} \subseteq \mathcal{M}$. We denote

$$\partial_i = \partial_{u^i} := \frac{\partial}{\partial u^i},$$

and by default take the induced bases $e_i = \partial_i$ on $T_p\mathcal{M}$ and $e^i = \mathrm{d}u^i$ on $T_p^*\mathcal{M}$, where $p \in \mathcal{U}$ and $1 \leq i \leq n$. We will often use multiple coordinate systems; for example, if also we have a coordinate system $\hat{u} = (\hat{u}^1, \ldots, \hat{u}^n)$ on $\mathcal{M}$ we denote

$$\widehat{\partial}_i = \partial_{\hat{u}^i} = \frac{\partial}{\partial \hat{u}^i}.$$

The spaces $C^\infty(\mathcal{M})$, $\mathfrak{X}(\mathcal{M})$, and $\Omega^1(\mathcal{M})$ denote the spaces of smooth functions, vector fields, and one-forms on $\mathcal{M}$, respectively.

2.1.2 Einstein summation

The Einstein summation convention is used throughout and is described as follows: an undefined index repeated once as a subscript and once as a superscript implies summation over this index. For example, a vector field may be written

$$X = X^i \partial_i := \sum_i X^i \partial_i.$$

2.1.3 Lie derivatives

The Lie derivative, or Lie bracket, of two vector fields on a differentiable manifold is denoted

$$\mathcal{L}_X Y = [X, Y] := X(Y) - Y(X) = X^i \partial_i \left(Y^j\right) \partial_j - Y^i \partial_i \left(X^j\right) \partial_j$$

for $X = X^i \partial_i, Y = Y^i \partial_i \in \mathfrak{X}(\mathcal{M})$.

The Lie derivative of a smooth function with respect to a vector field is defined as the directional derivative, so that

$$\mathcal{L}_X f := X(f) = X^i \partial_i(f)$$

for $X = X^i \partial_i \in \mathfrak{X}(\mathcal{M}), f \in C^\infty(\mathcal{M})$.

2.1.4 Metrics on a manifold

We work almost exclusively with pseudo-Riemannian manifolds $\mathcal{M}$ equipped with a metric η, which can equivalently be represented as an inner product $\langle \cdot, \cdot \rangle$. A metric is a symmetric nondegenerate $\mathbb{C}$-bilinear form on each tangent space $T_p\mathcal{M}$ varying smoothly with $p \in \mathcal{M}$. For pseudo-Riemannian manifolds, there is no requirement that the metric is positive-definite.

The metric can be represented in a coordinate system u by its Gram matrix (η_{ij}) with entries $\eta_{ij} = \langle \partial_i, \partial_j \rangle$. We say that η has (covariant) components η_{ij}.

The inverse of the Gram matrix of η describes the components of the inverse, or contravariant, metric η^{-1}. We denote the components of η^{-1} as η^{ij}.

The metric provides an isomorphism between the tangent space $T_p\mathcal{M}$ and the cotangent space $T_p^*\mathcal{M}$; any vector $v \in T_p\mathcal{M}$ uniquely determines a covector $\alpha \in T_p^*\mathcal{M}$ via $\langle v, w \rangle = \alpha(w)$ for all $w \in T_p\mathcal{M}$, and vice versa.

2.1.5 Subscripts and superscripts

Lower indices are typically used to denote covariant objects and upper indices used for contravariant objects.

A coordinate system u may be written using contravariant components u^i or covariant components u_i, where $u_i = \eta_{ij} u^j$ and $u^i = \eta^{ij} u_j$.

Note that coordinates u^i are preferentially represented with subscripts in concrete examples; this notation is limited to cases where the covariant components do not appear at all.

2.1.6 Connections and curvature

Given a pseudo-Riemmanian manifold $\mathcal{M}$ with metric η, the Levi-Civita connection is the unique affine connection ∇ that is compatible with the metric and is torsion-free. A connection ∇ is compatible with the metric (and is called a metric connection) if

$$\nabla_X \eta := \nabla_X \left(\eta(Y, Z)\right) - \eta\left(\nabla_X Y, Z\right) - \eta\left(Y, \nabla_X Z\right) = 0$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. The connection ∇ is torsion-free if the torsion tensor

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

vanishes for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

A metric is flat if there exist a set of coordinates t in which the entries of the matrix $\eta_{ij}(t) = \langle \partial_{t^i}, \partial_{t^j} \rangle$ are constant. The Levi-Civita connection for η in the flat coordinates t is simply

$$\nabla_X Y := X(Y) = X^i \frac{\partial Y^j}{\partial t^i} \partial_{t^j}$$

for $X = X^i \partial_{t^i}, Y = Y^i \partial_{t^i} \in \mathfrak{X}(\mathcal{M})$. A vector field $X = X^i \partial_{t^i}$ is said to be flat with respect to ∇ if

$$\nabla_Y X = 0$$

in the flat coordinates t, for all $Y \in \mathfrak{X}(\mathcal{M})$; that is, its components X^i must be constant.

2.1.7 Matrices

An indexed matrix M_i has entries $(M_i)_{jk} = M_{ijk}$. For a holomorphic function $F(t^1, \ldots, t^n)$, we denote by F_i the $n \times n$ matrix of third-order derivatives of F with $(j, k)^{th}$ entry

$$(F_i)_{jk} = F_{ijk} := \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}.$$

2.1.8 Sets

We include 0 in the set of natural numbers, so that $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$. We denote $A^\times = A \setminus \{0\}$ and $A_{\leq k} = \{a \in A \,|\, a \leq k\}$ for a set $A \subset \mathbb{R}$.

2.2 The WDVV equations

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations were first found in [15], [61] in the context of two-dimensional topological field theory, where they encode the associativity of an operator algebra. We refer to the original equations as the *ordinary* WDVV equations and introduce them later in § 2.2.2. We begin, instead, with a more general formulation of the WDVV equations obtained by Marshakov, Mironov, and Morosov.

Definition 2.2.1 (cf. [46])**.** Let $F = F(t)$ be a holomorphic function of n complex variables $t = (t^1, \dots, t^n)$, defined in an open set $\mathcal{U} \subset \mathbb{C}^n$. The *generalised WDVV equations* are the system of nonlinear partial differential equations

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i \ , \tag{2.2.1}$$

where $i, j, k \in \{1, \dots, n\}$.

One can think of $\mathcal{U}$ as (a chart of) some complex manifold with local coordinates $t^1, \dots, t^n$. We refer to $t^1, \dots, t^n$ as coordinates for this reason.

The system (2.2.1) may be rewritten as a particular instance of the equations

$$F_i Q^{-1} F_j = F_j Q^{-1} F_i \tag{2.2.2}$$

for a non-degenerate linear combination $Q = q^k F_k$ with functions $q^k = q^k\,(t^1, \dots, t^n)$. In fact, Marshakov et al. showed that these two formulations are equivalent for a given function.

Proposition 2.2.2 [45]**.** *If a function F satisfies (2.2.2) for some choice of $Q = q^k F_k$, it will satisfy the equations*

$$F_i B^{-1} F_j = F_j B^{-1} F_i$$

for any choice of invertible matrix $B = b^k F_k$, $b^k = b^k\,(t^1, \dots, t^n)$.

We refer to both (2.2.1) and (2.2.2) interchangeably as the generalised WDVV equations, and say that F is a solution of the generalised WDVV equations if F satisfies (2.2.2) for some Q.

Remark 2.2.3. The object $Q = (Q_{ij})$ is a matrix with entries $Q_{ij} = q^k F_{kij}$. We denote by Q^{ij} the entries of the inverse matrix Q^{-1}. These objects can be used to raise and lower indices so that we have, for example, $t_i = Q_{ij} t^j$ and $t^i = Q^{ij} t_j$.

Although Q is not required to be constant in (2.2.2), we typically deal with constant Q. For brevity, we introduce the following terminology.

Definition 2.2.4. The *WDVV equations* refer to the system (2.2.2) with constant Q.

The generalised WDVV equations are preserved under linear changes of variables and the addition of quadratic terms to $F(t)$. We formalise a notion of equivalence between solutions as follows.

Definition 2.2.5. Let $F(t)$, $\widehat{F}\left(\hat{t}\right)$ be two solutions of the generalised WDVV equations in n variables. We say that F and $\widehat{F}$ are *equivalent* if

$$F(t) = \alpha\widehat{F}\left(\hat{t}\left(t\right)\right) + \chi(t)$$

where $\chi(t)$ is a quadratic function, $\alpha \in \mathbb{C}$, and the relations

$$\hat{t}^i = \kappa^i_j t^j$$

are invertible with $\kappa^i_j \in \mathbb{C}$ for all $i, j \in \{1, \ldots, n\}$.

Although more restricted versions of the WDVV equations can be associated with richer geometric structures, all solutions of the generalised WDVV equations define an associative unital multiplication on the tangent spaces of their domain.

Definition 2.2.6. Given a solution $F(t)$ of the generalised WDVV equations, we define the symmetric *three-point (correlation) function*

$$c_{ijk}(t) := \frac{\partial^3 F(t)}{\partial t^i \partial t^j \partial t^k}$$

and *two-point function*

$$c^k_{ij}(t) := Q^{kl} c_{lij}(t). \tag{2.2.3}$$

Note that c^k_{ij} is symmetric only in its lower indices.

Proposition 2.2.7. *Let F be a solution of the generalised WDVV equations on $\mathcal{U} \subset \mathbb{C}^n$. We define the multiplication $\circ_p$ for $p \in \mathcal{U}$ as*

$$\partial_i \circ_p \partial_j = c^k_{ij}(p)\partial_k. \tag{2.2.4}$$

Then $\circ_p$ is commutative and associative with identity element $e = q^i\partial_i$.

Proof. We begin by rewriting the generalised WDVV equations in terms of c:

$$c_{ijk}Q^{kn}c_{lmn} = c_{ljk}Q^{kn}c_{imn} \quad \forall\, i, j, l, m. \tag{2.2.5}$$

The associativity condition for the multiplication,

$$\partial_i \circ_p (\partial_j \circ_p \partial_k) = (\partial_i \circ_p \partial_j) \circ_p \partial_k,$$

can be written

$$Q^{ml} c_{ilp} Q^{pq} c_{kjq} = Q^{ml} c_{klp} Q^{pq} c_{ijq},$$

which is equivalent to (2.2.5). Commutativity follows from the symmetry of c_{ij}^k.

To show that e is the identity, we require

$$e \circ_p \partial_i = \partial_i \quad \forall\, i. \tag{2.2.6}$$

We have that $e \circ_p \partial_i = q^j c_{ij}^k \partial_k$, so condition (2.2.6) is equivalent to

$$q^j c_{ij}^k = Q^{kl} q^j c_{ijl} = \delta_i^k.$$

Multiplying through by Q_{kl}, this becomes $q^j c_{jil} = Q_{il}$, which is true by definition of Q. □

Where $\mathcal{U}$ is (a chart of) a manifold $\mathcal{M}$, we thus have a pointwise multiplication $\circ_p$ on the tangent spaces of $T_p\mathcal{M}$. We often denote this simply as $\circ$. However, it is important to recall that $u \circ v$, where u, v are tangent vectors, is only defined if $u, v \in T_p\mathcal{M}$ for the same $p \in \mathcal{M}$.

2.2.1 Solutions with flat metrics

From this point on, we restrict our attention to solutions of the generalised WDVV equations (2.2.2) where $Q(t)$ is constant. Using definitions from Riemannian geometry, we call such a matrix $Q = \eta$ a flat metric.

Definition 2.2.8. Let $F(t)$ be a solution of the WDVV equations with $Q = \eta$. We define an inner product $\langle \cdot, \cdot \rangle$ as

$$\langle \partial_i, \partial_j \rangle = \eta_{ij}. \tag{2.2.7}$$

When $\mathcal{U}$ is (a chart of) a manifold $\mathcal{M}$, $\langle \cdot, \cdot \rangle$ is a nondegenerate $\mathbb{C}$-bilinear inner product on $T\mathcal{M}$. By definition, this makes η a flat metric on $\mathcal{M}$ with flat coordinates t. The Levi-Civita connection for η is denoted ∇.

Of particular relevance to this thesis are three classes of solutions of the WDVV equations: polynomial, rational, and trigonometric[1]. Polynomial solutions $F(t)$, as expected, are polynomial in $t^1, \ldots, t^n$; a large class of polynomial solutions is presented in § 2.3.4. Rational solutions include logarithmic terms, and are discussed in § 2.5.1. Trigonometric solutions feature the trilogarithm function $\mathrm{Li}_3(z)$, which is defined in § 2.5.2.

The WDVV equations are trivially satisfied in two dimensions; when $n = 2$, no choice of i, j, k can be made such that equations (2.2.1) are non-trivial. We now present a 3-dimensional example of each type of solution, while noting that 2-dimensional cases

[1] These terms make reference to the third-order derivatives of the solutions.

are sometimes studied as restrictions of higher-dimensional solutions or as examples of Frobenius manifolds.

Example 2.2.9 [18, Appendix A]**.** The function

$$F(t_1, t_2, t_3) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \frac{1}{4}t_2^2 t_3^2 + \frac{1}{60}t_3^5$$

is a polynomial solution, with metric given by

$$\eta = F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Example 2.2.10 [60]**.** The function

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i^2 \log x_i + \sum_{1 \leq i < j \leq 3} (x_i - x_j)^2 \log (x_i - x_j)$$

is a rational solution, with metric given by

$$\eta = \sum_{i=1}^{3} x_i F_i = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

Example 2.2.11 [25]**.** The function

$$\begin{aligned} F(y_1, y_2, y_3) = y_1^3 - y_1^2(y_2 + y_3) + 3y_1(y_2^2 + y_3^2) - 2y_1 y_2 y_3 - \frac{1}{3}y_2^3 - \frac{5}{3}y_3^3 \\ - \frac{1}{8}\Big[\mathrm{Li}_3\left(e^{-4y_2}\right) + \mathrm{Li}_3\left(e^{-4y_3}\right) + \mathrm{Li}_3\left(e^{-4(y_2 - y_3)}\right)\Big] \end{aligned}$$

is a trigonometric solution, with metric given by

$$\eta = F_1 = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

As we will later see, the solutions in Examples 2.2.9, 2.2.10, 2.2.11 are related to each other through various symmetries of the WDVV equations.

We can build on Proposition 2.2.7 for solutions with a flat metric to define a family of algebras with inner products at each point of the region $\mathcal{U}$. To this end, we introduce the concept of a Frobenius algebra as follows.

Definition 2.2.12 [18, Definition 1.1]**.** A (commutative) *Frobenius algebra* A is defined as the tuple $\left(A, \circ, e, \langle\cdot,\cdot\rangle\right)$ such that

1. A is a $\mathbb{C}$-algebra with associative (and commutative) multiplication $\circ : A \times A \to A$ and identity $e \in A$.

2. $\langle\cdot,\cdot\rangle : A \times A \to \mathbb{C}$ is a symmetric non-degenerate $\mathbb{C}$-bilinear inner product on A satisfying the Frobenius condition

$$\langle u \circ v, w\rangle = \langle u, v \circ w\rangle \quad \forall\, u, v, w \in A. \tag{2.2.8}$$

We consider a smooth, complex, n-dimensional manifold $\mathcal{M}$ with flat metric η, Levi-Civita connection ∇, and flat coordinates t defined on an open chart $\mathcal{U} \subseteq \mathcal{M}$. Let $\circ$ on $T\mathcal{M}$ denote a pointwise multiplication with identity e, with the restriction of $\circ$ to $T_p\mathcal{M}$ denoted $\circ_p$ for all $p \in \mathcal{U}$. We define the $(0,3)$-tensor

$$c\,(u, v, w) = \langle u \circ v, w\rangle, \tag{2.2.9}$$

where the inner product is given by $\langle u, v\rangle = \eta(u, v)$. Note that $T_p\mathcal{M}$ is a commutative Frobenius algebra if $c|_p$ is symmetric. The covariant derivative ∇c is

$$\left(\nabla_z c\right)(u, v, w) = \nabla_z\left(c(u, v, w)\right) - c\left(\nabla_z u, v, w\right) - c\left(u, \nabla_z v, w\right) - c\left(u, v, \nabla_z w\right),$$

for $u, v, w, z \in T\mathcal{M}$.

Theorem 2.2.13. *Let the tensor c given in (2.2.9) be symmetric and let $\left(\nabla_z c\right)(u, v, w)$ be symmetric in u, v, w, z. Then there exists a local solution $F = F(t)$ of the WDVV equations, defined on $\mathcal{U}$, such that its three-point function is given by*

$$c_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = c\left(\partial_i, \partial_j, \partial_k\right). \tag{2.2.10}$$

This solution is unique up to equivalence, as defined in Definition 2.2.5.

Conversely, let $F(t)$ be a solution of the WDVV equations defined on an open chart $\mathcal{U} \subseteq \mathcal{M}$. Then the multiplication (2.2.4) and inner product (2.2.7) define a commutative Frobenius algebra on $T_p\mathcal{M}$ for all $p \in \mathcal{U}$.

Proof. We repeatedly apply the Poincaré Lemma to show the existence of F as in (2.2.10). In flat coordinates for η, the tensor ∇c becomes

$$\left(\nabla_{\partial_l} c\right)\left(\partial_i, \partial_j, \partial_k\right) = \partial_l c\left(\partial_i, \partial_j, \partial_k\right).$$

Since this is symmetric, we have

$$\partial_l c\left(\partial_i, \partial_j, \partial_k\right) = \partial_k c\left(\partial_i, \partial_j, \partial_l\right)$$

for all k, l. By the Poincaré Lemma, there exists a local solution G_{ij}, unique up to constant terms, such that

$$c\left(\partial_i, \partial_j, \partial_k\right) = \partial_k G_{ij}.$$

Similarly, using the symmetry of c, two more applications of this method show that there exists locally some $F(t)$, unique up to quadratic terms, such that

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = c\left(\partial_i, \partial_j, \partial_k\right).$$

To check that this F satisfies the WDVV equations, we see from

$$c_{ijk} = c\left(\partial_i, \partial_j, \partial_k\right) = \langle \partial_i \circ \partial_j, \partial_k \rangle$$

that the multiplication can be written as

$$\partial_i \circ \partial_j = \eta^{kl} c_{ijl} \partial_k,$$

where η^{ij} are the entries of the inverse matrix $\left(\eta^{ij}\right) = \left(\eta_{ij}\right)^{-1}$. We can also rewrite the WDVV equations as

$$c_{ijk} \eta^{km} c_{lmn} = c_{ljk} \eta^{km} c_{imn};$$

this is then satisfied by associativity of $\circ$. Finally, we can write the identity as $e = q^k \partial_k$ for some functions $q^k(t)$. Then we have

$$q^k c_{ijk} = c(e, \partial_i, \partial_j) = \langle \partial_i, \partial_j \rangle = \eta_{ij},$$

so we see that the metric has components

$$\eta_{ij} = q^k c_{ijk} = q^k F_k.$$

For the converse statement, we already have from Proposition 2.2.7 that the multiplication $\circ_p$ defined by F is associative and unital for all $p \in \mathcal{U}$. It follows from (2.2.4) and (2.2.7) that

$$\langle \partial_i \circ_p \partial_j, \partial_k \rangle = c_{ijk}(p),$$

so we can see that the Frobenius condition (2.2.8) holds by symmetry of η and c_{ijk}. □

The components of the tensor $c_{ij}^k = \eta^{kl} c_{lij}$ introduced in Definition 2.2.6 are referred to as the *structure constants* of the associated Frobenius algebra. The indices of this object may be raised once more to define the *cotangent structure constants* $c_k^{ij} = \eta^{il} c_{lk}^j$.

We now introduce the Dubrovin connection, a one-parameter family of connections on a manifold $\mathcal{M}$. This is typically defined in the context of a Frobenius manifold. However, we already have the required ingredients when working with any solution of the generalised WDVV equations such that $Q = \eta$ is constant.

Definition 2.2.14 [18, Lecture 3]**.** Let $\mathcal{M}$ be a smooth manifold with multiplication $\circ$, flat metric η, and Levi-Civita connection ∇. The *Dubrovin connection* ${}^{\lambda}\nabla$ depending on spectral parameter $\lambda \in \mathbb{C}$ is defined as

$$ {}^{\lambda}\nabla_X Y = \nabla_X Y + \lambda X \circ Y $$

for $X, Y \in \mathfrak{X}(\mathcal{M})$.

Recall that a connection, denoted by ∇, on a (pseudo)-Riemannian manifold is flat if the Riemann curvature tensor

$$ R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} $$

vanishes for all $X, Y \in \mathfrak{X}(\mathcal{M})$. The Levi-Civita connection associated with a flat metric is always flat.

Proposition 2.2.15 [18, Lemma 3.1]**.** *The Dubrovin connection* ${}^{\lambda}\nabla$ *on* $\mathcal{M}$ *is flat if and only if there exists a local solution* $F = F(t)$ *of the WDVV equations such that its three-point function is given by*

$$ c_{ijk}(t) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = c\left(\partial_i, \partial_j, \partial_k\right), \tag{2.2.11} $$

where c *is the tensor defined in (2.2.9). This solution is unique up to equivalence.*

By Theorem 2.2.13, the flatness of the Dubrovin connection is also equivalent to the existence of commutative Frobenius algebra structures on the tangent spaces of $\mathcal{M}$, provided that ∇c is symmetric.

2.2.2 The ordinary WDVV equations

We now set up the original formulation of the WDVV equations, which appears in the context of Frobenius manifolds (see [18, Equation (1.14)]) and which we refer to as the ordinary WDVV equations. Note that the ordinary WDVV equations are obtained from the generalised WDVV equations (2.2.2) by setting $\eta = Q$ where $q^k = \delta_{1k}$, so that the multiplicative identity is given by $e = \partial_1$.

Definition 2.2.16. Let $F = F(t)$ be a holomorphic function of n complex variables $t = (t^1, \ldots, t^n)$, defined up to quadratic terms on some open set $\mathcal{U} \ni t$. The *ordinary*

WDVV equations are the system

$$F_i \eta^{-1} F_j = F_j \eta^{-1} F_i \tag{2.2.12}$$

where $\eta := F_1$ is a nondegenerate matrix with constant entries.

It is useful to distinguish between solutions to the ordinary WDVV equations in general and those which are quasi-homogeneous; the quasi-homogeneity condition which follows allows a solution to be associated with a Frobenius manifold. We refer to quasi-homogeneous solutions to the ordinary WDVV equations as prepotentials to highlight this distinction.

Definition 2.2.17. A *prepotential* $F(t)$ is a solution to the ordinary WDVV equations (2.2.12) such that

$$F\left(c^{d_1} t^1, \dots, c^{d_n} t^n\right) = c^{d_F} F\left(t^1, \dots, t^n\right)$$

for any $c \in \mathbb{C}^\times$, and some $d_1, \dots, d_n, d_F \in \mathbb{C}$. The numbers $d_1, \dots, d_n$ are called the *degrees* of F, and are often normalised so that $d_1 = 1$. The constant $d = 3 - d_F$ is called the *charge* of F.

The quasi-homogeneity of a prepotential $F = F(t)$ can be rewritten in terms of a vector field $E = E^i \partial_i$, known as the *Euler vector field*. The Euler field is such that

$$\mathcal{L}_E F = E^i \partial_i (F) = d_F F + \text{quadratic terms.} \tag{2.2.13}$$

2.3 Frobenius manifolds

In this section, we provide an overview of Frobenius manifolds and some related structures which are most relevant to this thesis.

We will find that the previous sections have already introduced everything needed to define a Frobenius manifold. The formal definition of these structures, and much of the surrounding theory, was developed by Boris Dubrovin to provide a geometric interpretation of the ordinary WDVV equations. In particular, we will see that there is a correspondence between prepotentials and Frobenius manifolds.

We denote by a subscript p the restriction of a pointwise-defined object on $T\mathcal{M}$ to the tangent space $T_p\mathcal{M}$.

Definition 2.3.1 [18, Definition 1.2]**.** A *Frobenius manifold* $\mathcal{M}$ is defined by the tuple $(\mathcal{M}, \circ, e, \eta, E)$, where $\mathcal{M}$ is a manifold, such that

1. Every tangent space $T_p\mathcal{M}$ has a commutative Frobenius algebra structure $(T_p\mathcal{M}, \circ_p, e_p, \langle \cdot, \cdot \rangle_p)$ depending smoothly on $p \in \mathcal{M}$.

2. The inner products $\langle\cdot,\cdot\rangle_p$, $p\in\mathcal{M}$, define a flat metric on $\mathcal{M}$, denoted η. We denote by ∇ the Levi-Civita connection for this metric.

3. There exists a flat vector field e which is the identity for $\circ$.

4. The tensor field c defined in (2.2.9) has a totally symmetric covariant derivative ∇c.

5. There exists a vector field $E\in\mathfrak{X}(\mathcal{M})$, called the Euler vector field, with the following properties:

   (a) $\nabla(\nabla E)=0$;

   (b) $\mathcal{L}_E(u\circ v)=u\circ v+\mathcal{L}_E u\circ v+u\circ\mathcal{L}_E v \quad \forall u,v\in T\mathcal{M}$;

   (c) $\mathcal{L}_E e=-e$;

   (d) $\mathcal{L}_E\eta=(2-d)\eta$, where $d\in\mathbb{C}$ is called the charge of the Frobenius manifold.

Property 2 of the definition of a Frobenius manifold means that there locally exist coordinates $t^1,\dots,t^n$ on $\mathcal{M}$ with respect to which η is constant. Property 5(b) may equivalently be replaced by the invariance requirement $\mathcal{L}_E c=(3-d)c$.

We have previously seen that solutions of the WDVV equations defined on a manifold are associated with a (flat) metric and Frobenius algebra structures on the tangent spaces. Combining this with the property of quasi-homogeneity lets us associate a Frobenius manifold structure to a prepotential, and vice versa.

Theorem 2.3.2 [18, Lemma 1.2]**.** *Any prepotential $F(t)$ on a chart $\mathcal{U}\subseteq\mathcal{M}$, with $d_1\neq 0$, defines a Frobenius manifold structure on $\mathcal{U}$ as follows:*

1. *The multiplication, identity field, and inner product are given by Proposition 2.2.7 and Definition 2.2.8.*

2. *The Euler vector field $E(t)$ is given by (2.2.13).*

Conversely, any Frobenius manifold has a local structure corresponding to some solution of the WDVV equations.

We can define a notion of equivalence for Frobenius manifolds as follows.

Definition 2.3.3 [18, Definition 1.3]**.** Let $\mathcal{M}$ and $\mathcal{N}$ be two Frobenius manifolds with metrics η, ξ and Euler fields $E_{\mathcal{M}}$, $E_{\mathcal{N}}$ respectively. We say that $\mathcal{M}$ and $\mathcal{N}$ are (locally) *equivalent* if there exists a (local) diffeomorphism $\phi:\mathcal{M}\to\mathcal{N}$ such that, over some open $\mathcal{U}\subseteq\mathcal{M}$ with $p\in\mathcal{U}$:

1. The differential $\phi_*:T_p\mathcal{M}\to T_{\phi(p)}\mathcal{N}$ is an algebra isomorphism.

2. $\xi_{\phi(p)}\big(\phi_*(u),\phi_*(v)\big)=\alpha\,\eta_p(u,v)$ for all $u,v\in T_p\mathcal{M}$ and some nonzero $\alpha\in\mathbb{C}$.

3. $\phi_*(E_{\mathcal{M}}) = E_{\mathcal{N}}$.

The final condition is an addition from [62, Def. 2.2.10]; we include this for the same reasoning as provided by Wright, namely that we expect the Euler vector field to be the same object under an equivalence of Frobenius manifolds. The first two conditions are enough to ensure that $\phi_*(E_{\mathcal{M}})$ is an Euler field for $\mathcal{N}$, but not strong enough to imply that $\phi_*(E_{\mathcal{M}}) = E_{\mathcal{N}}$.

Equivalence of prepotentials is given in Definition 2.2.5. We might expect the prepotential F of a Frobenius manifold $\mathcal{M}$ to be given by the pullback $\phi^*\widetilde{F}(t) = \widetilde{F}(\phi(t))$, $t \in \mathcal{M}$, where $\widetilde{F}$ is the prepotential of a Frobenius manifold $\mathcal{N}$ equivalent to $\mathcal{M}$; however, this is not always the case (see [18, p.17] for an example).

In two dimensions, the WDVV equations are trivially satisfied. However, the quasi-homogeneity condition and the form of the metric $\eta = F_1$ specify six unique (families of) prepotentials up to equivalence. For the derivation of the following prepotentials, see [18, Lecture 1] and [34, § 3.4].

Example 2.3.4. The prepotential $F = F(t_1, t_2)$, Euler field $E = E(t_1, t_2)$, and charge d for each of the $2D$ Frobenius manifolds are given in Table 2.1 below.

Prepotential	**Euler field**	**Charge**	**Parameter**	
$F = \frac{1}{2}t_1^2 t_2$	$E = t_1\partial_1$	$d = 1$		(2.3.1)
$F = \frac{1}{2}t_1^2 t_2 + e^{\frac{2}{r}t_2}$	$E = t_1\partial_1 + r\partial_2$	$d = 1$	$r \neq 0$	(2.3.2)
$F = \frac{1}{2}t_1^2 t_2 + \log t_2$	$E = t_1\partial_1 - 2t_2\partial_2$	$d = 3$		(2.3.3)
$F = \frac{1}{2}t_1^2 t_2 + t_2^2 \log t_2$	$E = t_1\partial_1 + 2t_2\partial_2$	$d = -1$		(2.3.4)
$F = \frac{1}{2}t_1^2 t_2 + t_2^k$	$E = t_1\partial_1 + \frac{2}{k-1}t_2\partial_2$	$d \neq -1, 1, 3$	$k = \frac{3-d}{1-d}$	(2.3.5)
$F = \frac{1}{6}t_1^3 + \frac{1}{2}t_1 t_2^2 + \frac{k}{6}t_2^3$	$E = t_1\partial_1 + t_2\partial_2$	$d = 0$	$k \neq 0$	(2.3.6)

Table 2.1: The two-dimensional Frobenius manifolds.

In cases (2.3.1)–(2.3.5), the metric with respect to (t_1, t_2) is

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the three-point functions c_{ijk} only differ when $i = j = k = 2$, we can write the multiplication $\circ$ for these five Frobenius manifolds as

$$\partial_i \circ \partial_j = \begin{cases} \partial_1 & i = j = 1; \\ \partial_2 & i \neq j; \\ c_{222}\partial_1 & i = j = 2. \end{cases} \tag{2.3.7}$$

In case (2.3.6), the metric is given by the identity matrix I_2. The multiplication for this Frobenius manifold differs from (2.3.7) in that

$$\partial_2 \circ \partial_2 = k\partial_2.$$

2.3.1 The intersection form

The Euler field of a Frobenius manifold $\mathcal{M}$ can be used to produce a new flat metric on $\mathcal{M}$, which we refer to as the intersection form. We follow Dubrovin — see [18, Lecture 3] — in defining this first as a contravariant metric: that is, an inner product on $T_p^*\mathcal{M}$, $p \in \mathcal{M}$. Recall that we may identify tangent and cotangent spaces on $\mathcal{M}$ via the inner product $\langle \cdot, \cdot \rangle$.

Definition 2.3.5. Let $\mathcal{M}$ be a Frobenius manifold with Euler field E, flat coordinates t for the metric η, and other arbitrary local coordinates x. We define the inner product $(\cdot, \cdot)^*$ on $T_p^*\mathcal{M}$, $p \in \mathcal{M}$, as

$$(\omega_1, \omega_2)^* := \iota_E (\omega_1 \circ \omega_2)$$

for $\omega_1, \omega_2 \in \Omega^1(\mathcal{M})$, where ι_E denotes contraction with the Euler field. The *intersection form* of $\mathcal{M}$ is the metric g with contravariant components

$$g^{ij}(x) = (\mathrm{d}x^i, \mathrm{d}x^j)^*.$$

In the flat coordinates t, this becomes

$$g^{ij}(t) = E^k(t) c_k^{ij}(t). \tag{2.3.8}$$

For the intersection form to describe a well-defined (covariant) metric at $p \in \mathcal{M}$, g must be invertible at this point.

Definition 2.3.6. The *discriminant locus* of a Frobenius manifold, denoted Σ, is the set of $p \in \mathcal{M}$ for which the matrix $(g^{ij})(p)$ is degenerate. Equivalently, Σ is the set of points at which E is not invertible as an operator of $\circ$-multiplication.

On $\mathcal{M} \setminus \Sigma$, the intersection form specifies a metric on the tangent bundle with covariant components g_{ij} such that

$$(g_{ij}) = \left(g^{ij}\right)^{-1}.$$

The inner product

$$(\partial_i, \partial_j) := g_{ij}$$

is then dual to $(\cdot,\cdot)^*$. We denote by g the metric on $T\mathcal{M}$, where it is well-defined, and use g^{-1} to denote the metric on $T^*\mathcal{M}$.

Remarkably, the intersection form is flat. A proof of this key result may be found in [18, Lecture 3]. The two flat metrics of a Frobenius manifold can be related using the multiplication and the Euler field.

Proposition 2.3.7 [18, Lecture 3]**.** *Let $\mathcal{M}$ be a Frobenius manifold with metric η defining the inner product $\langle\cdot,\cdot\rangle$, and intersection form g defined on $\mathcal{M}\setminus\Sigma$. Then we can relate η and g via*

$$(u, v) = \langle E^{-1} \circ u, v\rangle \tag{2.3.9}$$

for all $u, v \in T_p\mathcal{M}$, $p \in \mathcal{M}$.

Remark 2.3.8. We note briefly that the intersection form can be used to construct the monodromy group of an n-dimensional Frobenius manifold, following [18, Appendix G]. The metric g is locally Euclidean, so one can define a local isometry between $\mathbb{C}^n$ and the universal covering of $\mathcal{M}\setminus\Sigma$. This isometry is defined by a (multivalued) mapping between the flat coordinates of g, denoted x, and the flat coordinates t of the metric η. There exists a representation μ of the fundamental group, $\pi_1(\mathcal{M}\setminus\Sigma)$, on $\mathbb{C}^n$, so that

$$\mu : \pi_1(\mathcal{M}\setminus\Sigma) \to E(n)$$

where $E(n)$ is the group of isometries on $\mathbb{C}^n$. The monodromy group is then defined as the image $\mu(\pi_1(\mathcal{M}\setminus\Sigma))$.

2.3.2 Flat pencils of metrics

The two metrics η and g form a natural structure known as a flat pencil of metrics. This structure, together with additional properties, can be used to completely describe a Frobenius manifold: see [17] for the full construction.

Let $\mathcal{M}$ be a smooth n-dimensional manifold with a metric g and local coordinates $u^1, \ldots, u^n$ in which the metric has components g_{ij}. The Levi-Civita connection for g, denoted ∇, can be uniquely described by the Christoffel symbols Γ^k_{ij} such that

$$\begin{aligned}\nabla_k g_{ij} &= \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^m_{kj} g_{im} = 0,\\ \Gamma^k_{ij} &= \Gamma^k_{ji}.\end{aligned}$$

The metric g induces a contravariant metric g^{-1} on $\mathcal{M}$, or bilinear form $(\cdot,\cdot)^*$ on $T^*_p\mathcal{M}$, where g^{-1} has components $g^{ij} = (\mathrm{d}u^i, \mathrm{d}u^j)^*$ and the matrix (g^{ij}) is the inverse of (g_{ij}).

Then the Christoffel symbols can be defined as

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right),$$

and we introduce the contravariant Christoffel symbols,

$$\Gamma^{ij}_k := -g^{il} \Gamma^j_{lk}.$$

Definition 2.3.9. Let $\mathcal{M}$ be a smooth complex manifold with two non-proportional metrics g_1, g_2. The corresponding Levi-Civita connections, denoted ∇_1 and ∇_2, have associated contravariant Christoffel symbols Γ^{ij}_{1k}, Γ^{ij}_{2k}. We define a *pencil* of metrics as the contravariant metric with components

$$g^{ij} = g^{ij}_1 + \lambda g^{ij}_2, \tag{2.3.10}$$

where $\lambda \in \mathbb{C}$ and g_1^{-1}, g_2^{-1} are the contravariant metrics corresponding to g_1, g_2, respectively.

Definition 2.3.10 [18, Definition 3.1]**.** The two metrics g_1, g_2 are said to form a *flat pencil* if

1. The corresponding Levi-Civita connection for the metric (2.3.10) has contravariant Christoffel symbols
$$\Gamma^{ij}_k = \Gamma^{ij}_{1k} + \lambda \Gamma^{ij}_{2k}.$$
2. The metric given in (2.3.10) is flat for all $\lambda \in \mathbb{C}$.

Proposition 2.3.11 [18, Proposition 3.2]**.** *Let $\mathcal{M}$ be a Frobenius manifold with metric η and intersection form g. Then the metrics η and g form a flat pencil.*

2.3.3 Semisimplicity

We now introduce the notion of semisimplicity for Frobenius manifolds. Semisimple Frobenius manifolds form an important and well-studied class; this includes the Frobenius manifolds associated with Coxeter groups which we discuss in the following section.

Recall that an algebra A is called semisimple if it contains no nilpotents; that is, it contains no nonzero elements $a \in A$ such that $a^n = 0$ for some $n \in \mathbb{N}$.

Definition 2.3.12. A point p in a Frobenius manifold $\mathcal{M}$ is said to be *semisimple* if the Frobenius algebra defined on its tangent space $T_p\mathcal{M}$ is semisimple. The Frobenius manifold $\mathcal{M}$ is said to be *semisimple* (or massive) if a generic point on $\mathcal{M}$ is semisimple.

Remark 2.3.13. The Frobenius manifolds in Example 2.3.4 are all semisimple, with the exception of (2.3.1).

A semisimple Frobenius manifold comes equipped with a set of distinguished coordinates in which many of the geometric objects associated with the manifold have simple forms.

Proposition 2.3.14 [18, Lecture 3, Main lemma]**.** *Let $\mathcal{M}$ be an n-dimensional semisimple Frobenius manifold, with some semisimple point $p \in \mathcal{M}$. Then there exists a set of local coordinates $u^1, \dots, u^n$ on a neighbourhood of p such that*

$$\partial_i \circ \partial_j = \delta_{ij} \partial_i,$$

where

$$\partial_i = \frac{\partial}{\partial u^i}\,.$$

The coordinates u^i are called canonical coordinates. The canonical coordinates, as functions of the flat coordinates t of η, may be found as particular linearly independent solutions of the system of equations

$$\frac{\partial u}{\partial t^k} c^k_{ij}(t) = \frac{\partial u}{\partial t^i} \frac{\partial u}{\partial t^j}.$$

The canonical coordinates are also defined as the roots $u = u^i$ of the characteristic equation

$$\det\left(g^{ij}(t) - u\eta^{ij}(t)\right) = 0,$$

see [18, Proposition 3.3].

Example 2.3.15 [18, Lecture 3]**.** For the 2-dimensional Frobenius manifolds described by (2.3.3)–(2.3.5), the flat coordinates t and canonical coordinates u are related as follows:

$$t^1 = \frac{u^1 + u^2}{2}, \quad t^2 = \frac{\left(u^1 - u^2\right)^{2\epsilon+1}}{2(2\epsilon + 1)}. \tag{2.3.11}$$

The constant $\epsilon \in \mathbb{C}$ parametrises the Frobenius manifolds, with value $\epsilon = 1/2$ corresponding to (2.3.4), $\epsilon = -3/2$ corresponding to (2.3.3), and $\epsilon \neq \pm 1/2, -3/2$ corresponding to (2.3.5). The value $\epsilon = -1/2$ corresponds to the Frobenius manifold described by (2.3.2) when $r = 2$; in this case, the canonical coordinates are given by

$$t^1 = \frac{u^1 + u^2}{2}, \quad t^2 = \frac{1}{2} \log u^1 - u^2. \tag{2.3.12}$$

The Euler vector field, inner product, and intersection form on a semisimple Frobenius manifold $\mathcal{M}$ can be expressed in terms of the canonical coordinates, as follows.

Proposition 2.3.16 [18, Lecture 3]**.** *Let u^i be the canonical coordinates on the neighbourhood of some semisimple point $p \in \mathcal{M}$, with $\partial_i = \frac{\partial}{\partial u^i}$. Then,*

1. *The inner product $\langle \cdot, \cdot \rangle$ is diagonal in u:*

$$\langle \partial_i, \partial_j \rangle = \eta_{ii}(u)\delta_{ij}.$$

The components of η have the form

$$\eta_{ii}(u) = \frac{\partial}{\partial u^i} t_1(u).$$

2. *The unity vector field e has the form*

$$e = \sum_i \partial_i.$$

3. *The Euler vector field has the form*

$$E = u^i \partial_i.$$

4. *The intersection form is given by*

$$g^{ij}(u) = u^i \eta_{ii}^{-1} \delta_{ij}.$$

Semisimple Frobenius manifolds can be completely described by a function of one variable, known as a Landau-Ginzburg (LG) superpotential.

Definition 2.3.17 [18, Appendix I]**.** Let $\mathcal{M}$ be an n-dimensional semisimple Frobenius manifold with flat coordinates t and canonical coordinates u defined on a chart of $\mathcal{M}$. The *LG superpotential* of $\mathcal{M}$ is a holomorphic function $\lambda(p)$, where p depends on $t = (t^1, \dots, t^n)$, such that the canonical coordinates u are given by the critical points of $\lambda(p)$. That is,

$$u^i(t) = \lambda\left(q^i(t)\right),$$

where the q^i are such that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}p}\left(q^i(t)\right) = 0 \quad \text{for } i \in \{1, \dots, n\}.$$

Theorem 2.3.18 [18, Appendix I]**.** *Any semisimple Frobenius manifold $\mathcal{M}$ with charge $d < 1$ can be described by a superpotential $\lambda(p)$. In the flat coordinates t of $\mathcal{M}$, the metric*

η, intersection form g, and tensor field c are given by the residue formulas

$$\eta_{ij}(t) = \sum_{i=1}^{n} \underset{p=q^i}{res} \frac{\partial_i(\lambda)\,\partial_j(\lambda)}{\lambda'} dp,$$
$$g_{ij}(t) = \sum_{i=1}^{n} \underset{p=q^i}{res} \frac{\partial_i(\log\lambda)\,\partial_j(\log\lambda)}{(\log\lambda)'} dp,$$
$$c_{ijk} = \eta(\partial_i \circ \partial_j, \partial_k) = \sum_{i=1}^{n} \underset{p=q^i}{res} \frac{\partial_i(\lambda)\,\partial_j(\lambda)\,\partial_k(\lambda)}{\lambda'} dp,$$

where $\lambda' = \frac{d\lambda(p)}{dp}$.

2.3.4 Coxeter groups and their orbit spaces

A subclass of the semisimple Frobenius manifolds are closely related to Coxeter groups, otherwise known as reflection groups. Saito, for example in [54], established that the orbit spaces of Coxeter groups admit certain flat structures; this work was then used by Dubrovin to show that these orbit spaces are Frobenius manifolds with polynomial prepotentials.

2.3.4.1 Coxeter groups and root systems

In this section, we set V to be a real n-dimensional vector space and denote by $(\cdot,\cdot)$ the Euclidean inner product on V. We now make precise some of the important features of Coxeter groups, following Bourbaki [6].

We denote by $O(V)$ the orthogonal group over V.

Definition 2.3.19. A *reflection* along non-zero $\alpha \in V$ is the linear operator $s_\alpha \in O(V)$ which sends $\alpha \mapsto -\alpha$ and pointwise fixes the hyperplane orthogonal to α, denoted $H_\alpha := \{v \in V | (\alpha, v) = 0\}$.

The operator s_α can be written as

$$s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha, \tag{2.3.13}$$

for any $v \in V$.

For any $\alpha \in V$, there exists a decomposition of the vector space $V = \mathbb{R}_\alpha \oplus H_\alpha$, where $\mathbb{R}_\alpha = \{v \in V | v = k\alpha \text{ for } k \in \mathbb{R}\}$.

Definition 2.3.20. A *root system* is a finite set of non-zero vectors $\mathcal{R} \subseteq V$ such that

1. The roots $\alpha \in \mathcal{R}$ span V.

2. $\mathcal{R} \cap \mathbb{R}_\alpha = \{\alpha, -\alpha\}\ \forall\, \alpha \in \mathcal{R}$.

3. $s_\alpha(\mathcal{R}) = \mathcal{R}\ \forall\, \alpha \in \mathcal{R}$.

Definition 2.3.21. The *finite Coxeter group* W for the root system $\mathcal{R}$ is the finite reflection group generated by $\mathcal{R}$, that is $W = \{s_\alpha \,|\, \alpha \in \mathcal{R}\}$.

Root systems arise in the theory of Lie algebras; in particular, every semisimple Lie algebra has a crystallographic root system associated with it. We are interested in Weyl groups, or finite Coxeter groups with crystallographic root systems.

Definition 2.3.22. Let $\mathcal{R}$ be a root system for a finite Coxeter group W. We say that $\mathcal{R}$ is *crystallographic*, and call W a *Weyl group*, if

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \forall\, \alpha,\beta \in \mathcal{R}.$$

Definition 2.3.23. Let W be a Coxeter group with root system $\mathcal{R} \subseteq V$. Both W and $\mathcal{R}$ are said to be *irreducible* if there do not exist linear subspaces $V_1, V_2 \subset V$ with non-empty root systems $\mathcal{R}_1 \subset V_1$, $\mathcal{R}_2 \subset V_2$ such that $V = V_1 \oplus V_2$ and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.

Definition 2.3.24. Let W be a finite Coxeter group with root system $\mathcal{R}$ and arbitrary marked root $\gamma \in \mathcal{R}$. The hyperplane H_γ divides V into two half-spaces. We say that $\mathcal{R}_+ \subset \mathcal{R}$ is a set of *positive roots* if all $\alpha \in \mathcal{R}$ are in the same half-space as γ.
A set of roots $\Delta \subset \mathcal{R}_+$ is called a set of *simple roots* for W if

1. Δ is a basis for V.

2. Every root in $\mathcal{R}$ is a linear combination of $\alpha \in \Delta$ with only non-positive or only non-negative coefficients.

The rank of W is given by the cardinality of Δ.

Note that neither sets of positive roots nor sets of simple roots are uniquely defined for a given Coxeter group.

The finite irreducible Coxeter groups were completely classified by Coxeter [14]. We reproduce the full list, consisting of groups A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(\kappa)$ for $n, \kappa \in \mathbb{N}$, $\kappa \geq 2$. The groups of A-, B-, D-, E-, F-, or G-type are Weyl groups for their respective simple Lie algebras, and $I_2(\kappa)$ is the group of symmetries of a regular κ-gon. Note the isomorphisms $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$, $G_2 \cong I_2(6)$.

Definition 2.3.25. Let W be a Weyl group of rank n with a set of simple roots Δ. The *fundamental weights* are the vectors $\omega_1, \ldots, \omega_n \in V$ such that

$$\frac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}\ \forall\, \alpha_j \in \Delta.$$

Definition 2.3.26. Let W be an irreducible finite Coxeter group of rank n with a root system $\mathcal{R}$. The *Coxeter number* of W is $h = \frac{m}{n}$ where m is the number of roots in $\mathcal{R}$.

The Coxeter number is an integer which can be defined in several ways, including as the order of a Coxeter element. A Coxeter element is a product of all simple reflections in an irreducible Coxeter group W, and all Coxeter elements of W have the same order. For example, the Coxeter number of A_n is $h = n + 1$.

2.3.4.2 Frobenius manifold structures on the orbit spaces of Coxeter groups

This section follows the construction by Dubrovin in [18, Lecture 4], [19], and Saito in [54].

The action of a finite irreducible Coxeter group W can be extended to a linear action on the complexification of V, denoted $V^{\mathbb{C}} := V \otimes \mathbb{C} \cong \mathbb{C}^n$. The orbit space of W is denoted by M_W, that is

$$M_W = V^{\mathbb{C}}/W \cong \mathbb{C}^n/W.$$

Let $x^1, \ldots, x^n$ be the coordinates for an orthonormal basis of $V^{\mathbb{C}}$ with respect to (the bilinear extension of) the Euclidean metric $(\cdot, \cdot)$. We denote by $R = \mathbb{C}[x^1, \ldots, x^n]^W$ the algebra of polynomials in x which are invariant under the action of W; this action is defined as

$$(wp)(v) = p\left(w^{-1}v\right)$$

for $w \in W$, $p \in R$, $v \in V^{\mathbb{C}}$. By Chevalley's Theorem, R is generated by a set of homogeneous algebraically independent polynomials $y^1(x), \ldots, y^n(x) \in R$ such that $R \cong \mathbb{C}[y^1, \ldots, y^n]$ and the degrees $d_i := \deg\left(y^i\right)$ are uniquely defined by W. The y^i are called basic invariants of W. The degrees of the basic invariants can be ordered as

$$d_1 = h > d_2 \geq \cdots \geq d_{n-1} > d_n = 2$$

where h is the Coxeter number of W, and they also satisfy the condition $d_i + d_{n-i+1} = h + 2$ for all $i \in \{1, \ldots, n\}$.

The orbit space M_W has the structure of a manifold, with local coordinates given by basic invariants of W. Recall that the orbit of an element $v \in V^{\mathbb{C}}$ is $W(v) = \{wv | w \in W\}$. A regular orbit has cardinality $|W(v)| = |W|$, while an irregular orbit has $|W(v)| < |W|$.

Definition 2.3.27. The *discriminant*, $\mathrm{Discr} M_W$, is the union of all irregular orbits in M_W.

Theorem 2.3.28 [18, Theorem 4.1]**.** *Let W be a finite irreducible Coxeter group with Coxeter number h and a set of basic invariants $y^1, \ldots, y^n$. Then $\mathcal{M} = M_W$ is a Frobenius manifold such that*

1. *The unity vector field is* $e = \frac{\partial}{\partial y^1}$.

2. *The Euler vector field is* $E = \frac{1}{h}\sum_i d_i y^i \frac{\partial}{\partial y^i} = \frac{1}{h} x^i \frac{\partial}{\partial x^i}$.

3. *The intersection form* g^{-1} *is induced by the Euclidean metric on* V*. Its components are given by*

$$g^{ij}(y) = \left(dy^i, dy^j\right)^* = \sum_{k=1}^{n} \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^k}. \tag{2.3.14}$$

Such a Frobenius manifold structure on M_W *is unique up to equivalence.*

The intersection form g^{-1} defined by (2.3.14) is a complex-valued flat metric with flat coordinates x^i. It is non-degenerate on $M_W \setminus \mathrm{Discr} M_W$; in fact, the discriminant locus Σ of $\mathcal{M}$ coincides with $\mathrm{Discr} M_W$ [19].

Remark 2.3.29 [18, Appendix G]**.** The monodromy group of the Frobenius manifold $\mathcal{M} = M_W$ is given by the Coxeter group W.

Theorem 2.3.30 [54], [18]**.** *Let* $\mathcal{M} = M_W$ *be the Frobenius manifold defined in Theorem 2.3.28. Then there exists a set of basic invariants* $t^1, \dots, t^n$ *of* W*, known as the Saito coordinates, such that the metric* η*, known as the Saito metric, defined by*

$$\eta^{ij}(t) = \frac{\partial}{\partial t^1} g^{ij}(t)$$

is flat and anti-diagonal. The prepotential F *of* $\mathcal{M}$ *defined by structure constants*

$$c_{ij}^k(t) = \eta^{km}\partial_i\partial_j\partial_m F(t)$$

is polynomial in the Saito coordinates and has charge $d = 2h + 2$*. The intersection form may be rewritten as*

$$g^{ij}(t) = \frac{d_i + d_j - 2}{h}\eta^{ik}\eta^{jm}\partial_k\partial_m F(t).$$

Dubrovin showed that all Frobenius manifolds $\mathcal{M} = M_W$ are semisimple and have polynomial prepotentials [18, Lecture 4]. He also conjectured that all semisimple Frobenius manifolds with polynomial prepotentials, positive degrees, and charge $0 < d < 1$ are described by such a construction; this was later proven by Hertling [32].

Example 2.3.31 [18, Example 4.1]**.** The prepotentials associated with the dihedral groups $I_2(\kappa)$, that is the Frobenius manifold structures on $\mathbb{C}^2/I_2(\kappa)$, are described (up to equivalence) by equation (2.3.5) where $k = \kappa + 1$. In particular, $k = 4$ gives the prepotential for A_2 and $k = 5$ gives B_2.

Example 2.3.32 [18, Example 4.3 and Theorem A.1]**.** In three dimensions, there are exactly three semisimple polynomial solutions, corresponding to the three rank 3 finite

Coxeter groups A_3, B_3, and H_3. The prepotential for A_3 is given in Example 2.2.9. There is only one other 3-dimensional polynomial prepotential, given by

$$F(t) = \frac{1}{2}\left(t_1^2 t_3 + t_1 t_2^2\right) + t_2^4.$$

This has charge $d = 1$ and is not semisimple.

Remark 2.3.33. A similar construction was used by Dubrovin and Zhang [22] to show that the orbit spaces of extended affine Weyl groups also have a Frobenius manifold structure. Without going into detail, we note that a root system of rank n has associated with it a prepotential in $n + 1$ variables, and a distinguished coordinate direction. An example of such a prepotential is given by (2.3.2) for type A_1.

2.4 Symmetries of the WDVV equations

We have already introduced trivial examples of symmetries of the WDVV equations and Frobenius manifolds in the form of equivalences. We now define symmetries for both objects in general, and introduce three specific types of symmetry.

Definition 2.4.1. A *symmetry of the (generalised) WDVV equations* (2.2.2) is a transformation $(F, t, Q) \mapsto \left(\widetilde{F}, \tilde{t}, \widetilde{Q}\right)$ such that the (generalised) WDVV equations are satisfied by $\widetilde{F}\left(\tilde{t}\right)$, $\widetilde{Q}$ if $F = F(t)$ satisfies the (generalised) WDVV equations with metric Q.

This definition can easily be adapted for the ordinary WDVV equations; namely, F, $\widetilde{F}$ in Definition 2.4.1 must then both satisfy the ordinary WDVV equations (2.2.12).

We also define symmetries of Frobenius manifolds in the expected way.

Definition 2.4.2. A *symmetry of Frobenius manifolds* is a transformation taking

$$(\mathcal{M}, \circ, e, \eta, E) \mapsto \left(\widetilde{\mathcal{M}}, \tilde{\circ}, \tilde{e}, \tilde{\eta}, \widetilde{E}\right)$$

such that $\left(\widetilde{\mathcal{M}}, \tilde{\circ}, \tilde{e}, \tilde{\eta}, \widetilde{E}\right)$ is a Frobenius manifold if $(\mathcal{M}, \circ, e, \eta, E)$ is a Frobenius manifold.

Given a prepotential F with Euler field E, a symmetry S of the WDVV equations such that $S(F) = \widetilde{F}$ also defines a symmetry of Frobenius manifolds if $\widetilde{F}$ is a prepotential with Euler field $\widetilde{E} = S(E)$. Conversely, a symmetry of Frobenius manifolds automatically describes a symmetry of the WDVV equations.

Dubrovin introduced two symmetries of Frobenius manifolds [18, Appendix B], namely inversion and the Legendre-type transformation (or simply Legendre transformation). The multiplication on $\mathcal{M}$ is preserved by both kinds of symmetry.

Before discussing inversion and Legendre transformations, we present an overview of almost duality on Frobenius manifolds. This construction — also introduced by Dubrovin

[20] — is particular to Frobenius manifolds, but can be thought of as a symmetry of the WDVV equations.

2.4.1 Almost duality for Frobenius manifolds

The almost dual of a given Frobenius manifold is a new structure which itself satisfies most, but not all, of the properties of a Frobenius manifold; namely, the unity vector field for the almost dual is not flat. Although almost duality is therefore not a symmetry of Frobenius manifolds, we do obtain a new solution of the WDVV equations. In this way, it provides a symmetry of the WDVV equations if we restrict the input to the class of prepotentials. We shall see that almost duality relates the polynomial prepotentials associated with Coxeter groups to a large class of rational solutions.

We follow Dubrovin's formulation in [20], and we set $\mathcal{M}$ to be an arbitrary Frobenius manifold. The intersection form for $\mathcal{M}$ is denoted g, with flat coordinates x and discriminant locus Σ. Setting $\mathcal{M}^\star = \mathcal{M} \setminus \Sigma$, we define a new multiplication $\star$ on $T_p\mathcal{M}^\star$ by

$$u \star v = E^{-1} \circ u \circ v. \tag{2.4.1}$$

The commutativity and associativity of $\star$ follow immediately from the commutativity and associativity of $\circ$. It is also clear that the Euler field E is the identity for $\star$. We would like to show that the multiplication $\star$ and the intersection form g pointwise define a family of Frobenius algebras on $T\mathcal{M}^\star$. This can be done by using relation (2.3.9) to check the Frobenius condition: we find

$$(u, v \star w) = \langle E^{-1} \circ u, E^{-1} \circ v \circ w \rangle = \langle E^{-1} \circ \left(E^{-1} \circ u \circ v \right), w \rangle = (u \star v, w).$$

The multiplication $\star$ can be described using a new set of structure constants.

Definition 2.4.3. The *dual structure constants* denoted $\overset{\star}{c}{}_{ij}^{k}(x)$ are such that

$$\partial_i \star \partial_j = \overset{\star}{c}{}_{ij}^{k}(x) \partial_k.$$

The *dual three-point function* is defined as

$$\overset{\star}{c}_{ijk} := g_{kl} \overset{\star}{c}{}_{ij}^{l} = (\partial_i \star \partial_j, \partial_k).$$

Theorem 2.4.4 [20]**.** *Let $(\mathcal{M}, \circ, e, \eta, E)$ be a Frobenius manifold with intersection form g, discriminant locus Σ, and charge $d \neq 1$, and let $\mathcal{M}^\star = \mathcal{M} \setminus \Sigma$ with multiplication $\star$ defined by (2.4.1). Then $\mathcal{M}^\star = (\mathcal{M}^\star, \star, E, g, E)$ satisfies Definition 2.3.1, with the exception of property 3, and has charge d.*

We say that $\mathcal{M}^\star$ is the almost dual of $\mathcal{M}$.

Remark 2.4.5 [20]**.** In the flat coordinates x of g, the Euler field has the form

$$E(x) = \frac{1-d}{2} x^i \partial_i,$$

when $d \neq 1$.

A new solution of the WDVV equations is associated to the almost dual Frobenius manifold. Note that this is not a prepotential as in Definition 2.2.17 because it does not satisfy the ordinary WDVV equations; that is, the metric g is not equal to $F^\star_1$.

Proposition 2.4.6 [20]**.** *Let $x(t)$ be the flat coordinates for the intersection form, defined on $\mathcal{M}^\star$. Then there locally exists a function $F^\star(x)$, which we call the dual prepotential, such that the dual three-point function can be written as*

$$\overset{\star}{c}_{ijk}(x) = \frac{\partial^3 F^\star(x)}{\partial x^i \partial x^j \partial x^k} = g_{ia}(x) g_{jb}(x) \frac{\partial t^\gamma}{\partial x^k} \frac{\partial x^a}{\partial t^\alpha} \frac{\partial x^b}{\partial t^\beta} c^{\alpha\beta}_\gamma(t),$$

and $F^\star(x)$ satisfies the WDVV equations with metric g.

We now refer back to the polynomial Frobenius manifolds discussed in § 2.3.4. Since we have a Frobenius manifold structure on the orbit space of any finite irreducible Coxeter group W, one can obtain an almost dual structure, with a dual prepotential, for any W. Dubrovin showed that this dual prepotential is described by the following formula.

Theorem 2.4.7 [20, Theorem 5.2]**.** *Let W be a finite irreducible Coxeter group with a set of positive roots $\mathcal{R}_+$. Then the dual prepotential for the almost dual of the Frobenius manifold $\mathcal{M} = M_W$ is*

$$F^\star(x) = \frac{h}{2} \sum_{\alpha \in \mathcal{R}_+} (\alpha, x)^2 \log{(\alpha, x)}, \tag{2.4.2}$$

defined on $\mathbb{C}^n \setminus \bigcup_{\alpha \in \mathcal{R}_+} H_\alpha$, where the roots are normalised by $(\alpha, \alpha) = 2$.

Here we have an example of a symmetry of the WDVV equations mapping one class of solutions to a different class: a polynomial prepotential is mapped to a rational solution by almost duality.

Example 2.4.8 (cf. [20, Example 2])**.** The polynomial prepotential associated with A_3, given in Example 2.2.9, has the almost dual prepotential $F^\star(x) = F(x)$ where $F(x)$ is given, up to equivalence, in Example 2.2.10.

2.4.2 Inversion

One of the two symmetries of Frobenius manifolds introduced by Dubrovin is the inversion symmetry, denoted I. We follow the definition given in [18, Appendix B], but note that it also functions as a symmetry of the ordinary WDVV equations.

Definition 2.4.9. Let $F(t)$ be a solution to the ordinary WDVV equations with metric $\eta(t)$ such that $\eta = F_1$, and flat coordinates $t = (t^1, \ldots, t^n)$. The *inversion* of F is said to be the function

$$\widetilde{F}\left(\tilde{t}\right) = \left(\tilde{t}^n\right)^2 F + \frac{1}{2}\tilde{t}^1\tilde{t}_\sigma\tilde{t}^\sigma$$

with $\tilde{\eta}$ such that

$$\tilde{\eta}_{ij} = \eta_{ij}$$

and coordinates $\tilde{t}$ such that

$$\tilde{t}^i = \begin{cases} \frac{1}{2}\left(t^n\right)^{-1} t_\sigma t^\sigma & i = 1, \\ \left(t^n\right)^{-1} t^i & i \neq 1, n, \\ -\left(t^n\right)^{-1} & i = n. \end{cases}$$

Proposition 2.4.10 (c.f. [18, Appendix B])**.** *The inversion transformation in Definition 2.4.9, when applied to prepotentials, describes a symmetry of Frobenius manifolds.*

Inversion is then automatically a symmetry of the ordinary WDVV equations.

Example 2.4.11. Let $F(t)$ be the prepotential given in (2.3.3). The inversion of F is the function

$$\widetilde{F}(\tilde{t}) = \frac{1}{2}\tilde{t}_1^2\tilde{t}_2 - \tilde{t}_2^2 \log\left(-\tilde{t}_2\right)$$

with contravariant coordinates $\tilde{t}_1 = t_1$ and $\tilde{t}_2 = -t_2^{-1}$. We find that $\widetilde{F}\left(\tilde{t}\right)$ is equivalent to

$$\widehat{F}\left(\hat{t}\right) = \frac{1}{2}\hat{t}_1^2\hat{t}_2 + \hat{t}_2^2 \log\left(\hat{t}_2\right)$$

under the relations $\widehat{F} = -\widetilde{F}$, $\hat{t}_1 = \tilde{t}_1$, $\hat{t}_2 = -\tilde{t}_2$. Since $\widehat{F}$ is the prepotential for the Frobenius manifold described in (2.3.4), up to a change of notation, we can say that the two Frobenius manifolds associated with (2.3.3) and (2.3.4) are related by the inversion symmetry. We add to this mapping the following relations between differential operators,

$$\begin{aligned} \widehat{\partial}_1 &= \widetilde{\partial}_1 = \partial_1, \\ \widehat{\partial}_2 &= -\widetilde{\partial}_2 = -\left(t_2\right)^2 \partial_2, \end{aligned}$$

where $\widetilde{\partial}_i = \dfrac{\partial}{\partial \tilde{t}_i}$, $\widehat{\partial}_i = \dfrac{\partial}{\partial \hat{t}_i}$. Finally, we check that the Euler field remains the same under the coordinate transformation $t \mapsto \hat{t}$; that is,

$$E = t_1\partial_1 - 2t_2\partial_2 = \hat{t}_1\widehat{\partial}_1 + 2\hat{t}_2\widehat{\partial}_2 = \widehat{E}, \tag{2.4.3}$$

where E is the Euler field for F and $\widehat{E}$ is the Euler field for $\widehat{F}$.

2.4.3 Legendre transformations

As defined by Dubrovin, a Legendre-type transformation on a Frobenius manifold $\mathcal{M}$ produces a new Frobenius manifold structure on $\mathcal{M}$. The transformation produces a new prepotential, flat metric, and flat coordinates, but the multiplication $\circ$ and the Euler field remain geometrically unchanged.

In Dubrovin's definition, a Legendre transformation is generated by an invertible flat vector field. A vector field X on $\mathcal{M}$ is said to be invertible if it is an invertible element of the Frobenius algebra on $T_p\mathcal{M}$ for all $p \in \mathcal{M}$. That is, given the multiplication $\circ$ with identity e, X is invertible if there exists a vector field X^{-1} such that $X \circ X^{-1} = e$.

Dubrovin's definition has since been expanded to provide a symmetry of the WDVV equations by Strachan and Stedman [59], which was labelled the generalised Legendre transformation. In this case, the transformation is determined by an invertible vector field satisfying a condition which generalises flatness. We will call such a vector field a Legendre field, and refer to the corresponding generalised Legendre transformation simply as a Legendre transformation. This is consistent with terminology used by Liu et al [40].

Definition 2.4.12 [59, Definition 2.3]**.** Let $\mathcal{M}$ be a smooth manifold $\mathcal{M}$ with flat metric η, Levi-Civita connection ∇, and let $\circ$ be an associative pointwise multiplication on $T\mathcal{M}$, with identity e, such that $\circ$, η satisfy the Frobenius condition (2.2.8). A *Legendre field* $\delta \in \mathfrak{X}(\mathcal{M})$ is an invertible vector field that satisfies

$$X \circ \nabla_Y \delta = Y \circ \nabla_X \delta \tag{2.4.4}$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

We may set $Y = e$ in (2.4.4) to obtain the equivalent[2] condition

$$\nabla_X \delta = X \circ \nabla_e \delta, \tag{2.4.5}$$

which we call the *Legendre field condition.* Following [59, Definition 2.1], we define a new metric $\widehat{\eta}$ by

$$\widehat{\eta}(X, Y) = \eta(\delta \circ X, \delta \circ Y),$$

noting that this is non-degenerate since δ is invertible. By [59, Lemma 2.2], the Levi-Civita connection $\widehat{\nabla}$ is given by

$$\widehat{\nabla}_X Y = \delta^{-1} \circ \nabla_X (\delta \circ Y). \tag{2.4.6}$$

Given a solution of the WDVV equations, we can use Legendre fields to construct new solutions of the WDVV equations. Despite the fact that the following theorem was

[2]It is straightforward to check that a solution of (2.4.5) automatically satisfies (2.4.4).

originally given with reference only to solutions of the ordinary WDVV equations, the proof applies to any solution of the generalised WDVV equations with a flat metric η.

Theorem 2.4.13 [59, Theorem 4.1]. *Let $F(t)$ be a solution of the WDVV equations, defined in a chart of a smooth manifold $\mathcal{M}$ with flat metric η and flat coordinates t^i. If $\delta \in \mathfrak{X}(\mathcal{M})$ is a Legendre field, then $\widehat{F}\left(\hat{t}\right)$ satisfies the WDVV equations (2.2.2) with flat metric $\widehat{\eta}$ and flat coordinates $\hat{t}^i$, where*

$$\widehat{\eta}_{ij} = \eta_{ij}, \tag{2.4.7}$$

$$\frac{\partial \hat{t}^i}{\partial t^j} = \delta^k c^i_{kj}, \tag{2.4.8}$$

$$\frac{\partial^2 \widehat{F}}{\partial \hat{t}^i \partial \hat{t}^j} = \frac{\partial^2 F}{\partial t^i \partial t^j}. \tag{2.4.9}$$

In particular,

$$\hat{t}_i = \delta^k \partial_k \partial_i \left(F(t)\right)$$

when δ is flat.

Remark 2.4.14. It is not always possible to write $\widehat{F}\left(\hat{t}\right)$ explicitly. To do this, one needs to be able to write the second-order derivatives of $F(t)$ in terms of $\hat{t}$ so that the right-hand side of relation (2.4.9) can be integrated with respect to $\hat{t}$. Since the new coordinates $\hat{t} = \hat{t}(t)$ are found using (2.4.8), these relations need to be inverted to find $t\left(\hat{t}\right)$. This inversion step typically presents some difficulty and is not always possible.

Following the discussion in [59], we show that flat sections of the Dubrovin connection ${}^{\lambda}\nabla$, given by Definition 2.2.14, provide an infinite family of Legendre fields. We consider $S \in \mathfrak{X}(\mathcal{M})$ such that ${}^{\lambda}\nabla S = 0$, and take the series expansion of S given by

$$S = \sum_{i=0}^{\infty} \lambda^i s_i, \tag{2.4.10}$$

where λ is the spectral parameter of ${}^{\lambda}\nabla$ and $s_i \in \mathfrak{X}(\mathcal{M})$.

Proposition 2.4.15 [59]. *Let $\mathcal{M}$ be a smooth manifold with associated η, ∇, $\circ$ as in Definition 2.4.12. If S as in (2.4.10) is a flat section for the Dubrovin connection ${}^{\lambda}\nabla$, then s_i is a Legendre field for all i.*

Proof. Since S is a flat section for ${}^{\lambda}\nabla$, we have ${}^{\lambda}\nabla_X S = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$. Equivalently,

$$\left(\nabla_X + \lambda X \circ\right) S = \left(\nabla_X + \lambda X \circ\right) \sum_{i=0}^{\infty} \lambda^i s_i = 0,$$

so

$$\sum_{i=0}^{\infty} \lambda^i \nabla_X s_i + \sum_{i=0}^{\infty} \lambda^{i+1} (X \circ s_i) = 0. \tag{2.4.11}$$

Comparing coefficients of λ^i, we have

$$\begin{aligned} \nabla_X s_0 &= 0, \\ \nabla_X s_i &= -X \circ s_{i-1}, \quad i > 0. \end{aligned}$$

Therefore, $Y \circ \nabla_X s_i = -X \circ Y \circ s_{i-1} = X \circ \nabla_Y s_i$ when $i > 0$, and the Legendre field condition (2.4.4) is satisfied for all i. □

Following from the proof of Proposition 2.4.15, one can recursively construct an infinite sequence of Legendre fields indexed by $\alpha \in \mathbb{N}$, given any flat Legendre field δ_0 as a starting point. New Legendre fields δ_i are produced according to the relation[3]

$$\nabla_X \delta_\alpha = X \circ \delta_{\alpha-1}, \tag{2.4.12}$$

for all $\alpha > 0$, $X \in \mathfrak{X}(\mathcal{M})$. In coordinate notation, (2.4.12) can be written

$$\nabla_j \delta^i_\alpha = c^i_{jk} \delta^k_{\alpha-1}. \tag{2.4.13}$$

Remark 2.4.16. The objects we call Legendre fields are closely related to certain integrable hierarchies (recursively defined integrable systems) of hydrodynamic partial differential equations. Dubrovin showed that all Frobenius manifolds have an associated hierarchy of this type, known as the principal hierarchy of the manifold [18, Lecture 6]. The relation (2.4.13) appears in work by Lorenzoni, Pedroni, and Raimondo on generalisations of the principal hierarchy [41], where the PDEs being studied have the form

$$u_t = \delta \circ u_x. \tag{2.4.14}$$

In this setting, the recursion relation (2.4.13) is used to define higher flows, or consecutive systems of equations in the hierarchy. Legendre fields are exactly the generators δ of the flows (2.4.14); the Legendre field condition (2.4.4) describes a symmetry condition of the hierarchy. These connections are discussed in more detail in [40].

It has been shown that inverses and compositions of Legendre fields are also Legendre fields [59, Propositions 2.4 and 2.5]. We reproduce the former statement as follows.

Lemma 2.4.17 [59, Proposition 2.4]**.** *Let δ be a Legendre field for the Levi-Civita connection ∇. Then δ^{-1} is a Legendre field for $\widehat{\nabla}$ given by (2.4.6).*

[3] The sign of the term on the right-hand side is immaterial, as it depends only on the sign of the spectral parameter λ of the Dubrovin connection.

The composition of a Legendre field and its inverse act as the identity transformation.

Proposition 2.4.18 See [59, Proposition 2.5]. *If δ is a Legendre field which maps a solution F to $\widehat{F}$, the inverse field δ^{-1} maps $\widehat{F}$ to F.*

The Legendre-type transformation defined by Dubrovin is an example of the transformation as defined above, where the Legendre field is given by one of n flat vector fields for an n-dimensional Frobenius manifold. For later convenience, we introduce his notation as follows.

Notation 2.4.19 [18, Appendix B]. Let F be a solution of the WDVV equations in n variables. The Legendre transformation produced by ∂_γ for $\gamma \in \{1, \ldots, n\}$ is denoted by S_γ and we write $S_\gamma(F) = \widehat{F}$ where $\widehat{F}$ is given by Theorem 2.4.13.

When $\delta = \partial_\gamma$, we recover the property given in [18] that

$$\partial_i = \partial_\gamma \circ \widehat{\partial}_i \tag{2.4.15}$$

for all i. To see this, one should use (2.4.8) to rewrite

$$\partial_\gamma \circ \widehat{\partial}_j = c^k_{\gamma l} \frac{\partial t^l}{\partial \hat{t}^j} \partial_k.$$

From (2.4.15), we have $\widehat{\partial}_i = \partial_\gamma^{-1} \circ \partial_i$ and so it is straightforward to see that the unit vector field in the new coordinates under S_γ is given by $e = \widehat{\partial}_\gamma$. If $F(t)$ is a solution of the ordinary WDVV equations, so that $e = \partial_1$, then the transformation S_1 is the identity.

Example 2.4.20 [18, Example B.1]. Starting with the prepotential (2.3.2) with $r = 2$, the Legendre transformation S_2 produces new coordinates

$$\hat{t}_1 = e^{t_2},$$
$$\hat{t}_2 = t_1.$$

The new prepotential is therefore

$$\widehat{F} = \frac{1}{2}\hat{t}_1 \hat{t}_2^2 + \frac{1}{2}\hat{t}_1^2 \left(\log \hat{t}_1 - \frac{3}{2}\right),$$

which is equivalent to the prepotential in (2.3.4).

Note that, as mentioned in Remark 2.3.33, (2.3.2) is the Frobenius manifold associated with the extended affine Weyl group of type A_1.

Example 2.4.21. The A_3 almost dual prepotential in Example 2.2.10 is mapped to the trigonometric solution in Example 2.2.11 by the Legendre transformation S_1.

In the context of Frobenius manifolds, one can also consider homogeneous Legendre fields.

Definition 2.4.22. Given a Frobenius manifold $\mathcal{M}$ with Euler field E, a vector field $X \in \mathfrak{X}(\mathcal{M})$ is called *homogeneous* of degree $\mu \in \mathbb{C}$ if

$$\mathcal{L}_E X = \mu X. \tag{2.4.16}$$

Recently, Liu, Qu, and Zhang showed that homogeneous Legendre fields map quasi-homogeneous generalised Frobenius manifolds to each other [40]. A quasi-homogeneous generalised Frobenius manifold automatically satisfies all the properties of a Frobenius manifold except for flatness of the unity vector field; hence, they each locally define a quasi-homogeneous solution of the WDVV equations. Frobenius manifolds are considered a sub-class of quasi-homogeneous generalised Frobenius manifolds. In [40], it was also noted that a generic quasi-homogeneous generalised Frobenius manifold can be mapped to a Frobenius manifold by a Legendre transformation of the type S_γ.

2.4.3.1 Legendre fields for semisimple Frobenius manifolds

Recall that semisimple Frobenius manifolds admit a set of canonical coordinates, in which many of the manifold's associated structures have a simpler form than in flat coordinates. When written in canonical coordinates, the Legendre field condition (2.4.4) can be reduced to simple sets of equations. One such formulation is considered by Lorenzoni et al in [41], which we reproduce here.

Proposition 2.4.23 [41, Proposition 19]**.** *Let $\mathcal{M}$ be a Frobenius manifold with metric η, associated Levi-Civita connection ∇, multiplication $\circ$, and canonical coordinates u. The Legendre field condition (2.4.4) can be written in canonical coordinates as*

$$\partial_i \delta^j = \Gamma^j_{ij}(\delta^i - \delta^j), \quad i \neq j. \tag{2.4.17}$$

Here, δ^i are the components of the Legendre field δ in canonical coordinates, and Γ^i_{jk} denote the Christoffel symbols of the connection ∇.

We note that solutions of (2.4.17) for an n-dimensional manifold will depend on n arbitrary functions of a single variable [41].

An alternative formulation of the Legendre field condition for semisimple Frobenius manifolds was discussed in [59], where the reduced equation is written in terms of certain rotation coefficients of the metric η.

While such reduced forms of the Legendre field condition may be easier to solve, transforming between flat and canonical coordinates may pose problems. For this reason, we will focus on Legendre transformations in flat coordinates.

2.4.3.2 Twisted Legendre fields

An important result, first noted by Riley and Strachan [52] and expanded upon by Strachan and Stedman in [59], is the existence of so-called twisted Legendre fields for Frobenius manifolds. A twisted Legendre field can be constructed from a Legendre field to produce a transformation of the almost dual Frobenius manifold. If two Frobenius manifolds are connected by a Legendre field, the Legendre transformation produced by the twisted field connects the corresponding dual prepotentials. In general, a Legendre transformation applied to an almost dual Frobenius manifold is not guaranteed to map it to another almost dual Frobenius manifold.

Let us schematically denote by

$$F \xrightarrow{\delta} \widehat{F}$$

that the Legendre transformation defined by Legendre field δ on F produces the new solution $\widehat{F}$. As noted in [59], a prepotential F is mapped to another prepotential $\widehat{F}$ if δ is flat.

Theorem 2.4.24 [59, Theorem 5.1]**.** *Let F be a prepotential with Euler field E and multiplication $\circ$. Let δ be flat, and let $F^\star$, $\widehat{F}^\star$ denote the almost dual prepotentials of F, $\widehat{F}$ respectively. Then the field $E \circ \delta$ is a Legendre field for $F^\star$, and the following diagram commutes.*

$$\begin{array}{ccc} F & \xrightarrow{\delta} & \widehat{F} \\ \downarrow & & \downarrow \\ F^\star & \xrightarrow{E\circ\delta} & \widehat{F}^\star \end{array}$$

The field $E \circ \delta$ is called the twisted Legendre field.

It is important to note that the twisted field $E \circ \delta$ exists and is a Legendre field regardless of whether δ maps F to another prepotential or not. That is, one can define a twisted Legendre field for any Legendre field attached to a Frobenius manifold.

Example 2.4.25 [52]**.** Let F be the prepotential for the extended affine Weyl group of type A_1, given by (2.3.2) for $r = 2$. Taking $\delta = \partial_2$, we obtain the commuting diagram

$$\begin{array}{ccc} F & \xrightarrow{\delta} & \widehat{F} \\ \downarrow & & \downarrow \\ F^\star & \xrightarrow{E\circ\delta} & \widehat{F}^\star \end{array}$$

where, up to equivalence,

$$\begin{aligned}
\widehat{F} &= \frac{1}{2}\hat{t}_1\hat{t}_2^2 + \hat{t}_1^2 \log \hat{t}_1, \\
F^\star &= \frac{i}{3}x_1^3 + ix_1x_2^2 + \frac{1}{2}\left[\mathrm{Li}_3\left(e^{2x_2}\right) + \mathrm{Li}_3\left(e^{-2x_2}\right)\right], \\
\widehat{F}^\star &= \hat{x}_1^2 \log \hat{x}_1^2 + \hat{x}_2^2 \log \hat{x}_2^2 - (\hat{x}_1 - \hat{x}_2)^2 \log(\hat{x}_1 - \hat{x}_2)^2.
\end{aligned}$$

This is an early example of Legendre transformations producing a mapping between rational and trigonometric solutions. Note that the almost dual prepotential $F^\star$ is a trigonometric solution, as opposed to the rational solutions of the form (2.4.2) associated with the almost dual prepotentials of Coxeter groups. The Legendre transformation of the almost dual, $\widehat{F}^\star$, has a similar but inequivalent form to (2.4.2).

A homogeneous (Legendre) field produces a homogeneous twisted field; the degrees of the two fields can be related as follows.

Proposition 2.4.26. *Let $\mathcal{M}$ be a Frobenius manifold with Euler field E, and let δ be a homogeneous vector field of degree μ on $\mathcal{M}$. Then the twisted field $E \circ \delta$ is homogeneous of degree $\mu + 1$.*

Proof. By definition, we have $\mathcal{L}_E \delta = \mu\delta$. Then, using Property 5(b) of Definition 2.3.1, the Lie derivative of the twisted field is

$$\begin{aligned}
\mathcal{L}_E (E \circ \delta) &= \mathcal{L}_E(E) \circ \delta + E \circ \delta + E \circ \mathcal{L}_E(\delta) \\
&= (\mu + 1)E \circ \delta.
\end{aligned}$$

□

Flat Legendre fields are of particular interest to us, as they map Frobenius manifolds to each other, and produce simpler coordinate relations which tend to be easier to invert than those for non-flat fields (see Remark 2.4.14). However, a flat Legendre field does not always produce a flat twisted field, as shown in [59, Proposition 5.2]. We now expand on this result for homogeneous Legendre fields in general, giving the covariant derivative of a twisted Legendre field.

In the following, let $\mathcal{M}$ denote a Frobenius manifold with degree d, Euler field E, multiplication $\circ$, and Levi-Civita connection ∇ for the metric η.

Proposition 2.4.27. *Let δ be a homogeneous Legendre field of degree μ for a Frobenius manifold $\mathcal{M}$. Let $\delta^\star = E \circ \delta$ be the twisted Legendre field for the almost dual Frobenius manifold, where the intersection form g has Levi-Civita connection denoted $\nabla^\star$. Then*

$$\nabla^\star_X \delta^\star = \left(\mu + \frac{1}{2}(3 - d)\right) \delta^\star \star X.$$

Proof. The connections ∇, $\nabla^\star$ can be related by the following formula[4], see [59, 43]:

$$\nabla^\star_X Y = E \circ \nabla_X \left(E^{-1} \circ Y\right) - X \circ \nabla_{E^{-1} \circ Y} E + \frac{1}{2}(3-d) E^{-1} \circ X \circ Y \tag{2.4.18}$$

for all $X, Y \in \mathcal{M}$.

We set $Y = \delta^\star$ in (2.4.18) to obtain

$$\nabla_X \left(E \circ \delta\right) = E \circ \nabla_X \delta - X \circ \nabla_\delta E + \frac{1}{2}(3-d) X \circ \delta. \tag{2.4.19}$$

Since δ satisfies the Legendre field condition (2.4.5), we have

$$\nabla_X \delta = X \circ \nabla_e \delta. \tag{2.4.20}$$

Using the fact that ∇ is torsion-free and δ is homogeneous, we also have

$$\begin{aligned} \nabla_\delta E &= \nabla_E \delta - \mathcal{L}_E \delta \\ &= \nabla_E \delta - \mu\delta. \end{aligned} \tag{2.4.21}$$

The Legendre field condition with $X = E$ is used to substitute for $\nabla_E \delta$ in (2.4.21), which gives

$$\nabla_\delta E = E \circ \nabla_e \delta - \mu\delta. \tag{2.4.22}$$

Substituting (2.4.20), (2.4.22) back into (2.4.19), we obtain

$$\begin{aligned} \nabla^\star_X \delta^\star &= E \circ X \circ \nabla_e \delta - X \circ \left(E \circ \nabla_e \delta - \mu\delta\right) + \frac{1}{2}(3-d) X \circ \delta \\ &= \left(\mu + \frac{1}{2}(3-d)\right) \delta \circ X. \end{aligned}$$

By definition of the multiplication $\star$ in (2.4.1), we have

$$\delta \circ X = E^{-1} \circ E \circ \delta \circ X = \delta^\star \star X.$$

Then the required statement follows. □

[4]We note that equation (2.4.18) follows from straightforward but algebraically laborious manipulation of results in, for example, [32, § II.9]. The definition given in [32] of the second structure connection $\nabla^\star$ is unpacked using the symmetry of the tensor $\nabla \circ (X, Y, Z) := \nabla_X (Y \circ Z)$, the fact that ∇ is torsion-free, and properties of the Euler field.

2.4.3.3 Legendre field potentials

An alternative way to define a Legendre field is through its potential function. For a vector field $X \in \mathfrak{X}(\mathcal{M})$, the function $f : \mathcal{M} \to \mathbb{C}$ is called a *potential* for X if

$$X = \mathrm{grad} f = \eta^{ij} \partial_j(f)\, \partial_i,$$

where η is a flat metric on $\mathcal{M}$. Equivalently,

$$(\mathrm{d}f)\,(Y) = \eta(X, Y)$$

for all $Y \in \mathfrak{X}(\mathcal{M})$. Potentials are defined up to a constant term.

Potentials of Legendre fields were considered in [59], but only for Legendre fields coming from flat sections of the Dubrovin connection[5]. We show here that all Legendre fields admit a potential.

Theorem 2.4.28. *Let $\mathcal{M}$ be a smooth manifold with associated η, ∇, $\circ$ as in Definition 2.4.12. If $\delta \in \mathfrak{X}(\mathcal{M})$ is a Legendre field for ∇ then there exists a local potential h for δ.*

Proof. Let ω be the one-form such that $\omega\,(X) = \eta\,(\delta, X)$. The exterior derivative of ω is given by

$$\begin{aligned} \mathrm{d}\omega\,(X, Y) &= X\,(\omega(Y)) - Y\,(\omega(X)) - \omega([X, Y]) \\ &= X\,(\eta\,(\delta, Y)) - Y\,(\eta\,(\delta, X)) - \eta\,(\delta, [X, Y])\,. \end{aligned} \tag{2.4.23}$$

By the compatibility of the Levi-Civita connection with η, we have

$$\nabla_X\,(\eta(Y, Z)) = X\,(\eta(Y, Z)) = \eta\,(\nabla_X Y, Z) + \eta\,(Y, \nabla_X Z)\,,$$

so (2.4.23) can be rewritten as

$$\mathrm{d}\omega(X, Y) = \eta\,(\nabla_X \delta, Y) + \eta\,(\delta, \nabla_X Y) - \eta\,(\nabla_Y \delta, X) - \eta\,(\delta, \nabla_Y X) - \eta\,(\delta, [X, Y])\,. \tag{2.4.24}$$

Since the Levi-Civita connection is torsion-free, we also have

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

so

$$\eta\,(\delta, \nabla_X Y) - \eta\,(\delta, \nabla_Y X) - \eta\,(\delta, [X, Y]) = 0.$$

[5] This uses results from [18], where it was shown that the vector fields that define the principal hierarchy can be written in terms of Hamiltonian densities. The relationship between Legendre fields and integrable hierarchies is discussed in Remark 2.4.16

Substituting this back into (2.4.24), we get

$$\begin{aligned} d\omega(X,Y) &= \eta\left(\nabla_X\delta, Y\right) - \eta\left(\nabla_Y\delta, X\right) \\ &= \eta\left(Y\circ\nabla_X\delta, e\right) - \eta\left(X\circ\nabla_Y\delta, e\right) \end{aligned}$$

by the Frobenius property. Therefore, if the Legendre field condition (2.4.4) is satisfied, $d\omega = 0$. By the Poincaré Lemma, if $d\omega = 0$, there exists some function h such that $\omega = dh$. □

Certain related Legendre fields share the same potential. We first show that a Legendre field and its inverse have the same potential.

Proposition 2.4.29. *Let δ be a Legendre field on $\mathcal{M}$, with potential h. Then h is also a potential for δ^{-1}, in the flat coordinates for the new metric $\widehat{\eta}$ given by (2.4.3). That is*

$$\delta^{-1} = \widehat{\eta}^{ij}\widehat{\partial}_j(h)\widehat{\partial}_i.$$

Proof. By definition of a potential, we have

$$(dh)(X) = \eta(X,\delta)$$

for all $X \in \mathfrak{X}(\mathcal{M})$. By Proposition 2.4.18 and Theorem 2.4.28, there exists $\widehat{h} : \mathcal{M} \to \mathbb{C}$ such that

$$\left(d\widehat{h}\right)(X) = \widehat{\eta}\left(X,\delta^{-1}\right)$$

for all $X \in \mathfrak{X}(\mathcal{M})$. From the definition of $\widehat{\eta}$ in (2.4.3), and using properties of η, $\circ$, we get

$$\begin{aligned} \left(d\widehat{h}\right)(X) &= \eta\left(\delta\circ X, e\right) \\ &= \eta\left(X,\delta\right) \\ &= (dh)(X). \end{aligned}$$

Therefore $h = \widehat{h} + c$ for $c \in \mathbb{C}$. □

Similarly, a twisted Legendre field $E \circ \delta$ shares the potential of the original field δ.

Proposition 2.4.30. *Let δ be a Legendre field on a Frobenius manifold $\mathcal{M}$, and let h be its potential. Then the twisted field $\delta^\star = E\circ\delta$ also has potential h.*

Proof. As usual, we denote by η, g the two flat metrics on $\mathcal{M}$. Let δ be a Legendre field for $\mathcal{M}$ with potential h. Following the proof of Theorem 2.4.28, the one-form

$$\omega(X) = \eta(\delta, X)$$

is such that $\omega = \mathrm{d}h$ for some h.

Let $\tilde{\omega}$ be the one-form such that

$$\tilde{\omega}(X) = g(\delta^\star, X). \tag{2.4.25}$$

By the same arguments as in the proof of Theorem 2.4.28, and by Theorem 2.4.24, there exists some function $\tilde{h}$ such that $\tilde{\omega} = \mathrm{d}\tilde{h}$.

The two metrics η, g can be related by (2.3.9), which gives

$$g(X, Y) = \eta(E^{-1} \circ X, Y),$$

for $X, Y \in \mathfrak{X}(\mathcal{M})$. Then (2.4.25) becomes

$$\tilde{\omega}(X) = g(E \circ \delta, X) = \eta(\delta, X) = \omega(X),$$

which implies that $\mathrm{d}\tilde{h} = \mathrm{d}h$. Therefore $\tilde{h} = h$, up to a constant. □

As expected, a homogeneous Legendre field has a homogeneous potential. The degree of homogeneity of a Legendre field can be related to that of its potential as follows.

Theorem 2.4.31. *Let $\delta = \eta^{ij}\partial_j(h)\partial_i$ be a Legendre field with potential h on a Frobenius manifold with metric η, Euler field E, and charge d.*

If δ is homogeneous of degree $\mu \neq d - 2$, then $h + c$ for some $c \in \mathbb{C}$ is homogeneous of degree $\mu - d + 2$.

Conversely, if h is homogeneous of degree $\mu - d + 2$, then the Legendre field δ is homogeneous of degree μ.

Proof. For $\phi \in T_p^*\mathcal{M}$, let $\eta^{-1}(\phi) \in T_p\mathcal{M}$ be such that $\xi\left(\eta^{-1}(\phi)\right) = \eta^{-1}(\phi, \xi)$ for all $\xi \in T_p^*\mathcal{M}$. By definition of the potential, we can write $\delta = \eta^{-1}(\mathrm{d}h)$.

We take the Lie derivative along E of δ. In general,

$$\begin{aligned}\mathcal{L}_E(\delta) &= \mathcal{L}_E\left(\eta^{-1}(\mathrm{d}h)\right)\\ &= \mathcal{L}_E\left(\eta^{-1}\right)(\mathrm{d}h) + \eta^{-1}\big(\mathcal{L}_E(\mathrm{d}h)\big).\end{aligned}$$

By property 5(d) of Definition 2.3.1, we have

$$\mathcal{L}_E\left(\eta^{-1}\right) = (d-2)\eta^{-1},$$

and by the Cartan formula, we have

$$\begin{aligned}\mathcal{L}_E(\mathrm{d}h) &= \mathrm{d}\left(\iota_E \mathrm{d}h\right) + \iota_E\big(\mathrm{d}(\mathrm{d}h)\big)\\ &= \mathrm{d}\left(E(h)\right).\end{aligned}$$

Therefore,

$$\mathcal{L}_E(\delta) = (d-2)\delta + \eta^{-1}\big(\mathrm{d}E(h)\big). \tag{2.4.26}$$

Let δ be homogeneous of degree $\mu \neq d-2$. Then equation (2.4.16) holds for $X = \delta$, so by (2.4.26) we have

$$\mu\eta^{-1}\,(\mathrm{d}h) = (d-2)\eta^{-1}(\mathrm{d}h) + \eta^{-1}\big(\mathrm{d}E(h)\big),$$

which implies

$$\eta^{-1}\big((\mu - d + 2)\mathrm{d}h - \mathrm{d}E(h)\big) = 0. \tag{2.4.27}$$

Since η^{-1} is nondegenerate, the relation (2.4.27) is equivalent to

$$E(h) = (\mu - d + 2)h + a,$$

where $a \in \mathbb{C}$ is a constant. Then

$$\widetilde{h} = h - \frac{a}{\mu - d + 2}$$

is a potential for δ which satisfies

$$E(\widetilde{h}) = (\mu - d + 2)\widetilde{h}.$$

Conversely, let h be homogeneous of degree $\mu - d + 2$. Then

$$\mathcal{L}_E(h) = E(h) = (\mu - d + 2)h,$$

which we can substitute into (2.4.26) to recover condition (2.4.16). □

2.5 Rational and trigonometric solutions of the WDVV equations

In this section, we present an overview of some important known classes of solutions of the generalised WDVV equations. Recall that polynomial prepotentials are associated to Frobenius manifold structures on the orbit spaces of Coxeter groups (discussed in § 2.3.4), that these are mapped to rational solutions of the WDVV equations by almost duality (§ 2.4.1), and that we have seen some examples of rational solutions being mapped to trigonometric solutions via a Legendre transformation (§ 2.4.3).

We first discuss the wider class of rational solutions to the WDVV equations associated with (rational) $\vee$-systems, within which live the dual prepotentials given by Theorem 2.4.7. Then we discuss some families of trigonometric solutions appearing in the literature, which are also associated with (deformations of) root systems.

2.5.1 Rational solutions from $\vee$-systems

We consider solutions of the WDVV equations of the form

$$F_{\mathcal{A}}(x) = \sum_{\alpha \in \mathcal{A}} \alpha(x)^2 \log \alpha(x), \tag{2.5.1}$$

where $\mathcal{A} \subset V^*$ are a finite set of non-collinear covectors on a complex vector space V. Formula (2.5.1) is a generalisation of the formula given in Theorem 2.4.7 for the dual prepotential associated to a Coxeter group W.

The class of solutions represented by (2.5.1) was already known in four-dimensional Seiberg-Witten theory before Dubrovin's formulation of almost duality for Frobenius manifolds, with early examples appearing in [45]. Martini and Gragert [47] established that such functions satisfy the generalised WDVV equations when $\mathcal{A} = \mathcal{R}$ is the root system of a Weyl group. Veselov [60] then showed that solutions of the form (2.5.1) can be produced using real deformations of root systems known as $\vee$-systems, which include the root systems of Coxeter groups. The definition of $\vee$-systems was later expanded to complex spaces by Feigin and Veselov [27].

Let us assume that V is a complex n-dimensional vector space. Consider a symmetric bilinear form $G_{\mathcal{A}}$ defined on V as follows:

$$G_{\mathcal{A}}(u, v) := \sum_{\alpha \in \mathcal{A}} \alpha(u)\alpha(v),$$

where $u, v \in V$, and we assume that $G_{\mathcal{A}}$ is non-degenerate[6]. The form $G_{\mathcal{A}}$ defines an isomorphism $\varphi_{\mathcal{A}} : V \to V^*$ so we denote

$$\alpha^\vee := \varphi_{\mathcal{A}}^{-1}(\alpha)$$

for any $\alpha \in \mathcal{A}$. Thus we have $G_{\mathcal{A}}(\alpha^\vee, v) = \alpha(v)$ for any $v \in V$.

Moreover, given a function $F = F_{\mathcal{A}}$ as in (2.5.1), the matrix

$$\eta = \sum_i x^i F_i \tag{2.5.2}$$

is (proportional to) the matrix of $G_{\mathcal{A}}$. This can be seen by checking from (2.5.1) that the

[6]This non-degeneracy assumption implies that the elements of $\mathcal{A}$ span V^*.

third-order derivatives of F have the form

$$F_{ijk} = \sum_{\alpha \in \mathcal{A}} \frac{2\alpha(e_i)\alpha(e_j)\alpha(e_k)}{\alpha(x)},$$

where $e_1, \ldots, e_n$ is a basis in V.

Definition 2.5.1 [27]**.** A collection $\mathcal{A} \subset V^*$ is a $\vee$-*system* if the following $\vee$-condition is satisfied for any $\alpha \in \mathcal{A}$ and any 2-dimensional plane Π containing α:

$$\sum_{\beta \in \Pi \cap \mathcal{A}} \beta(\alpha^\vee)\beta^\vee = \lambda \alpha^\vee$$

where $\lambda = \lambda(\alpha, \Pi) \in \mathbb{C}$.

Note that a set of positive roots $\mathcal{A} = \mathcal{R}_+$ for any finite irreducible Coxeter group W is a $\vee$-system [60, Theorem 2] — an explicit proof is provided in [1]. The class of $\vee$-systems is closed under the operations of restriction to a subspace and taking subsystems [27]; this differs from the Coxeter case in that the restriction of a Coxeter root system is not in general a Coxeter root system.

The main result of this theory finds an equivalence between $\mathcal{A}$ satisfying the $\vee$-conditions and $F_{\mathcal{A}}$ satisfying the WDVV equations.

Theorem 2.5.2 [27, Theorem 1]**.** *The function $F_{\mathcal{A}}$ in (2.5.1) is a solution to the WDVV equations with metric η given by (2.5.2) if and only if $\mathcal{A}$ is a $\vee$-system.*

We take the domain of $F_{\mathcal{A}}$ to be $M_{\mathcal{A}} := V \backslash \bigcup_{\alpha \in \mathcal{A}} H_\alpha$, where we set $H_\alpha = \{v \in V | \alpha(v) = 0\}$. It was shown in [27] that the WDVV equations for $F_{\mathcal{A}}$ are equivalent to the associativity of the multiplication on the tangent space $T_pM_{\mathcal{A}}$ defined by

$$u \circ v = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(p)} \alpha^\vee. \tag{2.5.3}$$

Certain multi-parameter deformations of Coxeter root systems, generalising those originally appearing in the theory of generalised quantum Calogero-Moser systems, were shown to be $\vee$-systems. From Chalykh and Veselov's work [11], we define the n-parameter A_n-type $\vee$-system $A_n(k)$ as

$$A_n(k) = \left\{ \sqrt{k_i} e^i | 1 \leq i \leq n \right\} \cup \left\{ \sqrt{k_i k_j}(e^i - e^j) | 1 \leq i < j \leq n \right\} \tag{2.5.4}$$

and the $(n+1)$-parameter B_n-type $\vee$-system $B_n(k)$ as

$$B_n(k) = \left\{ \sqrt{2k_i(k_i + k_0)} e^i | 1 \leq i \leq n \right\} \cup \left\{ \sqrt{k_i k_j}(e^i \pm e^j) | 1 \leq i < j \leq n \right\}, \tag{2.5.5}$$

where $k_0 \in \mathbb{C}$, $k_1, \ldots, k_n \in \mathbb{C}^\times$.

Example 2.5.3. The deformed A_3-type $\vee$-system, where $A_n(k)$ is given by (2.5.4), produces the rational solution

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3} k_i x_i^2 \log x_i + \sum_{1 \leq i < j \leq 3} k_i k_j \left(x_i - x_j\right)^2 \log\left(x_i - x_j\right). \tag{2.5.6}$$

When all the parameters $k_i = 1$, this reduces to the almost dual prepotential associated with A_3, as written in Example 2.2.10.

We are interested in taking Legendre transformations of the rational solutions associated with the $\vee$-systems $A_n(k)$ and $B_n(k)$. Although there is so far no general statement describing how rational solutions transform, we will see that the A_n-type and B_n-type solutions can be mapped to trigonometric solutions.

2.5.2 Trigonometric solutions

The solutions discussed in this section all contain trilogarithmic as well as cubic terms. We will write them in terms of the function

$$f(z) = \frac{1}{6} z^3 - \frac{1}{4} \operatorname{Li}_3\left(e^{-2z}\right). \tag{2.5.7}$$

The polylogarithm Li_n is defined as

$$\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

for $|z| < 1$ and by analytic continuation elsewhere, so that $\operatorname{Li}_1(z) = -\log(1 - z)$. Its derivatives have the following property:

$$\frac{\mathrm{d} \operatorname{Li}_n\left(e^u\right)}{\mathrm{d}\, u} = \operatorname{Li}_{n-1}(e^u).$$

It is also useful to note the second-order derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\big(8 f(\kappa z/2)\big) = \kappa^3 z + 2\kappa^2 \log\left(1 - e^{-\kappa z}\right), \tag{2.5.8}$$

where $\kappa \in \mathbb{C}$.

We consider solutions $F = F_{\mathcal{A}}^{\text{trig}} : V \oplus U \to \mathbb{C}$ of the form

$$F_{\mathcal{A}}^{\text{trig}} = \sum_{\alpha \in \mathcal{A}} c_\alpha f\left(\alpha(x)\right) + Q(x, y),$$

where $x \in V \cong \mathbb{C}^n$, $y \in U \cong \mathbb{C}$, $\mathcal{A} \subset V^*$, $Q(x,y)$ is a cubic polynomial, and $c_\alpha \in \mathbb{C}$ denotes the multiplicity or deformation parameter associated with each covector. Solutions of this form for root systems appeared in [45], arising from five-dimensional Seiberg-Witten theory, and were described in greater detail by Hoevenaars and Martini [35], [36]. In [36], solutions were obtained for $\mathcal{A} = \mathcal{R}$, where $\mathcal{R}$ is the root system of a Weyl group and $c_\alpha = 1 \; \forall \alpha \in \mathcal{R}$. Various families of trigonometric solutions have been studied by Pavlov [49], Riley [52], Bryan and Gholampour [10], and Shen [56]; we will comment on these shortly. First, let us give a brief summary of work by Feigin and Alkadhem in [24], [2] which generalises many of these families of solutions.

2.5.2.1 Solutions from trigonometric $\vee$-systems

Let V be an n-dimensional complex vector space, with a collection of covectors $\mathcal{A} \subset V^*$ belonging to an n-dimensional lattice. The multiplicity function $c : \mathcal{A} \to \mathbb{C}$ gives the multiplicity $c_\alpha = c(\alpha)$ for each covector. We assume that the bilinear form on V given by

$$G_{(\mathcal{A},c)}(u,v) = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(u)\alpha(v),$$

where $u, v \in V$, is non-degenerate. An additional variable $t \in U \cong \mathbb{C}$ is introduced. We choose a basis $e_1, \ldots, e_{n+1}$ for $V \oplus U$ and coordinates $\xi_1, \ldots, \xi_{n+1}$ such that $\xi = (\xi^1, \ldots, \xi^n)$ represents a vector in V and $t = \xi_{n+1}$ represents a vector in U.

We consider the functions $F : V \oplus U \to \mathbb{C}$ of the form

$$F_{\mathcal{A}}^{\text{trig}} = \frac{1}{3}t^3 + \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(\xi)^2 t + \lambda \sum_{\alpha \in \mathcal{A}} c_\alpha f\left(\alpha(\xi)\right), \tag{2.5.9}$$

where $\lambda \in \mathbb{C}^\times$ and $f(z)$ is given by (2.5.7). The matrix $\eta = F_{n+1}$ can then be written

$$\eta = 2\begin{pmatrix} \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha \otimes \alpha & 0 \\ 0 & 1 \end{pmatrix},$$

where we denote by $\alpha \otimes \alpha$ the matrix with $(i,j)^{th}$ entry $\alpha_i \alpha_j$ for $\alpha = \alpha_i e^i$. As in the rational case, we can set $M_{\mathcal{A}}$ to be the complement of the hyperplanes defined by the kernel of $\alpha \in \mathcal{A}$. Then the associativity of the multiplication on $T\left(M_{\mathcal{A}} \oplus U\right)$, defined in the standard way via the third-order derivatives of $F_{\mathcal{A}}^{\text{trig}}$, is equivalent to the WDVV equations being satisfied for $F_{\mathcal{A}}^{\text{trig}}$ with metric $\eta = F_{n+1}$. The multiplicative identity element is given by $e = \partial_{n+1} = \frac{\partial}{\partial t}$.

As in the rational case, there exist a set of geometric conditions for $\mathcal{A}$ which are equivalent to $F_{\mathcal{A}}^{\text{trig}}$ satisfying the WDVV equations; see [2] for full details. These conditions include the requirement that $\mathcal{A}$ is a trigonometric $\vee$-system, which we now define. Let

$\mathcal{A}_\alpha = \{\beta \in \mathcal{A} \,|\, \beta = k\alpha \text{ for } k \in \mathbb{C}\}$. Then the complement $\mathcal{A} \setminus \mathcal{A}_\alpha$ can be represented as

$$\mathcal{A} \setminus \mathcal{A}_\alpha = \bigsqcup_{i=1}^{k} \Gamma_\alpha^i,$$

where $k = k(\alpha) \in \mathbb{N}$. Here, the α-strings Γ_α^i are disjoint collections of covectors such that if $\gamma \in \Gamma_\alpha^i$, then Γ_α^i contains all covectors $\pm\gamma + m\alpha \in \mathcal{A}$ for $m \in \mathbb{Z}$.

Definition 2.5.4 [2, Definition 2.7]**.** The pair $(\mathcal{A}, c)$ is a *trigonometric* $\vee$*-system* if

$$\sum_{\beta \in \Gamma_\alpha^i} c_\beta \alpha\left(\beta^\vee\right)\big(\alpha(u)\beta(v) - \alpha(v)\beta(u)\big) = 0$$

for all $\alpha \in \mathcal{A}$, $i \in \{1, \ldots, k(\alpha)\}$, $u, v \in V$.

Similarly to the rational case, root systems of Weyl groups give rise to a sub-class of trigonometric $\vee$-systems. In this case, the scalar λ in (2.5.9) represents a generalised version of the Coxeter number, as discussed in [2].

Proposition 2.5.5 [2, 24]**.** *Let $\mathcal{R}$ be an irreducible root system for Weyl group W, with a W-invariant multiplicity function c. Then $\mathcal{R}$ is a trigonometric $\vee$-system, and $F_{\mathcal{R}}^{trig}$ given by (2.5.9) satisfies the WDVV equations.*

2.5.2.2 Known families of trigonometric solutions

Hoevenaars and Martini [36] found solutions to the WDVV equations of the form

$$F(\xi_1, \ldots, \xi_n, t) = \frac{\gamma}{6}t^3 + \frac{\gamma}{2}t\langle \xi, \xi\rangle + \frac{1}{2}\sum_{\alpha \in \mathcal{R}} f\left(\alpha(\xi)\right),$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $\mathcal{R}$ is the root system of a Weyl group, and the parameter γ is fixed for each Weyl group. The value of γ was found in [36] for root systems of type A, B, C, D, E, and F.

Hoevenaars and Martini also considered trigonometric solutions with a small number of additional parameters in [35]; of particular relevance is an A-type family, which is dealt with in its most general form only in the preprint version of [35]. We reproduce here Theorem 2.1 from the preprint.

Theorem 2.5.6 [35]**.** *Let $F = F^{HM}$ be the function*

$$F^{HM}(y) = \sum_{1 \le i < j \le n} f\left(y_i - y_j\right) + \frac{a}{6}\left(\sum_{i=1}^{n} y_i\right)^3 + \frac{b}{2}\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{j=1}^{n} y_j^2\right) + \frac{c}{6}\sum_{i=1}^{n} y_i^3 \qquad (2.5.10)$$

with parameters a, b, c such that $nb+c \neq 0$ and $na+2b \neq 0$. Then the matrix $\eta = \sum_{k=1}^{n} F_k$ is constant and nondegenerate, and F satisfies the WDVV equations with respect to η if and only if the following relation holds:

$$b^3 n + 3b^2 c - ac^2 + an^2 + 3bn + c = 0.$$

We discuss a generalisation of this result in Chapter 4.

Pavlov obtained $(n+1)$-parametric families of trigonometric solutions of A_n, B_n, C_n type in [49]. These B_n-, C_n-type solutions were generalised in [2] to a BC_n-type solution of the form (2.5.9). Pavlov's A-type solutions also seem to be related to solutions found in [2], up to unresolved typos in [49].

In [51], Riley used the LG superpotentials of Frobenius manifold structures associated with the extended affine Weyl group of type A_n to find almost dual prepotentials. As noted in Example 2.4.25, the almost dual prepotential is a trigonometric solution. In fact, Riley showed that his solution in the undeformed case is a 2-parametric generalisation of the A-type solution obtained in [36]. By considering superpotentials restricted to the discriminant submanifold (that is, precisely the points where the intersection form is degenerate), Riley obtained an $(n+1)$-parameter family of A_n-type trigonometric solutions. We discuss these in more detail in Chapter 4.

Bryan and Gholampour found Frobenius algebras corresponding to trigonometric solutions for the ADE-type Weyl groups, by studying the quantum cohomology of resolutions of the ADE singularities $\mathbb{C}^2/G$ where G is a finite subgroup of $SU(2)$ [10]. In the same paper, they extended their result to produce a family of Frobenius algebras associated with arbitrary reduced irreducible root systems of rank n. These Frobenius algebras were reconstructed in [56], and the corresponding trigonometric solutions to the WDVV equations were shown to be of the form (2.5.9) in [2].

Shen's results in [56] produce trigonometric solutions, and their corresponding Frobenius algebra structures, from reduced irreducible root systems with multiplicities and additional parameters. Shen notes that these Frobenius algebras include those found by Bryan and Gholampour, and the solutions are, with the exception of his A-type configuration, equivalent to those found in [2]. Shen's A-type solutions are equivalent to those of the form (2.5.10); see comments in [56], and the analysis in Chapter 4.

The solutions of the form (2.5.9) found by Alkadhem and Feigin in [2] include a non-reduced system of BC_n-type that produces an $(n+3)$-parameter family of solutions, generalising earlier results of BCD-type. However, the A-type solutions considered in [56], [51], [35] contain additional cubic terms as compared to (2.5.9). In the notation of the previous section, these are "cross-terms" of the general form $\xi^i \xi^j \xi^k$ for $i, j, k \neq n+1$.

Example 2.5.7. Let $F(x)$ be the A_3 rational solution, as given in Example 2.2.10. Then the Legendre transformation S_1 maps F to the trigonometric solution given in Example 2.2.11, which may equivalently be written as

$$\widehat{F}(y) = y_1^3 - y_1^2(y_2 + y_3) + 3y_1(y_2^2 + y_3^2) - 2y_1y_2y_3 - \frac{5}{3}y_2^3 - \frac{5}{3}y_3^3 \\ + y_2y_3(y_2 + y_3) + \frac{1}{2}\big[f(2y_2) + f(2y_3) + f(2y_2 - 2y_3)\big].$$

Due to the cubic terms without y_1 (which here has the same role as t in (2.5.9)), this solution is not part of the family described by (2.5.9) for the A_2 case. However, it can be written in the form (2.5.10).

Chapter 3

Legendre fields in two dimensions

In this chapter, we consider arbitrary Legendre fields for each of the six 2-dimensional Frobenius manifolds listed in Example 2.3.4. We solve the Legendre field condition (2.4.5) for homogeneous Legendre fields in all six cases. In addition, we find twisted Legendre fields for the almost dual prepotential on $\mathbb{C}^2/A_2$; that is for the almost dual Frobenius manifold to (2.3.5) when $k = 4$. Throughout this chapter, subscripts are used to represent the contravariant components — denoted elsewhere in the thesis as t^i — of a given coordinate system t. The only exception to this is in Section 3.5, where we briefly discuss the Legendre field condition in canonical coordinates for semisimple Frobenius manifolds.

3.1 Legendre fields and their potentials

We discuss here some considerations which are common to five out of the six Frobenius manifolds. The Frobenius manifold described by (2.3.6) differs from the other five, in that its metric is diagonal instead of anti-diagonal. We treat this case separately.

Let $F(t_1, t_2)$ be the prepotential for one of the $2D$ Frobenius manifolds described by (2.3.1)–(2.3.5). Then the metric is given by

$$\eta(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we denote by ∇ the Levi-Civita connection for η. An arbitrary Legendre field

$$\delta = u\partial_1 + v\partial_2$$

satisfies the Legendre field condition

$$\nabla_X \delta = X \circ \nabla_e \delta \tag{3.1.1}$$

for all vector fields $X = x\partial_1 + y\partial_2$, where $u = u(t_1, t_2)$, $v = v(t_1, t_2)$, $x = x(t_1, t_2)$, $y = y(t_1, t_2)$ are some functions. We require throughout that $\delta \neq 0$.

Recall from Example 2.3.4 that the multiplication $\circ$ is defined as

$$\partial_i \circ \partial_j = \begin{cases} \partial_1 & i = j = 1; \\ \partial_2 & i \neq j; \\ c_{222}\partial_1 & i = j = 2; \end{cases} \tag{3.1.2}$$

where $c_{222} = F_{222}$ depends on each Frobenius manifold. If we calculate each side of (3.1.1), we obtain

$$\nabla_X \delta = x\partial_1(u)\partial_1 + y\partial_2(u)\partial_1 + x\partial_1(v)\partial_2 + y\partial_2(v)\partial_2 \tag{3.1.3}$$

and

$$\begin{aligned} X \circ \nabla_e \delta &= (x\partial_1 + y\partial_2) \circ (\partial_1(u)\partial_1 + \partial_1(v)\partial_2) \\ &= x\partial_1(u)\partial_1 + x\partial_1(v)\partial_2 + y\partial_1(u)\partial_2 + c_{222}y\partial_1(v)\partial_1, \end{aligned}$$

where we have used the multiplication given in (3.1.2). Considering the ∂_1 and ∂_2 components of condition (3.1.1) separately, we obtain the two differential equations

$$\frac{\partial u}{\partial t_2} = c_{222}\frac{\partial v}{\partial t_1}, \tag{3.1.4}$$

$$\frac{\partial u}{\partial t_1} = \frac{\partial v}{\partial t_2}. \tag{3.1.5}$$

We can rewrite (3.1.4) in terms of the potential for δ, which is a function $h = h(t_1, t_2)$ such that

$$\delta = \eta^{ij}\partial_j\left(h(t_1, t_2)\right)\partial_i, \tag{3.1.6}$$

or equivalently, $u = \partial_2 h$ and $v = \partial_1 h$. Then (3.1.4) becomes a second-order PDE for h:

$$\frac{\partial^2 h}{\partial t_2^2} = c_{222}\frac{\partial^2 h}{\partial t_1^2}.$$

Note that h is only defined up to a constant.

3.1.1 Homogeneous Legendre fields

As equations (3.1.4) and (3.1.5) may be very hard to solve in the general case, one can impose homogeneity on δ to find a smaller set of solutions. In this case, δ satisfies

$$\mathcal{L}_E \delta = \mu\delta \tag{3.1.7}$$

where E is the Euler field and $\mu \in \mathbb{C}$.

For cases (2.3.1), (2.3.3), (2.3.4), (2.3.5), the Euler field has the same form, namely

$$E = t_1\partial_1 + \alpha t_2 \partial_2 \tag{3.1.8}$$

for some $\alpha \in \mathbb{C}$ depending on the Frobenius manifold in question. We therefore find

$$\begin{aligned}\mathcal{L}_E\delta &= E(\delta) - \delta(E)\\ &= t_1\partial_1(u)\partial_1 + t_1\partial_1(v)\partial_2 + \alpha t_2\partial_2(u)\partial_1 + \alpha t_2\partial_2(v)\partial_2 - u\partial_1 - \alpha v\partial_2.\end{aligned}$$

Then, the homogeneity condition given by equation (3.1.7) becomes

$$t_1\frac{\partial u}{\partial t_1} + \alpha t_2\frac{\partial u}{\partial t_2} = (\mu+1)\,u, \tag{3.1.9}$$

and

$$t_1\frac{\partial v}{\partial t_1} + \alpha t_2\frac{\partial v}{\partial t_2} = (\mu+\alpha)\,v. \tag{3.1.10}$$

It follows, by applying the method of characteristics, that

$$\begin{cases} u\,(t_1,t_2) = t_1^{\mu+1}A(z),\\ v\,(t_1,t_2) = t_1^{\mu+\alpha}B(z),\end{cases} \tag{3.1.11}$$

where $A(z)$, $B(z)$ are functions of $z = t_1^{-\alpha}t_2$. With these substitutions, the Legendre field condition (3.1.4), (3.1.5) becomes the two equations

$$t_1^{2(1-\alpha)}A'(z) = c_{222}(\mu+\alpha)B(z) - c_{222}\alpha z B'(z),$$

and

$$B'(z) = (\mu+1)A(z) - \alpha z A'(z).$$

In cases (2.3.3), (2.3.4), (2.3.5), these equations can be used to produce a hypergeometric differential equation for each of the functions $A(z)$ and $B(z)$. In the trivial case, (2.3.1), equations (3.1.4) and (3.1.5) can be solved directly. In case (2.3.2), the Euler field has a different form, however, a very similar procedure can be used which also results in hypergeometric differential equations for $A(z)$ and $B(z)$.

3.2 Hypergeometric differential equations

We refer to the Digital Library of Mathematical Functions [16] and Erdélyi [23] for the material in this section. Care must be taken when using standard formulas in the literature, as we will be considering hypergeometric differential equations with non-generic parameters.

The hypergeometric differential equation with parameters $a, b, c \in \mathbb{C}$ is the following linear second-order ordinary differential equation for a function $A(z)$:

$$z(1-z)A''(z) + [c - (a+b+1)z]A'(z) - abA(z) = 0. \tag{3.2.1}$$

Since this equation has three regular singular points — at $z = 0, 1, \infty$ — special solutions are considered in the neighbourhoods of each singular point. These solutions are written in terms of the hypergeometric function.

The (ordinary) hypergeometric function is denoted ${}_2F_1\left(a, b; c; z\right)$ and can be written as the power series,

$${}_2F_1\left(a, b; c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{3.2.2}$$

where $(q)_n$ is the (rising) Pochhammer symbol defined by

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)\ldots(q+n-1) & n > 0. \end{cases}$$

When $c \neq 0, -1, -2, -3, \ldots,$ the hypergeometric function is well-defined on $|z| < 1$ and is a solution to the hypergeometric differential equation with the same set of parameters. Note that both the function and the differential equation are symmetric in a and b.

To describe general solutions of the hypergeometric differential equation, we also use the digamma function. The digamma function is denoted $\psi(z)$ and defined as

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the gamma function. We will later use the following properties of the digamma function [23, § 1.7.1]:

$$\psi(1+n) = \sum_{k=1}^{n} \frac{1}{k} - \gamma_{EM} \tag{3.2.3}$$

if $n \in \mathbb{N}^\times$, where γ_{EM} denotes the Euler-Mascheroni constant, and

$$\psi(1+z) = \psi(z) + \frac{1}{z}, \tag{3.2.4}$$

for any $z \in \mathbb{C}^\times$.

3.2.1 Solutions around 0

For non-integer c, all solutions of (3.2.1) in the neighbourhood of $z = 0$ can be given as linear combinations of (essentially) hypergeometric functions. More precisely, from [16, 15.10.2], we have $A(z) = \alpha y_1 + \beta y_2$ for $\alpha, \beta \in \mathbb{C}$, where

$$\begin{aligned} y_1 &= {}_2F_1\left(a, b; c; z\right), \\ y_2 &= z^{1-c}{}_2F_1\left(a-c+1, b-c+1; 2-c; z\right). \end{aligned}$$

We will need to consider solutions of (3.2.1) when $c \in \mathbb{Z}$, which we split into two cases: (i) $c \in \mathbb{N}^\times$, and (ii) $c \in \mathbb{Z}_{\leq 0}$.

In case (i), the function y_1 is well-defined but y_2 is either not well-defined ($c > 1$) or equal to y_1 ($c = 1$). Let $y_3 = G\left(a, b; c; z\right)$, where we define

$$\begin{aligned} G\left(a, b; c; z\right) = {}_2F_1\left(a, b; c; z\right)\log(z) &- \sum_{k=1}^{c-1} \frac{(c-1)!(k-1)!}{(c-k-1)!(1-a)_k(1-b)_k}(-z)^{-k} \\ &+ \sum_{k=0}^{\Lambda} \frac{(a)_k(b)_k}{(c)_k k!} S_k(a,b;c) z^k + H\left(a, b; c; z\right), \end{aligned} \tag{3.2.5}$$

and Λ, S, H are given below. Let $\tilde{a} = \max\left(a, b\right)$ and $\tilde{b} = \min\left(a, b\right)$. Then we define

$$H\left(a, b; c; z\right) = \begin{cases} 0 & a, b \notin \mathbb{Z}_{<c}, \\ (-1)^{-a}(-a)! \displaystyle\sum_{k=1-a}^{\infty} \frac{(k+a-1)!(b)_k}{(c)_k k!} z^k & a \in \mathbb{Z}_{\leq 0},\ b \notin \mathbb{Z}_{<c}, \\ (-1)^{-\tilde{a}}\left(-\tilde{a}\right)! \displaystyle\sum_{k=1-\tilde{a}}^{-\tilde{b}} \frac{(k+\tilde{a}-1)!\left(\tilde{b}\right)_k}{(c)_k k!} z^k & a, b \in \mathbb{Z}_{\leq 0}; \end{cases} \tag{3.2.6}$$

$$S_k\left(a, b; c\right) = \begin{cases} \psi(a+k) + \psi(b+k) - \psi(1+k) - \psi(c+k) & a, b \notin \mathbb{Z}_{<c}, \\ \psi(1-a-k) + \psi(b+k) - \psi(1+k) - \psi(c+k) & a \in \mathbb{Z}_{\leq 0},\ b \notin \mathbb{Z}_{<c}, \\ \psi(1-a-k) + \psi(1-b-k) - \psi(1+k) - \psi(c+k) & a, b \in \mathbb{Z}_{\leq 0}; \end{cases}$$

as well as the summation upper limit

$$\Lambda = \begin{cases} \infty & a, b \notin \mathbb{Z}_{<c}, \\ -a & a \in \mathbb{Z}_{\leq 0},\ b \notin \mathbb{Z}_{<c}, \\ -\tilde{a} & a, b \in \mathbb{Z}_{\leq 0}. \end{cases}$$

These formulas combine [16, Equations 15.10.8–15.10.9]. Note that if $a \notin \mathbb{Z}_{<c}$ and $b \in \mathbb{Z}_{\leq 0}$ then solution y_3 can be written as $y_3 = G(b, a; c; z)$. The general solution to (3.2.1) around $z = 0$ is $A(z) = \alpha y_1 + \beta y_3$, $\alpha, \beta \in \mathbb{C}$, with exceptions when a or $b \in \mathbb{N}^{\times}_{<c}$. These exceptions correspond to degenerate cases dealt with in [23, § 2.2], which are not covered by the above considerations.

In case (ii), y_2 is well-defined while y_1 is not. A second solution is then given by $y_4 = z^{1-c}G(a+1-c, b+1-c; 2-c; z)$, so that the general solution around $z = 0$ is $A(z) = \alpha y_2 + \beta y_4$, $\alpha, \beta \in \mathbb{C}$.

3.2.2 Solutions around the other singular points

The solutions to equation (3.2.1) around 1 (respectively, ∞) can be found by transforming the equation under coordinate shift $z_1 = 1 - z$ (respectively, $z_\infty = \frac{1}{z}$) and solving it around $z_1 = 0$ for the function $A_1(z_1) = A(z)$ (respectively, solving around $z_\infty = 0$ for $A_\infty(z_\infty) = A(z)$). Since (3.2.1) remains a hypergeometric equation under these transformations, the formulas in §3.2.1 can be used to solve for $A_1(z_1)$ and $A_\infty(z_\infty)$.

3.3 Legendre fields for Frobenius manifolds

We now analyse the Legendre field equations for each of the $2D$ Frobenius manifolds (2.3.1)–(2.3.6) in Sections 3.3.1–3.3.6, labelled Cases 1–6 respectively. In Cases 2, 4, and 5, corresponding to Frobenius manifolds (2.3.2), (2.3.4), and (2.3.5) respectively, solutions to the hypergeometric differential equations associated with homogeneous Legendre fields are only found around the singular point 0. In Case 3, corresponding to (2.3.3), solutions are found around all three singular points and the monodromy of these solutions is discussed.

3.3.1 Case 1

The trivial 2D Frobenius manifold has prepotential F given by

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 \tag{3.3.1}$$

and Euler field

$$E = t_1\partial_1.$$

To complete the description of the multiplication given by (3.1.2), we see that $c_{222} = 0$. We consider an arbitrary Legendre field δ with potential h, so that

$$\delta = u(t_1,t_2)\partial_1 + v(t_1,t_2)\partial_2 = \partial_2(h)\partial_1 + \partial_1(h)\partial_2. \tag{3.3.2}$$

Condition (3.1.4) then becomes

$$\frac{\partial u}{\partial t_2} = 0. \tag{3.3.3}$$

Theorem 3.3.1. *A Legendre field δ, as in (3.3.2), has a potential h of the form*

$$h\left(t_1,t_2\right) = t_2h_1\left(t_1\right) + h_2\left(t_1\right), \tag{3.3.4}$$

where h_1 and h_2 are some functions of t_1. The field has components

$$u\left(t_1\right) = h_1\left(t_1\right),$$
$$v\left(t_1,t_2\right) = t_2h_1'(t_1) + h_2'\left(t_1\right).$$

Proof. By (3.3.3), we have $u = h_1(t_1)$. By definition of the potential as in (3.1.6), we also have $u = \partial_2\left(h\right)$ and $v = \partial_1(h)$. Integrating the expression $\partial_2\left(h\right) = h_1(t_1)$, we obtain formula (3.3.4). Substituting (3.3.4) into $v = \partial_1(h)$ produces the formula for v. □

3.3.1.1 Homogeneous Legendre fields

We now consider the case when δ in (3.3.2) is homogeneous of degree $\mu \in \mathbb{C}$.

Theorem 3.3.2. *Let δ given by (3.3.2) be an arbitrary homogeneous Legendre field of degree μ. Then its potential is*

$$h\left(t_1,t_2\right) = t_1^{\mu+1}\left(c_1t_2 + c_2\right)$$

and its components are given by

$$u\left(t_1\right) = c_1t_1^{\mu+1},$$
$$v\left(t_1,t_2\right) = (\mu+1)t_1^{\mu}\left(c_1t_2 + c_2\right),$$

where $c_1, c_2 \in \mathbb{C}$ are free parameters.

Proof. Imposing homogeneity on δ via (3.1.9), (3.1.10) produces the equations

$$t_1 u'(t_1) = (\mu+1)u(t_1), \tag{3.3.5}$$

$$t_1 \frac{\partial v(t_1,t_2)}{\partial t_1} = \mu v(t_1,t_2). \tag{3.3.6}$$

Equation (3.3.5) gives $u = c_1 t_1^{\mu+1}$. By Theorem 3.3.1, we have the general form of h, with $h_1 = u$, along with

$$v = c_1(\mu+1)t_1^{\mu} t_2 + h_2'(t_1) \tag{3.3.7}$$

for some function $h_2(t_1)$. The substitution of (3.3.7) into (3.3.6) gives

$$t_1 h_2''(t_1) = \mu h_2'(t_1),$$

which can be solved for

$$h_2(t_1) = c_2 t_1^{\mu+1}.$$

□

3.3.2 Case 2

The Frobenius manifold with prepotential given by

$$F(t_1,t_2) = \frac{1}{2}t_1^2 t_2 + e^{\frac{2}{r}t_2} \tag{3.3.8}$$

has Euler field

$$E = t_1\partial_1 + r\partial_2,$$

and

$$c_{222} = \frac{8}{r^3}e^{\frac{2}{r}t_2}.$$

Condition (3.1.4) applied to $\delta = u\partial_1 + v\partial_2$ then becomes

$$\frac{\partial u}{\partial t_2} = \frac{8}{r^3}e^{\frac{2}{r}t_2}\frac{\partial v}{\partial t_1}. \tag{3.3.9}$$

3.3.2.1 Homogeneous Legendre fields

Note that the following results partially reproduce and expand on Example 4.2 in [59], which deals with the case $r = 2$.

Let us impose homogeneity on δ. The homogeneity condition (3.1.7) produces the equations

$$t_1 \frac{\partial u}{\partial t_1} + r \frac{\partial u}{\partial t_2} = (\mu + 1)\, u,$$
$$t_1 \frac{\partial v}{\partial t_1} + r \frac{\partial v}{\partial t_2} = \mu v.$$

Using the method of characteristics, we get

$$\begin{aligned} u\,(t_1, t_2) &= t_1^{\mu+1} U(\omega), \\ v\,(t_1, t_2) &= t_1^{\mu} V(\omega), \end{aligned} \tag{3.3.10}$$

where $\omega = \frac{8}{r} t_1^{-2} e^{\frac{2}{r} t_2}$ and $U(\omega), V(\omega)$ are some functions. With these substitutions, the Legendre field condition given by (3.1.5) and (3.3.9) becomes the two ordinary differential equations,

$$U'(\omega) = \frac{\mu}{2r} V(\omega) - \frac{\omega}{r} V'(\omega), \tag{3.3.11}$$
$$V'(\omega) = \frac{(\mu+1)r}{2\omega} U(\omega) - rU'(\omega). \tag{3.3.12}$$

Theorem 3.3.3. *Let $\delta = u\partial_1 + v\partial_2$, with u, v given by (3.3.10), be an arbitrary homogeneous Legendre field of degree μ. Then $U(\omega)$ satisfies*

$$\omega(1-\omega)U''(\omega) + \frac{2\mu - 1}{2} \omega U'(\omega) - \frac{\mu(\mu+1)}{4} U(\omega) = 0, \tag{3.3.13}$$

and $V(\omega)$ satisfies

$$\omega(1-\omega)V''(\omega) + \left(1 + \frac{2\mu - 3}{2}\omega\right) V'(\omega) - \frac{\mu(\mu-1)}{4} V(\omega) = 0. \tag{3.3.14}$$

Proof. Differentiating (3.3.11) and (3.3.12) with respect to ω produces

$$U''(\omega) = \frac{\mu - 2}{2r} V'(\omega) - \frac{\omega}{r} V''(\omega), \tag{3.3.15}$$
$$V''(\omega) = -\frac{(\mu+1)r}{2\omega^2} U(\omega) + \frac{(\mu+1)r}{2\omega} U'(\omega) - rU''(\omega). \tag{3.3.16}$$

Equations (3.3.12) and (3.3.16) can be used to eliminate V in (3.3.15) and produce the hypergeometric equation (3.3.13). Similarly, U can be eliminated from (3.3.16) to produce (3.3.14). □

Equations (3.3.13) and (3.3.14) are instances of the hypergeometric equation (3.2.1) with $(a, b, c, z) = \left(-\frac{\mu}{2}, -\frac{\mu+1}{2}, 0, \omega\right)$ and $(a, b, c, z) = \left(-\frac{\mu}{2}, \frac{1-\mu}{2}, 1, \omega\right)$, respectively. All homogeneous Legendre fields for F can be described in terms of solutions to (3.3.13) and

(3.3.14). We now find solutions around $\omega = 0$ for general $\mu \notin \{1, 0, -1\}$; these solutions are divided into two cases depending on whether $\mu \notin \mathbb{N}$ or $\mu \in \mathbb{N}$.

Theorem 3.3.4. *Let $\delta = u\partial_1 + v\partial_2$, with u, v given by (3.3.10), be a homogeneous Legendre field of degree $\mu \notin \mathbb{N} \cup \{-1\}$. Then $U = c_1 y_1 + c_2 y_2$ and $V = \frac{2r}{\mu}(c_1 y_3 + c_2 y_4)$ in the neighbourhood of $\omega = 0$, where*

$$y_1(\omega) = \omega\, {}_2F_1\left(\frac{2-\mu}{2}, \frac{1-\mu}{2}; 2; \omega\right),$$

$$y_2(\omega) = y_1(\omega)\log\omega + \frac{4}{\mu(\mu+1)} + \sum_{k-0}^{\infty} \frac{\left(\frac{2-\mu}{2}\right)_k \left(\frac{1-\mu}{2}\right)_k}{(k-1)!k!}\omega^{k+1} \times$$
$$\left[\psi\left(\frac{2-\mu}{2}+k\right) + \psi\left(\frac{1-\mu}{2}+k\right) - \psi(1+k) - \psi(2+k)\right],$$

$$y_3(\omega) = {}_2F_1\left(-\frac{\mu}{2}, \frac{1-\mu}{2}; 1; \omega\right),$$

$$y_4(\omega) = y_3(\omega)\log\omega + \sum_{k=0}^{\infty} \frac{\left(-\frac{\mu}{2}\right)_k \left(\frac{1-\mu}{2}\right)_k}{(k!)^2}\omega^k \left[\psi\left(k-\frac{\mu}{2}\right) + \psi\left(\frac{1-\mu}{2}+k\right) - 2\psi(1+k)\right],$$

and $c_1, c_2 \in \mathbb{C}$ are free parameters.

Proof. To solve (3.3.13), we refer to case (ii) in § 3.2.1, since $c = 0$. A first solution is given by $y_1 = \omega\, {}_2F_1\left(\frac{2-\mu}{2}, \frac{1-\mu}{2}; 2; \omega\right)$ and a second solution is $y_2 = \omega G\left(\frac{2-\mu}{2}, \frac{1-\mu}{2}; 2; \omega\right)$ where the function G is defined in (3.2.5). We have $\mu \notin \mathbb{N} \cup \{-1\}$, so $\frac{2-\mu}{2}, \frac{1-\mu}{2} \neq 1, 0, -1, \ldots$, and the correction term $H\left(\frac{2-\mu}{2}, \frac{1-\mu}{2}; 2; \omega\right)$ given in (3.2.6) is equal to zero.

For (3.3.14), we have $c = 1$ so case (i) in § 3.2.1 is applicable. A first solution is then $y_3 = {}_2F_1\left(-\frac{\mu}{2}, \frac{1-\mu}{2}; 1; \omega\right)$ and a second solution is $y_4 = G\left(-\frac{\mu}{2}, \frac{1-\mu}{2}; 1; \omega\right)$. Since $\mu \notin \mathbb{N} \cup \{-1\}$, we have $-\frac{\mu}{2}, \frac{1-\mu}{2} \neq 0, -1, -2, \ldots$, and the correction term $H\left(-\frac{\mu}{2}, \frac{1-\mu}{2}; 1; \omega\right)$ is again equal to zero.

We get that $U(\omega) = c_1 y_1 + c_2 y_2$ and $V(\omega) = c_3 y_3 + c_4 y_4$ for y_1, y_2, y_3, y_4 as above and some $c_1, c_2, c_3, c_4 \in \mathbb{C}$. Let us now relate these constants by using (3.3.11) to compare similar terms in the series expansions with respect to ω on both sides of the equation. First, we consider terms of the form $\log\omega$ arising in (3.3.11). Using the definition of the hypergeometric function, (3.2.2), and noting that no terms of the form $\alpha\log\omega$, $\alpha \in \mathbb{C}$, can appear in the expansion of $\omega V'(\omega)$, we find

$$c_2\log\omega + g_1(\omega) = \frac{\mu c_4}{2r}\log\omega + g_2(\omega)$$

where the $g_i(\omega)$ are analytic at $\omega = 0$, which implies that

$$c_4 = \frac{2r}{\mu}c_2. \tag{3.3.17}$$

Next, we consider the constant terms arising on both sides of (3.3.11). Note that the constant terms in $U'(\omega)$ are produced by the linear and $\omega \log \omega$ terms in $U(\omega)$. We have

$$U'(\omega) = c_1 + c_2 f(\mu) + h_1(\omega), \tag{3.3.18}$$

where

$$f(\mu) = 1 + \psi\left(\frac{2-\mu}{2}\right) + \psi\left(\frac{1-\mu}{2}\right) - \psi(1) - \psi(2), \tag{3.3.19}$$

and $h_i(\omega)$ are linear combinations of terms of the form ω^k, $k \geq 1$, and $\omega^l \log \omega, l \geq 0$. We also have

$$\omega V'(\omega) = c_4 + h_2(\omega), \tag{3.3.20}$$

and

$$\begin{aligned} V(\omega) &= c_3 + c_4\left[\psi\left(-\frac{\mu}{2}\right) + \psi\left(\frac{1-\mu}{2}\right) - 2\psi(1)\right] + h_3(\omega) \\ &= c_3 + c_4\left[f(\mu) + \frac{2}{\mu}\right] + h_3(\omega), \end{aligned} \tag{3.3.21}$$

where we have used (3.2.4) and (3.3.19) to rewrite the digamma functions. Substituting (3.3.17), (3.3.18), (3.3.20), and (3.3.21) into (3.3.11) produces

$$c_3 = \frac{2r}{\mu} c_1.$$

□

The analogous statement for $\mu \in \mathbb{N}$ is as follows.

Theorem 3.3.5. *Let $\delta = u\partial_1 + v\partial_2$ with u, v as in (3.3.10) be an arbitrary homogeneous Legendre field of degree $\mu \in \mathbb{N} \setminus \{0, 1\}$. Define $m, l \in \mathbb{Z}_{\leq 0}$ and $p, q \notin \mathbb{Z}$ such that*

$$(m, p, l, q) = \begin{cases} \left(\frac{2-\mu}{2}, \frac{1-\mu}{2}, -\frac{\mu}{2}, \frac{1-\mu}{2}\right) & \text{if } \mu \text{ is even} \\ \left(\frac{1-\mu}{2}, \frac{2-\mu}{2}, \frac{1-\mu}{2}, -\frac{\mu}{2}\right) & \text{if } \mu \text{ is odd.} \end{cases}$$

Then, $U = c_1 y_1 + c_2 y_2$ *and* $V = \frac{2r}{\mu}\left(c_1 y_3 + c_2 y_4\right)$ *in the neighbourhood of* $\omega = 0$*, where*

$$
\begin{aligned}
y_1(\omega) &= \omega_2 F_1\left(m, p; 2; \omega\right), \\
y_2(\omega) &= y_1(\omega)\log\omega + \frac{4}{\mu(\mu+1)} + (-1)^{-m}(-m)! \sum_{k=1-m}^{\infty} \frac{(k+m-1)!(p)_k}{(k-1)!k!}\omega^{k+1} \\
&\quad + \sum_{k-0}^{-m} \frac{(m)_k\,(p)_k}{(k-1)!k!}\omega^{k+1}\left[\psi\left(1-m-k\right) + \psi\left(p+k\right) - \psi\left(1+k\right) - \psi\left(2+k\right)\right], \\
y_3(\omega) &= {}_2F_1\left(l, q; 1; \omega\right), \\
y_4(\omega) &= y_3(\omega)\log\omega + (-1)^{-l}(-l)! \sum_{k=1-l}^{\infty} \frac{(k+l-1)!(q)_k}{(k!)^2}\omega^k \\
&\quad + \sum_{k-0}^{-l} \frac{(l)_k\,(q)_k}{(k!)^2}\omega^k\left[\psi\left(1-l-k\right) + \psi\left(q+k\right) - 2\psi\left(1+k\right)\right],
\end{aligned}
$$

and $c_1, c_2 \in \mathbb{C}$ *are free parameters.*

Proof. We use § 3.2.1 to find the following solutions. To solve (3.3.13), we use case (ii) and find $y_1 = \omega\,{}_2F_1\left(\frac{2-\mu}{2}, \frac{1-\mu}{2}; 2; \omega\right)$. Note that $\{a+1, b+1\} = \left\{\frac{2-\mu}{2}, \frac{1-\mu}{2}\right\} = \{m, p\}$ where $m \in \mathbb{Z}_{\le 0}$, $p \notin \mathbb{Z}$. Hence, $y_2 = \omega G\left(m, p; 2; \omega\right)$, and we get $U(\omega) = c_1 y_1 + c_2 y_2$ for $c_1, c_2 \in \mathbb{C}$.

Similarly, using case (i) to solve (3.3.14) leads to $y_3 = {}_2F_1\left(-\frac{\mu}{2}, \frac{1-\mu}{2}; 1; \omega\right)$ and $y_4 = G\left(l, q; 1; \omega\right)$ where $\left\{-\frac{\mu}{2}, \frac{1-\mu}{2}\right\} = \{l, q\}$ with $l \in \mathbb{Z}_{\le 0}$, $q \notin \mathbb{Z}$. We have $V(\omega) = c_3 y_3 + c_4 y_4$ for $c_3, c_4 \in \mathbb{C}$.

Using the same method as in the proof of Theorem 3.3.4, we find

$$c_3 = \frac{2r}{\mu}c_1 \text{ and } c_4 = \frac{2r}{\mu}c_2.$$

□

We now deal with the three values of μ excluded in Theorems 3.3.4 and 3.3.5: namely $\mu \in \{-1, 0, 1\}$. For these values, the last term in one or both of (3.3.13), (3.3.14) vanishes. We find the solutions $U(\omega)$, $V(\omega)$ as well as the components and potential of the Legendre field in these cases.

Theorem 3.3.6. *Let* $\delta = u\partial_1 + v\partial_2$*, with* u, v *given by (3.3.10), be a homogeneous Legendre field of degree* $\mu \in \{-1, 0, 1\}$*. Then the components* u *and* v *of* δ *are given in Table 3.1, where*

$$s = \sqrt{1 - \frac{8}{rt_1^2}e^{\frac{2}{r}t_2}}$$

and $c_1, c_2 \in \mathbb{C}$*. The potentials* h *such that* $u = \partial_2(h)$ *and* $v = \partial_1(h)$ *are given in Table 3.2.*

Proof. Case 1: $\mu = -1$.

Table 3.1: Components of a homogeneous Legendre field in Case 2.

Degree μ	**First component** u	**Second component** v
-1	$\frac{c_1}{s} + c_2$	$-\frac{c_1 r}{t_1 s}$
0	$c_1 t_1 s$	$-c_1 r \operatorname{arctanh}(s)$
1	$\frac{c_1}{2r}\left(t_1^2 s - \frac{8}{r} e^{\frac{2}{r}t_2} \operatorname{arctanh}(s)\right) + \frac{4c_2}{r^2} e^{\frac{2}{r}t_2}$	$c_1 t_1\big(s - \operatorname{arctanh}(s)\big) + c_2 t_1$

Components $u = u(t_1, t_2)$, $v = v(t_1, t_2)$ for a Legendre field $\delta = u\partial_1 + v\partial_2$ of degree μ, with $c_1, c_2 \in \mathbb{C}$.

Table 3.2: Potential function of a homogeneous Legendre field in Case 2.

Degree μ	**Potential** h
-1	$-c_1 r \operatorname{arctanh}(s) + c_2 t_2$
0	$c_1 r t_1\big(s - \operatorname{arctanh}(s)\big)$
1	$\frac{c_1}{4}\left(3t_1^2 s - 2\left(t_1^2 + \frac{4}{r} e^{\frac{2}{r}t_2}\right) \operatorname{arctanh}(s)\right) + \frac{c_2}{2}\left(t_1^2 + \frac{4}{r} e^{\frac{2}{r}t_2}\right)$

Potential $h = h(t_1, t_2)$ for a Legendre field δ of degree μ. Constants $c_1, c_2 \in \mathbb{C}$ are as in Table 3.1.

Equation (3.3.13) becomes

$$2\omega(\omega - 1)U''(\omega) + 3\omega U'(\omega) = 0.$$

Substituting $Y(\omega) = U'(\omega)$, we can rearrange this to get

$$\int \frac{Y'(\omega)}{Y(\omega)} \mathrm{d}\omega = \frac{3}{2} \int \frac{1}{1-\omega} \mathrm{d}\omega$$

which has the solution

$$Y(\omega) = \frac{c_1}{(1-\omega)^{3/2}},$$

with $c_1 \in \mathbb{C}$. After rescaling the constant c_1, this gives us

$$U(\omega) = \frac{c_1}{\sqrt{1-\omega}} + c_2,$$

for $c_1, c_2 \in \mathbb{C}$. Substituting $U(\omega)$ into (3.3.11), (3.3.12) produces

$$V(\omega) = -\frac{c_1 r}{\sqrt{1-\omega}}.$$

By definition of U, V in (3.3.10), we have $u = U(\omega)$ and $v = t_1^{-1} V(\omega)$ where $\omega = \frac{8}{r} t_1^{-2} e^{\frac{2}{r}t_2}$.

The formulas for u, v, h can then be verified.

Case 2: $\mu = 0$.

Equation (3.3.13) becomes

$$2\omega(\omega-1)U''(\omega) + \omega U'(\omega) = 0.$$

By the same methods as in the previous case, we find

$$U(\omega) = c_1\sqrt{1-\omega} + c_2,$$

which we substitute into (3.3.12) to obtain

$$V(\omega) = -c_1 r \text{arctanh}\left(\sqrt{1-\omega}\right) + \frac{c_2 r}{2}\log\omega.$$

However, for U, V to satisfy (3.3.11), we must set $c_2 = 0$: so we have

$$U(\omega) = c_1\sqrt{1-\omega}, V(\omega) = -c_1 r \text{arctanh}\left(\sqrt{1-\omega}\right).$$

From the relations in (3.3.10), the given formulas for u, v, h may be verified.

Case 3: $\mu = 1$.

In this case we start with equation (3.3.14), which becomes

$$2\omega(1-\omega)V''(\omega) + (2-\omega)V'(\omega) = 0.$$

Setting $Y(\omega) = V'(\omega)$, this produces

$$\int \frac{Y'(\omega)}{Y(\omega)}\mathrm{d}\omega = \frac{1}{2}\int \frac{\omega-2}{\omega-\omega^2}\mathrm{d}\omega \tag{3.3.22}$$

The substitution $x = \omega - \omega^2$ can be used to integrate the right-hand side (3.3.22) as follows:

$$\frac{1}{2}\int \frac{\omega-2}{\omega-\omega^2}\mathrm{d}\omega = -\frac{3}{4}\int \frac{1}{\omega-\omega^2}\mathrm{d}\omega - \frac{1}{4}\int \frac{1}{x}\mathrm{d}x = \log\frac{c_1\sqrt{1-\omega}}{\omega}.$$

Equation (3.3.22) therefore has the solution

$$Y(\omega) = \frac{c_1\sqrt{1-\omega}}{\omega},$$

which gives us

$$V(\omega) = c_1\left(\sqrt{1-\omega} - \text{arctanh}\left(\sqrt{1-\omega}\right)\right) + c_2.$$

Substitution into (3.3.11), (3.3.12) produces

$$U(\omega) = \frac{c_1}{2r}\left(\sqrt{1-\omega} - \omega \mathrm{arctanh}\left(\sqrt{1-\omega}\right)\right) + \frac{c_2}{2r}\omega.$$

Once again, the relations given by (3.3.10) can be used to check the formulas provided for u, v, h. □

3.3.3 Case 3

The Frobenius manifold with prepotential

$$F(t_1, t_2) = \frac{1}{2}t_1^2 t_2 + \log t_2 \tag{3.3.23}$$

has

$$c_{222} = \frac{2}{t_2^3}$$

and Euler field

$$E = t_1\partial_1 - 2t_2\partial_2,$$

that is $\alpha = -2$. Condition (3.1.4) therefore becomes

$$\frac{\partial u}{\partial t_2} = \frac{2}{t_2^3}\frac{\partial v}{\partial t_1}. \tag{3.3.24}$$

3.3.3.1 Homogeneous Legendre fields

For the homogeneous case, we represent the Legendre field δ as

$$\delta = u(t_1, t_2)\partial_1 + v(t_1, t_2)\partial_2 = t_1^{\mu+1}U(\omega)\partial_1 + t_1^{\mu-2}V(\omega)\partial_2, \tag{3.3.25}$$

where $\omega = 8t_1^{-2}t_2^{-1}$, μ is the degree of homogeneity, and U, V are related to A, B in (3.1.11) by $U(\omega) = A(z)$ and $V(\omega) = B(z)$. The Legendre field condition, given by equations (3.1.4) and (3.1.5), can then be restated as

$$U'(\omega) = \frac{2-\mu}{32}\omega V(\omega) + \frac{\omega^2}{16}V'(\omega), \tag{3.3.26}$$

$$V'(\omega) = -\frac{8(\mu+1)}{\omega^2}U(\omega) + \frac{16}{\omega}U'(\omega). \tag{3.3.27}$$

Theorem 3.3.7. *Let δ given by (3.3.25) be an arbitrary homogeneous Legendre field. Then $U(\omega)$ satisfies*

$$\omega(1-\omega)U''(\omega) + \left(\frac{2\mu-1}{2}\omega - 1\right)U'(\omega) - \frac{\mu(\mu+1)}{4}U(\omega) = 0, \tag{3.3.28}$$

and $V(\omega)$ satisfies

$$\omega(1-\omega)V''(\omega)+\left(2-\frac{7-2\mu}{2}\omega\right)V'(\omega)-\frac{(\mu-2)(\mu-3)}{4}V(\omega)=0. \tag{3.3.29}$$

Proof. Differentiating (3.3.26) and (3.3.27) with respect to ω produces

$$U''(\omega)=\frac{2-\mu}{32}V(\omega)+\frac{6-\mu}{32}\omega V'(\omega)+\frac{\omega^2}{16}V''(\omega), \tag{3.3.30}$$

$$V''(\omega)=\frac{16(\mu+1)}{\omega^3}U(\omega)-\frac{8(\mu+3)}{\omega^2}U'(\omega)+\frac{16}{\omega}U''(\omega). \tag{3.3.31}$$

Equations (3.3.26), (3.3.27) and (3.3.31) can be used to eliminate V in (3.3.30) and produce the hypergeometric equation in terms of U, with parameters $\{a,b\}=\{-\frac{\mu}{2},-\frac{\mu+1}{2}\}$, $c=-1$. Similarly, U can be eliminated from (3.3.31) to produce the hypergeometric equation for V with parameters $\{a,b\}=\{\frac{2-\mu}{2},\frac{3-\mu}{2}\}$, $c=2$. $\square$

Equations (3.3.28) and (3.3.29) are instances of the hypergeometric equation, (3.2.1), with $(a,b,c,z)=\left(-\frac{\mu}{2},-\frac{\mu+1}{2},-1,\omega\right)$ and $(a,b,c,z)=\left(\frac{2-\mu}{2},\frac{3-\mu}{2},2,\omega\right)$ respectively. We can now describe all homogeneous Legendre fields in terms of solutions to equations (3.3.28) and (3.3.29). We first consider solutions for $\mu\notin\{-1,0,1,2,3\}$ in the neighbourhoods of the singular points $\omega=0,1,\infty$; these are dealt with in Theorems 3.3.8–3.3.11. In these cases, we use formulas given by [16, §15.10], as partially reproduced above in §3.2.1.

When $\mu\in\{-1,0,1,2,3\}$, either the last term vanishes in one of (3.3.28), (3.3.29), or the resulting parameters $\{a,b,c\}$ for one or both differential equations correspond to one of the degenerate cases discussed in [23, §2.2]. We deal with these special cases separately in Theorem 3.3.14.

We begin by finding solutions around $\omega=0$ for $\mu\notin\{-1,0,1,2,3\}$. Solutions here are split into the two cases $\mu\notin\mathbb{N}$ and $\mu\in\mathbb{N}$, dealt with respectively in Theorems 3.3.8 and 3.3.9. The treatment of solutions around 1 and ∞, in Theorems 3.3.10 and 3.3.11 respectively, is more condensed.

Theorem 3.3.8. *Let δ given by (3.3.25), with $\mu\notin\mathbb{N}\cup\{-1\}$, be a homogeneous Legendre* *field. Then,* $U = c_1 y_1 + c_2 y_2$ *and* $V = c_1 y_3 + c_2 y_4$ *in the neighbourhood of* $\omega = 0$*, where*

$$y_1(\omega) = \frac{2-\mu}{64}\omega^2\, {}_2F_1\left(\frac{4-\mu}{2}, \frac{3-\mu}{2}; 3; \omega\right),$$

$$y_2(\omega) = y_1(\omega)\log\omega + \frac{1}{2(\mu-1)}\left(\frac{1}{\mu(\mu+1)} - \frac{\omega}{4}\right) + \frac{2-\mu}{64}\sum_{k=0}^{\infty}\frac{\left(\frac{4-\mu}{2}\right)_k\left(\frac{3-\mu}{2}\right)_k}{(3)_k k!}\omega^{k+2}\times$$
$$\left[\psi\left(\frac{4-\mu}{2}+k\right) + \psi\left(\frac{3-\mu}{2}+k\right) - \psi(1+k) - \psi(3+k)\right],$$

$$y_3(\omega) = {}_2F_1\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; 2; \omega\right),$$

$$y_4(\omega) = y_3(\omega)\log\omega + \frac{4}{\mu(\mu-1)\omega} + \sum_{k-0}^{\infty}\frac{\left(\frac{2-\mu}{2}\right)_k\left(\frac{3-\mu}{2}\right)_k}{(k-1)!k!}\omega^k\times$$
$$\left[\psi\left(\frac{2-\mu}{2}+k\right) + \psi\left(\frac{3-\mu}{2}+k\right) - \psi(1+k) - \psi(2+k)\right],$$

and $c_1, c_2 \in \mathbb{C}$ *are free parameters.*

Proof. To solve (3.3.28), we refer to case (ii) in § 3.2.1, since $c = -1$. A first solution is given by $y_1 = \omega^2\, {}_2F_1\left(\frac{4-\mu}{2}, \frac{3-\mu}{2}; 3; \omega\right)$ and the second solution is $y_2 = \omega^2 G\left(\frac{4-\mu}{2}, \frac{3-\mu}{2}; 3; \omega\right)$ where the function G is defined in (3.2.5). We have $\mu \notin \mathbb{N} \cup \{-1\}$, so $\frac{4-\mu}{2}, \frac{3-\mu}{2} \neq 2, 1, 0, -1, \ldots$ and the correction term given in (3.2.6) is equal to zero.

For (3.3.29), we have $c = 2$ so case (i) in § 3.2.1 is applicable. A first solution is then $y_3 = {}_2F_1\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; 2; \omega\right)$ and a second solution is $y_4 = G\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; 2; \omega\right)$. Since $\mu \notin \mathbb{N} \cup \{-1\}$, we have $a, b \neq 1, 0, -1, -2, \ldots$, so the correction term H is again equal to zero.

Writing $U(\omega) = c_1 y_1 + c_2 y_2$ and $V(\omega) = c_3 y_3 + c_4 y_4$ for y_1, y_2, y_3, y_4 as above and arbitrary $c_1, c_2, c_3, c_4 \in \mathbb{C}$, we can relate these constants by using (3.3.26) to compare coefficients of terms in the series expansions with respect to ω on both sides of the equation. First, we consider terms of the form $\omega\log\omega$ arising in (3.3.26). Using the definition of the hypergeometric function, (3.2.2), and noting that no terms of the form $\omega\log\omega$ can appear in the expansion of $\omega^2 V'(\omega)$, we find

$$2c_2\omega\log\omega = \frac{2-\mu}{32}c_4\omega\log\omega,$$

equivalently,

$$c_2 = \frac{2-\mu}{64}c_4. \tag{3.3.32}$$

Next, we consider linear terms in ω arising on both sides of (3.3.26). We have

$$U'(\omega) = \Big[2c_1 + c_2\big(2f(\mu)+1\big)\Big]\omega + \text{nonlinear terms}, \tag{3.3.33}$$

where we define

$$f(\mu) = \psi\left(\frac{4-\mu}{2}\right) + \psi\left(\frac{3-\mu}{2}\right) - \psi(1) - \psi(3). \tag{3.3.34}$$

Note that the linear terms in $U'(\omega)$ are produced by the quadratic and $\omega^2 \log \omega$ terms in $U(\omega)$. We also have

$$\omega^2 V'(\omega) = c_4 \omega + \text{nonlinear terms},$$

and

$$\begin{aligned} \omega V(\omega) &= \left[c_3 + c_4\left(\psi\left(\frac{2-\mu}{2}\right) + \psi\left(\frac{3-\mu}{2}\right) - \psi(1) - \psi(2)\right)\right]\omega + \text{nonlinear terms} \\ &= \left[c_3 + c_4\left(f(\mu) - \frac{2}{2-\mu} + \frac{1}{2}\right)\right]\omega + \text{nonlinear terms}, \end{aligned} \tag{3.3.35}$$

where we have used (3.2.3), (3.2.4), and (3.3.34) to rewrite the digamma functions. Substituting (3.3.32), (3.3.33), and (3.3.35) into (3.3.26) produces

$$c_1 = \frac{2-\mu}{64} c_3.$$

Constants c_3 and c_4 are renamed and functions y_1, y_2 are rescaled in the theorem statement.

□

In the next statement, we deal with most of the remaining values of μ not covered by Theorem 3.3.8.

Theorem 3.3.9. *Let δ given by (3.3.25), with $\mu \in \mathbb{N} \setminus \{0,1,2,3\}$, be a homogeneous Legendre field. Define $m, l \in \mathbb{Z}_{\leq 0}$ and $p, q \notin \mathbb{Z}$ such that*

$$(m,p,l,q) = \begin{cases} \left(\frac{4-\mu}{2}, \frac{3-\mu}{2}, \frac{2-\mu}{2}, \frac{3-\mu}{2}\right) & \textit{if } \mu \textit{ is even} \\ \left(\frac{3-\mu}{2}, \frac{4-\mu}{2}, \frac{3-\mu}{2}, \frac{2-\mu}{2}\right) & \textit{if } \mu \textit{ is odd.} \end{cases}$$

Then, $U = c_1y_1 + c_2y_2$ *and* $V = c_1y_3 + c_2y_4$ *in the neighbourhood of* $\omega = 0$*, where*

$$\begin{aligned}
y_1(\omega) &= \frac{2-\mu}{64}\omega^2\, {}_2F_1(m,p;3;\omega),\\
y_2(\omega) &= y_1(\omega)\log\omega + \frac{1}{\mu-1}\left(\frac{1}{\mu(\mu+1)} - \frac{\omega}{8}\right)\\
&\quad + \frac{2-\mu}{64}\sum_{k=0}^{-m}\frac{(m)_k(p)_k}{(3)_k k!}\omega^{k+2}\big[\psi(1-m-k)+\psi(p+k)-\psi(1+k)-\psi(3+k)\big]\\
&\qquad + \frac{(2-\mu)(-1)^{-m}(-m)!}{64}\sum_{k=1-m}^{\infty}\omega^{k+2}\frac{(k+m-1)!(p)_k}{(3)_k k!},\\
y_3(\omega) &= {}_2F_1(l,q;2;\omega),\\
y_4(\omega) &= y_3(\omega)\log\omega + \frac{4}{\mu(\mu-1)\omega} + (-1)^{-l}(-l)!\sum_{k=1-l}^{\infty}\omega^k\frac{(k+l-1)!(q)_k}{(k-1)!k!}\\
&\quad + \sum_{k-0}^{-l}\frac{(l)_k(q)_k}{(k-1)!k!}\omega^k\big[\psi(1-l-k)+\psi(q+k)-\psi(1+k)-\psi(2+k)\big],
\end{aligned}$$

and c_1*,* $c_2 \in \mathbb{C}$ *are free parameters.*

Proof. We use § 3.2.1 to find the following solutions. To solve (3.3.28), we use case (ii) and find $y_1 = \omega^2\,{}_2F_1\left(\frac{4-\mu}{2}, \frac{3-\mu}{2}; 3; \omega\right)$ as before. Since $\mu = 4, 5, 6, 7, \ldots$ and $\{a+2, b+2\} = \left\{\frac{4-\mu}{2}, \frac{3-\mu}{2}\right\}$, we always have $\left\{\frac{4-\mu}{2}, \frac{3-\mu}{2}\right\} = \{m, p\}$ for some $m \in \mathbb{Z}_{\leq 0}$, $p \notin \mathbb{Z}$. Thus, $y_2 = \omega^2 G(m, p; 3; \omega)$, and we write $U(\omega) = c_1y_1 + c_2y_2$ for $c_1, c_2 \in \mathbb{C}$.

Similarly, using case (i) to solve (3.3.29) leads to $y_3 = {}_2F_1\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; 2; \omega\right)$ and $y_4 = G(l, q; 2; \omega)$ where $\left\{\frac{2-\mu}{2}, \frac{3-\mu}{2}\right\} = \{l, q\}$ for some $l \in \mathbb{Z}_{\leq 0}$, $q \notin \mathbb{Z}$. We have $V(\omega) = c_3y_3 + c_4y_4$ for $c_3, c_4 \in \mathbb{C}$.

Using the same methods as in the proof of Theorem 3.3.8, we find

$$c_1 = \frac{2-\mu}{64}c_3 \text{ and } c_2 = \frac{2-\mu}{64}c_4.$$

Constants c_3 and c_4 are renamed and functions y_1, y_2 are rescaled in the theorem statement. □

We now consider the solutions to (3.3.28) and (3.3.29) in the neighbourhoods of the singular points 1 and ∞. Around $\omega = 1$, we have three cases: μ is not a half-integer, μ is a half-integer less than or equal to 1/2, and μ is a half-integer greater than 1/2.

Theorem 3.3.10. *Let* δ *given by (3.3.25) be a homogeneous Legendre field of degree* $\mu \notin \{-1, 0, 1, 2, 3\}$*. Then, the formulas for* U*,* V *in the neighbourhood of* $\omega = 1$ *are as follows, with* $c_1^{(1)}, c_2^{(1)} \in \mathbb{C}$ *and* G *defined by (3.2.5).*

Case 1: $\mu \neq \frac{2k+1}{2}$ *for* $k \in \mathbb{Z}$.

$$\begin{aligned} U(\omega) &= c_1^{(1)}\,{}_2F_1\left(-\frac{\mu}{2}, -\frac{\mu+1}{2}; \frac{3-2\mu}{2}; 1-\omega\right) \\ &\quad + c_2^{(1)}(1-\omega)^{\frac{2\mu-1}{2}}\,{}_2F_1\left(\frac{\mu-2}{2}, \frac{\mu-1}{2}; \frac{2\mu+1}{2}; 1-\omega\right), \\ V(\omega) &= \frac{16c_1^{(1)}(\mu+1)}{2-\mu}\,{}_2F_1\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; \frac{3-2\mu}{2}; 1-\omega\right) \\ &\quad + 16c_2^{(1)}(1-\omega)^{\frac{2\mu-1}{2}}\,{}_2F_1\left(\frac{\mu+1}{2}, \frac{\mu+2}{2}; \frac{2\mu+1}{2}; 1-\omega\right). \end{aligned}$$

Case 2: $\mu = \frac{2k+1}{2}$ *for* $k \in \mathbb{Z}_{\leq 0}$.

$$\begin{aligned} U(\omega) &= c_1^{(1)}\,{}_2F_1\left(-\frac{\mu}{2}, -\frac{\mu+1}{2}; \frac{3-2\mu}{2}; 1-\omega\right) + c_2^{(1)}G\left(-\frac{\mu}{2}, -\frac{\mu+1}{2}; \frac{3-2\mu}{2}; 1-\omega\right), \\ V(\omega) &= \frac{16c_1^{(1)}(\mu+1)}{2-\mu}\,{}_2F_1\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; \frac{3-2\mu}{2}; 1-\omega\right) \\ &\quad + \frac{16c_2^{(1)}(\mu+1)}{2-\mu}G\left(\frac{2-\mu}{2}, \frac{3-\mu}{2}; \frac{3-2\mu}{2}; 1-\omega\right). \end{aligned}$$

Case 3: $\mu = \frac{2k+1}{2}$ *for* $k \in \mathbb{N}^\times$.

$$\begin{aligned} U(\omega) &= c_1^{(1)}(1-\omega)^{\frac{2\mu-1}{2}}\,{}_2F_1\left(\frac{\mu-2}{2}, \frac{\mu-1}{2}; \frac{2\mu+1}{2}; 1-\omega\right) \\ &\quad + c_2^{(1)}(1-\omega)^{\frac{2\mu-1}{2}}G\left(\frac{\mu-2}{2}, \frac{\mu-1}{2}; \frac{2\mu+1}{2}; 1-\omega\right), \\ V(\omega) &= \frac{16c_1^{(1)}(\mu+1)}{2-\mu}(1-\omega)^{\frac{2\mu-1}{2}}\,{}_2F_1\left(\frac{\mu+1}{2}, \frac{\mu+2}{2}; \frac{2\mu+1}{2}; 1-\omega\right) \\ &\quad + \frac{16c_2^{(1)}(\mu+1)}{2-\mu}(1-\omega)^{\frac{2\mu-1}{2}}G\left(\frac{\mu+1}{2}, \frac{\mu+2}{2}; \frac{2\mu+1}{2}; 1-\omega\right). \end{aligned}$$

Proof. We define the coordinate shift $\omega_1 = 1-\omega$, with $U(\omega) = U_1(\omega_1)$ and $V(\omega) = V_1(\omega_1)$. Under this transformation, equation (3.3.28) becomes a hypergeometric differential equation for $U_1(\omega_1)$ with parameters $(a, b, c, z) = \left(-\frac{\mu}{2}, -\frac{\mu+1}{2}, \frac{3-2\mu}{2}, \omega_1\right)$, and equation (3.3.29) becomes a hypergeometric differential equation for $V_1(\omega_1)$ with $(a, b, c, z) = \left(\frac{2-\mu}{2}, \frac{3-\mu}{2}, \frac{3-2\mu}{2}, \omega_1\right)$. We may now solve both equations around $\omega_1 = 0$ using the formulas in § 3.2.1, before rewriting them in terms of ω. Since $c = \frac{3-2\mu}{2}$ for both equations, the solutions for each can be split into the same three cases dependent on the value of μ.

To relate the constants in the general solutions for U and V, it is simplest to work in ω_1 and rewrite equations (3.3.26) and (3.3.27) in this variable. Respectively, these equations become

$$U_1'(\omega_1) = \frac{\mu-2}{32}(1-\omega_1)V_1(\omega_1) + \frac{(1-\omega_1)^2}{16}V_1'(\omega_1),$$
$$(1-\omega_1)^2 V_1'(\omega_1) = 8(\mu+1)U_1(\omega_1) + 16(1-\omega_1)U_1'(\omega_1).$$

Using the same methods as in the proof of Theorem 3.3.8 produces the required statements. □

Finally, the solutions to (3.3.28) and (3.3.29) around ∞ can be written with one expression for all μ.

Theorem 3.3.11. *Let δ be as in (3.3.25), with $\mu \notin \{-1,0,1,2,3\}$. Then, the formulas for $U(\omega)$, $V(\omega)$ in the neighbourhood of $\omega = \infty$ are as follows, with $c_1^{(\infty)}, c_2^{(\infty)} \in \mathbb{C}$.*

$$U(\omega) = c_1^{(\infty)}\omega^{\frac{\mu}{2}}{}_2F_1\left(-\frac{\mu}{2}, \frac{4-\mu}{2}; \frac{3}{2}; \frac{1}{\omega}\right) + c_2^{(\infty)}\omega^{\frac{\mu+1}{2}}{}_2F_1\left(-\frac{\mu+1}{2}, \frac{3-\mu}{2}; \frac{1}{2}; \frac{1}{\omega}\right),$$
$$V(\omega) = \frac{16c_1^{(\infty)}}{2-\mu}\omega^{\frac{\mu-2}{2}}{}_2F_1\left(\frac{2-\mu}{2}, -\frac{\mu}{2}; \frac{1}{2}; \frac{1}{\omega}\right)$$
$$- 16c_2^{(\infty)}(\mu+1)\omega^{\frac{\mu-3}{2}}{}_2F_1\left(\frac{3-\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; \frac{1}{\omega}\right).$$

Proof. We may either transform equations (3.3.28) and (3.3.29) with $\omega_\infty = \omega^{-1}$, $U(\omega) = U_\infty(\omega_\infty)$, $V(\omega) = V_\infty(\omega_\infty)$ and solve the resulting equations around $\omega_\infty = 0$ using § 3.2.1; or, we can refer to [16, formulas 15.10.6 and 15.10.7]. Either way, we obtain the general solutions

$$U(\omega) = c_1\omega^{\frac{\mu}{2}}{}_2F_1\left(-\frac{\mu}{2}, \frac{4-\mu}{2}; \frac{3}{2}; \frac{1}{\omega}\right) + c_2\omega^{\frac{\mu+1}{2}}{}_2F_1\left(-\frac{\mu+1}{2}, \frac{3-\mu}{2}; \frac{1}{2}; \frac{1}{\omega}\right),$$
$$V(\omega) = c_3\omega^{\frac{\mu-2}{2}}{}_2F_1\left(\frac{2-\mu}{2}, -\frac{\mu}{2}; \frac{1}{2}; \frac{1}{\omega}\right) + c_4\omega^{\frac{\mu-3}{2}}{}_2F_1\left(\frac{3-\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; \frac{1}{\omega}\right),$$

for any $\mu \notin \{-1,0,1,2,3\}$.

We use the expansion of (3.3.26) and (3.3.27) in ω to relate c_1, c_2, c_3, c_4 by the same methods as in the proof of Theorem 3.3.8. In particular, comparing terms of degree $\frac{\mu}{2}$ in (3.3.27), recalling that by definition $\mu \neq 0, 2$, produces

$$c_3 = \frac{16}{2-\mu}c_1.$$

Comparing terms of degree $\frac{\mu-1}{2}$ in (3.3.26), given $\mu \neq -1, 3$, produces

$$c_4 = -16(\mu+1)c_2.$$

We rewrite $c_1 = c_1^{(\infty)}$, $c_2 = c_2^{(\infty)}$ in the theorem statement. □

We now consider the monodromy of these solutions for all $\mu \notin \{-1, 0, 1, 2, 3\}$. Since V is similar in structure to U around all the singular points, and in all cases, it has the same monodromy as U. In the following, we let

$$\begin{aligned} U(\omega) &= c_1^{(i)} y_1^{(i)}(\omega) + c_2^{(i)} y_2^{(i)}(\omega), \\ V(\omega) &= c_1^{(i)} y_3^{(i)}(\omega) + c_2^{(i)} y_4^{(i)}(\omega) \end{aligned} \tag{3.3.36}$$

denote solutions to (3.3.28), (3.3.29) respectively, around the singular point i. We set $c_j^{(0)} = c_j$ and $y_j^{(0)} = y_j$.

Corollary 3.3.12. *We have U, V as in (3.3.36) given by Theorems 3.3.8 – 3.3.11. The monodromy matrices for U, with basis $y_1^{(i)}$, $y_2^{(i)}$, and V, with basis $y_3^{(i)}$, $y_4^{(i)}$, around 0, 1, ∞ are denoted by M_0, M_1, M_∞ respectively. The matrices are as follows:*

$$M_0 = \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix};$$

$$M_1 = \begin{cases} \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix} & \text{if } \mu \text{ is a half-integer,} \\ \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\pi i \mu} \end{pmatrix} & \text{if } \mu \text{ is not a half-integer;} \end{cases}$$

$$M_\infty = \begin{cases} \begin{pmatrix} e^{-\pi i \mu} & 0 \\ 0 & -e^{\pi i \mu} \end{pmatrix} & \text{if } \mu \text{ is not an integer,} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \mu \text{ is even,} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mu \text{ is odd.} \end{cases}$$

Proof. We wish to write M_0 such that $\begin{pmatrix} y_1(\omega) \\ y_2(\omega) \end{pmatrix} \mapsto M_0 \begin{pmatrix} y_1(\omega) \\ y_2(\omega) \end{pmatrix}$ under the transformation $\omega \mapsto e^{\phi i}\omega$ as ϕ completes a single period, going from 0 to 2π. Although the expressions for U are split into two cases around $\omega = 0$ (given in Theorems 3.3.8 and 3.3.9), in both cases we have that y_1 consists entirely of analytic terms and y_2 has the form of $\log(\omega) \times$ analytic terms $+$ analytic terms. This leads to the stated expression for M_0. It can also be easily verified that the same is true with V, y_3, y_4 replacing U, y_1, y_2.

To find M_1, we consider the monodromy around $\omega_1 = 1 - \omega = 0$ for the formulas given in Theorem 3.3.10. If μ is not a half-integer (Case 1 in Theorem 3.3.10), we have that $y_1^{(1)}$ consists entirely of analytic terms and $y_2^{(1)}$ has the form $\omega_1^{\frac{2\mu-1}{2}} \times$ analytic terms + analytic terms. Meanwhile, if μ is a half-integer (Cases 2 and 3 in Theorem 3.3.10), $y_1^{(1)}$ again is analytic while $y_2^{(1)}$ has the form $\log(\omega_1) \times$ analytic terms + analytic terms.

For M_∞, we consider the monodromy around $\omega_\infty = \omega^{-1} = 0$ using the formulas in Theorem 3.3.11. Here, $y_1^{(\infty)}$ has the form $\omega_\infty^{-\frac{\mu}{2}} \times$ analytic terms + analytic terms and $y_2^{(\infty)}$ has the form $\omega_\infty^{-\frac{\mu+1}{2}} \times$ analytic terms + analytic terms. □

It is desirable to be able to express the solutions around the different singular points in relation to each other (over appropriate domains). Such connection formulas are readily available in the literature for the "generic" solutions of a hypergeometric differential equation, i.e. where c is non-integer; for example, see [16, § 15.10(ii)]. However, we are concerned with special values of c where solutions include correction terms. We refer to Haraoka [30] for a complete set of connection relations between solutions[1] in all cases, including "non-generic" parameters. These relations also allow the free parameters $c_1, c_2, c_1^{(1)}, c_2^{(1)} c_1^{(\infty)}, c_2^{(\infty)}$ to be related to each other.

Example 3.3.13 *Relating solutions and monodromy around $\omega = 0$, $\omega = 1$ for specific μ.* For the purposes of this example, we only consider solutions to equation (3.3.28), which has parameters $a = -\frac{\mu}{2}$, $b = -\frac{\mu+1}{2}$, $c = -1$. We take $\mu \notin \mathbb{N}$ and $\mu \notin \mathbb{Z} + \frac{1}{2}$ so that the solution to (3.3.28) around $\omega = 0$, denoted $U_0(\omega)$, is given by Theorem 3.3.8 and the solution around $\omega = 1$, denoted $U_1(\omega)$, is given by Case 1 in Theorem 3.3.10. Using the notation in [30], we rewrite these solutions as

$$\begin{aligned} U_0(\omega) &= c_1 y_2 + c_2\Big(\widehat{y}_1 + \big(\psi(a+1-c) + \psi(b+1-c) - \psi(1) - \psi(2-c)\big) y_2\Big), \\ U_1(\omega) &= c_1^{(1)} y_3 + c_2^{(1)} y_4, \end{aligned}$$

where functions $y_2, \widehat{y}_1, y_3, y_4$ are given by [30, formulas (3.3), (3.13), (3.4), (3.5)] respectively and $c_1, c_2, c_1^{(1)}, c_2^{(1)}$ are as written in Theorems 3.3.8 and 3.3.10. We define

$$\widetilde{y}_1 = \widehat{y}_1 + \big(\psi(a+1-c) + \psi(b+1-c) - \psi(1) - \psi(2-c)\big) y_2$$

so that the solution $U_0 = c_1 y_2 + c_2 \widetilde{y}_1$.

By [30, Theorem 4.1, Case (ii-2)], we have the relations

$$y_2 = \gamma_1 y_3 + \gamma_2 y_4, \tag{3.3.37}$$

$$\widetilde{y}_1 = \gamma_1 \beta y_3, \tag{3.3.38}$$

[1]Some of the solutions given in [30] are expressed in terms of different bases of fundamental solutions to those defined in § 3.2.1. However, they are equivalent to the solutions given here.

with constants

$$\begin{aligned}
\gamma_1 &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\
&= \frac{\Gamma(3)\Gamma\left(\frac{2\mu-1}{2}\right)}{\Gamma\left(\frac{\mu+2}{2}\right)\Gamma\left(\frac{\mu+3}{2}\right)}, \\
\gamma_2 &= \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \\
&= \frac{\Gamma(3)\Gamma\left(\frac{1-2\mu}{2}\right)}{\Gamma\left(\frac{4-\mu}{2}\right)\Gamma\left(\frac{3-\mu}{2}\right)}, \\
\beta &= \psi(a+1-c) - \psi(c-a) + \psi(b+1-c) - \psi(c-b) \\
&= \psi\left(\frac{4-\mu}{2}\right) - \psi\left(\frac{\mu-2}{2}\right) + \psi\left(\frac{3-\mu}{2}\right) - \psi\left(\frac{\mu-1}{2}\right).
\end{aligned}$$

These relations hold in the domain $\mathbb{C}\setminus\big((-\infty,0]\cup[1,\infty)\big)$. Here, $\Gamma(z)$ denotes the gamma function. When $\mu\notin\mathbb{Z}$, we can further simplify β as follows:

$$\begin{aligned}
\beta &= -\pi\big(\cot(\pi a) + \cot(\pi b)\big) \\
&= 2\pi\cot(\mu\pi).
\end{aligned}$$

For this, we use the property of the digamma function, given in [23, § 1.7], that

$$\psi(1-z) - \psi(z) = \pi\cot(\pi z),$$

as well as standard trigonometric identities.

From equations (3.3.37) and (3.3.38), we can define transformation matrices

$$\begin{aligned}
T_{01} &= \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_1\beta & 0 \end{pmatrix}, \\
T_{10} = T_{01}^{-1} &= \begin{pmatrix} 0 & \frac{1}{\gamma_1\beta} \\ \frac{1}{\gamma_2} & -\frac{1}{\gamma_2\beta} \end{pmatrix}
\end{aligned}$$

such that $\begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix} = T_{01}\begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$ and $\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = T_{10}\begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix}$ in the domain $\mathbb{C}\setminus\big((-\infty,0]\cup[1,\infty)\big)$. We can now refer to Corollary 3.3.12 for the appropriate monodromy matrices M_0, M_1, where analytic continuation around $\omega = 0$ and $\omega = 1$ respectively sends

$$\begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix} \mapsto M_0 \begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix} = M_0 T_{01} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix},$$
$$\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \mapsto M_1 \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = M_1 T_{10} \begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix}.$$

By composing transformations, we find that the effect of taking the analytic continuation of $U_0(\omega)$ around both $\omega = 0$ and $\omega = 1$ can be represented by the matrix

$$M_0 T_{01} M_1 T_{10} = \begin{pmatrix} e^{-2\pi i \mu} & \frac{1}{\beta}\left(1 - e^{-2\pi i \mu}\right) \\ 2\pi i e^{-2\pi i \mu} & \frac{2\pi i}{\beta}\left(1 - e^{-2\pi i \mu}\right) + 1 \end{pmatrix}$$

such that $\begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix} \mapsto M_0 T_{01} M_1 T_{10} \begin{pmatrix} y_2 \\ \widetilde{y}_1 \end{pmatrix}$.

We now return to the task of finding solutions for equations (3.3.28) and (3.3.29) for the special cases of μ not covered by the previous discussion — namely $\mu \in \{-1, 0, 1, 2, 3\}$. In this cases, we also use U, V to find the components and potential of the Legendre field for each value of μ.

Theorem 3.3.14. *Let δ as in (3.3.25) be a Legendre field of degree $\mu \in \{-1, 0, 1, 2, 3\}$. Then the components $u = u(t_1, t_2)$ and $v = v(t_1, t_2)$ of δ are given in Tables 3.3 and 3.4 respectively, where*

$$s = \sqrt{1 - \frac{8}{t_1^2 t_2}}$$

and $c_1, c_2 \in \mathbb{C}$. The potentials $h = h(t_1, t_2)$ such that $u = \partial_2(h)$ and $v = \partial_1(h)$ are given in Table 3.5.

Proof. We consider the five values of μ successively.

Case 1: $\mu = -1$.

The equation (3.3.28) becomes

$$2\omega(\omega - 1)U''(\omega) + (3\omega + 2)U'(\omega) = 0.$$

Substituting $Y(\omega) = U'(\omega)$, we can rearrange this ODE to get

$$\int \frac{Y'(\omega)}{Y(\omega)} \mathrm{d}\omega = \int \frac{3\omega + 2}{2(\omega - \omega^2)} \mathrm{d}\omega. \tag{3.3.39}$$

The right-hand side of (3.3.39) can be integrated, using the substitution $x = \omega - \omega^2$, as

Table 3.3: First component of a homogeneous Legendre field in Case 3.

Degree μ	**First component** u
-1	$c_1\left(\frac{12}{t_1^2t_2}-1\right)s^{-3}+c_2$
0	$c_1t_1\left(1-\frac{4}{t_1^2t_2}\right)s^{-1}+c_2t_1$
1	$c_1\left(t_1^2-\frac{4}{t_2}\right)+c_2t_1^2s$
2	$c_1t_1^3s^3$
3	$c_1\left[\frac{t_1^4}{4}\left(1-\frac{20}{t_1^2t_2}\right)s+\frac{24}{t_2^2}\text{arctanh}\,(s)\right]-\frac{c_2}{t_2^2}$

First component $u=u(t_1,t_2)$ of a Legendre field $\delta=u\partial_1+v\partial_2$ of degree μ. Constants $c_1,c_2\in\mathbb{C}$ are as in Table 3.4.

Table 3.4: Second component of a homogeneous Legendre field in Case 3.

Degree μ	**Second component** v
-1	$8c_1t_1^{-3}s^{-3}$
0	$c_1t_2s^{-1}+c_2t_2$
1	$2c_1t_1t_2+2c_2t_1t_2s$
2	$3c_1\big(t_1^2t_2s-8\text{arctanh}\,(s)\big)+c_2$
3	$c_1\left[t_1^3t_2\left(1+\frac{16}{t_1^2t_2}\right)s-24t_1\text{arctanh}\,(s)\right]+c_2t_1$

Second component $v=v(t_1,t_2)$ of a Legendre field $\delta=u\partial_1+v\partial_2$ of degree μ. Constants $c_1,c_2\in\mathbb{C}$ are as in Table 3.3.

follows:

$$\int\frac{3\omega+2}{2(\omega-\omega^2)}\mathrm{d}\omega=\frac{7}{4}\int\frac{1}{\omega-\omega^2}\mathrm{d}\omega-\frac{3}{4}\int\frac{1}{x}\mathrm{d}x=\log\frac{c_1\omega}{(1-\omega)^{5/2}}.$$

Thus, (3.3.39) has the solution

$$Y(\omega)=\frac{c_1\omega}{(1-\omega)^{5/2}},$$

and we obtain

$$U(\omega)=\frac{c_1(3\omega-2)}{(1-\omega)^{3/2}}+c_2.$$

Since this is a closed-form solution, it is easiest to solve (3.3.26) and (3.3.27) directly for $V(\omega)$. We find

$$V(\omega)=\frac{16c_1}{(1-\omega)^{3/2}}.$$

Table 3.5: Potential function of a homogeneous Legendre field in Case 3.

Degree μ	**Potential** h
-1	$c_2t_2 - c_1t_2s^{-1}$
0	$c_1t_1t_2s + c_2t_1t_2$
1	$c_1\left(t_1^2t_2 - 4\log t_2\right) + c_2\left(t_1^2t_2s - 8\text{arctanh}\,(s)\right)$
2	$c_1t_1\left[\left(t_1^2t_2 + 16\right)s - 24\text{arctanh}\,(s)\right] + c_2t_1$
3	$c_1\left[\frac{t_1^4t_2}{4}\left(1+\frac{52}{t_1^2t_2}\right)s - 12\left(t_1^2+\frac{2}{t_2}\right)\text{arctanh}\,(s)\right] + c_2\left(\frac{t_1^2}{2}+\frac{1}{t_2}\right)$

Potential $h = h(t_1, t_2)$ for a Legendre field δ of degree μ. Constants $c_1, c_2 \in \mathbb{C}$ are as in Tables 3.3, 3.4.

From (3.3.25), we have $u(t_1, t_2) = U(\omega)$ and $v(t_1, t_2) = t_1^{-3}V(\omega)$, where $\omega = 8t_1^{-2}t_2^{-1}$, which (after rescaling c_1) lead to the formulas for $u(t_1, t_2)$ and $v(t_1, t_2)$ as in the theorem statement. Finally, the formula for the potential $h(t_1, t_2)$ can be easily verified using (3.1.6).

Case 2: $\mu = 0$.

The equation (3.3.28) becomes

$$2\omega(\omega - 1)U''(\omega) + (\omega + 2)U'(\omega) = 0.$$

The same method as used in the case $\mu = -1$ leads to the equation:

$$\int \frac{Y'(\omega)}{Y(\omega)}\mathrm{d}\omega = \log \frac{c_1\omega}{(1-\omega)^{3/2}},$$

where $Y(\omega) = U'(\omega)$. This gives us

$$U(\omega) = \frac{c_1(2-\omega)}{\sqrt{1-\omega}} + c_2.$$

We can either substitute this into (3.3.26), (3.3.27) to find $V(\omega)$, or refer to [23, § 2.2.2], case 17 with $m = 0 = l$, for the appropriate solution to (3.3.29). Either method produces

$$V(\omega) = \frac{16c_1}{\omega\sqrt{1-\omega}} + \frac{8c_2}{\omega}.$$

In this case, (3.3.25) gives us $u(t_1, t_2) = t_1U(\omega)$ and $v(t_1, t_2) = t_1^{-2}V(\omega)$. From these relations and the definition of the potential $h(t_1, t_2)$ in (3.1.6), the given formulas can be verified.

Case 3: $\mu = 1$.

Neither ODE (3.3.28) or (3.3.29) changes in form, but the general formulas given in theorems 3.3.8 and 3.3.9 are not applicable since they result in singularities. We refer to the degenerate cases dealt with in [23, § 2.2.2]. The solution to (3.3.28) is given in case 12 (ibid) with $m = 1$ and $n = 0 = l$, while the solution to (3.3.29) is given in case 17 (ibid) with $m = 0 = l$. Both expressions can be written in terms of elementary functions, and we obtain

$$U(\omega) = c_1(2-\omega) + c_2\sqrt{1-\omega}$$
$$V(\omega) = \frac{32c_1}{\omega} + \frac{16c_2\sqrt{1-\omega}}{\omega},$$

where (3.3.26) has been used to relate constants. We have $u(t_1,t_2) = t_1^2 U(\omega)$ and $v(t_1,t_2) = t_1^{-1}V(\omega)$ from (3.3.25), from which the formulas in the table follow.

Case 4: $\mu = 2$.

The solution to (3.3.28) can be obtained from [23, § 2.2.2], case 12 with $m = 1$ and $n = 0 = l$. We obtain

$$U(\omega) = c_1(2-3\omega) + c_2(1-\omega)^{3/2}. \tag{3.3.40}$$

The ODE (3.3.29) becomes

$$\omega(1-\omega)V''(\omega) + (2 - \frac{3}{2}\omega)V'(\omega) = 0.$$

We use the same method as in the case $\mu = -1$ to arrive at the equation

$$V(\omega) = \int \frac{c_3\sqrt{1-\omega}}{\omega^2}\mathrm{d}\omega.$$

The substitution $x = \sqrt{1-\omega}$, with the standard integral $\int (1-x^2)^{-1}\mathrm{d}x = \operatorname{arctanh}(x)$, lets us solve this integral for

$$V(\omega) = c_3\left(\frac{\sqrt{1-\omega}}{\omega} - \operatorname{arctanh}\left(\sqrt{1-\omega}\right)\right) + c_4.$$

Equation (3.3.26) can be used to relate this to $U(\omega)$ in (3.3.40), leading to

$$c_1 = 0 \text{ and } c_3 = 24c_2.$$

Finally, we use (3.3.25), which gives $u(t_1,t_2) = t_1^3 U(\omega)$ and $v(t_1,t_2) = V(\omega)$, and (3.1.6) to verify the formulas in the statement.

Case 5: $\mu = 3$.

Equation (3.3.29) becomes

$$\omega(1-\omega)V''(\omega) + (2 - \frac{\omega}{2})V'(\omega) = 0,$$

which can be solved using the same methods as in the case $\mu = 2$. This produces

$$V(\omega) = c_1\left(\frac{2\omega+1}{\omega}\sqrt{1-\omega} - 3\text{arctanh}\left(\sqrt{1-\omega}\right)\right) + c_2.$$

To obtain $U(\omega)$, note that the formula for $U(\omega)$ in Theorem 3.3.9 is applicable, as $\mu = 3$ does not produce singularities. The formula may be written in closed form for this case, but it is more convenient to find this expression by substituting V into (3.3.26). By doing this, we find

$$U'(\omega) = \frac{3c_1}{32}\left(\omega\,\text{arctanh}\left(\sqrt{1-\omega}\right) - \sqrt{1-\omega}\right) - \frac{c_2\omega}{32},$$

hence

$$U(\omega) = \frac{c_1}{64}\left(3\omega^2\,\text{arctanh}\left(\sqrt{1-\omega}\right) + (2-5\omega)\sqrt{1-\omega}\right) - \frac{c_2\omega^2}{64}.$$

We use $u(t_1,t_2) = t_1^4 U(\omega)$ and $v(t_1,t_2) = t_1 V(\omega)$ from (3.3.25), and (3.1.6) to arrive at the formulas in the statement, where c_1 and c_2 have been rescaled. □

3.3.4 Case 4

Next, we consider the Frobenius manifold with prepotential

$$\widehat{F}(\hat{t}) = \frac{1}{2}\hat{t}_1^2\hat{t}_2 + \hat{t}_2^2\log\hat{t}_2, \tag{3.3.41}$$

that is, $\widehat{F}$ is given by (2.3.4) up to a change of notation. For the purposes of this section, we denote this prepotential as $\widehat{F}$ to distinguish it from the one given by F in (3.3.23). The corresponding Frobenius manifolds are related by the inversion symmetry, as shown in Example 2.4.11.

We find

$$\hat{c}_{222} = \widehat{F}_{222} = \frac{2}{\hat{t}_2},$$

and the Euler vector field is

$$\widehat{E} = \hat{t}_1\widehat{\partial}_1 + 2\hat{t}_2\widehat{\partial}_2,$$

where we set $\widehat{\partial}_i = \partial_{\hat{t}_i}$. Condition (3.1.4) for the arbitrary Legendre field $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ now becomes

$$\frac{\partial\hat{u}}{\partial\hat{t}_2} = \frac{2}{\hat{t}_2}\frac{\partial\hat{v}}{\partial\hat{t}_1}. \tag{3.3.42}$$

3.3.4.1 Inversion relations

We saw in Example 2.4.11 that $\widehat{F}\left(\hat{t}\right)$ is related to the Frobenius manifold described by (3.3.23), referred to as $F(t)$. Under the relations established in this example, the Legendre fields of F can be used to describe those of $\widehat{F}$. For brevity, we say that δ is a Legendre field for a prepotential F if it is a Legendre field for the Levi-Civita connection of the Frobenius manifold associated with F.

Recall from Example 2.4.11 that the coordinates t and $\hat{t}$ are related by

$$\begin{aligned} \hat{t}_1 &= t_1, \quad \hat{t}_2 = \frac{1}{t_2}; \\ \widehat{\partial}_1 &= \partial_1, \quad \widehat{\partial}_2 = -\left(t_2\right)^2 \partial_2, \end{aligned} \tag{3.3.43}$$

where $\widehat{\partial}_i = \frac{\partial}{\partial \hat{t}_i}$.

Theorem 3.3.15. *Let F be the prepotential given by (3.3.23) and $\widehat{F}$ by (3.3.41). There is a one-to-one correspondence between the Legendre fields of F and those of $\widehat{F}$, provided by the following statements.*

1. *Let δ be a Legendre field for F, with potential h. Then the vector field*

$$\widehat{\delta} = t_2 \delta - h \partial_1 \tag{3.3.44}$$

is a Legendre field for $\widehat{F}$, and has potential

$$\hat{h} = -\hat{t}_2 h. \tag{3.3.45}$$

2. *Let $\widehat{\delta}$ be a Legendre field for $\widehat{F}$, with potential $\hat{h}$. Then the vector field*

$$\delta = \hat{t}_2 \widehat{\delta} - \hat{h} \widehat{\partial}_1 \tag{3.3.46}$$

is a Legendre field for F, and has potential

$$h = -t_2 \hat{h}. \tag{3.3.47}$$

3. *If $\widehat{\delta}$ is defined by (3.3.44), then*

$$\hat{t}_2 \widehat{\delta} - \hat{h} \widehat{\partial}_1 = \delta.$$

Proof. Let $\delta = \partial_2(h)\partial_1 + \partial_1(h)\partial_2$ be an arbitrary Legendre field for F. Since δ satisfies the Legendre field condition for F, its components $u = \partial_2(h)$ and $v = \partial_1(h)$ satisfy the differential equations (3.3.24) and (3.1.5). Under the coordinate transformation $t \mapsto \hat{t}$, these two equations become

$$2\left(\hat{t}_2\right)^3 \widehat{\partial}_2(h) + \left(\hat{t}_2\right)^4 \widehat{\partial}_2\left(\widehat{\partial}_2(h)\right) = 2\left(\hat{t}_2\right)^3 \widehat{\partial}_1\left(\widehat{\partial}_1(h)\right), \tag{3.3.48}$$

$$\widehat{\partial}_1\left(\widehat{\partial}_2(h)\right) = \widehat{\partial}_2\left(\widehat{\partial}_1(h)\right). \tag{3.3.49}$$

To show that $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ given by (3.3.44) is a Legendre field for $\widehat{F}$, we must show that it satisfies the Legendre field condition for $\widehat{F}$; equivalently, that its components satisfy

$$\widehat{\partial}_2(\hat{u}) = \frac{2}{\hat{t}_2}\widehat{\partial}_1(\hat{v}), \tag{3.3.50}$$

$$\widehat{\partial}_1(\hat{u}) = \widehat{\partial}_2(\hat{v}), \tag{3.3.51}$$

which are equations (3.3.42) and (3.1.5) in the new notation. Since from (3.3.44) we have

$$\hat{u} = -\hat{t}_2\widehat{\partial}_2(h) - h, \tag{3.3.52}$$

$$\hat{v} = -\hat{t}_2\widehat{\partial}_1(h), \tag{3.3.53}$$

we use (3.3.48) to show that the left-hand side of (3.3.50) becomes

$$\begin{aligned}\widehat{\partial}_2(\hat{u}) &= -2\widehat{\partial}_2(h) - \hat{t}_2\widehat{\partial}_2\left(\widehat{\partial}_2(h)\right)\\ &= -2\widehat{\partial}_1\left(\widehat{\partial}_1(h)\right).\end{aligned}$$

This expression is equivalent to the right-hand side of (3.3.50), and so $\widehat{\delta}$ satisfies this first condition. Similarly, (3.3.51) is equivalent to equation (3.3.49), which is already satisfied. This proves that the $\widehat{\delta}$ constructed in Statement 1 is a Legendre field.

We now check that (3.3.45) describes the potential of $\widehat{\delta}$. Recall that we should have $\hat{u} = \widehat{\partial}_2(\hat{h})$, $\hat{v} = \widehat{\partial}_1(\hat{h})$. Substituting $\widehat{h} = -\hat{t}_2 h$, we recover equations (3.3.52), (3.3.53), which hold by definition of $\widehat{\delta}$. Statement 1 is therefore proven.

To prove Statement 2, we can use the same methods. We know that $\widehat{\delta}$ satisfies equations (3.3.50) and (3.3.51), which become

$$\begin{aligned}2(t_2)^3\partial_2(\hat{h}) + (t_2)^4\partial_2\left(\partial_2(\hat{h})\right) &= 2t_2\partial_1\left(\partial_1(\hat{h})\right),\\ \partial_1\left(\partial_2(\hat{h})\right) &= \partial_2\left(\partial_1(\hat{h})\right)\end{aligned}$$

under the transformation $\hat{t} \mapsto t$. A new vector field with potential (3.3.47) has components

$$\begin{aligned}u &= \partial_2(h) = -t_2\partial_2(\hat{h}) - \hat{h},\\ v &= \partial_1(h) = -t_2\partial_1(\hat{h}),\end{aligned}$$

which coincides with the vector field δ given by (3.3.46). This implies Legendre field condition (3.1.5), and the condition (3.3.24) is easy to check as well.

To prove Statement 3, we begin with δ, a Legendre field for F. We construct $\widehat{\delta}$ as in

(3.3.44), which then gives us

$$\begin{aligned}\hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1 &= \hat{t}_2\left(t_2\delta - h\partial_1\right) - \hat{h}\widehat{\partial}_1 \\ &= \delta - \hat{t}_2 h\partial_1 - \left(-\hat{t}_2 h\right)\partial_1 \\ &= \delta\end{aligned}$$

as required, where we have used relation (3.3.45). □

Therefore, Legendre fields for $\widehat{F}$ can be found using the Legendre fields for F; although we note that the potential for a given Legendre field is also required. Since we described all homogeneous Legendre fields for F in §3.3.3, we check whether the homogeneity of a Legendre field carries over under the relations given in Theorem 3.3.15.

Lemma 3.3.16. *Let δ be a homogeneous Legendre field for F of degree μ with potential h. Then $\widehat{\delta}$, $\hat{h}$ as given by (3.3.44), (3.3.45) satisfy the relation*

$$\hat{t}_2\mathcal{L}_{\hat{E}}\left(\widehat{\delta}\right) - \mathcal{L}_{\hat{E}}\left(\hat{h}\widehat{\partial}_1\right) = (\mu-2)\hat{t}_2\widehat{\delta} - \mu\hat{h}\widehat{\partial}_1. \tag{3.3.54}$$

Furthermore, let $\widehat{\delta}$ be a homogeneous Legendre field for $\widehat{F}$ of degree $\hat{\mu}$ with potential $\hat{h}$. Then δ, h as given by (3.3.46), (3.3.47) satisfy the relation

$$t_2\mathcal{L}_E\left(\delta\right) - \mathcal{L}_E\left(h\partial_1\right) = (\hat{\mu}+2)t_2\delta - \hat{\mu}h\partial_1.$$

Proof. The field δ is homogeneous if it satisfies equation (3.1.7). By Theorem 3.3.15, we also have that $\delta = \hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1$, so

$$\begin{aligned}\mathcal{L}_E(\delta) &= \mathcal{L}_E\left(\hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1\right) \\ &= \hat{E}\left(\hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1\right) - \left(\hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1\right)\left(\hat{E}\right)\end{aligned}$$

where the Euler fields are given by (2.4.3). Continuing, we have

$$\begin{aligned}\mathcal{L}_E(\delta) &= \hat{t}_2\hat{E}\left(\widehat{\delta}\right) + 2\hat{t}_2\widehat{\delta} - \hat{E}\left(\hat{h}\widehat{\partial}_1\right) - \hat{t}_2\widehat{\delta}\left(\hat{E}\right) + \hat{h}\widehat{\partial}_1\left(\hat{E}\right) \\ &= \hat{t}_2\mathcal{L}_{\hat{E}}\left(\widehat{\delta}\right) + 2\hat{t}_2\widehat{\delta} - \mathcal{L}_{\hat{E}}\left(\hat{h}\widehat{\partial}_1\right).\end{aligned}$$

By equation (3.1.7), we obtain

$$\hat{t}_2\mathcal{L}_{\hat{E}}\left(\widehat{\delta}\right) + 2\hat{t}_2\widehat{\delta} - \mathcal{L}_{\hat{E}}\left(\hat{h}\widehat{\partial}_1\right) = \mu\left(\hat{t}_2\widehat{\delta} - \hat{h}\widehat{\partial}_1\right)$$

from which the first statement follows.

The proof of the second statement follows similarly. □

By Theorem 2.4.31, homogeneous Legendre fields for a Frobenius manifold with charge d have homogeneous potentials, provided that the Legendre field in question has degree $\mu \neq d-2$. We note that the charge for the manifold with prepotential F is $d=3$, and the charge for the manifold with prepotential $\widehat{F}$ is $\hat{d}=-1$.

Corollary 3.3.17. *Let δ be a homogeneous Legendre field for F of degree $\mu \neq 1$, and let h be its homogeneous potential. Then $\widehat{\delta}$ as in (3.3.44) is homogeneous of degree $\hat{\mu}=\mu-2$.*

Let $\widehat{\delta}$ be a homogeneous Legendre field for $\widehat{F}$ of degree $\hat{\mu} \neq -3$, and let $\hat{h}$ be its homogeneous potential. Then δ as in (3.3.46) is homogeneous of degree $\mu=\hat{\mu}+2$.

Proof. By Lemma 3.3.16, it is enough to check that $\hat{h}\widehat{\partial}_1$ is homogeneous of degree μ. Recalling that $\hat{h}=-\hat{t}_2 h$ by Theorem 3.3.15, we have

$$\begin{aligned}\mathcal{L}_{\hat{E}}\left(\hat{h}\widehat{\partial}_1\right) &= \mathcal{L}_E\left(-\frac{1}{t_2}h\partial_1\right)\\ &= -\frac{1}{t_2}E(h)\partial_1 - \frac{2}{t_2}h\partial_1 + \frac{1}{t_2}h\partial_1(E).\end{aligned} \tag{3.3.55}$$

We therefore have $E(h)=(\mu-1)h$, which we can substitute into (3.3.55) to get

$$\mathcal{L}_{\hat{E}}\left(\hat{h}\widehat{\partial}_1\right) = \mu\hat{h}\widehat{\partial}_1.$$

The second statement can be verified by the same method. □

In principle, this means we can use results from §3.3.3 to describe homogeneous Legendre fields for $\widehat{F}$. The construction for $\widehat{\delta}$ in (3.3.44) still requires knowledge of the potential h, so in practice we will only use results from Theorem 3.3.14.

3.3.4.2 Homogeneous Legendre fields

Imposing homogeneity on $\widehat{\delta}=\hat{u}\widehat{\partial}_1+\hat{v}\widehat{\partial}_2$ of degree $\hat{\mu}$, as in § 3.1.1, gives us

$$\widehat{\delta}=\hat{u}\left(\hat{t}_1,\hat{t}_2\right)\widehat{\partial}_1+\hat{v}\left(\hat{t}_1,\hat{t}_2\right)\widehat{\partial}_2=\hat{t}_1^{\hat{\mu}+1}\widehat{U}(\hat{\omega})\widehat{\partial}_1+\hat{t}_1^{\hat{\mu}+2}\widehat{V}(\hat{\omega})\widehat{\partial}_2, \tag{3.3.56}$$

where $\hat{\omega}=8\hat{t}_1^{-2}\hat{t}_2$; compare ansatz (3.1.11). We see also that $\hat{\omega}=\omega=8t_1^{-2}t_2^{-1}$, where ω is as defined in §3.3.3, by the coordinate transformation in (3.3.43). The Legendre field condition, represented by equation (3.3.42), is then equivalent to the two differential equations

$$\widehat{U}'(\omega)=\frac{2(\hat{\mu}+2)}{\omega}\widehat{V}(\omega)-4\widehat{V}'(\omega), \tag{3.3.57}$$

$$\widehat{V}'(\omega)=\frac{\hat{\mu}+1}{8}\widehat{U}(\omega)-\frac{\omega}{4}\widehat{U}'(\omega). \tag{3.3.58}$$

Theorem 3.3.18. *Let $\widehat{\delta}$ be an arbitrary homogeneous Legendre field, as in (3.3.56). Then $\widehat{U}(\hat{\omega}) = \widehat{U}(\omega)$ satisfies*

$$\omega(1-\omega)\widehat{U}''(\omega) + \left(\frac{2\hat{\mu}-1}{2}\omega + 1\right)\widehat{U}'(\omega) - \frac{\hat{\mu}(\hat{\mu}+1)}{4}\widehat{U}(\omega) = 0, \tag{3.3.59}$$

and $\widehat{V}(\hat{\omega}) = \widehat{V}(\omega)$ satisfies

$$\omega(1-\omega)\widehat{V}''(\omega) + \frac{2\hat{\mu}+1}{2}\omega\widehat{V}'(\omega) - \frac{(\hat{\mu}+1)(\hat{\mu}+2)}{4}\widehat{V}(\omega) = 0. \tag{3.3.60}$$

Equations (3.3.59) and (3.3.60) are instances of the hypergeometric equation (3.2.1), with $(a, b, c, z) = \left(-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}, 1, \omega\right)$ and $(a, b, c, z) = \left(-\frac{\hat{\mu}+1}{2}, -\frac{\hat{\mu}+2}{2}, 0, \omega\right)$, respectively.

Proof. Differentiating (3.3.57) and (3.3.58) with respect to ω produces

$$\widehat{U}''(\omega) = -\frac{2(\hat{\mu}+2)}{\omega^2}\widehat{V}(\omega) + \frac{2(\hat{\mu}+2)}{\omega}\widehat{V}'(\omega) - 4\widehat{V}''(\omega), \tag{3.3.61}$$

$$\widehat{V}''(\omega) = \frac{\hat{\mu}-1}{8}\widehat{U}'(\omega) - \frac{\omega}{4}\widehat{U}''(\omega). \tag{3.3.62}$$

Equations (3.3.57), (3.3.58) and (3.3.62) can be used to eliminate $\widehat{V}$ in (3.3.30) and produce the hypergeometric equation in terms of $\widehat{U}$, with parameters $\{a, b\} = \{-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}\}$, $c = 1$. Similarly, $\widehat{U}$ can be eliminated from (3.3.62) to produce the hypergeometric equation for $\widehat{V}$ with parameters $\{a, b\} = \{-\frac{\hat{\mu}+1}{2}, -\frac{\hat{\mu}+2}{2}\}$, $c = 0$. □

We have δ and h for $\mu \in \{-1, 0, 1, 2, 3\}$ from Theorem 3.3.14. By Theorem 3.3.15 and Corollary 3.3.17, we can use these to construct the homogeneous fields $\widehat{\delta}$ for $\hat{\mu} \in \{-3, -2, 0, 1\}$. We obtain the field of degree $\hat{\mu} = -1$ directly, by solving (3.3.59) and (3.3.60).

Theorem 3.3.19. *Let $\widehat{\delta}$ given by (3.3.56) be a homogeneous Legendre field of degree $\hat{\mu} \in \{-3, -2, -1, 0, 1\}$. Then the components $\hat{u}$ and $\hat{v}$ of $\widehat{\delta}$ are given in Tables 3.6 and 3.7 respectively, where*

$$s = \sqrt{1 - \frac{8\hat{t}_2}{\hat{t}_1^2}}$$

and $c_1, c_2 \in \mathbb{C}$. The potentials $\hat{h} = \hat{h}\left(\hat{t}_1, \hat{t}_2\right)$ such that $\hat{u} = \widehat{\partial}_2\hat{h}$ and $\hat{v} = \widehat{\partial}_1\hat{h}$ are given in Table 3.8.

Proof. We consider two different cases for the value of $\hat{\mu}$.

Case 1: $\hat{\mu} \neq -1$.

We refer to Theorem 3.3.14 for Legendre fields δ of F, of degree of homogeneity $\mu \in \{-1, 0, 2, 3\}$. By Theorem 3.3.15 and Corollary 3.3.17, these fields correspond to Legendre

Table 3.6: First component of a homogeneous Legendre field in Case 4.

Degree $\hat{\mu}$	First component $\hat{u}$
-3	$4c_1\hat{t}_1^{-2}s^{-3}$
-2	$c_1\hat{t}_1^{-1}s^{-1}$
-1	$c_1\text{arctanh}\,(s) + c_2$
0	$24c_1\hat{t}_1\big(s - \text{arctanh}\,(s)\big) + c_2\hat{t}_1$
1	$c_1\left[18\hat{t}_1^2 s - 12\left(\hat{t}_1^2 + 4\hat{t}_2\right)\text{arctanh}\,(s)\right] + c_2\left(\hat{t}_1^2 + 4\hat{t}_2\right)$

First component $\hat{u} = \hat{u}(\hat{t}_1, \hat{t}_2)$ of a Legendre field $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ of degree $\hat{\mu}$. Constants $c_1, c_2 \in \mathbb{C}$ are as in Table 3.7.

Table 3.7: Second component of a homogeneous Legendre field in Case 4.

Degree $\hat{\mu}$	Second component $\hat{v}$
-3	$-8c_1\hat{t}_1^{-3}\hat{t}_2 s^{-3}$
-2	$c_2 - c_1 s^{-1}$
-1	$-\dfrac{c_1}{4}\hat{t}_1 s$
0	$3c_1\big(\hat{t}_1^2 s - 8\hat{t}_2\text{arctanh}\,(s)\big) + c_2\hat{t}_2$
1	$c_1\hat{t}_1\left[\left(\hat{t}_1^2 + 16\hat{t}_2\right)s - 24\hat{t}_2\text{arctanh}\,(s)\right] + 2c_2\hat{t}_1\hat{t}_2$

Second component $\hat{v} = \hat{v}(\hat{t}_1, \hat{t}_2)$ of a Legendre field $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ of degree $\hat{\mu}$. Constants $c_1, c_2 \in \mathbb{C}$ are as in Table 3.6.

fields $\widehat{\delta}$ for $\widehat{F}$ of degree of homogeneity $\hat{\mu} = \mu - 2$, where

$$\begin{aligned}\widehat{\delta} &= t_2\delta - h\partial_1 \\ &= \left(\hat{t}_2^{-1}u - h\right)\widehat{\partial}_1 - \hat{t}_2 v\widehat{\partial}_2\end{aligned}$$

under the coordinate transformations given in (3.3.43). The Legendre fields with $\hat{\mu} \in \{-3, -2, 0, 1\}$ therefore have their potential and components given by

$$\begin{aligned}\hat{h} &= -\hat{t}_2 h, \\ \hat{u} &= \hat{t}_2^{-1}u - h, \\ \hat{v} &= -\hat{t}_2 v,\end{aligned} \tag{3.3.63}$$

where u, v, h can be read off from Tables 3.3, 3.4, and 3.5.

Case 2: $\hat{\mu} = -1$.

While we can use the relations in (3.3.63) to produce a new Legendre field $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ from the Legendre field δ of degree of homogeneity $\mu = 1$, this new field will not necessarily

Table 3.8: Potential function of a homogeneous Legendre field in Case 4.

Degree $\hat{\mu}$	**Potential** $\hat{h}$
-3	$c_1 s^{-1} - c_2$
-2	$c_2\hat{t}_1 - c_1\hat{t}_1 s$
-1	$c_1\left[\hat{t}_2\text{arctanh}\left(s\right) - \frac{\hat{t}_1^2 s}{8}\right] + c_2\hat{t}_2$
0	$c_1\hat{t}_1\left[\left(\hat{t}_1^2 + 16\hat{t}_2\right)s - 24\hat{t}_2\text{arctanh}\left(s\right)\right] + c_2\hat{t}_1\hat{t}_2$
1	$c_1\left[\frac{\hat{t}_1^2 s}{4}\left(\hat{t}_1^2 + 52\hat{t}_2\right) - 12\hat{t}_2\left(\hat{t}_1^2 + 2\hat{t}_2\right)\text{arctanh}\left(s\right)\right] + c_2\hat{t}_2\left(\hat{t}_1^2 + 2\hat{t}_2\right)$

Potential $\hat{h} = \hat{h}(\hat{t}_1, \hat{t}_2)$ for a Legendre field $\widehat{\delta}$ of degree $\hat{\mu}$ such that $\widehat{\delta} = \text{grad}\hat{h}$. Constants $c_1, c_2 \in \mathbb{C}$ are as in Tables 3.6, 3.7.

be homogeneous. Indeed, by this method we obtain

$$\begin{aligned}
\hat{u} &= 4c_1\left(1 + \log\hat{t}_2\right) + 8c_2\text{arctanh}\left(s\right),\\
\hat{v} &= 2c_1\hat{t}_1 - 2c_2\hat{t}_1 s,\\
\hat{h} &= c_1\left(\hat{t}_1^2 + 4\hat{t}_2\log\hat{t}_2\right) + c_2\left[8\text{arctanh}\left(s\right) - \hat{t}_1^2 s\right],
\end{aligned}$$

where $\widehat{\delta} = \hat{u}\widehat{\partial}_1 + \hat{v}\widehat{\partial}_2$ is a Legendre field for $\widehat{F}$ — as can be easily verified by substitution into (3.3.42). However, this $\widehat{\delta}$ is only homogeneous (of degree $\hat{\mu} = -1$ as expected) for $c_1 = 0$.

We instead solve differential equations (3.3.59) and (3.3.60) for $\widehat{U}(\omega)$, $\widehat{V}(\omega)$ such that $\hat{u} = \widehat{U}$, $\hat{v} = \hat{t}_1\widehat{V}$ by (3.3.56). Recall that $\hat{\omega} = 8\hat{t}_1^{-2}\hat{t}_2$. When $\hat{\mu} = -1$, equation (3.3.59) becomes

$$2\omega(\omega - 1)\widehat{U}''(\omega) + (3\omega - 2)\widehat{U}'(\omega) = 0.$$

Substituting $Y(\omega) = \widehat{U}'(\omega)$, we can rearrange this ODE to get

$$\int \frac{Y'(\omega)}{Y(\omega)}\text{d}\omega = \int \frac{3\omega - 2}{2(\omega - \omega^2)}\text{d}\omega,$$

which can be solved for

$$Y(\omega) = \frac{c_1}{\omega\sqrt{1 - \omega}}.$$

Thus, after integration and rescaling constants, we arrive at

$$\widehat{U}(\omega) = c_1\text{arctanh}\left(\sqrt{1 - \omega}\right) + c_2, \tag{3.3.64}$$

equivalently,

$$\hat{u}\left(\hat{t}_1, \hat{t}_2\right) = c_1 \text{arctanh}\left(s\right) + c_2.$$

Substituting (3.3.64) into (3.3.58) gives us

$$\widehat{V}(\omega) = -\frac{c_1}{4}\sqrt{1-\omega},$$

so

$$\hat{v}\left(\hat{t}_1, \hat{t}_2\right) = -\frac{c_1}{4}\hat{t}_1 s.$$

The potential $\hat{h}$ for $\hat{\mu} = -1$ given in Table 3.8 can then be verified through differentiation. □

To use the formulation in Theorem 3.3.15 we need the potential h, which we do not have for generic μ; instead we now solve (3.3.59) and (3.3.60) directly for $\hat{\mu} \notin \{-3, -2, -1, 0, 1\}$. We only provide solutions around $\omega = 0$, which are split into the same two cases as in § 3.3.3.

Theorem 3.3.20. *Let $\widehat{\delta}$ given by (3.3.56) be a homogeneous Legendre field of degree $\hat{\mu} \notin \mathbb{N} \cup \{-1, -2, -3\}$. Then $\widehat{U} = c_1 y_1 + c_2 y_2$, $\widehat{V} = \frac{\hat{\mu}+1}{8}\left(c_1 y_3 + c_2 y_4\right)$ in the neighbourhood of $\omega = 0$, where:*

$$\begin{aligned}
y_1(\omega) &= {}_2F_1\left(-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}; 1; \omega\right),\\
y_2(\omega) &= y_1(\omega)\log\omega + \sum_{k=0}^{\infty}\frac{\left(-\frac{\hat{\mu}}{2}\right)_k\left(-\frac{\hat{\mu}+1}{2}\right)_k}{(k!)^2}\omega^k \times\\
&\qquad \left[\psi\left(k-\frac{\hat{\mu}}{2}\right) + \psi\left(k - \frac{\hat{\mu}+1}{2}\right) - 2\psi(1+k)\right],\\
y_3(\omega) &= \omega\, {}_2F_1\left(\frac{1-\hat{\mu}}{2}, -\frac{\hat{\mu}}{2}; 2; \omega\right),\\
y_4(\omega) &= y_3\log\omega + \frac{4}{(1+\hat{\mu})(2+\hat{\mu})} + \sum_{k=0}^{\infty}\frac{\left(\frac{1-\hat{\mu}}{2}\right)_k\left(-\frac{\hat{\mu}}{2}\right)_k}{(k-1)!k!}\omega^{k+1}\times\\
&\qquad \left[\psi\left(\frac{1-\hat{\mu}}{2}+k\right) + \psi\left(k-\frac{\hat{\mu}}{2}\right) - \psi\left(2+k\right) - \psi\left(1+k\right)\right].
\end{aligned}$$

Proof. We solve (3.3.59) with case (i) from §3.2.1, taking $c = 1$. We find y_1 as written in the theorem statement, and $y_2 = G\left(-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}; 1; \omega\right)$ where G is defined in (3.2.5). Since we are considering $\hat{\mu} \notin \mathbb{N}$, parameters $a, b \notin \mathbb{Z}_{\leq 0}$ and the correction term (3.2.6) is equal to zero.

To solve (3.3.60), we use case (ii) in §3.2.1 since $c = 0$. The first solution, y_3, is as written, and the second solution is $y_4 = \omega G\left(\frac{1-\hat{\mu}}{2}, -\frac{\hat{\mu}}{2}; 2; \omega\right)$ with correction term H again equal to zero; we then have $\widehat{V}(\omega) = c_3 y_3 + c_4 y_4$.

The constants c_1, c_2, c_3, c_4 can be related by comparing terms in (3.3.57) and (3.3.58), using the same methods as in the proof of Theorem 3.3.8. We find $c_3 = \frac{\hat{\mu}+1}{8} c_1$ and $c_4 = \frac{\hat{\mu}+1}{8} c_2$. □

Theorem 3.3.21. *Let $\widehat{\delta}$ given by (3.3.56) be a homogeneous Legendre field of degree $\hat{\mu} \in \mathbb{N} \setminus \{0,1\}$. Define $m, l \in \mathbb{Z}_{<0}$ and $p, q \notin \mathbb{Z}$ such that*

$$(m,p,l,q) = \begin{cases} \left(-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}, -\frac{\hat{\mu}}{2}, \frac{1-\hat{\mu}}{2}\right) & \text{if } \hat{\mu} \text{ is even} \\ \left(-\frac{\hat{\mu}+1}{2}, -\frac{\hat{\mu}}{2}, \frac{1-\hat{\mu}}{2}, -\frac{\hat{\mu}}{2}\right) & \text{if } \hat{\mu} \text{ is odd.} \end{cases}$$

Then $\widehat{U} = c_1 y_1 + c_2 y_2$, $\widehat{V} = \frac{\hat{\mu}+1}{8}\left(c_1 y_3 + c_2 y_4\right)$ in the neighbourhood of $\omega = 0$, where

$$\begin{aligned} y_1(\omega) &= {}_2F_1\left(m, p; 1; \omega\right), \\ y_2(\omega) &= y_1(\omega)\log\omega + (-1)^m(-m)! \sum_{k=1-m}^{\infty} \frac{(k+m-1)!(p)_k}{(k!)^2}\omega^k \\ &\quad + \sum_{k=0}^{-m} \frac{(m)_k\,(p)_k}{(k!)^2}\omega^k\left[\psi\left(1-m-k\right) + \psi\left(p+k\right) - 2\psi(1+k)\right], \\ y_3(\omega) &= \omega\, {}_2F_1\left(l, q; 2; \omega\right), \\ y_4(\omega) &= y_3 \log\omega + \frac{4}{(1+\hat{\mu})(2+\hat{\mu})} + (-1)^{-l}(-l)! \sum_{k=1-l}^{\infty} \frac{(k+l-1)!(q)_k}{k!(k+1)!}\omega^{k+1} \\ &\quad + \sum_{k=0}^{-l} \frac{(l)_k\,(q)_k}{(k+1)!k!}\omega^{k+1}\left[\psi\left(1-l-k\right) + \psi\left(q+k\right) - \psi\left(2+k\right) - \psi\left(1+k\right)\right]. \end{aligned}$$

Proof. From §3.2.1, case (i), we find $y_1(\omega)$ as written. Since $\hat{\mu} \in \mathbb{N}$, we have $\{a,b\} = \{-\frac{\hat{\mu}}{2}, -\frac{\hat{\mu}+1}{2}\} = \{m,p\}$ for some $m \in \mathbb{Z}_{<0}$, $p \notin \mathbb{Z}$. Hence, we can write $y_2(\omega) = G\left(m,p;1;\omega\right)$ and $\widehat{U}(\omega) = c_1 y_1(\omega) + c_2 y_2(\omega)$.

Using case (ii) to solve (3.3.60) gives us y_3 as written and $y_4(\omega) = G\left(l,q;2;\omega\right)$, where $\{a,b\} = \{-\frac{\hat{\mu}}{2}, \frac{1-\hat{\mu}}{2}\} = \{l,q\}$ for some $l \in \mathbb{Z}_{<0}$, $q \notin \mathbb{Z}$. We then have $\widehat{V}(\omega) = c_3 y_3(\omega) + c_4 y_4(\omega)$.

The constants c_1, c_2, c_3, c_4 can be related by comparing terms in (3.3.57) and (3.3.58), using the same methods as in the proof of Theorem 3.3.8. We find $c_3 = \frac{\hat{\mu}+1}{8} c_1$ and $c_4 = \frac{\hat{\mu}+1}{8} c_2$. □

3.3.5 Case 5

Consider the prepotential given by

$$F(t_1, t_2) = \frac{1}{2}t_1^2 t_2 + t_2^k \tag{3.3.65}$$

with Euler field

$$E = t_1\partial_1 + (1-d)t_2\partial_2 = t_1\partial_1 + \frac{2}{k-1}t_2\partial_2,$$

where we recall $d \neq -1, 1, 3$ and $k = \frac{3-d}{1-d} \neq 0, 1, 2$. Then

$$c_{222} = F_{222} = k(k-1)(k-2)t_2^{k-3},$$

so equation (3.1.4) becomes

$$\frac{\partial u}{\partial t_2} = k(k-1)(k-2)t_2^{k-3}\frac{\partial v}{\partial t_1}. \tag{3.3.66}$$

3.3.5.1 Homogeneous Legendre fields

We impose homogeneity on a Legendre field δ so that we may use (3.1.11) to write

$$\delta = u\left(t_1, t_2\right)\partial_1 + v\left(t_1, t_2\right)\partial_2 = t_1^{\mu+1}U(\omega)\partial_1 + t_1^{\mu+\frac{2}{k-1}}V(\omega)\partial_2 \tag{3.3.67}$$

with $\omega = \frac{4k(k-2)}{k-1}t_1^{-2}t_2^{k-1}$. The Legendre field condition, given by (3.1.4) and (3.3.66), can then be represented by the equations

$$U'(\omega) = \frac{2+\mu(k-1)}{4}\left(\frac{(k-1)\omega}{4k(k-2)}\right)^{\frac{1}{1-k}}V(\omega) - 2k(k-2)\left(\frac{(k-1)\omega}{4k(k-2)}\right)^{\frac{k-2}{k-1}}V'(\omega), \tag{3.3.68}$$

$$V'(\omega) = \frac{\mu+1}{4k(k-2)}\left(\frac{(k-1)\omega}{4k(k-2)}\right)^{\frac{2-k}{k-1}}U(\omega) - \frac{2}{k-1}\left(\frac{(k-1)\omega}{4k(k-2)}\right)^{\frac{1}{k-1}}U'(\omega). \tag{3.3.69}$$

Theorem 3.3.22. *Let δ given by (3.3.67) be a homogeneous Legendre field. Then $U(\omega)$ satisfies*

$$\omega(1-\omega)U''(\omega) + \left(\frac{1}{k-1} + \frac{2\mu-1}{2}\omega\right)U'(\omega) - \frac{\mu(\mu+1)}{4}U(\omega) = 0, \tag{3.3.70}$$

and $V(\omega)$ satisfies

$$0 = \omega(1-\omega)V''(\omega) + \left(\frac{k-2}{k-1} + \frac{7-3k+2\mu(k-1)}{2(k-1)}\omega\right)V'(\omega) - \frac{\big(\mu(k-1)+3-k\big)\big(\mu(k-1)+2\big)}{4(k-1)^2}V(\omega). \tag{3.3.71}$$

Equations (3.3.70) and (3.3.71) are instances of the hypergeometric equation, (3.2.1), with $(a,b,c,z) = \left(-\frac{\mu}{2}, -\frac{\mu+1}{2}, \frac{1}{k-1}, \omega\right)$ and $(a,b,c,z) = \left(\frac{k-3}{2(k-1)} - \frac{\mu}{2}, \frac{k-3}{2(k-1)} - \frac{\mu+1}{2}, \frac{k-2}{k-1}, \omega\right)$, respectively.

Proof. To start with, we will work in terms of $A(z) = U(\omega)$, $B(z) = V(\omega)$, with $z = t_1^{\frac{2}{1-k}} t_2$. Substituting the ansatzes (3.1.11) into the Legendre field condition (3.1.4), (3.1.5) with $\alpha = \frac{2}{k-1}$, we find

$$z^{3-k}A'(z) = k(k-2)(2+\mu(k-1))B(z) - 2k(k-2)zB'(z), \tag{3.3.72}$$

$$B'(z) = (\mu+1)A(z) + \frac{2}{1-k}zA'(z). \tag{3.3.73}$$

Differentiating (3.3.72) and (3.3.73) with respect to z produces

$$(3-k)z^{2-k}A'(z) + z^{3-k}A''(z) = \mu k(k-1)(k-2)B'(z) - 2k(k-2)zB''(z), \tag{3.3.74}$$

$$B''(z) = \left(\mu + \frac{3-k}{1-k}\right)A'(z) + \frac{2}{1-k}zA''(z). \tag{3.3.75}$$

Using (3.3.73) and (3.3.75) to eliminate $B'(z)$ and $B''(z)$ in (3.3.74), we obtain the equation

$$\mu(\mu+1)k(k-1)^2(k-2)A(z) - 2k(k-2)(2\mu(k-1)+k-3)zA'(z) + (k-1)(k-3)z^{2-k}A'(z) + 4k(k-2)z^2A''(z) - (k-1)z^{3-k}A''(z) = 0. \tag{3.3.76}$$

Under the change of variables

$$\omega = \frac{4k(k-2)}{k-1}z^{k-1} \tag{3.3.77}$$

with $U(\omega) = A(z)$, equation (3.3.76) becomes (3.3.70) after rescaling by a factor of $-4k(k-1)^2(k-2)$.

We can rearrange (3.3.75) to obtain

$$A''(z) = \frac{1-k}{2z}B''(z) + \left(\frac{\mu(k-1)}{2z} + \frac{k-3}{2z}\right)A'(z),$$

which can substituted along with (3.3.72) into (3.3.74) to get

$$k(k-2)\big(\mu(k-1)+3-k\big)\big(\mu(k-1)+2\big)B(z)+2k(k-2)\big(k-3-2\mu(k-1)\big)zB'(z) \\ +z^2\left(4k(k-2)-(k-1)z^{1-k}\right)B''(z)=0. \quad (3.3.78)$$

After the same change of variables (3.3.77) with $V(\omega) = B(z)$, and some rescaling, the equation (3.3.78) becomes (3.3.71). □

We will consider solutions to the equations (3.3.70) and (3.3.71) around $\omega = 0$ for generic k, that is

$$k \in \mathbb{C} \setminus \left(\{0,1,2\} \cup \left\{ \frac{n \pm 1}{n} | n \in \mathbb{N}^{\times} \right\} \right). \quad (3.3.79)$$

Note that the set described in (3.3.79) contains $\mathbb{Z} \setminus \{0,1,2\}$. If k lies outside the set (3.3.79), the parameter c for both equations (3.3.70), (3.3.71) is an integer, meaning that these equations must be solved using different methods.

Theorem 3.3.23. *Let δ given by (3.3.67) be a homogeneous Legendre field with arbitrary μ, and k as in (3.3.79). Then, the formulas for $U(\omega)$, $V(\omega)$ in the neighbourhood of $\omega = 0$ are as follows, where $c_1, c_2 \in \mathbb{C}$.*

$$U(\omega) = c_1 y_1(\omega) + c_2 y_2(\omega),$$
$$V(\omega) = \frac{4c_2(k-2)}{2(k-1)+\mu(k-1)^2}\left(\frac{k-1}{4k(k-2)}\right)^{\frac{1}{k-1}} y_3(\omega) + c_1(\mu+1)\left(\frac{k-1}{4k(k-2)}\right)^{\frac{1}{k-1}} y_4(\omega),$$

where

$$y_1(\omega) = {}_2F_1\left(-\frac{\mu}{2}, -\frac{\mu+1}{2}; \frac{1}{k-1}; \omega\right),$$
$$y_2(\omega) = \omega^{\frac{k-2}{k-1}}{}_2F_1\left(\frac{k-2}{k-1} - \frac{\mu}{2}, \frac{k-2}{k-1} - \frac{\mu+1}{2}; \frac{2k-3}{k-1}; \omega\right),$$
$$y_3(\omega) = {}_2F_1\left(\frac{k-3}{2(k-1)} - \frac{\mu}{2}, \frac{k-3}{2(k-1)} - \frac{\mu+1}{2}; \frac{k-2}{k-1}; \omega\right),$$
$$y_4(\omega) = \omega^{\frac{1}{k-1}}{}_2F_1\left(\frac{1-\mu}{2}, -\frac{\mu}{2}; \frac{k}{k-1}; \omega\right).$$

Proof. Starting with (3.3.70), we see that the parameter $c = \frac{1}{k-1}$ is never an integer for the allowed values of k. As per the discussion in §3.2.1, the fundamental solutions for U are given by $y_1 = {}_2F_1(a,b;c;\omega)$ and $y_2 = \omega^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; \omega)$ with $\{a,b\} = \left\{-\frac{\mu}{2}, -\frac{\mu+1}{2}\right\}$.

Similarly, for equation (3.3.71) we have $c = \frac{k-2}{k-1}$, which also cannot be an integer. A pair of linearly independent solutions for V is then $y_3 = {}_2F_1(a,b;c;\omega)$ and $y_4 = \omega^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; \omega)$ with $\{a,b\} = \left\{\frac{k-3}{2(k-1)} - \frac{\mu}{2}, \frac{k-3}{2(k-1)} - \frac{\mu+1}{2}\right\}$.

Writing $U = c_1 y_1 + c_2 y_2$ and $V = c_3 y_3 + c_4 y_4$, we can relate c_1, c_2, c_3, c_4 via equations (3.3.68) and (3.3.69) with the same methods as in the proof of Theorem 3.3.8. The relations can be most easily seen by considering constant terms and terms of degree $-\frac{1}{k-1}$ in the expansion of both sides of (3.3.68) with respect to ω. □

3.3.6 Case 6

The Frobenius manifold with prepotential

$$F(t_1, t_2) = \frac{1}{6}t_1^3 + \frac{1}{2}t_1 t_2^2 + \frac{k}{6}t_2^3, \tag{3.3.80}$$

where $k \neq 0$, has metric $\eta = I_2$ and Euler field

$$E = t_1\partial_1 + t_2\partial_2.$$

The multiplication is given by

$$\partial_i \circ \partial_j = \begin{cases} \partial_1 & i = j = 1; \\ \partial_2 & i \neq j; \\ k\partial_2 & i = j = 2. \end{cases} \tag{3.3.81}$$

As in Section 3.1, we consider a Legendre field $\delta = u\partial_1 + v\partial_2$ which satisfies the Legendre field condition (3.1.1). We can rewrite δ in terms of its potential $h = h(t_1, t_2)$ such that $\delta = \eta^{ij}\partial_j(h)\partial_i$. The components u, v are then given by $u = \partial_1(h)$ and $v = \partial_2(h)$.

Theorem 3.3.24. *Let δ be a Legendre field with components $u = u(t_1, t_2)$, $v = v(t_1, t_2)$ and potential h such that $u = \partial_1(h)$ and $v = \partial_2(h)$. The potential h has the form*

$$h(t_1, t_2) = c_1\left(t_1^2 + t_2^2\right) + c_2 t_1 + c_3 t_2 + c_4 \tag{3.3.82}$$

for some $c_i \in \mathbb{C}$.

Proof. Let $X = x\partial_1 + y\partial_2$ for some functions $x = x(t_1, t_2)$, $y = y(t_1, t_2)$. Using the multiplication (3.3.81), the right-hand side of (3.1.1) becomes

$$\begin{aligned} X \circ \nabla_e \delta &= (x\partial_1 + y\partial_2) \circ (\partial_1(u)\partial_1 + \partial_1(v)\partial_2) \\ &= x\partial_1(u)\partial_1 + y\partial_1(u)\partial_2 + x\partial_1(v)\partial_2 + ky\partial_1(v)\partial_2. \end{aligned} \tag{3.3.83}$$

Equating (3.1.3) and (3.3.83), we compare the ∂_1 and ∂_2 components to obtain the equations

$$\frac{\partial u}{\partial t_2} = 0, \tag{3.3.84}$$

$$\frac{\partial v}{\partial t_2} = \frac{\partial u}{\partial t_1} + k\frac{\partial v}{\partial t_1}. \tag{3.3.85}$$

We rewrite (3.3.84), (3.3.85) in terms of h, so that they respectively become

$$\frac{\partial^2 h}{\partial t_1 \partial t_2} = 0, \tag{3.3.86}$$

$$\frac{\partial^2 h}{\partial t_2^2} = \frac{\partial^2 h}{\partial t_1^2} + k\frac{\partial^2 h}{\partial t_1 \partial t_2}. \tag{3.3.87}$$

Substituting (3.3.86) into (3.3.87), we obtain the wave equation,

$$\frac{\partial^2 h}{\partial t_2^2} = \frac{\partial^2 h}{\partial t_1^2}. \tag{3.3.88}$$

Together, (3.3.86) and (3.3.88) imply that $h = G_1(t_1) + G_2(t_2)$ with $G_1''(t_1) = G_2''(t_2)$ for some functions G_1, G_2; that is, h is quadratic in t_1, t_2 with no term of the form $t_1 t_2$. □

3.3.6.1 Homogeneous Legendre fields

We now impose homogeneity on δ, so that condition (3.1.7) is satisfied.

Theorem 3.3.25. *Let $\delta = u\partial_1 + v\partial_2$ be a homogeneous Legendre field of degree μ. Then δ must have degree $\mu \in \{0, -1\}$. Furthermore, the potential h of δ is given by*

$$h = c_1\left(t_1^{\mu+2} + t_2^{\mu+2}\right) + c_2$$

for some $c_1, c_2 \in \mathbb{C}$, and the components of δ are

$$u = (\mu + 2)c_1 t_1^{\mu+1},$$
$$v = (\mu + 2)c_1 t_2^{\mu+1}.$$

Proof. As in § 3.1.1, the left-hand side of the homogeneity condition (3.1.7) is

$$\begin{aligned}\mathcal{L}_E\delta &= E(\delta) - \delta(E) \\ &= t_1\partial_1(u)\partial_1 + t_1\partial_1(v)\partial_2 + t_2\partial_2(u)\partial_1 + t_2\partial_2(v)\partial_2 - u\partial_1 - v\partial_2.\end{aligned} \tag{3.3.89}$$

By Theorem 3.3.24, we have

$$\partial_2(u) = 0 \tag{3.3.90}$$

and

$$\partial_1(v) = 0. \tag{3.3.91}$$

Using (3.3.90) and (3.3.91), we see that (3.3.89) becomes

$$\mathcal{L}_E\delta = t_1\partial_1(u)\partial_1 + t_2\partial_2(v)\partial_2 - u\partial_1 - v\partial_2. \tag{3.3.92}$$

We substitute (3.3.92) into the homogeneity condition (3.1.7) and compare components to obtain the equations

$$t_1\partial_1(u) = (\mu+1)u,$$
$$t_2\partial_2(v) = (\mu+1)v.$$

These equations are both easily solved to find

$$u = \alpha t_1^{\mu+1}, \tag{3.3.93}$$
$$v = \beta t_2^{\mu+1}, \tag{3.3.94}$$

for some $\alpha, \beta \in \mathbb{C}$. However, by Theorem 3.3.24, we also have that

$$u = 2c_1t_1 + c_2,$$
$$v = 2c_1t_2 + c_3$$

for some $c_i \in \mathbb{C}$. Therefore μ must equal either 0 or -1, and the form of the potential follows from (3.3.93), (3.3.94). □

3.4 Twisted Legendre fields for the A_2 almost dual Frobenius manifold

When $k = 4$, the Frobenius manifold structure associated with (3.3.65) is defined on the orbit space $\mathbb{C}^2/A_2$ [18]. Recall that the prepotential is

$$F(t) = \frac{1}{2}t_1^2t_2 + t_2^4 \tag{3.4.1}$$

and the Euler field is

$$E(t) = E^i\partial_{t_i} = t_1\partial_{t_1} + \frac{2}{3}t_2\partial_{t_2}. \tag{3.4.2}$$

The almost dual prepotential, $F^{\star}(z)$, is equivalent to the rational solution

$$F_{A_2}^{\mathrm{rat}}(x) = \sum_{\alpha \in A_2} \alpha(x)^2 \log \alpha(x).$$

We use the realisation $A_2 = \left\{ \sqrt{\gamma} e^1, \sqrt{\gamma} e^2, \sqrt{\gamma}\left(e^1 - e^2\right) \right\}$ with arbitrary $\gamma \in \mathbb{C}^{\times}$, and basis $e^1, e^2 \in \left(\mathbb{C}^2\right)^*$ being dual to the basis $\{e_1 = \partial x_1, e_2 = \partial x_2\} \subset \mathbb{C}^2$ for $x = (x_1, x_2)$. Relations between coordinate systems are made explicit in §3.4.1, where we construct the intersection form and the almost dual prepotential. By Theorem 2.4.24, if δ is a Legendre field for $F(t)$, the twisted field $E \circ \delta$ is a Legendre field for $F^{\star}$. We saw in Proposition 2.4.26 that a homogenous δ produces a homogeneous twisted field, so the general results from §3.3.5 may be used to produce Legendre fields for the almost dual.

The focus of this section is on the question “When is the twisted Legendre field flat?”; that is, on finding a Legendre field δ such that the twisted field $E \circ \delta$ is flat (with respect to the Levi-Civita connection of the intersection form). We compute the twisted Legendre field for arbitrary δ, then use this expression along with Theorem 3.3.23 to find the family of Legendre fields δ for which $E \circ \delta$ is flat.

3.4.1 Almost dual structures on $\mathbb{C}^2/A_2$

Following [18, Example 4.1] with a change of notation, the Frobenius structure on $\mathbb{C}^2/A_2$ has associated prepotential

$$\widetilde{F}(\tilde{t}) = \frac{1}{2}\tilde{t}_1^2 \tilde{t}_2 + 81 \tilde{t}_2^4$$

in the Saito coordinates $\tilde{t}$, where $\tilde{t}_1 = z^3 + \overline{z}^3$, $\tilde{t}_2 = \frac{1}{6} z\overline{z}$. The flat coordinates of the intersection form are z_1, z_2 such that $z = z_1 + i z_2$. Under some rescaling, the prepotential (3.4.1) is given by $F = \frac{1}{9}\widetilde{F}$ with $\tilde{t}_1 = 9t_1$, $\tilde{t}_2 = t_2$; we then have

$$\begin{aligned} t_1 &= \frac{1}{9}\left(z^3 + \overline{z}^3\right) = \frac{2}{9}\left(z_1^3 - 3z_1 z_2^2\right), \\ t_2 &= \frac{1}{6} z\overline{z} = \frac{1}{6}\left(z_1^2 + z_2^2\right). \end{aligned} \tag{3.4.3}$$

The cotangent structure constants are given by

$$c_k^{ij}(t) = \eta^{ip}\eta^{jq} c_{pqk}(t) = \begin{cases} 24t_2 & (i,j,k) = (1,1,2) \\ 1 & (i,j,k) = (2,1,1), (1,2,1), (2,2,2) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.4}$$

Then, using formulas (2.3.8), (3.4.2), the intersection form g^{-1} is given by

$$g^{-1}(t) = \begin{pmatrix} 16t_2^2 & t_1 \\ t_1 & \frac{2}{3}t_2 \end{pmatrix},$$

and the metric g is given by

$$g(t) = \frac{1}{32t_2^3 - 3t_1^2} \begin{pmatrix} 2t_2 & -3t_1 \\ -3t_1 & 48t_2^2 \end{pmatrix}. \tag{3.4.5}$$

Recall that the discriminant locus Σ is

$$\Sigma = \{t \,|\, \det g^{-1}(t) = 0\} \tag{3.4.6}$$

where

$$\det g^{-1} = \frac{32}{3}t_2^3 - t_1^2. \tag{3.4.7}$$

The almost dual prepotential is defined on $\mathbb{C}^2 \setminus \Sigma$.

For consistency with notation used in Chapter 5, we introduce another set of coordinates $x_1, x_2 \in \mathbb{C}$ defined by

$$\begin{aligned} z &= \sqrt{\frac{\gamma}{3}}\left(\left(1 - i\sqrt{3}\right) x_1 + \left(1 + i\sqrt{3}\right) x_2 \right), \\ \overline{z} &= \sqrt{\frac{\gamma}{3}}\left(\left(1 + i\sqrt{3}\right) x_1 + \left(1 - i\sqrt{3}\right) x_2 \right), \end{aligned} \tag{3.4.8}$$

for z as in (3.4.3) and arbitrary $\gamma \in \mathbb{C}^\times$. Since z_1, z_2 are flat coordinates for the intersection form g and are related to x_1, x_2 by a linear transformation, the x coordinates are also flat coordinates for g.

Using (3.4.3), (3.4.8), we have

$$\begin{aligned} t_1 &= -\frac{8\sqrt{3}\gamma^{3/2}}{81}\left(2x_1^3 - 3x_1^2x_2 - 3x_1x_2^2 + 2x_2^3\right), \\ t_2 &= \frac{2\gamma}{9}\left(x_1^2 - x_1x_2 + x_2^2\right). \end{aligned} \tag{3.4.9}$$

The Jacobian of $t(x)$, given by

$$\begin{aligned} J_t(x) &= \begin{pmatrix} \frac{\partial t_1}{\partial x_1} & \frac{\partial t_1}{\partial x_2} \\ \frac{\partial t_2}{\partial x_1} & \frac{\partial t_2}{\partial x_2} \end{pmatrix} \\ &= \frac{2\gamma}{9}\begin{pmatrix} 4\sqrt{\frac{\gamma}{3}}\left(x_2^2 + 2x_1x_2 - 2x_1^2\right) & 4\sqrt{\frac{\gamma}{3}}\left(x_1^2 + 2x_1x_2 - 2x_2^2\right) \\ 2x_1 - x_2 & 2x_2 - x_1 \end{pmatrix}, \end{aligned} \tag{3.4.10}$$

has determinant $\det J_t(x) = \frac{16\gamma^{5/2}}{9\sqrt{3}} x_1 x_2 (x_2 - x_1)$. By (3.4.6), (3.4.7), the discriminant locus is defined in the x coordinates as

$$\Sigma = \left\{ x \mid \frac{64\gamma^3}{81} x_1^2 x_2^2 (x_1 - x_2)^2 = 0 \right\},$$

so $J_t(x)$ is degenerate only on Σ and we may consider its inverse on the complement of Σ. By the Inverse Function Theorem, the inverse of $J_t(x)$ is the Jacobian of $x(t)$: that is,

$$\begin{aligned} (J_t(x))^{-1} = J_x(t) &= \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \end{pmatrix} \\ &= \frac{1}{2\gamma x_1 x_2 (x_2 - x_1)} \begin{pmatrix} \frac{\sqrt{3}}{4\sqrt{\gamma}}(2x_2 - x_1) & (2x_2^2 - 2x_1 x_2 - x_1^2) \\ \frac{\sqrt{3}}{4\sqrt{\gamma}}(x_2 - 2x_1) & (x_2^2 + 2x_1 x_2 - 2x_1^2) \end{pmatrix}. \end{aligned} \tag{3.4.11}$$

We are now equipped to describe the almost dual structures on $\mathbb{C}^2/A_2 \setminus \Sigma$ in the x coordinates.

Proposition 3.4.1 (cf. [20]). *Let $F(t)$ be as written in (3.4.1). The almost dual prepotential $F^\star$ can be written*

$$F^\star(x) = F_{A_2}^{rat}(x) = \gamma x_1^2 \log x_1 + \gamma x_2^2 \log x_2 + \gamma (x_1 - x_2)^2 \log (x_1 - x_2)^2, \tag{3.4.12}$$

where x_1, x_2 are defined in (3.4.9).

Proof. By Proposition 2.4.6, $F^\star(x)$ satisfies

$$\frac{\partial^3 F^\star(x)}{\partial x_i \partial x_j \partial x_k} = g_{ia}(x) g_{jb}(x) \frac{\partial t_\gamma}{\partial x_k} \frac{\partial x_a}{\partial t_\alpha} \frac{\partial x_b}{\partial t_\beta} c_\gamma^{\alpha\beta}(t), \tag{3.4.13}$$

where the metric g transforms tensorially via

$$g_{ij}(x) = \frac{\partial t_\alpha}{\partial x_i} \frac{\partial t_\beta}{\partial x_j} g_{\alpha\beta}(t). \tag{3.4.14}$$

Using (3.4.5), (3.4.10) with the expression (3.4.14), we find

$$g(x) = \frac{2\gamma}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{3.4.15}$$

We now use (3.4.4), (3.4.9), (3.4.10), (3.4.11), (3.4.15) to make substitutions in (3.4.13).

This leads to the third-order derivatives of $F^\star(x)$ as follows:

$$F^\star_{ijk}(x) = \begin{cases} \frac{2\gamma}{x_1(x_1-x_2)}(2x_1 - x_2) & i=j=k=1, \\ \frac{2\gamma}{x_2-x_1} & i=j=1, k=2, \\ \frac{2\gamma}{x_1-x_2} & i=j=2, k=1, \\ \frac{2\gamma}{x_2(x_1-x_2)}(x_1 - 2x_2) & i=j=k=2. \end{cases} \tag{3.4.16}$$

The derivatives in (3.4.16) are equal to the third-order derivatives of the ansatz (3.4.12), proving the statement. □

Remark 3.4.2. The charge of F is given by $d = \frac{k-3}{k-1} = \frac{1}{3}$. By Remark 2.4.5, we expect that the Euler field in x is

$$E(x) = \frac{1}{3}x_1\partial_{x_1} + \frac{1}{3}x_2\partial_{x_2}. \tag{3.4.17}$$

This agrees with the results of computing $E(x) = E^i \frac{\partial x_j}{\partial t_i}\partial_{x_j}$, where $E(t) = E^i\partial_{t_i}$ is given by (3.4.2) and the partial derivatives $\frac{\partial x_i}{\partial t_j}$ are given by (3.4.11).

Proposition 3.4.3. *Given an arbitrary vector field $\delta = u(t)\partial_{t_1} + v(t)\partial_{t_2}$ for $F(t)$, the twisted field $E \circ \delta$ may be written in coordinates t, z, x respectively as*

$$(E \circ \delta)(t) = \left(t_1 u + 16t_2^2 v\right)\partial_{t_1} + \left(\frac{2}{3}t_2 u + t_1 v\right)\partial_{t_2}, \tag{3.4.18}$$

$$(E \circ \delta)(z) = \frac{1}{3}\left(zu + 2\overline{z}^2 v\right)\partial_z + \frac{1}{3}\left(\overline{z}u + 2z^2 v\right)\partial_{\overline{z}}, \tag{3.4.19}$$

$$\begin{aligned}(E \circ \delta)(x) = {} & \frac{x_1}{9}\left(3u - 4\sqrt{3\gamma}(x_1 - 2x_2)v\right)\partial_{x_1} \\ & + \frac{x_2}{9}\left(3u - 4\sqrt{3\gamma}(x_2 - 2x_1)v\right)\partial_{x_2}.\end{aligned} \tag{3.4.20}$$

Proof. By (3.1.2), the multiplication is described by the relations $\partial_{t_1} \circ \partial_{t_i} = \partial_{t_i}$ and $\partial_{t_2} \circ \partial_{t_2} = 24t_2\partial_{t_1}$. Then

$$(E \circ \delta)(t) = \left(t_1\partial_{t_1} + \frac{2}{3}t_2\partial_{t_2}\right) \circ \left(u\partial_{t_1} + v\partial_{t_2}\right)$$

produces expression (3.4.18).

Denoting $(E \circ \delta)(t) = \phi^i\partial_{t_i}$, we find the twisted field under the coordinate transformations $t \to z$ and $t \to x$ using

$$(E \circ \delta)(z) = \phi^i \frac{\partial z}{\partial t_i}\partial_z + \phi^j \frac{\partial \overline{z}}{\partial t_j}\partial_{\overline{z}}, \tag{3.4.21}$$

$$(E \circ \delta)(x) = \phi^i \frac{\partial x_j}{\partial t_i}\partial_{x_j}. \tag{3.4.22}$$

From the relations $t(z)$ given in (3.4.3), we find the Jacobian

$$J_t(z) = \begin{pmatrix} \dfrac{\partial t_1}{\partial z} & \dfrac{\partial t_1}{\partial \overline{z}} \\ \dfrac{\partial t_2}{\partial z} & \dfrac{\partial t_2}{\partial \overline{z}} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} z^2 & \overline{z}^2 \\ \frac{1}{2}\overline{z} & \frac{1}{2}z \end{pmatrix},$$

which may be inverted on the complement of Σ to find

$$J_z(t) = \begin{pmatrix} \dfrac{\partial z}{\partial t_1} & \dfrac{\partial z}{\partial t_2} \\ \dfrac{\partial \overline{z}}{\partial t_1} & \dfrac{\partial \overline{z}}{\partial t_2} \end{pmatrix} = \frac{3}{z^3 - \overline{z}^3}\begin{pmatrix} z & -2\overline{z}^2 \\ -\overline{z} & 2z^2 \end{pmatrix}.$$

Substituting the components of (3.4.18) into (3.4.21), under the coordinate transformation $t \to z$ in (3.4.3), we obtain

$$\begin{aligned}(E \circ \delta)(z) = \frac{1}{9}\left(\left(u\left(z^3+\overline{z}^3\right) + 4vz^2\overline{z}^2\right)\frac{\partial z}{\partial t_1} + \left(uz\overline{z} + v\left(z^3+\overline{z}^3\right)\right)\frac{\partial z}{\partial t_2}\right)\partial_z \\ + \frac{1}{9}\left(\left(u\left(z^3+\overline{z}^3\right) + 4vz^2\overline{z}^2\right)\frac{\partial \overline{z}}{\partial t_1} + \left(uz\overline{z} + v\left(z^3+\overline{z}^3\right)\right)\frac{\partial \overline{z}}{\partial t_2}\right)\partial_{\overline{z}},\end{aligned}$$

which becomes the required expression (3.4.19) using the partial derivatives given by $J_z(t)$.

By the same method as above, the partial derivatives $\frac{\partial x_i}{\partial t_j}$ given by $J_x(t)$ in (3.4.11) and components of $(E \circ \delta)(t)$ given by (3.4.18), under $t \to x$ as in (3.4.9), may be substituted into (3.4.22) to obtain the expression (3.4.20). □

3.4.2 Flat twisted Legendre fields

Given an arbitrary vector field δ for F, the twisted field $E \circ \delta$ is flat for the Levi-Civita connection of the intersection form if and only if

$$E \circ \delta = \alpha \partial_{x_1} + \beta \partial_{x_2}, \tag{3.4.23}$$

or equivalently,

$$E \circ \delta = k_1 \partial_z + k_2 \partial_{\overline{z}} \tag{3.4.24}$$

for some $\alpha, \beta, k_1, k_2 \in \mathbb{C}$. For the purposes of this section, we refer to such a field simply as a flat twisted field.

Lemma 3.4.4. *Let $E \circ \delta$ be a flat twisted field. Then α, β, k_1, k_2 as in (3.4.23), (3.4.24)* *are related by*

$$k_1 = \sqrt{\frac{\gamma}{3}}\left(\alpha\left(1 - i\sqrt{3}\right) + \beta\left(1 + i\sqrt{3}\right)\right), \tag{3.4.25}$$

$$k_2 = \sqrt{\frac{\gamma}{3}}\left(\alpha\left(1 + i\sqrt{3}\right) + \beta\left(1 - i\sqrt{3}\right)\right). \tag{3.4.26}$$

Proof. From the relations defining $z \to x$ in (3.4.8), we have the Jacobian

$$J_z(x) = \begin{pmatrix} \frac{\partial z}{\partial x_1} & \frac{\partial z}{\partial x_2} \\ \frac{\partial \overline{z}}{\partial x_1} & \frac{\partial \overline{z}}{\partial x_2} \end{pmatrix} = \sqrt{\frac{\gamma}{3}}\begin{pmatrix} 1 - i\sqrt{3} & 1 + i\sqrt{3} \\ 1 + i\sqrt{3} & 1 - i\sqrt{3} \end{pmatrix}.$$

Then

$$\begin{aligned} E \circ \delta &= \alpha \partial x_1 + \beta \partial x_2 \\ &= \alpha \frac{\partial z}{\partial x_1}\frac{\partial}{\partial z} + \alpha \frac{\partial \overline{z}}{\partial x_1}\frac{\partial}{\partial \overline{z}} + \beta \frac{\partial z}{\partial x_1}\frac{\partial}{\partial z} + \beta \frac{\partial \overline{z}}{\partial x_1}\frac{\partial}{\partial \overline{z}} \\ &= \sqrt{\frac{\gamma}{3}}\left(\alpha\left(1 - i\sqrt{3}\right) + \beta\left(1 + i\sqrt{3}\right)\right)\partial z + \sqrt{\frac{\gamma}{3}}\left(\alpha\left(1 + i\sqrt{3}\right) + \beta\left(1 - i\sqrt{3}\right)\right)\partial \overline{z} \\ &= k_1 \partial z + k_2 \partial \overline{z}. \end{aligned}$$

□

Proposition 3.4.5. *Let $\delta_F = u_F \partial t_1 + v_F \partial t_2$ be a vector field for F such that $E \circ \delta_F$ is flat, with $E \circ \delta_F = \alpha \partial x_1 + \beta \partial x_2$ for some $\alpha, \beta \in \mathbb{C}$. Then*

$$u_F(x) = \frac{\alpha x_2(2x_1 - x_2) + \beta x_1(x_1 - 2x_2)}{x_1 x_2 (x_1 - x_2)}, \tag{3.4.27}$$

$$v_F(x) = \frac{\sqrt{3}(\beta x_1 - \alpha x_2)}{4\sqrt{\gamma}x_1 x_2 (x_1 - x_2)}, \tag{3.4.28}$$

and the potential for δ_F is given by

$$h_F(x) = \frac{2\alpha\gamma}{3}(2x_1 - x_2) + \frac{2\beta\gamma}{3}(2x_2 - x_1). \tag{3.4.29}$$

Proof. Equating formulas (3.4.20) and (3.4.23) for $\delta = \delta_F$, we get

$$\alpha = \frac{x_1}{9}\left(3u_F - 4\sqrt{3\gamma}(x_1 - 2x_2)v_F\right), \tag{3.4.30}$$

$$\beta = \frac{x_2}{9}\left(3u_F - 4\sqrt{3\gamma}(x_2 - 2x_1)v_F\right)\partial x_2. \tag{3.4.31}$$

Rearranging (3.4.30) produces

$$u_F = \frac{3\alpha}{x_1} + 4\sqrt{\frac{\gamma}{3}}(x_1 - 2x_2)v_F, \tag{3.4.32}$$

which can be substituted into (3.4.31) to find $v_F(x)$ as in (3.4.28). Similarly, replacing $v_F(x)$ in (3.4.32) with (3.4.28) produces $u_F(x)$ as in (3.4.27).

Recall that the potential for δ_F is a function h_F such that

$$u_F = \frac{\partial h_F}{\partial t_2} = \frac{\partial x_1}{\partial t_2}\frac{\partial h_F}{\partial x_1} + \frac{\partial x_2}{\partial t_2}\frac{\partial h_F}{\partial x_2}, \tag{3.4.33}$$

$$v_F = \frac{\partial h_F}{\partial t_1} = \frac{\partial x_1}{\partial t_1}\frac{\partial h_F}{\partial x_1} + \frac{\partial x_2}{\partial t_1}\frac{\partial h_F}{\partial x_2}. \tag{3.4.34}$$

Referring back to $J_x(t)$ in (3.4.11) for the partial derivatives $\frac{\partial x_i}{\partial t_j}$, we check that the function (3.4.29) satisfies these equations. From (3.4.33), we have

$$\begin{aligned} \frac{\partial h_F}{\partial t_2} &= \frac{1}{3x_1x_2(x_2 - x_1)}\Big(\left(2x_2^2 - 2x_1x_2 - x_1^2\right)(2\alpha - \beta) + \left(x_2^2 + 2x_1x_2 - 2x_1^2\right)(2\beta - \alpha) \Big) \\ &= \frac{1}{x_1x_2(x_2 - x_1)}\Big(\alpha\left(x_2^2 - 2x_1x_2\right) + \beta\left(2x_1x_2 - x_1^2\right)\Big) \\ &= u_F(x) \end{aligned}$$

for $u_F(x)$ given by (3.4.27). Similarly, (3.4.34) becomes

$$\begin{aligned} \frac{\partial h_F}{\partial t_1} &= \frac{1}{4\sqrt{3\gamma}x_1x_2(x_2 - x_2)}\Big((2x_2 - x_1)(2\alpha - \beta) + (x_2 - 2x_1)(2\beta - \alpha)\Big) \\ &= \frac{\sqrt{3}}{4\sqrt{\gamma}x_1x_2(x_2 - x_2)}(\alpha x_2 - \beta x_1) \\ &= v_F(x) \end{aligned}$$

for $v_F(x)$ given by (3.4.28). □

Since we have the Euler field in x given by (3.4.17), it is straightforward to see that

$$\mathcal{L}_E\left(\alpha\partial x_1 + \beta\partial x_2\right) = -\frac{1}{3}\left(\alpha\partial x_1 + \beta\partial x_2\right), \tag{3.4.35}$$

that is, a flat field $\alpha\partial x_1 + \beta\partial x_2$ is homogeneous of degree $-\frac{1}{3}$. Therefore, by Proposition 2.4.26, any Legendre field δ such that $E \circ \delta$ is flat must be homogeneous of degree $\mu = -\frac{4}{3}$. This allows us to connect the Legendre fields obtained in Proposition 3.4.5 with those considered in Theorem 3.3.23.

Remark 3.4.6. The relations for $t(z)$ given in (3.4.3) can be inverted to find

$$\begin{aligned} z &= \left(\frac{9}{2}\left(t_1 + \sqrt{t_1^2 - \frac{32}{3}t_2^3}\right)\right)^{1/3}, \\ \overline{z} &= \left(\frac{9}{2}\left(t_1 - \sqrt{t_1^2 - \frac{32}{3}t_2^3}\right)\right)^{1/3}. \end{aligned} \tag{3.4.36}$$

From these expressions, it is clear that the coordinates $z, \overline{z} \in \mathbb{C}^2$ describe a 6-fold cover over the chart of $\mathbb{C}^2/A_2$ with coordinates t. This is to be expected, as $|A_2| = 6$. The monodromy group of the Frobenius manifold, which can be constructed via the mapping $(t_1, t_2) \mapsto (z, \overline{z})$, is the group A_2 (see [18, Appendix G]).

As (3.4.36) indicates, we will need to deal with multivaluedness arising from coordinate transformations. For the following results, we restrict our attention to the open set $U \subset \mathbb{C}^2 \setminus \Sigma$ defined by $0 < \mathrm{Re}\,(x_2 - x_1) < \epsilon$ for small $\epsilon > 0$. Note that $0 \notin U$.

We reintroduce, from Theorem 3.3.23, the variable ω. By the coordinate change $t \to z$ given in (3.4.3), we have

$$\omega = \frac{32t_2^3}{3t_1^2} = \frac{4z^3\overline{z}^3}{\left(z^3 + \overline{z}^3\right)^2},$$

and

$$\sqrt{1-\omega} = \sqrt{\frac{z^6 - 2z^3\overline{z}^3 + \overline{z}^6}{\left(z^3 + \overline{z}^3\right)^2}} = \sqrt{\left(\frac{z^3 - \overline{z}^3}{z^3 + \overline{z}^3}\right)^2}.$$

In the set U, we can continuously define

$$\sqrt{1-\omega} = \frac{z^3 - \overline{z}^3}{z^3 + \overline{z}^3}. \tag{3.4.37}$$

When taking cube roots in U, we set $\left(z^3\right)^{1/3} = z$ and $\left(\overline{z}^3\right)^{1/3} = \overline{z}$. We also require consistency of the choice of branches for $(9t_1)^{1/3} = \left(z^3 + \overline{z}^3\right)^{1/3}$ and $(9t_1)^{2/3} = \left(z^3 + \overline{z}^3\right)^{2/3}$ in the sense that

$$\left(z^3 + \overline{z}^3\right)^{1/3}\left(z^3 + \overline{z}^3\right)^{2/3} = z^3 + \overline{z}^3.$$

Theorem 3.4.7. *Let $\delta_F = u_F\partial_{t_1} + v_F\partial_{t_2}$ be a Legendre field over U with*

$$\begin{aligned} u_F(t) &= \frac{c_1\left(1+\sqrt{1-\omega}\right)^{2/3} + 2^{4/3}c_2\left(1-\sqrt{1-\omega}\right)^{2/3}}{(4t_1)^{1/3}\sqrt{1-\omega}}, \\ v_F(t) &= -\frac{c_1\left(1-\sqrt{1-\omega}\right)^{1/3} + 2^{4/3}c_2\left(1+\sqrt{1-\omega}\right)^{2/3}}{(12t_1)^{2/3}\sqrt{1-\omega}}, \end{aligned}$$

where $\omega = \frac{32}{3} t_2^3 t_1^{-2}$, *and* $c_1, c_2 \in \mathbb{C}$. *Then*

$$E \circ \delta_F = k_1 \partial_z + k_2 \partial_{\overline{z}} = \alpha \partial_{x_1} + \beta \partial_{x_2} \tag{3.4.38}$$

for $k_1, k_2, \alpha, \beta \in \mathbb{C}$ *given by equations*

$$\begin{aligned} c_1 &= 3^{1/3} k_1 = \frac{\sqrt{\gamma}}{3^{1/6}} \left(\alpha \left(1 - i\sqrt{3}\right) + \beta \left(1 + i\sqrt{3}\right)\right), \\ c_2 &= -\left(\frac{3}{16}\right)^{1/3} k_2 = -\frac{\sqrt{\gamma}}{2^{4/3} \cdot 3^{1/6}} \left(\alpha \left(1 + i\sqrt{3}\right) + \beta \left(1 - i\sqrt{3}\right)\right). \end{aligned}$$

Proof. Theorem 3.3.23 describes all homogeneous Legendre fields for F, denoted here as $\delta_H = u_H \partial_{t_1} + v_H \partial_{t_1}$. Recall from (3.4.35) and Proposition 2.4.26 that a field δ_H of degree $\mu = -\frac{4}{3}$ produces a flat twisted field. Setting $k = 4$, $\mu = -4/3$ in Theorem 3.3.23, the components of δ_H are

$$u_H = t_1^{-1/3} U(\omega), \tag{3.4.39}$$

$$v_H = t_1^{-2/3} V(\omega), \tag{3.4.40}$$

where $\omega = \dfrac{32 t_2^3}{3 t_1^2}$ and

$$\begin{aligned} U(\omega) &= c_1\, {}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{1}{3}; \omega\right) + c_2 \omega^{2/3} {}_2F_1\left(\frac{4}{3}, \frac{5}{6}; \frac{5}{3}; \omega\right), \\ V(\omega) &= -\frac{c_1}{2 \cdot 6^{2/3}} \omega^{1/3} {}_2F_1\left(\frac{7}{6}, \frac{2}{3}; \frac{4}{3}; \omega\right) - c_2 \left(\frac{2\omega}{9}\right)^{1/3} {}_2F_1\left(\frac{5}{6}, \frac{1}{3}; \frac{2}{3}; \omega\right) \end{aligned}$$

for $c_1, c_2 \in \mathbb{C}$. Referring to case 104 in [50, § 7.3.1], we have the representation

$${}_2F_1\left(\kappa, \kappa + \frac{1}{2}; 2\kappa; y\right) = \frac{1}{\sqrt{1-y}} \left(\frac{2}{1 + \sqrt{1-y}}\right)^{2\kappa - 1},$$

for $\kappa \in \mathbb{R}$. In terms of elementary functions, $U(\omega), V(\omega)$ can therefore be expressed as

$$U(\omega) = \frac{c_1 \left(1 + \sqrt{1-\omega}\right)^{2/3}}{2^{2/3} \sqrt{1-\omega}} + \frac{c_2 (2\omega)^{2/3}}{\left(1 + \sqrt{1-\omega}\right)^{2/3} \sqrt{1-\omega}}, \tag{3.4.41}$$

$$V(\omega) = -\frac{c_1 \omega^{1/3}}{2^{4/3} \cdot 3^{2/3} \left(1 + \sqrt{1-\omega}\right)^{1/3} \sqrt{1-\omega}} - \frac{c_2 \left(1 + \sqrt{1-\omega}\right)^{1/3}}{3^{2/3} \sqrt{1-\omega}}. \tag{3.4.42}$$

Noting that

$$\omega = \left(1 + \sqrt{1-\omega}\right)\left(1 - \sqrt{1-\omega}\right),$$

expressions (3.4.41) and (3.4.42) can be rewritten as

$$U(\omega) = \frac{c_1\left(1+\sqrt{1-\omega}\right)^{2/3} + 2^{4/3}c_2\left(1-\sqrt{1-\omega}\right)^{2/3}}{2^{2/3}\sqrt{1-\omega}}, \tag{3.4.43}$$

$$V(\omega) = -\frac{c_1\left(1-\sqrt{1-\omega}\right)^{1/3} + 2^{4/3}c_2\left(1+\sqrt{1-\omega}\right)^{1/3}}{12^{2/3}\sqrt{1-\omega}}. \tag{3.4.44}$$

By formula (3.4.37), we have

$$1+\sqrt{1-\omega} = \frac{2z^3}{z^3+\overline{z}^3},$$
$$1-\sqrt{1-\omega} = \frac{2\overline{z}^3}{z^3+\overline{z}^3}.$$

The expressions (3.4.43), (3.4.44) can then be rewritten in $z, \overline{z}$ as

$$U(z,\overline{z}) = \frac{c_1z^2 + 2^{4/3}c_2\overline{z}^2}{\left(z^3-\overline{z}^3\right)}\left(z^3+\overline{z}^3\right)^{1/3},$$
$$V(z,\overline{z}) = -\frac{2^{1/3}c_1\overline{z} + 2^{5/3}c_2z}{12^{2/3}\left(z^3-\overline{z}^3\right)}\left(z^3+\overline{z}^3\right)^{2/3}.$$

From (3.4.39), (3.4.40), the components of δ_H in z are therefore

$$u_H(z) = 3^{2/3}\frac{c_1z^2 + 2^{4/3}c_2\overline{z}^2}{\left(z^3-\overline{z}^3\right)}, \tag{3.4.45}$$

$$v_H(z) = -3^{2/3}\frac{c_1\overline{z} + 2^{4/3}c_2z}{2\left(z^3-\overline{z}^3\right)}. \tag{3.4.46}$$

The constants c_1, c_2 here are arbitrary; we must relate them to k_1, k_2, α, β as in (3.4.38).

Let $\delta = \delta_F = u_F\partial_{t_1} + v_F\partial_{t_2}$ be such that $E \circ \delta_F = k_1\partial_z + k_2\partial_{\overline{z}}$. By Proposition 3.4.3, we have

$$3k_1 = zu_F + 2\overline{z}^2v_F,$$
$$3k_2 = \overline{z}u_F + 2z^2v_F,$$

which can be solved for

$$u_F(z,\overline{z}) = \frac{3(k_1z^2 - k_2\overline{z}^2)}{z^3-\overline{z}^3},$$
$$v_F(z,\overline{z}) = \frac{3(k_1\overline{z} - k_2z)}{2\left(\overline{z}^3-z^3\right)}.$$

We set $u_F = u_H$, $v_F = v_H$, where u_H, v_H are given by (3.4.45), (3.4.46), to find

$$c_1 = 3^{1/3} k_1 \text{ and } c_2 = -\left(\frac{3}{16}\right)^{1/3} k_2. \tag{3.4.47}$$

Finally, by Lemma 3.4.4 we have $k_1(\alpha, \beta)$, $k_2(\alpha, \beta)$ where $\delta_F = \alpha\partial_{x_1} + \beta\partial_{x_2}$. Substituting (3.4.25), (3.4.26) into (3.4.47) produces $c_1(\alpha, \beta)$, $c_2(\alpha, \beta)$ as in the statement. □

3.5 Legendre fields in canonical coordinates

As noted in § 2.4.3.1, the Legendre field condition can be reformulated as a much simpler set of equations in the case of semisimple Frobenius manifolds, by using the canonical coordinates of such manifolds. Since finding (and transforming to and from) the canonical coordinates is in general non-trivial, we have so far chosen to work in the flat coordinates of η. However, using canonical coordinates may allow us to obtain general solutions for the Legendre field condition.

For the two-dimensional semisimple Frobenius manifolds with anti-diagonal metric — discussed above in Cases 2–5 — the relationship between canonical and flat coordinates is known. The flat coordinates t expressed as functions of the canonical coordinates u are given in Example 2.3.15. We find that in all of these cases, the Legendre field condition (2.4.4) reduces to a single second-order partial differential equation of Euler-Darboux-Poisson type. Throughout this section, we work in canonical coordinates u and set $\partial_i = \partial_{u^i}$, $\eta_{ij} = \eta_{ij}(u)$, $\eta^{ij} = \eta^{ij}(u)$.

Proposition 3.5.1. *Let $\mathcal{M}$ be one of the Frobenius manifolds described by (2.3.2)–(2.3.5), with metric η, Levi-Civita connection ∇, multiplication $\circ$ and canonical coordinates u. Then the Legendre field condition (2.4.4) reduces to the following:*

$$\frac{\partial x}{\partial u^2} = \frac{\partial y}{\partial u^1} = \frac{\epsilon(x-y)}{u^1 - u^2}, \tag{3.5.1}$$

where $x = x(u^1, u^2)$, $y = y(u^1, u^2)$ are the components of the Legendre field δ in canonical coordinates, and $\epsilon \in \mathbb{C}$ is the same as in Example 2.3.15.

Proof. By Proposition 2.4.23, the Legendre field condition in canonical coordinates is equivalent to the two equations

$$\frac{\partial x}{\partial u^2} = \Gamma^1_{21}(y - x), \tag{3.5.2}$$

$$\frac{\partial y}{\partial u^1} = \Gamma^2_{12}(x - y), \tag{3.5.3}$$

where Γ^i_{jk} are the Christoffel symbols for ∇. By definition, Christoffel symbols can be derived from the entries of the metric η as follows:

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{im}\left(\frac{\partial \eta_{mk}}{\partial u^l} + \frac{\partial \eta_{ml}}{\partial u^k} - \frac{\partial \eta_{kl}}{\partial u^m}\right). \tag{3.5.4}$$

Note that we require the entries of η, η^{-1} in terms of u.

By Property 1 of Theorem 2.3.16, we have the following expressions for the covariant components of η:

$$\eta_{ij}(u) = \begin{cases} \partial_i\left(t_1\right) = \partial_i\left(\eta_{1j}(t)\, t^j\right) & i = j \\ \eta_{ij}(u) = 0 & i \neq j. \end{cases} \tag{3.5.5}$$

Since we have

$$\eta(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all the Frobenius manifolds in question, (3.5.5) becomes

$$\eta^{ij}(u) = \begin{cases} \partial_i\left(t^2\right) & i = j \\ 0 & i \neq j, \end{cases} \tag{3.5.6}$$

and we also have

$$\eta^{ij}(u) = \begin{cases} \frac{1}{\partial_i(t^2)} & i = j \\ 0 & i \neq j. \end{cases} \tag{3.5.7}$$

Substituting (3.5.6), (3.5.7) into (3.5.4), we find

$$\Gamma^1_{21} = \frac{1}{2}\left(\frac{\partial t^2}{\partial u^1}\right)^{-1}\frac{\partial^2 t^2}{\partial u^1 \partial u^2}, \tag{3.5.8}$$

$$\Gamma^2_{12} = \frac{1}{2}\left(\frac{\partial t^2}{\partial u^2}\right)^{-1}\frac{\partial^2 t^2}{\partial u^1 \partial u^2}. \tag{3.5.9}$$

Recall from Example 2.3.15 that the manifolds described by (2.3.3)–(2.3.5) share the same expression for $t^2(u)$; we will consider these as one case first. From (2.3.11), we have

$$t^2 = \frac{\left(u^1 - u^2\right)^{2\epsilon+1}}{2(2\epsilon+1)}, \tag{3.5.10}$$

where $\epsilon \in \mathbb{C}$, $\epsilon \neq -1/2$. Substituting (3.5.10) into (3.5.8), (3.5.9) gives us

$$\Gamma^2_{12} = \epsilon\left(u^1 - u^2\right)^{-1} = -\Gamma^1_{21}.$$

With this, the right-hand side of both (3.5.2), (3.5.3) evaluates to

$$\frac{\epsilon\,(x-y)}{(u^1-u^2)}$$

as required.

For the Frobenius manifold described by (2.3.2), the expression for $t^2(u)$ is

$$t^2 = \frac{1}{2}\log u^1 - u^2.$$

Substituting this into (3.5.8), (3.5.9), we find

$$\Gamma^2_{12} = -1/2\left(u^1-u^2\right)^{-1} = -\Gamma^1_{21}.$$

Therefore (3.5.2), (3.5.3) produce the required statement with $\epsilon = -1/2$. □

We can now rewrite (3.5.1) in terms of the potential h, recalling that

$$\delta = \mathrm{grad}h = \eta^{ij}\partial_j(h)\partial_i. \tag{3.5.11}$$

Corollary 3.5.2. *Let $\delta = x\partial_1 + y\partial_2$ be a Legendre field for one of the Frobenius manifolds described by (2.3.2)–(2.3.5). Then its potential $h = h\left(u^1,u^2\right)$ satisfies the equation*

$$\left(u^1-u^2\right)\frac{\partial^2 h}{\partial u^1\partial u^2} = \epsilon\left(\frac{\partial h}{\partial u^2} - \frac{\partial h}{\partial u^1}\right), \tag{3.5.12}$$

where $\epsilon \in \mathbb{C}$ is the same as in Example 2.3.15.

Proof. Following from the proof of the previous proposition, we see that the contravariant components of the metric are given in all cases by

$$\eta^{ij} = \begin{cases} 2\left(u^1-u^2\right)^{-2\epsilon} & i=j=1 \\ -2\left(u^1-u^2\right)^{-2\epsilon} & i=j=2 \\ 0 & i\neq j. \end{cases}$$

Using definition (3.5.11), we have

$$x = 2\left(u^1-u^2\right)^{-2\epsilon}\frac{\partial h}{\partial u^1},$$
$$y = -2\left(u^1-u^2\right)^{-2\epsilon}\frac{\partial h}{\partial u^2},$$

and therefore

$$\frac{\partial x}{\partial u^2} = 4\epsilon \left(u^1 - u^2\right)^{-2\epsilon-1} \frac{\partial h}{\partial u^1} + 2\left(u^1 - u^2\right)^{2\epsilon} \frac{\partial^2 h}{\partial u^1 \partial u^2}, \tag{3.5.13}$$

$$\frac{\partial y}{\partial u^1} = 4\epsilon \left(u^1 - u^2\right)^{-2\epsilon-1} \frac{\partial h}{\partial u^2} - 2\left(u^1 - u^2\right)^{2\epsilon} \frac{\partial^2 h}{\partial u^1 \partial u^2}. \tag{3.5.14}$$

By Proposition 3.5.1, we can equate (3.5.13) and (3.5.14) to obtain the required second-order PDE for h. □

Equation (3.5.12) is of Euler-Darboux-Poisson type. For specific values of ϵ, a general solution can be found computationally or by using recurrence formulae presented in [12]. For example, when $\epsilon = -1$, it can be verified that a general solution to (3.5.12) is

$$h = \frac{G_1(u^1) + G_2(u^2)}{u^1 - u^2},$$

for arbitrary functions G_1, G_2.

Chapter 4

Multi-parameter A_{n-1}-type trigonometric solutions

In this chapter, we generalise solutions found by Hoevenaars and Martini; see Theorem 2.1 in the preprint version of [35] for full details[1]. Solutions of this form were also considered by Shen [56] and Riley [51]. We reproduce here results published in [25].

We introduce multi-parameter deformations of solutions of the form (2.5.10). The corresponding function $F = F^{\text{trig}}_{A_{n-1}}$ depends on the n-tuple $m = (m_1, \dots, m_n)$, $m_i \in \mathbb{C}^\times \, \forall \, i$, as well as on three parameters $a, b, c \in \mathbb{C}$. The non-polynomial part of $F^{\text{trig}}_{A_{n-1}}$ is expressed via the function $f(z)$ given by formula (2.5.7). Relations between a, b, c and $M = \sum m_i$ are needed to ensure that $F^{\text{trig}}_{A_{n-1}}$ solves the WDVV equations. We will first consider a generic case before looking at other situations.

Theorem 4.0.1. *Let $F = F^{\text{trig}}_{A_{n-1}}$ denote the function*

$$F^{\text{trig}}_{A_{n-1}}(y) = \sum_{1 \le i < j \le n} m_i m_j f(y_i - y_j) + \frac{a}{6} \left(\sum_{i=1}^n m_i y_i \right)^3 + \frac{b}{2} \left(\sum_{i=1}^n m_i y_i \right) \left(\sum_{j=1}^n m_j y_j^2 \right) + \frac{c}{6} \sum_{i=1}^n m_i y_i^3, \quad (4.0.1)$$

with $m_i \in \mathbb{C}^\times \, \forall \, i \in \{1, \dots, n\}$ and $a, b, c \in \mathbb{C}$. Suppose additionally that

$$bM + c \neq 0, \quad (4.0.2)$$

and

$$aM^2 + 3bM + c \neq 0. \quad (4.0.3)$$

[1]Note that this theorem does not appear in the published version.

Then $\eta = \sum_{k=1}^{n} F_k$ is a constant non-degenerate matrix. Furthermore, F solves the WDVV equations (2.2.2) for $Q = \eta$ if the following relation holds:

$$b^3 M + 3b^2 c - ac^2 + aM^2 + 3bM + c = 0. \tag{4.0.4}$$

Conversely, the WDVV equations (2.2.2) imply relation (4.0.4) when $n \geq 3$.

In the case when $m_i = 1$ for all i, Theorem 4.0.1 is equivalent to Theorem 2.5.6, which appears in the preprint of [35].

The function (4.0.1) can also be rewritten in the form

$$F^{\text{trig}}_{A_{n-1}}(y) = \sum_{1\leq i<j\leq n} m_i m_j f\,(y_i - y_j) + a \sum_{1\leq i<j<k\leq n} m_i m_j m_k y_i y_j y_k + \frac{1}{2}\sum_{i\neq j}^{n}(am_i + b)m_i m_j y_i^2 y_j + \frac{1}{6}\sum_{i=1}^{n}(am_i^2 + 3bm_i + c)m_i y_i^3. \tag{4.0.5}$$

4.1 Proof of Theorem 4.0.1

From the definition of $f(z)$ in (2.5.7) we have that $f'''(z) = \coth z$, and so

$$f'''(-z) = -f'''(z).$$

For convenience, we will use the following shorthand throughout the calculations:

$$\beta_{ij} = \begin{cases} \coth\,(y_i - y_j) & i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Let δ_{ij} be the Kronecker delta function. We also define

$$\delta_{ijr} = \delta_{ij}\delta_{jr} = \begin{cases} 1 & i = j = r, \\ 0 & \text{otherwise.} \end{cases}$$

First, we compute the matrices of third order derivatives of F. As before, let F_k be the $n \times n$ matrix with $(r, s)^{\text{th}}$ entry

$$F_{krs} = \frac{\partial^3 F}{\partial y_k \partial y_r \partial y_s}.$$

Lemma 4.1.1. *The matrix F_k for the function $F = F^{\text{trig}}_{A_{n-1}}$ given by (4.0.1) can be written as*

$$F_k = W_k + V_k,$$

where the $n \times n$ matrices W_k and V_k have $(r,s)^{th}$ entries respectively given by

$$\begin{aligned} W_{krs} &= a m_k m_r m_s, \\ V_{krs} &= \delta_{krs} m_k \left(\sum_{q=1}^{n} m_q \beta_{kq} + c \right) + \delta_{kr} m_k m_s \beta_{sk} + \delta_{ks} m_k m_r \beta_{rk} + \delta_{rs} m_k m_r \beta_{kr} \\ &\qquad + \delta_{kr} b m_k m_s + \delta_{ks} b m_k m_r + \delta_{rs} b m_k m_r. \end{aligned}$$

Proof. Considering each term in (4.0.1) individually, we can set out the following collection of statements:

$$\begin{aligned} \partial_k \partial_r \partial_s \left(\sum_{i<j} m_i m_j f(y_i - y_j) \right) &= \delta_{krs} \sum_{q=1}^{n} m_k m_q \beta_{kq} + \delta_{kr}(1 - \delta_{rs}) m_k m_s \beta_{sk} \\ &\quad + \delta_{ks}(1-\delta_{rs}) m_k m_r \beta_{rk} + \delta_{rs}(1-\delta_{kr}) m_k m_r \beta_{kr}, \quad (4.1.1) \\ \partial_k \partial_r \partial_s \left(\frac{a}{6} \left(\sum_{i=1}^{n} m_i y_i \right)^3 \right) &= a m_k m_r m_s, \quad (4.1.2) \\ \partial_k \partial_r \partial_s \left(\frac{b}{2} \left(\sum_{i=1}^{n} m_i y_i \right) \left(\sum_{j=1}^{n} m_j y_j^2 \right) \right) &= 3b \delta_{krs} m_k^2 + \delta_{kr}(1-\delta_{rs}) b m_k m_s \\ &\quad + \delta_{ks}(1-\delta_{rs}) b m_k m_r + \delta_{rs}(1-\delta_{kr}) b m_r m_k, \quad (4.1.3) \end{aligned}$$

and

$$\partial_k \partial_r \partial_s \left(\frac{c}{6} \sum_{i=1}^{n} m_i y_i^3 \right) = \delta_{krs} c m_k. \quad (4.1.4)$$

The expression (4.1.2) is equal to W_{krs}, and the sum of (4.1.1), (4.1.3) and (4.1.4) equals V_{krs}. Note that we can use the fact that $\delta_{ab}\beta_{ab} = 0$ to simplify (4.1.1) as required. □

For the WDVV equations to hold, we now need to check that the linear combination $\eta = \sum_{k=1}^{n} F_k$ is a constant, non-degenerate matrix.

Lemma 4.1.2. *The $n \times n$ matrix $\eta = \sum_{k=1}^{n} F_k$ has $(r,s)^{th}$ entry*

$$\eta_{rs} = (aM + 2b) m_r m_s + \delta_{rs}(bM + c) m_r.$$

In particular, the $(r,s)^{th}$ entry of the matrix $\sum_{k=1}^{n} V_k$ is given by

$$\sum_{k=1}^{n} V_{krs} = 2b m_r m_s + \delta_{rs}(bM + c) m_r. \quad (4.1.5)$$

Proof. By Lemma 4.1.1, we have $\eta = \sum_k W_k + \sum_k V_k$. It is clear that

$$\sum_{k=1}^{n} W_{krs} = aMm_r m_s,$$

Additionally, we have that

$$\sum_{k=1}^{n} V_{krs} = \delta_{rs} m_r \left(\sum_{q=1}^{n} m_q \beta_{rq} + c \right) + m_r m_s \beta_{sr} + m_s m_r \beta_{rs}$$
$$+ \delta_{rs} m_r \sum_{k=1}^{n} m_k \beta_{kr} + 2bm_r m_s + \delta_{rs} bMm_r,$$

which can be simplified to the form (4.1.5). □

Note that the matrix η can be represented as

$$\eta = A + uv^{\mathrm{T}},$$

where the $n \times n$ matrix A has the entries

$$A_{rs} = \delta_{rs}(bM + c)m_r \tag{4.1.6}$$

and $u = (u_1, \dots, u_n)^T$, $v = (v_1, \dots, v_n)^T$ are n-dimensional column vectors such that

$$u_i = m_i \text{ and } v_i = (aM + 2b)m_i.$$

Lemma 4.1.3. *Suppose that $bM + c \neq 0$, $aM^2 + 3bM + c \neq 0$, and $m_i \neq 0 \; \forall \, i$. Then the matrix η from Lemma 4.1.2 has determinant*

$$\det \eta = \left(aM^2 + 3bM + c\right) (bM + c)^{n-1} \prod_{i=1}^{n} m_i,$$

and the inverse matrix η^{-1} has entries

$$\eta^{rs} = \frac{\delta_{rs}}{(bM + c)m_r} - \frac{aM + 2b}{(aM^2 + 3bM + c)(bM + c)}. \tag{4.1.7}$$

Proof. Since we can write $\eta = A + uv^{\mathrm{T}}$, this implies that

$$\det \eta = \det(A + uv^T). \tag{4.1.8}$$

By applying the Matrix Determinant Lemma (see Theorem 18.1.1 in [31]), we have

$$\det \eta = (1 + v^T A^{-1} u) \det A.$$

The matrix A, given in (4.1.6), has determinant

$$\det A = (bM + c)^n \prod_{i=1}^{n} m_i. \tag{4.1.9}$$

As A is a diagonal matrix, it is straightforward to see that A^{-1} exists and has the entries

$$(A^{-1})_{rs} = \frac{\delta_{rs}}{(bM + c)m_r}. \tag{4.1.10}$$

We can now compute

$$1 + v^T A^{-1} u = 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(aM + 2b)m_i \delta_{ij}}{(bM + c)} = \frac{aM^2 + 3bM + c}{bM + c}. \tag{4.1.11}$$

It follows from (4.1.8), (4.1.9) and (4.1.11) that

$$\det \eta = \left(aM^2 + 3bM + c\right)(bM + c)^{n-1} \prod_{i=1}^{n} m_i.$$

The condition $aM^2 + 3bM + c \neq 0$ is equivalent to $1 + v^T A^{-1} u \neq 0$. It is then easy to check (cf. the Sherman-Morrison formula, see e.g. [29]) that

$$\eta^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}.$$

We have

$$\begin{aligned} \left(A^{-1} u v^T A^{-1}\right)_{rs} &= \frac{aM + 2b}{(bM + c)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ri}\delta_{js}}{m_r m_s} m_i m_j \\ &= \frac{aM + 2b}{(bM + c)^2}, \end{aligned} \tag{4.1.12}$$

and then formulas (4.1.10)–(4.1.12) imply (4.1.7). □

For the WDVV equations (2.2.2) to hold we need

$$F_i \eta^{-1} F_j - F_j \eta^{-1} F_i = 0. \tag{4.1.13}$$

From Lemma 4.1.3, we have for non-singular η

$$\begin{aligned}\left(F_i\eta^{-1}F_j\right)_{rs} &= \sum_{k=1}^{n}\sum_{l=1}^{n}\left[F_{irk}\left(\kappa+\frac{\delta_{kl}}{(bM+c)m_k}\right)F_{jls}\right]\\ &= \kappa\sum_{k=1}^{n}F_{irk}\sum_{l=1}^{n}F_{jls}+\frac{1}{bM+c}\sum_{k=1}^{n}\frac{1}{m_k}F_{irk}F_{jks}\\ &= \kappa\,\eta_{ri}\eta_{js}+\frac{1}{bM+c}\sum_{k=1}^{n}\frac{1}{m_k}\Big[W_{irk}W_{jks}+W_{irk}V_{jks} \qquad (4.1.14)\\ &\qquad + V_{irk}W_{jks}+V_{irk}V_{jks}\Big],\end{aligned}$$

where

$$\kappa = -\frac{aM+2b}{(aM^2+3bM+c)(bM+c)}.$$

We will consider contributions to (4.1.13) from each of the five terms in (4.1.14) individually.

For a four-index expression M_{ijkl}, we denote by $M_{[ij]kl}$ the anti-symmetrisation $M_{[ij]kl} = M_{ijkl} - M_{jikl}$.

Lemma 4.1.4. *We have*

$$\begin{aligned}\left(F_i\eta^{-1}F_j - F_j\eta^{-1}F_i\right)_{rs} &= \lambda_{[ij]rs}+\kappa(bM+c)^2 m_r m_s(\delta_{ir}\delta_{js}-\delta_{jr}\delta_{is})\\ &\quad + \big(\kappa(aM+2b)(bM+c)+a\big)m_r m_s\big(m_j(\delta_{ir}-\delta_{is})+m_i(\delta_{js}-\delta_{jr})\big),\end{aligned}$$

where

$$\lambda_{ijrs} = \frac{1}{bM+c}\sum_{k=1}^{n}\frac{1}{m_k}\left(V_i\right)_{rk}\left(V_j\right)_{ks}. \qquad (4.1.15)$$

Proof. We start with the first term in (4.1.14). From Lemma 4.1.2 we have

$$\begin{aligned}\eta_{ri}\eta_{js} &= (aM+2b)^2 m_r m_s m_i m_j + \delta_{ir}(aM+2b)(bM+c)m_r m_j m_s\\ &\quad + \delta_{js}(aM+2b)(bM+c)m_s m_r m_i + \delta_{ir}\delta_{js}(bM+c)^2 m_r m_s,\end{aligned}$$

so the antisymmetrisation becomes

$$\begin{aligned}\eta_{ri}\eta_{js}-\eta_{rj}\eta_{is} &= (bM+c)^2 m_r m_s(\delta_{ir}\delta_{js}-\delta_{jr}\delta_{is})\\ &\quad + (aM+2b)(bM+c)m_r m_s\big(m_j(\delta_{ir}-\delta_{is})+m_i(\delta_{js}-\delta_{jr})\big). \qquad (4.1.16)\end{aligned}$$

Next, we use Lemma 4.1.1 to consider the other terms in (4.1.14). We have

$$\sum_{k=1}^{n}\frac{1}{m_k}W_{irk}W_{jks} = a^2 m_i m_r m_j m_s M,$$

which leads to

$$\sum_{k=1}^{n} \frac{1}{m_k} \big[W_{irk} W_{jks} - W_{jrk} W_{iks} \big] = 0. \tag{4.1.17}$$

We continue with

$$\sum_{k=1}^{n} \frac{1}{m_k} W_{irk} V_{jks} = a m_i m_r \sum_{k=1}^{n} V_{kjs},$$

since it follows from Lemma 4.1.1 that V_{kjs} is symmetric in k, j, and s. By Lemma 4.1.2,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{m_k} W_{irk} V_{jks} &= a m_i m_r \big(2b m_j m_s + \delta_{js} (bM + c) m_j \big) \\ &= 2ab m_i m_j m_r m_s + \delta_{js} a (bM + c) m_i m_j m_r,\end{aligned}$$

and so we find

$$\sum_{k=1}^{n} \frac{1}{m_k} \big[W_{irk} V_{jks} - W_{jrk} V_{iks} \big] = a(bM + c) m_r m_s \left(\delta_{js} m_i - \delta_{is} m_j \right). \tag{4.1.18}$$

By the symmetry of V_{ijk} and W_{ijk}, we similarly obtain

$$\sum_{k=1}^{n} \frac{1}{m_k} \big[V_{irk} W_{jks} - V_{jrk} W_{iks} \big] = a(bM + c) m_r m_s (\delta_{ir} m_j - \delta_{jr} m_i). \tag{4.1.19}$$

The antisymmetrisation of (4.1.14) is then found by combining (4.1.16), (4.1.17), (4.1.18), and (4.1.19) with $\lambda_{[ij]rs}$ to represent the remaining terms. □

To find the antisymmetrisation of λ_{ijrs}, given in (4.1.15), we now separately consider terms in the expansion of this expression which are (without the use of any identities) quadratic, linear and constant in β_{pq}. It will turn out by the use of various identities that all terms in $\lambda_{[ij]rs}$ are constant.

Lemma 4.1.5. *We have*

$$\begin{aligned}\lambda_{[ij]rs} = \frac{m_r m_s}{bM + c} \Bigg(&\left(M \left(b^2 + 1 \right) + 2bc \right) \left(\delta_{ir} \delta_{js} - \delta_{jr} \delta_{is} \right) \\ &+ \left(1 - b^2 \right) m_i (\delta_{jr} - \delta_{js}) + \left(1 - b^2 \right) m_j (\delta_{is} - \delta_{ir}) \Bigg).\end{aligned}$$

Proof. Let $A_{[ij]rs}$ be the terms in $\lambda_{[ij]rs}$ which are quadratic in β_{pq}. That is,

$$A_{ijrs} = \sum_{k=1}^{n} m_k \left(\delta_{kir} \sum_{q=1}^{n} m_q \beta_{kq} + \delta_{ki} m_r \beta_{rk} + \delta_{kr} m_i \beta_{ik} + \delta_{ir} m_i \beta_{ki} \right) \times \left(\delta_{kjs} \sum_{q=1}^{n} m_q \beta_{kq} + \delta_{kj} m_s \beta_{sk} + \delta_{ks} m_j \beta_{jk} + \delta_{js} m_j \beta_{kj} \right),$$

where we have used the expression for λ_{ijrs} given in (4.1.15), and for V_q given in Lemma 4.1.1. Terms in the expansion of A_{ijrs} with a factor of δ_{ij} will cancel after antisymmetrisation in i, j, so we find

$$A_{[ij]rs} = \tilde{A}_{[ij]rs},$$

where

$$\begin{aligned}
\tilde{A}_{ijrs} = {} & \delta_{irs} m_i m_j \beta_{ji} \left(\sum_{q=1}^{n} m_q \beta_{iq} \right) + \delta_{ir} \delta_{js} m_i m_j \beta_{ij} \left(\sum_{q=1}^{n} m_q \beta_{iq} \right) \\
& + \delta_{is} m_i m_j m_r \beta_{ri} \beta_{ji} + \delta_{js} m_i m_j m_r \beta_{ri} \beta_{ij} + \delta_{jrs} m_i m_j \beta_{ij} \left(\sum_{q=1}^{n} m_q \beta_{jq} \right) \\
& + \delta_{jr} m_i m_j m_s \beta_{ij} \beta_{sj} + \delta_{rs} m_i m_j m_r \beta_{ir} \beta_{jr} + \delta_{js} m_i m_j m_r \beta_{ir} \beta_{rj} \\
& + \delta_{ir} \delta_{js} m_i m_j \beta_{ji} \left(\sum_{q=1}^{n} m_q \beta_{jq} \right) + \delta_{ir} m_i m_j m_s \beta_{ji} \beta_{sj} \\
& + \delta_{ir} m_i m_j m_s \beta_{si} \beta_{js} + \delta_{ir} \delta_{js} m_i m_j \left(\sum_{k=1}^{n} m_k \beta_{ki} \beta_{kj} \right).
\end{aligned}$$

Note that the antisymmetrisation of

$$\delta_{irs} m_i m_j \beta_{ji} \left(\sum_{q=1}^{n} m_q \beta_{iq} \right) + \delta_{jrs} m_i m_j \beta_{ij} \left(\sum_{q=1}^{n} m_q \beta_{jq} \right) + \delta_{rs} m_i m_j m_r \beta_{ir} \beta_{jr}$$

is equal to zero. Hence,

$$A_{[ij]rs} = \tilde{A}_{[ij]rs} = \tilde{A}^{(1)}_{[ij]rs} + \tilde{A}^{(2)}_{[ij]rs} + \tilde{A}^{(3)}_{[ij]rs}, \tag{4.1.20}$$

where

$$\begin{aligned}
\tilde{A}^{(1)}_{ijrs} &= \delta_{ir} \delta_{js} m_i m_j \sum_{q=1}^{n} m_q \left(\beta_{ij} \beta_{iq} + \beta_{ij} \beta_{qj} + \beta_{iq} \beta_{jq} \right), \\
\tilde{A}^{(2)}_{ijrs} &= \delta_{jr} m_i m_j m_s \left(\beta_{ij} \beta_{sj} + \beta_{ij} \beta_{is} + \beta_{is} \beta_{js} \right),
\end{aligned}$$

and

$$\tilde{A}^{(3)}_{ijrs} = \delta_{is} m_i m_j m_r \left(\beta_{ir}\beta_{ij} + \beta_{rj}\beta_{ij} + \beta_{ir}\beta_{jr}\right).$$

We now apply the following identity:

$$\beta_{ij}\beta_{ik} + \beta_{ij}\beta_{kj} + \beta_{ik}\beta_{jk} = 1,$$

where i, j, k are distinct from each other. We get

$$\begin{aligned}\tilde{A}^{(1)}_{[ij]rs} = (1-\delta_{ij})(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) m_i m_j \left(M - m_i - m_j\right) \\ + (1-\delta_{ij})(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) m_i m_j (m_i + m_j)\beta_{ij}^2 \\ + \delta_{ij}(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) m_i^2 \left(\sum_{q=1}^{n} m_q \beta_{iq}^2\right),\end{aligned}$$

and so

$$\tilde{A}^{(1)}_{[ij]rs} = (\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) m_i m_j \left(M - m_i - m_j + (m_i + m_j)\beta_{ij}^2\right) \tag{4.1.21}$$

since $\delta_{ij}(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) = 0$. Furthermore,

$$\begin{aligned}\tilde{A}^{(2)}_{ijrs} = \delta_{jr}\delta_{ij}(1-\delta_{is}) m_i^2 m_s \beta_{is}^2 + \delta_{jr}\delta_{is}(1-\delta_{ij}) m_i^2 m_j \beta_{ij}^2 \\ + \delta_{jr}\delta_{js}(1-\delta_{ij}) m_i m_j^2 \beta_{ij}^2 + \delta_{jr}(1-\delta_{ij})(1-\delta_{is})(1-\delta_{js}) m_i m_j m_s,\end{aligned}$$

which becomes

$$\begin{aligned}\tilde{A}^{(2)}_{ijrs} = \delta_{ijr} m_i^2 m_s \beta_{is}^2 + \delta_{jr}\delta_{is} m_i^2 m_j \beta_{ij}^2 + \delta_{jrs} m_i m_j^2 \beta_{ij}^2 \\ + \delta_{jr}(1-\delta_{ij})(1-\delta_{is})(1-\delta_{js}) m_i m_j m_s.\end{aligned} \tag{4.1.22}$$

Similarly,

$$\begin{aligned}\tilde{A}^{(3)}_{ijrs} = \delta_{ijs} m_j^2 m_r \beta_{jr}^2 + \delta_{is}\delta_{jr} m_j^2 m_i \beta_{ji}^2 + \delta_{irs} m_j m_i^2 \beta_{ji}^2 \\ + \delta_{is}(1-\delta_{ij})(1-\delta_{jr})(1-\delta_{ir}) m_i m_j m_r.\end{aligned} \tag{4.1.23}$$

Note that the first term in the right-hand side of each of (4.1.22) and (4.1.23) becomes zero after antisymmetrisation over i and j, and that the second term in each of $\tilde{A}^{(2)}_{ijrs}$, $-\tilde{A}^{(2)}_{jirs}$, $\tilde{A}^{(3)}_{ijrs}$, and $-\tilde{A}^{(3)}_{jirs}$ cancels with the β_{ij}^2 terms in (4.1.21). Moreover, the third term in the right-hand side of (4.1.22) cancels with the third term in (4.1.23) after antisymmetrisation.

Therefore

$$\tilde{A}^{(1)}_{[ij]rs} + \tilde{A}^{(2)}_{[ij]rs} + \tilde{A}^{(3)}_{[ij]rs} = (\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is})m_i m_j\,(M - m_i - m_j) + \alpha_{[ij]rs}, \tag{4.1.24}$$

where

$$\alpha_{ijrs} = \delta_{jr}(1-\delta_{ir})(1-\delta_{is})(1-\delta_{rs})m_i m_r m_s \\ + \delta_{is}(1-\delta_{jr})(1-\delta_{js})(1-\delta_{rs})m_j m_r m_s.$$

Note that

$$\alpha_{[ij]rs} = (\delta_{jr} - \delta_{js})\,(1-\delta_{ir})(1-\delta_{is})m_i m_r m_s \\ + (\delta_{is} - \delta_{ir})\,(1-\delta_{jr})(1-\delta_{js})m_j m_r m_s. \tag{4.1.25}$$

It then follows from formulas (4.1.20), (4.1.24) and (4.1.25) that

$$A_{[ij]rs} = M m_r m_s(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) + m_r m_s\big(m_i(\delta_{jr} - \delta_{js}) + m_j(\delta_{is} - \delta_{ir})\big). \tag{4.1.26}$$

We now consider those terms in $\lambda_{[ij]rs}$ which are linear in β_{pq}. By Lemmas 4.1.4 and 4.1.1 the sum of these terms is given by $B_{[ij]rs}$, where

$$B_{ijrs} = \sum_{k=1}^{n} m_k \left(\delta_{kir}\sum_{q=1}^{n} m_q\beta_{kq} + \delta_{ki}m_r\beta_{rk} + \delta_{kr}m_i\beta_{ik} + \delta_{ir}m_i\beta_{ki}\right) \\ \times (\delta_{kjs}c + \delta_{kj}bm_s + \delta_{ks}bm_j + \delta_{js}bm_j) \\ + \sum_{k=1}^{n} m_k \left(\delta_{kjs}\sum_{q=1}^{n} m_q\beta_{kq} + \delta_{kj}m_s\beta_{sk} + \delta_{ks}m_j\beta_{jk} + \delta_{js}m_j\beta_{kj}\right) \\ \times (\delta_{kir}c + \delta_{ki}bm_r + \delta_{kr}bm_i + \delta_{ir}bm_i)\,.$$

By expanding, and cancelling terms which are symmetric in i and j or in r and s, we see that

$$B_{[ij]rs} = \tilde{B}_{[ij][rs]},$$

where

$$\tilde{B}_{ijrs} = m_i m_j\Big(\delta_{is}bm_r\beta_{ri} + \delta_{jr}bm_s\beta_{ir} + \delta_{ir}\delta_{js}c\beta_{ji} + \delta_{ir}bm_s\beta_{ji} + \delta_{ir}bm_s\beta_{si}\Big),$$

and

$$\tilde{B}_{[ij][rs]} = \tilde{B}_{[ij]rs} - \tilde{B}_{[ij]sr}.$$

Note that the term $\delta_{ir}\delta_{js}cm_i m_j\beta_{ji}$ in $\tilde{B}_{ijrs}$ cancels with the corresponding term in $\tilde{B}_{jisr}$, and hence terms proportional to c vanish in $\tilde{B}_{[ij][rs]}$. The terms

$$\delta_{is} b m_i m_j m_r \beta_{ri} + \delta_{ir} b m_i m_j m_s \beta_{si} = b m_i m_j \beta_{rs} \left(\delta_{is} m_r - \delta_{ir} m_s\right)$$

are together symmetric under the swap of r and s, hence the corresponding terms cancel in $\tilde{B}_{ij[rs]}$. Finally, the sum of the remaining two terms is symmetric under the swap of i and j, hence the corresponding terms cancel in $\tilde{B}_{[ij]rs}$. Overall, we get that

$$B_{[ij]rs} = \tilde{B}_{[ij][rs]} = 0.$$

Finally, we consider the contribution to $\lambda_{[ij]rs}$ from constant terms, denoted $C_{[ij]rs}$ with

$$C_{ijrs} = \sum_{k=1}^{n} m_k \left(\delta_{kir} c + \delta_{ki} b m_r + \delta_{kr} b m_i + \delta_{ir} b m_i\right) \\ \times \left(\delta_{kjs} c + \delta_{kj} b m_s + \delta_{ks} b m_j + \delta_{js} b m_j\right).$$

We can again omit terms symmetric in i and j to find $C_{[ij]rs} = \tilde{C}_{[ij]rs}$, where

$$\tilde{C}_{ijrs} = b m_i m_j \big(\delta_{irs} c + \delta_{ir}\delta_{js}\left(2c + bM\right) + \delta_{is} b m_r + 2\delta_{js} b m_r + \delta_{jrs} c + \delta_{jr} b m_s \\ + \delta_{rs} b m_r + 2\delta_{ir} b m_s\big).$$

Note that the sum of the terms

$$b m_i m_j \left(\delta_{irs} c + \delta_{jrs} c + \delta_{is} b m_r + \delta_{js} b m_r + \delta_{jr} b m_s + \delta_{ir} b m_s + \delta_{rs} b m_r\right)$$

is symmetric in i and j, hence vanishes after the antisymmetrisation. Therefore

$$C_{[ij]rs} = \tilde{C}_{[ij]rs} = b m_r m_s (bM + 2c)(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}) \\ + b^2 m_j m_r m_s (\delta_{ir} - \delta_{is}) + b^2 m_i m_r m_s (\delta_{js} - \delta_{jr}). \quad (4.1.27)$$

To finish the proof, we combine expressions (4.1.26)–(4.1.27) to obtain the required statement for

$$\lambda_{[ij]rs} = \frac{1}{bM + c}\left(A_{[ij]rs} + B_{[ij]rs} + C_{[ij]rs}\right).$$

□

We are now ready to prove Theorem 4.0.1.

Proof of Theorem 4.0.1. By Lemmas 4.1.4 and 4.1.5

$$\begin{aligned}\left(F_i\eta^{-1}F_j\right)_{rs} - \left(F_j\eta^{-1}F_i\right)_{rs} =&\\ m_r m_s(\delta_{jr}\delta_{is} - \delta_{ir}\delta_{js})&\left(\frac{(aM+2b)(bM+c)}{aM^2+3bM+c} - \frac{M+b^2M+2bc}{bM+c}\right)\\ + m_r m_s\big(m_j\left(\delta_{is}-\delta_{ir}\right) + m_i(\delta_{jr}-\delta_{js})\big)&\left(\frac{(aM+2b)^2}{aM^2+3bM+c} - a - \frac{b^2-1}{bM+c}\right)\\ &= \frac{m_r m_s \gamma\big(M(\delta_{ir}\delta_{js}-\delta_{jr}\delta_{is}) + m_j\left(\delta_{is}-\delta_{ir}\right) + m_i(\delta_{jr}-\delta_{js})\big)}{\left(aM^2+3bM+c\right)\left(bM+c\right)}, \qquad (4.1.28)\end{aligned}$$

where

$$\gamma = b^3M + 3b^2c - ac^2 + aM^2 + 3bM + c.$$

Therefore the WDVV equations hold when $\gamma = 0$. It is also easy to see that γ has to be zero when $n \geq 3$ if expression (4.1.28) vanishes for all i, j, r, s. □

Remark 4.1.6. In the special case $b = -\frac{1}{2}aM$, the metric η is diagonal, the condition (4.0.3) reduces to (4.0.2), and the relation (4.0.4) can be rearranged to the form

$$a^2M^2 - 4ac + 4 = 0.$$

4.2 Other cases

Theorem 4.0.1 relies on the two conditions (4.0.2) and (4.0.3). We now consider the case when condition (4.0.2) does not hold. The first statement is as follows.

Theorem 4.2.1. *Suppose $bM + c = 0$. Let Q be the diagonal $n \times n$ matrix with entries $Q_{rs} = \delta_{rs}m_r$. Then the equations*

$$F_iQ^{-1}F_j = F_jQ^{-1}F_i \qquad (4.2.1)$$

hold for all $i, j \in \{1, \ldots, n\}$ if $b = \pm 1$. Moreover, equations (4.2.1) imply $b = \pm 1$ provided that $n \geq 3$.

Proof. Note that

$$\left(F_iQ^{-1}F_j\right)_{rs} = \sum_{k=1}^{n} \frac{1}{m_k} F_{irk}F_{jks},$$

and $F_i = W_i + V_i$ as in Lemma 4.1.1. By formula (4.1.14) and Lemmas 4.1.4, 4.1.5, we find that

$$\begin{aligned}\left(F_i Q^{-1} F_j\right)_{rs} - \left(F_j Q^{-1} F_i\right)_{rs} &= \sum_{k=1}^{n} \frac{1}{m_k} \Big[(V_i)_{rk} (V_j)_{ks} - (V_j)_{rk} (V_i)_{ks} \Big] \\ &= \left(M + b^2 M + 2bc\right) m_r m_s \left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}\right) \\ &\quad + \left(b^2 - 1\right) m_r m_s \big(m_i(\delta_{js} - \delta_{jr}) + m_j(\delta_{ir} - \delta_{is})\big).\end{aligned}$$

Since $c = -bM$, we get $M + b^2 M + 2bc = M(-b^2 + 1)$ and the statement follows. □

We also have the following statement on a solution of the WDVV equations.

Theorem 4.2.2. *Suppose that $bM + c = 0$, $b = \pm 1$, and $aM + 2b \neq 0$. Then the matrix Q from Theorem 4.2.1 can be represented as*

$$Q = \kappa^{-1} \sum_{k=1}^{n} h_k F_k,$$

where

$$h_k = -(aM + 2b)e^{2by_k} + a \sum_{q=1}^{n} m_q e^{2by_q} \tag{4.2.2}$$

and

$$\kappa = -2b(aM + 2b) \sum_{q=1}^{n} m_q e^{2by_q}.$$

Proof. By Lemma 4.1.1 and substituting $c = -bM$, we have

$$\begin{aligned}F_{krs} &= a m_k m_r m_s + \delta_{krs} m_k \left(\sum_{q=1}^{n} m_q \beta_{kq} - bM \right) + \delta_{kr} m_k m_s \beta_{sk} \\ &\quad + \delta_{ks} m_k m_r \beta_{rk} + \delta_{rs} m_k m_r \beta_{kr} + b m_k \left(\delta_{kr} m_s + \delta_{ks} m_r + \delta_{rs} m_r\right).\end{aligned}$$

We now compute the linear combination $B = \sum_{k=1}^{n} h_k F_k$, where h_k is defined in (4.2.2). By expanding and cancelling terms, we find that the matrix entries of B are

$$\begin{aligned}B_{rs} &= -\delta_{rs}(aM + 2b) m_r \sum_{k=1}^{n} m_k \beta_{rk} \left(e^{2by_r} - e^{2by_k}\right) + \delta_{rs}(aM + 2b) b m_r M e^{2by_r} \\ &\quad + (aM + 2b) m_r m_s \beta_{rs} \left(e^{2by_r} - e^{2by_s}\right) - (aM + 2b) b m_r m_s \left(e^{2by_r} + e^{2by_s}\right) \\ &\quad - \delta_{rs}(aM + 2b) b m_r \sum_{k=1}^{n} m_k e^{2by_k}.\end{aligned} \tag{4.2.3}$$

Observe that

$$\beta_{ij} \left(e^{2by_i} - e^{2by_j}\right) = \begin{cases} b\left(e^{2by_i} + e^{2by_j}\right) & i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Then formula (4.2.3) can be simplified to $B_{rs} = \kappa Q_{rs}$ as required. □

Theorems 4.2.1 and 4.2.2 generalise observations from the arXiv version of [35] to the case of arbitrary (non-zero) parameters m_i.

4.3 Comparison with known solutions

4.3.1 Riley's work

In [51], Riley produced $(n+1)$-parameter trigonometric functions related to the extended affine Weyl group of type A_{n-1}. As the formula given in the relevant theorem, [51, Theorem 4.12], appears to contain typos, we provide a corrected version below. We then show that these functions are included in the family described by (4.0.1).

Let V be an n-dimensional vector space. We consider the function

$$F_k^R(\phi) = \frac{1}{2}\sum_{p=1}^{n}\sum_{q\neq p} k_p k_q \mathrm{Li}_3\left(e^{i(\phi_p-\phi_q)}\right) + \sum_{p=1}^{n} A_p^R \phi_p^3 + \sum_{p=1}^{n}\sum_{q\neq p} B_{pq}^R \phi_p^2\phi_q + \sum_{p=1}^{n-2}\sum_{q>p}\sum_{r>q} C_{pqr}^R \phi_p\phi_q\phi_r, \quad (4.3.1)$$

where $\phi = (\phi_1, \ldots, \phi_n) \in V$, $k = (k_1, \ldots, k_n) \in \mathbb{C}^n$, and

$$A_p^R = \frac{ik_p^2}{3} - \frac{ik_p^3}{6\mu} - \frac{ik_p\mu}{6} + \frac{i}{12}\sum_{q\neq p} k_p k_q, \quad (4.3.2)$$

$$B_{pq}^R = \frac{ik_pk_q}{4} - \frac{ik_p^2k_q}{2\mu}, \quad (4.3.3)$$

$$C_{pqr}^R = -\frac{ik_pk_qk_r}{\mu}, \quad (4.3.4)$$

for $\mu \in \mathbb{C}$. When $\mu \in \{1, \ldots, n-1\}$, this corresponds to a choice of marked simple root (or marked node in a Dynkin diagram) in the construction of the extended affine Weyl group.

Remark 4.3.1. The expression for the parameter A_p^R given by (4.3.2) contains corrections to the analogous expression in [51, Theorem 4.12]. Without these corrections, the result is inconsistent with [51, Lemma 4.11], which provides third-order derivatives of (4.3.1). It can also be verified that the function (4.3.1) as defined here is a solution of the ordinary WDVV equations (2.2.12), while the function given in [51, Theorem 4.12] is not.

Let $F = F_k^R(\phi)$ given by (4.3.1) and $\widetilde{F} = F_{A_{n-1}}^{\mathrm{trig}}(y)$ given by (4.0.5), with the parameters a, b, c, m, M as defined in Theorem 4.0.1.

Theorem 4.3.2. *Define a, b, c, and $m = (m_1, \dots, m_n)$ by*

$$\begin{aligned} m_i &= k_i \quad \forall\, i \in \{1, \dots, n\}, \\ a &= -\frac{2}{\mu}, \\ b &= 1, \\ c &= \sum_{i}^{n} k_i - 2\mu. \end{aligned} \tag{4.3.5}$$

Then functions F and $\widetilde{F}$ satisfy

$$-\frac{1}{4}F(\phi) = \widetilde{F}(y) \text{ up to quadratic terms,}$$

where the coordinate system y is given by

$$y_j = \frac{i}{2}\phi_j \quad \forall\, j \in \{1, \dots, n\}.$$

Conditions (4.0.2) and (4.0.3) are both equivalent to the condition

$$\mu \neq \sum_{i=1}^{n} k_i.$$

Condition (4.0.4) is satisfied for all values of k, μ.

Proof. The function F can be rewritten as

$$F = \frac{1}{2}\sum_{p\neq q}^{n} k_p k_q \Big(\mathrm{Li}_3\left(e^{i(\phi_p-\phi_q)}\right) + \mathrm{Li}_3\left(e^{i(\phi_q-\phi_p)}\right)\Big) + \sum_{p=1}^{n} A_p^R \phi_p^3 + \sum_{p\neq q}^{n} B_{pq}^R \phi_p^2 \phi_q + \sum_{1\le p<q<r\le n} C_{pqr}^R \phi_p\phi_q\phi_r. \tag{4.3.6}$$

From the definition of $f(z)$ in (2.5.7), we have

$$-8f\left(\frac{iz}{2}\right) = \frac{i}{6}z^3 + 2\mathrm{Li}_3\left(e^{-iz}\right)$$

and

$$\frac{\partial^3}{\partial z^3}\left(-8f\left(\frac{iz}{2}\right)\right) = \frac{i\left(e^{iz}+1\right)}{e^{iz}-1} = \frac{\partial^3}{\partial z^3}\Big(\mathrm{Li}_3\left(e^{iz}\right) + \mathrm{Li}_3\left(e^{-iz}\right)\Big).$$

Therefore,

$$\mathrm{Li}_3\left(e^{iz}\right) + \mathrm{Li}_3\left(e^{-iz}\right) = -8f\left(\frac{iz}{2}\right) \text{ up to quadratic terms,}$$

and (4.3.6) becomes

$$F = -4\sum_{p\neq q}^{n} k_p k_q f\left(\frac{i}{2}(\phi_p - \phi_q)\right) + \sum_{p=1}^{n} A_p^R \phi_p^3 + \sum_{p\neq q}^{n} B_{pq}^R \phi_p^2 \phi_q + \sum_{1\leq p<q<r\leq n} C_{pqr}^R \phi_p \phi_q \phi_r$$

up to quadratic terms. By comparison of the terms in $-\frac{1}{4}F$ and $\widetilde{F}$ under the change of variables $y_j = \frac{i}{2}\phi_j$, it is straightforward to recover the relations given by (4.3.5).

We then find that

$$bM + c = 2\sum_{i=1}^{n} k_i - 2\mu,$$

and

$$\begin{aligned} aM^2 + 3bM + c &= -\frac{2}{\mu}\left(\sum_{i=1}^{n} k_i\right) + 4\sum_{i=1}^{n} k_i - 2\mu \\ &= -\frac{2}{\mu}\left(\sum_{i=1}^{n} k_i - \mu\right)^2. \end{aligned}$$

Conditions (4.0.2), (4.0.3) are therefore both equivalent to

$$\mu \neq \sum_{i=1}^{n} k_i.$$

The left-hand side of condition (4.0.4) becomes

$$5\sum_{i=1}^{n} k_i + 3\left(\sum_{i=1}^{n} k_i - 2\mu\right) + \frac{2}{\mu}\left(\sum_{i=1}^{n} k_i - 2\mu\right)^2 - \frac{2}{\mu}\left(\sum_{i=1}^{n} k_i\right)^2 - 2\mu = 0,$$

without any further restrictions. □

4.3.2 Shen's work

Shen found trigonometric solutions to the WDVV equations in [56] associated with the reduced irreducible root system $\mathcal{R}$ of a rank n Weyl group W. These structures include a multiplicity parameter $\kappa = (k_\alpha)_{\alpha\in\mathcal{R}} \in \mathbb{C}^n$; however, this κ is required to be W-invariant. In the type A case that we consider here, this means that κ is described by the single value $k = k_\alpha$ for all $\alpha \in \mathcal{R}$. An additional parameter $k' \in \mathbb{C}$ appears in this construction, so that it becomes a 2-parameter system.

The type A solution is obtained explicitly in [56, § 4]; we reproduce this as follows. We consider a function $\Phi = \Phi(x, \xi) : V \times \mathbb{C} \to \mathbb{C}$, where $\xi \in \mathbb{C}$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, and

$V \subset \mathbb{C}^n$ is the hyperplane defined by $\sum_{i=1}^n x_i = 0$. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, with dual basis $\{e^1, \ldots, e^n\}$. Let $\mathcal{R}_+$ be the set of positive roots of A_{n-1} given by $\mathcal{R}_+ = \{e^i - e^j \,|\, 1 \leq i < j \leq n\}$. Given $\alpha = e^i - e^j$ for some i, j, we define

$$\alpha' := e^i + e^j - \frac{2}{n} \sum_{l=1}^n e^l.$$

For the following expression, see formula (4.1) in [56]. The function Φ associated with A_{n-1} is given by

$$\Phi = \frac{\xi^3}{6} - K^S \xi \sum_{\alpha \in \mathcal{R}_+} \alpha(x)^2 + nkK^S \sum_{\alpha \in \mathcal{R}_+} q\left(\alpha(x)\right) + nk'K^S \sum_{\alpha \in \mathcal{R}_+} \frac{\alpha(x)^2 \alpha'(x)}{6}, \tag{4.3.7}$$

where $k, k' \in \mathbb{C}$, and

$$\begin{aligned} K^S &= \frac{1}{8}\left(k^2 - k'^2\right), \\ q(z) &= \frac{1}{6} z^3 - 2\mathrm{Li}_3\left(e^{-z}\right) \\ &= 8f\left(\frac{z}{2}\right), \end{aligned}$$

where $f(z)$ is given by (2.5.7).

Remark 4.3.3. The function (4.3.7) includes corrections to the last term in formula (4.1) in [56]. Namely, a factor of $\frac{1}{2}$ in [56] has been changed here to $\frac{1}{6}$. Without this change, the function does not satisfy the ordinary WDVV equations (2.2.12).

We now set $\xi = \sum_{i=1}^n x_i$ and let $F = F(x) = \Phi\left(x, \xi(x)\right)$ so that $F : \mathbb{C}^n \to \mathbb{C}$. As before, let $\widetilde{F} = F_m^{\mathrm{trig}}(y)$ be given by (4.0.5).

Theorem 4.3.4. *Define a, b, c and m_i, $1 \leq i \leq n$, by*

$$\begin{aligned} m_i &= 1 \quad \forall\, i \in \{1, \ldots, n\}, \\ a &= \frac{8 + (6 + 2k')\left(k^2 - k'^2\right)}{nk\left(k^2 - k'^2\right)}, \\ b &= -\frac{2 + k'}{k}, \\ c &= \frac{nk'}{k}, \end{aligned} \tag{4.3.8}$$

where $k \neq 0$, $k^2 \neq k'^2$. Then functions F and $\widetilde{F}$ satisfy

$$\frac{1}{nk\left(k^2 - k'^2\right)} F(x) = \widetilde{F}(y),$$

where the coordinate system y is given by

$$y_i = \frac{1}{2}x_i \quad \forall\, i \in \{1,\ldots,n\}.$$

Conditions (4.0.2), (4.0.3), (4.0.4) are all satisfied.

Proof. We start by rewriting (4.3.7) as

$$F(x) = 8nkK^S \sum_{1\le i<j\le n} f\left(\frac{1}{2}(x_i - x_j)\right) - K^S \left(\sum_{i=1}^n x_i\right)\left(\sum_{1\le i<j\le n} x_i^2 - 2x_ix_j + x_j^2\right)$$
$$+ \frac{1}{6}\left(\sum_{i=1}^n x_i\right)^3 + \frac{k'}{6}K^S \sum_{1\le i<j\le n} \left(x_i^2 - 2x_ix_j + x_j^2\right)\left((n-2)(x_i+x_j) - 2\sum_{l\ne i,j} x_l\right).$$

Rescaling this by $\frac{1}{nk\left(k^2-k'^2\right)}$, we get

$$\frac{1}{nk\left(k^2-k'^2\right)}F(x) = \sum_{1\le i<j\le n} f\left(\frac{1}{2}(x_i - x_j)\right) + \frac{1}{6nk\left(k^2-k'^2\right)}A^S$$
$$- \frac{1}{8nk}B^S + \frac{k'}{48nk}C^S, \quad (4.3.9)$$

where

$$A^S = \left(\sum_{i=1}^n x_i\right)^3 = \sum_{i=1}^n x_i^3 + 3\sum_{i\ne j}^n x_i^2x_j + 6\sum_{1\le i<j<l\le n} x_ix_jx_l, \quad (4.3.10)$$

$$B^S = \left(\sum_{i=1}^n x_i\right)\left(\sum_{1\le i<j\le n} x_i^2 - 2x_ix_j + x_j^2\right), \quad (4.3.11)$$

$$C^S = \sum_{1\le i<j\le n} \left(x_i^2 - 2x_ix_j + x_j^2\right)\left((n-2)(x_i+x_j) - 2\sum_{l\ne i,j}^n x_l\right). \quad (4.3.12)$$

To simplify (4.3.11), note that

$$\sum_{1\le i<j\le n} \left(x_i^2 + x_j^2\right) = \frac{1}{2}\left(\sum_{i=1}^n\sum_{j=1}^n \left(x_i^2 + x_j^2\right) - \sum_{i=1}^n 2x_i^2\right) = (n-1)\sum_{i=1}^n x_i^2.$$

We then have

$$B^S = \left(\sum_{i=1}^n x_i\right)\left((n-1)\sum_{i=1}^n x_i^2 - \sum_{i\ne j}^n x_ix_j\right)$$
$$= (n-1)\sum_{i=1}^n x_i^3 + (n-3)\sum_{i\ne j}^n x_i^2x_j - 6\sum_{1\le i<j<l\le n} x_ix_jx_k. \quad (4.3.13)$$

The expression (4.3.12) can similarly be expanded and simplified, as follows:

$$\begin{aligned} C^S &= (n-2)\sum_{1\le i<j\le n}\left(x_i^3 - x_i^2x_j - x_ix_j^2 + x_j^3\right) - 2(n-2)\sum_{i\neq j}^n x_i^2x_j + 4\sum_{\substack{1\le i<j\le n\\ l\neq i,j}} x_ix_jx_l \\ &= (n-1)(n-2)\sum_{i=1}^n x_i^3 - 3(n-2)\sum_{i\neq j}^n x_i^2x_j + 12\sum_{1\le i<j<l\le n} x_ix_jx_l, \end{aligned} \tag{4.3.14}$$

where we additionally used the formula

$$\sum_{\substack{1\le i<j\le n\\ l\neq i,j}} x_i^2x_l + x_j^2x_l = \sum_{j\neq i,l}\sum_{l\neq i}\sum_{i=1}^n x_i^2x_l = (n-2)\sum_{i\neq j}^n x_i^2x_j.$$

Substituting the expressions for A^S, B^S, C^S given by (4.3.10), (4.3.13), (4.3.14) back into (4.3.9), we have

$$\begin{aligned} \frac{1}{nk\left(k^2-k'^2\right)}F(x) &= \sum_{1\le i<j\le n} f\left(\frac{1}{2}(x_i-x_j)\right) + \left(\frac{1}{nk\left(k^2-k'^2\right)} + \frac{6+2k'}{8nk}\right)\sum_{1\le i<j<l\le n} x_ix_jx_l \\ &+ \left(\frac{1}{2nk\left(k^2-k'^2\right)} - \frac{2(n-3)+k'(n-2)}{16nk}\right)\sum_{i\neq j}^n x_i^2x_j \\ &+ \frac{1+(n-1)\left(k'(n-2)-6\right)\left(k^2-k'^2\right)}{48nk\left(k^2-k'^2\right)}\sum_{i=1}^n x_i^3. \end{aligned} \tag{4.3.15}$$

Comparing the cubic terms in (4.3.15) with those in (4.0.5), under the change of variables $y_i = \frac{1}{2}x_i$, produces the relations given in (4.3.8).

Since $m_i = 1\ \forall\ i$, we have $M = n$. Using the values of a, b, c in (4.3.8), the left-hand side of condition (4.0.2) becomes

$$bM + c = -\frac{2n}{k},$$

and the left-hand side of (4.0.3) becomes

$$\begin{aligned} aM^2 + 3bM + c &= \frac{8n}{k\left(k^2-k'^2\right)} + \frac{(6+2k')n}{k} - \frac{(6+3k')n}{k} + \frac{nk'}{k} \\ &= \frac{8n}{k\left(k^2-k'^2\right)}, \end{aligned}$$

so both conditions hold. Finally, the left-hand side of (4.0.4) is

$$b^3M + 3b^2c - ac^2 + aM^2 + 3bM + c = -\frac{n\left(8 + 12k' + 6k'^2 + k'^3\right)}{k^3} + \frac{3nk'\left(4 + 4k' + k'^2\right)}{k^3} - \frac{8nk'^2}{k^3\left(k^2 - k'^2\right)} - \frac{nk'^2(6 + 2k')}{k^3} + \frac{8n}{k\left(k^2 - k'^2\right)},$$

which is equal to zero as all the terms cancel. □

Chapter 5

Legendre transformations of A_n-type rational solutions

In this chapter, we apply Legendre transformations to the $(n+1)$-parameter family of rational solutions associated with the A_n-type $\vee$-system found in [11]. The transformations considered produce families of A_{n-1}-type trigonometric solutions, of the form considered in Chapter 4. We follow [25] for the results in Section 5.2.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, with coordinates $x = (x^1, \ldots, x^n) \in \mathbb{C}^n$. Let $k_1, \ldots, k_n \in \mathbb{C}^\times$ be such that $K := \sum_{i=1}^n k_i \neq -1$. The multi-parameter A_n-type $\vee$-system is the collection of vectors [11]

$$A_n(k) = \left\{\sqrt{k_i} e_i | 1 \leq i \leq n\right\} \cup \left\{\sqrt{k_i k_j}(e_i - e_j) | 1 \leq i < j \leq n\right\}. \tag{5.0.1}$$

The corresponding solutions of the WDVV equations are

$$F = F^{\text{rat}}_{A_n(k)} = \sum_{i=1}^n k_i \left(x^i\right)^2 \log\left(x^i\right) + \sum_{1 \leq i < j \leq n} k_i k_j \left(x^i - x^j\right)^2 \log\left(x^i - x^j\right). \tag{5.0.2}$$

The function F satisfies the WDVV equations (2.2.2) with a choice of metric η as in (2.5.2); that is,

$$\eta = \sum_{i=1}^n x^i F_i,$$

which has constant entries

$$\eta_{ij} = \begin{cases} 2k_i(K - k_i + 1), & i = j, \\ -2k_i k_j, & \text{otherwise.} \end{cases}$$

Then the contravariant metric, denoted η^{-1}, has entries given by

$$\eta^{ij} = \begin{cases} \frac{k_i+1}{2k_i(K+1)}, & i=j, \\ \frac{1}{2(K+1)}, & \text{otherwise.} \end{cases} \tag{5.0.3}$$

5.1 Roots and fundamental weights of A_n as Legendre fields

Since the choice of Legendre field amounts to selecting a new distinguished direction, roots and fundamental weights are a natural choice of Legendre field for a solution associated with a root system. As fundamental weights are not defined for the deformed system (5.0.1) with $k_i \neq 1$, we refer here to the geometry of the Coxeter case, where $k_i = 1$.

To recover the Coxeter A_n root system from (5.0.1), we set $k_i = 1 \ \forall \ i \in \{1, \dots, n\}$. We choose the set of simple roots $\Delta = \{\beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n\} \subset \mathbb{C}^n$ where $\beta_i = e_i - e_{i+1}$, $1 \leq i < n$, and $\beta_n = e_n$. This realisation of A_n can be obtained from a realisation in $(n+1)$ dimensions — see, for example, [6, Plate I] — as follows. Let $f_1, \dots, f_{n+1}$ denote the standard basis of $\mathbb{C}^{n+1}$ with inner product $(f_i, f_j) = \delta_{ij}$. A set of simple roots for A_n in $\mathbb{C}^{n+1}$ is

$$\widetilde{\Delta} = \{f_1 - f_2, \dots, f_{n-1} - f_n, f_n - f_{n+1}\}.$$

Taking the orthogonal projection of $\widetilde{\Delta}$ onto the hyperplane $\Pi = \{x \in \mathbb{C}^{n+1} \,|\, x^{n+1} = 0\} \cong \mathbb{C}^n$, we can set a basis in the hyperplane Π to be

$$e_i = f_i - f_{n+1}, \quad 1 \leq i \leq n.$$

The projection of the inner product onto $\mathbb{C}^n$ is then given by $(e_i, e_j) = \delta_{ij} + 1$.

Referring to Definition 2.3.25, it can be verified that the first fundamental weight of A_n in the n-dimensional realisation is $\omega_1 = e_1 - \frac{1}{n+1}\sum_{k=1}^n e_k$ and the last fundamental weight is $\omega_n = \frac{1}{n+1}\sum_{k=1}^n e_k$.

We will first consider the Legendre transformation of (5.0.2) along an arbitrary root of the form e_γ. That is, we set the Legendre field to be $\delta = \partial_\gamma$ for some $\gamma \in \{1, \dots, n\}$. The Legendre transformation produced by this field is denoted S_γ, following Notation 2.4.19. We then consider the Legendre transformation of (5.0.2) via a rescaling of the last fundamental weight of A_n; that is, we choose the Legendre field $\delta = \kappa \sum_{k=1}^n \partial_k$.

Remark 5.1.1. For $n = 2$, we saw in Section 3.4 how the flat Legendre fields considered here can be obtained from the twisting of homogeneous Legendre fields for the Frobenius manifold $\mathbb{C}^2/A_2$.

5.2 Legendre transformation along a root of A_n

We now compute the result of applying a Legendre transformation S_γ to F for any $\gamma \in \{1, \ldots, n\}$. We set $\widehat{F} = S_\gamma(F)$.

Note that since the WDVV equations are unaffected by the addition of quadratic terms to a function F, we may disregard such terms during calculations. By definition, the new coordinates $\hat{x}_\alpha$ are found using second-order derivatives of F. For simplicity, constant terms that appear from these calculations can be absorbed with the addition of appropriate quadratic terms to F, and the flat coordinates $\hat{x}_\alpha$ redefined. Since the addition of such terms to F is equivalent to a linear coordinate transformation of the resulting $\widehat{F}$, we will find and discuss $\widehat{F}$ up to such transformations. We will use the notation $A \doteq B$ to denote the equality of A and B up to constant terms.

Under the Legendre transformation S_γ, the new flat coordinates $\hat{x}_\alpha$ can be chosen (after constant coordinate shifts) as follows:

$$\hat{x}_\alpha = \begin{cases} 2k_\gamma \log(x^\gamma) + 2k_\gamma \sum_{i\neq\gamma} k_i \log\left(x^\gamma - x^i\right) & \alpha = \gamma, \\ -2k_\gamma k_\alpha \log\left(x^\gamma - x^\alpha\right) & \text{otherwise.} \end{cases} \tag{5.2.1}$$

The original flat coordinates x^α now must be rewritten in terms of the new contravariant coordinates $\hat{x}^\alpha$.

Lemma 5.2.1. *The flat coordinates x^α can be expressed as follows:*

$$x^\alpha = \left(1 - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right) x^\gamma \text{ for } \alpha \neq \gamma, \tag{5.2.2}$$

$$\log\left(x^\gamma\right) = \sum_{i=1}^{n} \frac{k_i}{k_\gamma} \hat{x}^i. \tag{5.2.3}$$

Proof. The new contravariant coordinates $\hat{x}^\alpha$ can be found by using the inverse metric η^{-1} (5.0.3) to raise the indices of the covariant coordinates in (5.2.1). Since the Legendre transformation does not affect the metric, we have

$$\eta_{\alpha\beta} = \widehat{\eta}_{\alpha\beta},$$

which implies that

$$\hat{x}^\alpha = \widehat{\eta}^{\alpha\beta}\hat{x}_\beta = \eta^{\alpha\beta}\hat{x}_\beta.$$

The contravariant coordinates are therefore

$$\hat{x}^\alpha = \begin{cases} \frac{k_\gamma+1}{K+1}\log\left(x^\gamma\right) + \frac{1}{K+1}\sum_{i\neq\gamma} k_i \log\left(x^\gamma - x^i\right) & \alpha=\gamma, \\ \frac{k_\gamma}{K+1}\log\left(x^\gamma\right) - \frac{k_\gamma}{K+1}\log\left(x^\gamma - x^\alpha\right) & \text{otherwise.} \end{cases} \tag{5.2.4}$$

Considering the case when $\alpha \neq \gamma$, we have

$$\hat{x}^\alpha = \frac{k_\gamma}{K+1}\log\left(\frac{x^\gamma}{x^\gamma - x^\alpha}\right),$$

which can be rearranged to expression (5.2.2). In the case where $\alpha = \gamma$ in (5.2.4), we see

$$\hat{x}^\gamma = \frac{k_\gamma+1}{K+1}\log\left(x^\gamma\right) + \frac{1}{K+1}\sum_{i\neq\gamma} k_i \log\left(x^\gamma - x^i\right).$$

Using (5.2.2) to substitute for x^i in the summation term gives

$$\begin{aligned}\hat{x}^\gamma &= \frac{k_\gamma+1}{K+1}\log\left(x^\gamma\right) + \frac{1}{K+1}\sum_{i\neq\gamma} k_i \left(\log\left(x^\gamma\right) + \log\left(e^{-\frac{K+1}{k_\gamma}\hat{x}^i}\right)\right) \\ &= \log\left(x^\gamma\right) - \sum_{i\neq\gamma}\frac{k_i}{k_\gamma}\hat{x}^i.\end{aligned}$$

This is easily rearranged to find (5.2.3). □

We may now find the second-order derivatives of $\widehat{F}$.

Lemma 5.2.2. *The second-order derivatives of $\widehat{F}$, written $\widehat{F}_{\alpha\beta}$, can be expressed by the following formulas for all $\alpha, \beta \in \{1, \ldots, n\}$.*

Case 1: $\alpha = \gamma$.

$$\widehat{F}_{\gamma\beta} \doteq 2k_\beta(K - k_\beta + 1)\hat{x}^\beta - 2k_\beta \sum_{i\neq\beta} k_i \hat{x}^i.$$

Case 2: $\alpha = \beta \neq \gamma$.

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq 2k_\alpha(K+1)\hat{x}^\gamma &+ \frac{2k_\alpha}{k_\gamma}(k_\alpha - k_\gamma)(K+1)\hat{x}^\alpha - \frac{2k_\alpha^2}{k_\gamma}\sum_{i=1}^n k_i\hat{x}^i \\ &+ 2k_\alpha \log\left(1 - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right) + 2k_\alpha \sum_{i\neq\alpha,\gamma} k_i \log\left(1 - e^{-\frac{K+1}{k_\gamma}\left(\hat{x}^\alpha - \hat{x}^i\right)}\right).\end{aligned}$$

Case 3: α, β, γ distinct.

$$\widehat{F}_{\alpha\beta} \doteq -\frac{2k_\alpha k_\beta}{k_\gamma}\sum_{i=1}^n k_i\hat{x}^i - 2k_\alpha k_\beta \log\left(1 - e^{-\frac{K+1}{k_\gamma}(\hat{x}^\alpha - \hat{x}^\beta)}\right) + \frac{2k_\alpha k_\beta(K+1)}{k_\gamma}\hat{x}^\beta.$$

Proof. Case 1: $\alpha = \gamma$.
Using previously defined properties of the Legendre transformation S_γ and the coordinate system $\hat{x}$, we can see that

$$\widehat{F}_{\gamma\beta} = F_{\gamma\beta} \doteq \hat{x}_\beta = \widehat{\eta}_{\alpha\beta}\hat{x}^\alpha = \eta_{\alpha\beta}\hat{x}^\alpha,$$

which leads to the statement.

Case 2: $\alpha = \beta \neq \gamma$.
First we calculate the relevant second-order derivatives of F:

$$\widehat{F}_{\alpha\alpha} = F_{\alpha\alpha} \doteq 2k_\alpha \log(x^\alpha) + 2k_\alpha \sum_{i\neq\alpha} k_i \log\left(x^\alpha - x^i\right).$$

Using expression (5.2.2) to make substitutions, we find that

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq{}& 2k_\alpha(K - k_\alpha + 1)\log(x^\gamma) + 2k_\alpha \log\left(1 - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right)\\ &- 2k_\alpha(K+1)\hat{x}^\alpha + 2k_\alpha \sum_{i\neq\alpha,\gamma} k_i \log\left(e^{-\frac{K+1}{k_\gamma}\hat{x}^i} - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right).\end{aligned}$$

We continue as follows, using expression (5.2.3) to substitute for $\log(x^\gamma)$:

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq{}& 2k_\alpha(K - k_\alpha + 1)\hat{x}^\gamma + \frac{2k_\alpha}{k_\gamma}\left((k_\alpha - k_\gamma)(K+1) - k_\alpha^2\right)\hat{x}^\alpha\\ &+ 2k_\alpha \log\left(1 - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right) + 2k_\alpha \sum_{i\neq\alpha,\gamma} k_i \log\left(1 - e^{-\frac{K+1}{k_\gamma}\left(\hat{x}^\alpha - \hat{x}^i\right)}\right)\\ &+ 2k_\alpha \sum_{i\neq\alpha,\gamma} k_i \log\left(e^{-\frac{K+1}{k_\gamma}\hat{x}^i}\right) + 2k_\alpha(K - k_\alpha + 1)\sum_{i=1}^{n} \frac{k_i}{k_\gamma}\hat{x}^i.\end{aligned}$$

With some rearranging, this leads to the required statement.

Case 3: α, β, γ distinct.
Here we again use the formulae in Lemma 5.2.1 to show that

$$\begin{aligned}\widehat{F}_{\alpha\beta} &= F_{\alpha\beta}\\ &\doteq -2k_\alpha k_\beta \log\left(x^\alpha - x^\beta\right)\\ &= -2k_\alpha k_\beta \log(x^\gamma) - 2k_\alpha k_\beta \log\left(e^{-\frac{K+1}{k_\gamma}\hat{x}^\beta} - e^{-\frac{K+1}{k_\gamma}\hat{x}^\alpha}\right)\\ &= -2k_\alpha k_\beta \sum_{i=1}^{n} \frac{k_i}{k_\gamma}\hat{x}^i - 2k_\alpha k_\beta \log\left(1 - e^{-\frac{K+1}{k_\gamma}(\hat{x}^\alpha - \hat{x}^\beta)}\right) + \frac{2k_\alpha k_\beta(K+1)}{k_\gamma}\hat{x}^\beta,\end{aligned}$$

which has the required form. ∎

Theorem 5.2.3. *The Legendre transform S_γ of the function $F = F^{rat}_{A_n(k)}$, given by (5.0.2),* *has the form*

$$\widehat{F} = \frac{\eta_{\gamma\gamma}}{6}(\hat{x}^\gamma)^3 + (\hat{x}^\gamma)^2 \sum_{i\neq\gamma} \frac{\eta_{i\gamma}}{2}\hat{x}^i + \hat{x}^\gamma \sum_{i\neq\gamma} \frac{\eta_{ii}}{2}(\hat{x}^i)^2 - \sum_{1\leq i<j<l\leq n} \frac{2k_ik_jk_l}{k_\gamma}\hat{x}^i\hat{x}^j\hat{x}^l$$
$$+ \sum_{i\neq\gamma}\left(\frac{k_i(\eta_{ii}-\eta_{\gamma\gamma})}{12k_\gamma} - \frac{\eta_{ii}^2}{24k_ik_\gamma} + \frac{\eta_{i\gamma}}{12}\right)(\hat{x}^i)^3 + \sum_{\substack{i\neq j\\ i,j\neq\gamma}} \frac{k_j\eta_{ii}+k_i\eta_{ij}}{4k_\gamma}(\hat{x}^i)^2\hat{x}^j$$
$$+ \sum_{i\neq\gamma} \frac{8k_\gamma^2k_i}{(K+1)^2} f\left(\frac{K+1}{2k_\gamma}\hat{x}^i\right) + \sum_{\substack{1\leq i<j\leq n\\ i,j\neq\gamma}} \frac{8k_\gamma^2k_ik_j}{(K+1)^2} f\left(\frac{K+1}{2k_\gamma}(\hat{x}^i-\hat{x}^j)\right) \quad (5.2.5)$$

up to quadratic terms and coordinate shifts, where $f(z)$ is defined in (2.5.7).

Proof. This can be checked by taking (5.2.5) as an ansatz and comparing its second-order derivatives with the expressions given in Lemma 5.2.2. In particular, we use (2.5.8) to rewrite the logarithmic terms. □

We compare this solution of the WDVV equations with other families of trigonometric solutions in §5.4.

5.3 Legendre transformation via the last fundamental weight of A_n

We compute the result of applying the Legendre transformation produced by the field $\delta = \kappa\sum_{i=1}^n \partial_i$ to the rational solution F given by (5.0.2), where $\kappa \in \mathbb{C}^\times$. We denote by $\widetilde{F}$ the resulting solution of the WDVV equations, up to equivalence.

By Theorem 2.4.13, the new coordinates $\tilde{x}$ are defined by

$$\tilde{x}_\alpha = \kappa\sum_{i=1}^n \partial_i\partial_\alpha F(x).$$

As in the previous section, we disregard constant terms and set

$$\tilde{x}_\alpha = 2\kappa k_\alpha \log x^\alpha \quad \text{for } 1 \leq \alpha \leq n. \quad (5.3.1)$$

Lemma 5.3.1. *The flat coordinates x^α are*

$$x^\alpha = \exp\left(\frac{K+1}{\kappa}\tilde{x}^\alpha - \frac{1}{\kappa}\sum_{i=1}^n k_i\tilde{x}^i\right) \quad \textit{for } 1 \leq \alpha \leq n. \quad (5.3.2)$$

Proof. Using the metric in (5.0.3) to raise the indices of $\tilde{x}_\alpha$ in (5.3.1), we have

$$\begin{aligned}\tilde{x}^\alpha &= \eta^{\alpha\beta}\tilde{x}_\beta \\ &= \frac{\kappa}{K+1}\sum_{i\neq\alpha}^{n} k_i \log x^i + \frac{\kappa(k_\alpha+1)}{K+1}\log x^\alpha \\ &= \frac{\kappa}{K+1}\sum_{i=1}^{n} k_i \log x^i + \frac{\kappa}{K+1}\log x^\alpha \end{aligned} \tag{5.3.3}$$

for all $1 \leq \alpha \leq n$. Rearranging (5.3.3), we find

$$\frac{K+1}{\kappa}\tilde{x}^\alpha = \log\left(x^\alpha \prod_{i=1}^{n}\left(x^i\right)^{k_i}\right),$$

so we have

$$x^\alpha = \exp\left(\frac{K+1}{\kappa}\tilde{x}^\alpha\right)\prod_{i=1}^{n}\left(x^i\right)^{-k_i}. \tag{5.3.4}$$

For all β, we then obtain

$$\frac{x^\alpha}{x^\beta} = \exp\left(\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^\beta\right)\right),$$

or equivalently,

$$x^\alpha = x^\beta \exp\left(\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^\beta\right)\right). \tag{5.3.5}$$

Substituting (5.3.5) into the right-hand side of (5.3.4), we have

$$\begin{aligned} x^\alpha &= \exp\left(\frac{K+1}{\kappa}\tilde{x}^\alpha\right)\prod_{i=1}^{n}\left(\left(x^\alpha\right)^{-k_i}\exp\left(\frac{k_i(K+1)}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^i\right)\right)\right) \\ &= \left(x^\alpha\right)^{-K}\exp\left(\frac{K+1}{\kappa}\left(\tilde{x}^\alpha + \sum_{i=1}^{n} k_i\left(\tilde{x}^\alpha - \tilde{x}^i\right)\right)\right) \\ &= \left(x^\alpha\right)^{-K}\exp\left(\frac{(K+1)^2}{\kappa}\tilde{x}^\alpha - \frac{K+1}{\kappa}\sum_{i=1}^{n} k_i\tilde{x}^i\right). \end{aligned} \tag{5.3.6}$$

The final expression, (5.3.6), can be rearranged to find the stated formula. □

Lemma 5.3.2. *The second-order derivatives $\widetilde{F}_{\alpha\beta}$ of $\widetilde{F}$ can be expressed by the following formulas for all $\alpha, \beta \in \{1, \ldots, n\}$.*

Case 1: $\alpha = \beta$.

$$\widetilde{F}_{\alpha\alpha} \doteq \frac{2k_\alpha \left(K - k_\alpha + 1\right)^2}{\kappa} \tilde{x}^\alpha - \frac{2k_\alpha \left(K - k_\alpha + 1\right)}{\kappa} \sum_{i \neq \alpha}^{n} k_i \tilde{x}^i + 2k_\alpha \sum_{i \neq \alpha}^{n} k_i \log\left(1 - e^{-\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^i\right)}\right).$$

Case 2: $\alpha \neq \beta$.

$$\widetilde{F}_{\alpha\beta} \doteq \frac{2k_\alpha k_\beta}{\kappa} \sum_{i \neq \alpha}^{n} k_i \tilde{x}^i - \frac{2k_\alpha k_\beta}{\kappa} \left(K - k_\alpha + 1\right) \tilde{x}^\alpha - 2k_\alpha k_\beta \log\left(1 - e^{-\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^\beta\right)}\right).$$

Proof. By definition of the Legendre transformation, we have $\widetilde{F}_{\alpha\beta} = F_{\alpha\beta}$ for all α, β.

Case 1: $\alpha = \beta$.

We have

$$\widetilde{F}_{\alpha\alpha} \doteq 2k_\alpha \log x^\alpha + 2k_\alpha \sum_{i \neq \alpha} k_i \log\left(x^\alpha - x^i\right).$$

Expression (5.3.2) can be used to make substitutions, as follows:

$$\begin{aligned}
\widetilde{F}_{\alpha\alpha} &\doteq \frac{2k_\alpha(K+1)}{\kappa}\tilde{x}^\alpha - \frac{2k_\alpha}{\kappa}\sum_{i=1}^{n} k_i \tilde{x}^i + 2k_\alpha \sum_{i \neq \alpha} \log\left(e^{\frac{K+1}{\kappa}\tilde{x}^\alpha} - e^{\frac{K+1}{\kappa}\tilde{x}^i}\right) \\
&\qquad - \frac{2k_\alpha}{\kappa} \sum_{i \neq \alpha} k_i \sum_{j=1}^{n} k_j \tilde{x}^j \\
&= \frac{2k_\alpha(K+1)}{\kappa}\tilde{x}^\alpha - \frac{2k_\alpha(K - k_\alpha + 1)}{\kappa} \sum_{i=1}^{n} k_i \tilde{x}^i + 2k_\alpha \sum_{i \neq \alpha} \frac{K+1}{\kappa}\tilde{x}^\alpha \\
&\qquad 2k_\alpha \sum_{i \neq \alpha} \log\left(1 - e^{-\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^i\right)}\right) \\
&= \frac{2k_\alpha \left(K - k_\alpha + 1\right)^2}{\kappa}\tilde{x}^\alpha - \frac{2k_\alpha \left(K - k_\alpha + 1\right)}{\kappa} \sum_{i \neq \alpha}^{n} k_i \tilde{x}^i \\
&\qquad + 2k_\alpha \sum_{i \neq \alpha}^{n} k_i \log\left(1 - e^{-\frac{K+1}{\kappa}\left(\tilde{x}^\alpha - \tilde{x}^i\right)}\right).
\end{aligned}$$

Case 2: $\alpha \neq \beta$.

We have

$$\widetilde{F}_{\alpha\beta} \doteq -2k_\alpha k_\beta \log\left(x^\alpha - x^\beta\right).$$

Using (5.3.2) to change variable from x to $\tilde{x}$ again, we find

$$\widetilde{F}_{\alpha\beta} \doteq \frac{2k_\alpha k_\beta}{\kappa} \sum_{i=1}^{n} k_i \tilde{x}^i - 2k_\alpha k_\beta \log\left(e^{\frac{K+1}{\kappa}\tilde{x}^\alpha} - e^{\frac{K+1}{\kappa}\tilde{x}^\beta}\right),$$

from which the required statement follows. □

Theorem 5.3.3. *The Legendre transform of $F = F^{rat}_{A_n(k)}$ via the Legendre field $\kappa \sum_{i=1}^n \partial_i$ results in the function*

$$\widetilde{F} = \frac{8\kappa^2}{(K+1)^2} \sum_{1\le i<j\le n} k_i k_j f\left(\frac{K+1}{2\kappa}\left(\tilde{x}^i - \tilde{x}^j\right)\right) - \sum_{i\neq j} \frac{k_i k_j}{2\kappa}(K - 2k_i + 1)\left(\tilde{x}^i\right)^2 \tilde{x}^j$$
$$+ \sum_{i=1}^n \frac{k_i}{6\kappa}\left(2(K-k_i+1)^2 - (K+1)(K-k_i)\right)\left(\tilde{x}^i\right)^3 + \sum_{1\le i<j<l\le n} \frac{2k_i k_j k_k}{\kappa} \tilde{x}^i \tilde{x}^j \tilde{x}^l, \quad (5.3.7)$$

up to quadratic terms and coordinates shifts, where $f(z)$ is defined in (2.5.7).

Proof. This may be verified by comparing the second-order derivatives of (5.3.7) with the expressions given in Lemma 5.3.2. We use (2.5.8) to rewrite the logarithmic terms. □

5.4 Relating families of A_{n-1}-type trigonometric solutions

In Sections 5.2 and 5.3, we found two families of trigonometric solutions by applying different Legendre transformations to a rational solution $F^{\text{rat}}_{A_n(k)}$, of the form (5.0.2). We now show that in both cases, these trigonometric solutions belong to the family $F^{\text{trig}}_{A_{n-1}}$ from Theorem 4.0.1, subject to some linear change of variables and a choice of parameters m_i, a, b, c. We will deal with the most general case, where the deformation parameters k_i have arbitrary (non-zero) values. In general, the solutions obtained in Theorems 5.2.3 and 5.3.3 are not equivalent to each other. However, they are equivalent for specific values of the deformation parameters.

In the case when $k_i = 1$ for all i, the solution $F^{\text{rat}}_{A_n(k)}$ corresponds to the Coxeter root system A_n. In this case, the Legendre transformations produced by Legendre fields ∂_γ or $\kappa \sum_{i=1}^n \partial_i$ all produce solutions which are equivalent to that found by Hoevenaars and Martini, discussed in Chapter 4.

Throughout this section, we set $F = F^{\text{trig}}_{A_{n-1}}(y)$, as written in (4.0.5). Let $\widehat{F} = S_\gamma\left(F^{\text{rat}}_{A_n(k)}\right)$ for some $\gamma \in \{1, \dots, n\}$, given explicitly in Theorem 5.2.3. The coordinate system $\hat{x}$ is given by the Legendre transformation S_γ, see formulas (5.2.1).

Theorem 5.4.1. *Define $a, b, c,$ and $m = (m_1, \ldots, m_n)$ by*

$$\begin{aligned} m_\alpha &= \begin{cases} 1 & \alpha = \gamma, \\ k_\alpha & \text{otherwise;} \end{cases} \\ a &= -\frac{2}{K+1}; \\ b &= 1; \\ c &= -(K + k_\gamma + 1), \end{aligned} \tag{5.4.1}$$

where $K = \sum_{i=1}^{n} k_i$. Then functions $\widehat{F}$ and F satisfy

$$\frac{(K+1)^2}{8k_\gamma^2} \widehat{F}(\hat{x}) = F(y) \text{ up to quadratic terms,} \tag{5.4.2}$$

where the coordinate system y is given by

$$y_\alpha = \begin{cases} -\frac{K+1}{2k_\gamma} \hat{x}^\gamma & \alpha = \gamma, \\ \frac{K+1}{2k_\gamma} (\hat{x}^\alpha - \hat{x}^\gamma) & \text{otherwise.} \end{cases} \tag{5.4.3}$$

Proof. Inverting the expressions in (5.4.3) gives us

$$\hat{x}^\alpha = \begin{cases} -\frac{2k_\gamma}{K+1} y_\gamma & \alpha = \gamma, \\ \frac{2k_\gamma}{K+1} (y_\alpha - y_\gamma) & \text{otherwise.} \end{cases}$$

We rearrange the sum of the terms in $\widehat{F}$ which are proportional to $\hat{x}^i \hat{x}^j \hat{x}^l$ as

$$-\sum_{i<j<l} \frac{2k_i k_j k_l}{k_\gamma} \hat{x}^i \hat{x}^j \hat{x}^l = -2\hat{x}^\gamma \sum_{\substack{i<j \\ i,j \neq \gamma}} k_i k_j \hat{x}^i \hat{x}^j - \sum_{\substack{i<j<l \\ i,j,l \neq \gamma}} \frac{2k_i k_j k_l}{k_\gamma} \hat{x}^i \hat{x}^j \hat{x}^l.$$

We then make the coordinate transformation $\hat{x} \to y$ in the expression for $\widehat{F}$ to obtain

$$\begin{aligned} \widehat{F} &= \sum_{i \neq \gamma} \frac{8k_\gamma^2 k_i}{(K+1)^2} f(y_i - y_\gamma) + \sum_{\substack{i<j \\ i,j \neq \gamma}} \frac{8k_\gamma^2 k_i k_j}{(K+1)^2} f(y_i - y_j) + \text{cubic terms} \\ &= \frac{8k_\gamma^2}{(K+1)^2} \sum_{1 \leq i<j \leq n} m_i m_j f(y_i - y_j) + \text{cubic terms,} \end{aligned}$$

since the m_α are defined by (5.4.1). We now compare coefficients of the various cubic terms in F and $\frac{(K+1)^2}{8k_\gamma^2} \widehat{F}$ to show that relation (5.4.2) is satisfied.

After the change of variables $\hat{x} \to y$ in $\widehat{F}$ and some algebraic manipulation, we find that the sum of the cubic terms of the form $y_i y_j y_k$ in $\frac{(K+1)^2}{8k_\gamma^2}\widehat{F}$, with i, j, k distinct, is equal to

$$-\frac{2}{K+1}\sum_{\substack{i<j\\ i,j\neq\gamma}} k_i k_j y_\gamma y_i y_j - \frac{2}{K+1}\sum_{\substack{i<j<l\\ i,j,l\neq\gamma}} k_i k_j k_l y_i y_j y_l.$$

This is the same as the second term in (4.0.5), since $m_\gamma = 1$ and $a = -\frac{2}{K+1}$.

Next, we consider terms of the form $y_i^2 y_j$, with $i \neq j$, in $\frac{(K+1)^2}{8k_\gamma^2}\widehat{F}$. The coefficient of the sum of the terms proportional to $y_\gamma^2 y_i$, for $i \neq \gamma$, in $\frac{(K+1)^2}{8k_\gamma^2}\widehat{F}$ is equal to

$$\frac{k_\gamma}{K+1}\left[-k_i k_\gamma + 2k_i(K+1-k_i) - 2k_i(K-k_i-k_\gamma) - \frac{2k_i}{k_\gamma}\sum_{\substack{j<l\\ j,l\neq i,\gamma}} k_j k_l\right]$$
$$+\frac{k_i}{2(K+1)}\big[2k_i(K+1-k_i) - (K+1)(K+1+k_\gamma-k_i)\big]$$
$$+\frac{k_i}{2(K+1)}\sum_{j\neq i,\gamma} k_j\left(3+3K-4k_i-2k_j\right). \quad (5.4.4)$$

By replacing the first sum in this expression with

$$-\frac{k_i}{k_\gamma}\left((K-k_i-k_\gamma)^2 - \sum_{j\neq i,\gamma} k_j^2\right)$$

and some straightforward but substantial manipulations, the expression (5.4.4) can be simplified to $\frac{k_i(K-1)}{2(K+1)}$. All terms of the form $y_i^2 y_j$, with $i \neq j$, in $\frac{(K+1)^2}{8k_\gamma^2}\widehat{F}$ sum to

$$y_\gamma^2\sum_{i\neq\gamma}\frac{k_i(K-1)}{2(K+1)}y_i + y_\gamma\sum_{i\neq\gamma}\frac{k_i(K-2k_i+1)}{2(K+1)}y_i^2 + \sum_{\substack{i\neq j\\ i,j\neq\gamma}}\frac{k_i k_j(K-2k_i+1)}{2(K+1)}y_i^2 y_j,$$

which is equal to the third term in (4.0.5) since we have set $b = 1$.

In computing the coefficient of terms proportional to y_γ^3, it is useful to note that

$$\sum_{\substack{i<j<l\\ i,j,l\neq\gamma}} k_i k_j k_l = \frac{1}{6}(K-k_\gamma)^3 - \frac{1}{2}(K-k_\gamma)\sum_{i\neq\gamma}k_i^2 + \frac{1}{3}\sum_{i\neq\gamma}k_i^3.$$

The sum of all terms of the form y_i^3 in $\frac{(K+1)^2}{8k_\gamma^2}\widehat{F}$ can be expressed as

$$\left(\frac{K(1-K)}{6(K+1)} - \frac{k_\gamma}{6}\right)y_\gamma^3 + \sum_{i\neq\gamma}\left(-\frac{k_i^3}{3(K+1)} + \frac{k_i^2}{2} - \frac{k_i(K+k_\gamma+1)}{6}\right)y_i^3.$$

This agrees with the last term in (4.0.5), as $c = -(K+k_\gamma+1)$. $\square$

Let $\widetilde{F}$ be the solution obtained from $F^{\text{rat}}_{A_n(k)}$ by the Legendre transformation in the direction $\kappa\sum_{i=1}^{n}\partial_i$, given explicitly in (5.3.7).

Theorem 5.4.2. *Define $a, b, c,$ and $m = (m_1, \dots, m_n)$ by*

$$\begin{aligned} m_\alpha &= k_\alpha \quad \text{for } 1 \le \alpha \le n; \\ a &= -\frac{2}{K+1}; \\ b &= 1; \\ c &= -(K+2), \end{aligned} \tag{5.4.5}$$

where $K = \sum_{i=1}^{n} k_i$. Then functions $\widetilde{F}$ and F satisfy

$$\frac{(K+1)^2}{8\kappa^2}\widetilde{F}(\tilde{x}) = F(y) \text{ up to quadratic terms,} \tag{5.4.6}$$

where the coordinate system y is given by

$$y_\alpha = -\frac{K+1}{2\kappa}\tilde{x}^\alpha. \tag{5.4.7}$$

Proof. We follow the method used in the proof of Theorem 5.4.1. Since less involved algebraic manipulation is needed here, fewer details are given.

By considering the second-order derivatives of $\widetilde{F}$, given in Lemma 5.3.2, it is clear that swapping the ordering of $\tilde{x}^i, \tilde{x}^j$ in the trilogarithmic terms of $\widetilde{F}$ will not affect the equivalence class of $\widetilde{F}$. That is, we may alternatively write

$$\widetilde{F} = \frac{8\kappa^2}{(K+1)^2}\sum_{1\le i<j\le n} k_i k_j f\left(\frac{K+1}{2\kappa}\left(\tilde{x}^j - \tilde{x}^i\right)\right) + \text{cubic and lower-order terms.}$$

Relation (5.4.6) is therefore satisfied when considering only the trilogarithmic terms in $\frac{(K+1)^2}{8\kappa^2}\widetilde{F}$ and F, after making the change of variables $y \to \tilde{x}$ in F and taking the values of the parameters given in (5.4.5). We now proceed to compare the cubic terms in these functions.

It is immediate that terms of the form $\tilde{x}^i\tilde{x}^j\tilde{x}^l$ with i, j, l distinct are equal in $\frac{(K+1)^2}{8\kappa^2}\widetilde{F}$ and F.

Similarly, equating terms of the form $\left(\tilde{x}^i\right)^2\tilde{x}^j$ with i, j distinct in $\frac{(K+1)^2}{8\kappa^2}\widetilde{F}$ with the analogous terms in F leads to

$$-\frac{(K+1)^2}{16\kappa^3}k_i k_j(K - 2k_i + 1) = -\frac{(K+1)^3}{16\kappa^3}(ak_i + b)k_i k_j.$$

As required, this is satisfied for the values $a = -\frac{2}{K+1}$, $b = 1$.

Finally, equating terms of the form $(\tilde{x}^i)^3$ produces

$$\frac{(K+1)^2 k_i}{48\kappa^3}\left(2(K-k_i+1)^2-(K+1)(K-k_i)\right)=-\frac{(K+1)^3 k_i}{48\kappa^3}\left(ak_i^2+3bk_i+c\right),$$

which is also satisfied for the parameters given in (5.4.5). □

In general, the family of solutions F is larger than both families $\widehat{F}$ and $\widetilde{F}$. Note that the choice of parameters a, b, c is the same in both cases. For the values in (5.4.1) or (5.4.5), we have

$$bM + c = k_\gamma \neq 0$$

and

$$aM^2 + 3bM + c = -\frac{2k_\gamma^2}{K+1} \neq 0.$$

By Theorem 4.0.1, it follows that condition (4.0.4) holds, which can also be checked directly for this choice of parameters.

From the comparisons in Theorems 5.4.1 and 5.4.2, the relationship between $\widehat{F}$ and $\widetilde{F}$ is straightforward to see.

Corollary 5.4.3. *Let $\widehat{F}(\hat{x})$ be as written in (5.2.5) and $\widetilde{F}(\tilde{x})$ as in (5.3.7). Let $k_\gamma = 1 = \kappa$, and define*

$$\hat{x}^\alpha = \begin{cases} \tilde{x}^\gamma & \alpha = \gamma, \\ \tilde{x}^\gamma - \tilde{x}^\alpha & \text{otherwise}. \end{cases} \tag{5.4.8}$$

Then $\widetilde{F}(\tilde{x}) = \widehat{F}(\hat{x})$, up to quadratic terms.

Proof. By Theorems 5.4.1 and 5.4.2, both $\widehat{F}$ and $\widetilde{F}$ are particular examples of solutions of the form F. We check the requirements for all relations (5.4.1), (5.4.2), (5.4.3), (5.4.5), (5.4.6), (5.4.7) to hold simultaneously, with the same set of deformation parameters k_i for both $\widehat{F}$ and $\widetilde{F}$. Equating (5.4.1) and (5.4.5), we require $k_\gamma = 1$. Equating (5.4.2) and (5.4.6), we require $\kappa = k_\gamma$. Equating (5.4.3) and (5.4.7), we require the coordinate relations given by (5.4.8). □

In particular we see that in the Coxeter case, when $k_i = 1$ for all $i \in \{1, \dots, n\}$, the Legendre transformation defined by an arbitrary root e_γ produces the same result as the Legendre transformation defined by the last fundamental weight.

Chapter 6

Legendre transformations of B_n-type rational solutions

In this chapter, we apply Legendre transformations to the $(n+1)$-parameter family of rational solutions associated with the B_n-type $\vee$-system found in [11]. The transformations considered produce a known family of BC_{n-1}-type trigonometric solutions. We follow [25].

The B_n-type $\vee$-system described in (2.5.5) produces the following family of rational solutions of the WDVV equations. We have

$$F = F^{\text{rat}}_{B_n(k)} = \sum_{i=1}^{n} 2k_i(k_0+k_i)(x^i)^2\log(x^i) + \sum_{1\leq i<j\leq n} k_ik_j(x^i \pm x^j)^2\log(x^i \pm x^j), \quad (6.0.1)$$

with parameters $k_0 \in \mathbb{C}$, $k_1, \ldots, k_n \in \mathbb{C}^{\times}$ and where the sum of the parameters

$$K := \sum_{i=0}^{n} k_i \neq 0.$$

The metric for this system, $\eta = x^iF_i$ as defined in (2.5.2), has entries

$$\eta_{ij} = 4k_iK\delta_{ij}$$

while the contravariant metric, denoted η^{-1}, has entries given by

$$\eta^{ij} = \frac{\delta_{ij}}{4k_iK}. \quad (6.0.2)$$

6.1 Legendre transformation results

We now apply the Legendre transformation S_γ, using Notation 2.4.19, to $F = F^{\text{rat}}_{B_n(k)}$ for an arbitrary $\gamma \in \{1, \ldots, n\}$. Throughout this section, we set $\widehat{F} = S_\gamma(F)$. As discussed previously, solutions to the WDVV equations are only defined up to quadratic terms and so we may omit, for example, constant terms arising from second-order derivatives of such solutions. Using the definition of a Legendre transformation, we choose the set of new (covariant) coordinates $\hat{x}_\alpha$ as follows

$$\hat{x}_\alpha = \begin{cases} 4k_\gamma(k_\gamma + k_0)\log(x^\gamma) + 2k_\gamma \sum_{\substack{i=1 \\ i\neq\gamma}}^{n} k_i \log\left((x^\gamma)^2 - (x^i)^2\right) & \alpha = \gamma, \\ 2k_\gamma k_\alpha \log\left(\frac{x^\gamma + x^\alpha}{x^\gamma - x^\alpha}\right) & \text{otherwise.} \end{cases} \tag{6.1.1}$$

To continue with the transformation, we now must find expressions for coordinates x^α in terms of $\hat{x}^\alpha$.

Lemma 6.1.1. *The flat coordinates x^α can be expressed as follows:*

$$x^\alpha = x^\gamma \coth\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right) \text{ for } \alpha \neq \gamma, \tag{6.1.2}$$

$$\log(x^\gamma) = \hat{x}^\gamma - \sum_{\substack{i=1 \\ i\neq\gamma}}^{n} \frac{k_i}{2K} \log\left(1 - \coth^2\left(\frac{K}{k_\gamma}\hat{x}^i\right)\right). \tag{6.1.3}$$

Proof. First, we use the inverse metric as in (6.0.2) to raise the indices of the covariant coordinates $\hat{x}_\alpha$ in (6.1.1) via the formula

$$\hat{x}^\alpha = \widehat{\eta}^{\alpha\beta}\hat{x}_\beta = \eta^{\alpha\beta}\hat{x}_\beta.$$

We obtain

$$\hat{x}^\alpha = \begin{cases} \frac{k_0+k_\gamma}{K}\log(x^\gamma) + \sum_{i\neq 0,\gamma} \frac{k_i}{2K}\log\left((x^\gamma)^2 - (x^i)^2\right) & \alpha = \gamma, \\ \frac{k_\gamma}{2K}\log\left(\frac{x^\gamma + x^\alpha}{x^\gamma - x^\alpha}\right) & \text{otherwise.} \end{cases} \tag{6.1.4}$$

In the case where $\alpha \neq \gamma$, we can invert the expression for $\hat{x}^\alpha$ to obtain

$$e^{\frac{2K}{k_\gamma}\hat{x}^\alpha} = \frac{x^\gamma + x^\alpha}{x^\gamma - x^\alpha},$$

which can be rearranged to find

$$x^\alpha = x^\gamma\left(\frac{e^{2K\hat{x}^\alpha/k_\gamma} - 1}{e^{2K\hat{x}^\alpha/k_\gamma} + 1}\right) = x^\gamma \coth\left(K\hat{x}^\alpha/k_\gamma\right).$$

In the case where $\alpha = \gamma$ in (6.1.4), we can use expression (6.1.2) to substitute for x^i

with $i \neq \gamma$ as follows:

$$\begin{aligned}\hat{x}^\gamma &= \frac{k_0 + k_\gamma}{K}\log(x^\gamma) + \sum_{i\neq 0,\gamma}\frac{k_i}{2K}\log\left((x^\gamma)^2(1-\coth^2\left(K\hat{x}^i/k_\gamma\right)\right) \\ &= \frac{1}{K}\left(k_0 + k_\gamma + \sum_{i\neq 0,\gamma} k_i\right)\log(x^\gamma) + \sum_{i\neq 0,\gamma}\frac{k_i}{2K}\log\left((1-\coth^2\left(K\hat{x}^i/k_\gamma\right)\right).\end{aligned}$$

From this, we obtain (6.1.3). □

From the definition of a Legendre transform, we have that

$$F_{\alpha\beta} = \widehat{F}_{\alpha\beta}.$$

We will use this, along with Lemma 6.1.1, to compute all the second-order derivatives of $\widehat{F}$ in terms of $\hat{x}$.

Lemma 6.1.2. *The second-order derivatives $\widehat{F}_{\alpha\beta}$ can be expressed, up to constant terms, by the following formulae for all $\alpha, \beta \in \{1, \ldots, n\}$.*
Case 1: $\alpha = \gamma$.

$$\widehat{F}_{\gamma\beta} = 4k_\beta K\hat{x}^\beta.$$

Case 2: $\alpha = \beta \neq \gamma$.

$$\begin{aligned}\widehat{F}_{\alpha\alpha} &\doteq 4k_\alpha K\hat{x}^\gamma + \frac{4k_\alpha K}{k_\gamma}(K - k_0 - 2k_\alpha)\hat{x}^\alpha \\ &\quad + 4k_\alpha(k_0 + K)\log\left(1 - e^{2K\hat{x}^\alpha/k_\gamma}\right) + 4k_\alpha(k_\alpha - K)\log\left(1 - e^{4K\hat{x}^\alpha/k_\gamma}\right) \\ &\quad + 2k_\alpha\sum_{\substack{i=1\\ i\neq\alpha,\gamma}}^{n} k_i\left[\log\left(1 - e^{2K(\hat{x}^i - \hat{x}^\alpha)/k_\gamma}\right) + \log\left(1 - e^{2K(\hat{x}^i + \hat{x}^\alpha)/k_\gamma}\right) - \frac{2K}{k_\gamma}\hat{x}^i\right].\end{aligned}$$

Case 3: α, β, γ distinct.

$$\widehat{F}_{\alpha\beta} \doteq 2k_\alpha k_\beta\left[\log\left(1 - e^{2K(\hat{x}^\alpha + \hat{x}^\beta)/k_\gamma}\right) - \log\left(1 - e^{2K(\hat{x}^\alpha - \hat{x}^\beta)/k_\gamma}\right) - \frac{2K}{k_\gamma}\hat{x}^\beta\right].$$

Proof. Case 1: $\alpha = \gamma$.
By definition of the new coordinate system, we have

$$\widehat{F}_{\gamma\beta} \doteq \hat{x}_\beta = \eta_{\beta\alpha}\hat{x}^\alpha = \eta_{\beta\beta}\hat{x}^\beta,$$

since the metric η is diagonal.

Case 2: $\alpha = \beta$ and $\alpha \neq \gamma$.

Calculating $\widehat{F}_{\alpha\alpha}$ directly, we have

$$\widehat{F}_{\alpha\alpha} = F_{\alpha\alpha} \doteq 4k_\alpha(k_\alpha + k_0)\log(x^\alpha) + 2k_\alpha \sum_{\substack{i=1 \\ i\neq\alpha}}^{n} k_i \log\left((x^i)^2 - (x^\alpha)^2\right).$$

We can use expression (6.1.2) to make substitutions, and collect terms in $\log(x^\gamma)$ as follows:

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq 4k_\alpha K\log(x^\gamma) &+ 4k_\alpha(k_\alpha + k_0)\log\left(\coth\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right)\right) \\ &+ 2k_\gamma k_\alpha \log\left(1 - \coth^2\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right)\right) \\ &+ 2k_\alpha \sum_{\substack{i=1 \\ i\neq\alpha,\gamma}}^{n} k_i \log\left(\coth^2\left(\frac{K}{k_\gamma}\hat{x}^i\right) - \coth^2\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right)\right).\end{aligned}$$

Next, we use (6.1.3) to substitute for the terms in $\log(x^\gamma)$:

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq 4k_\alpha K\hat{x}^\gamma &+ 4k_\alpha(k_\alpha + k_0)\log\left(\coth\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right)\right) \\ &+ 2k_\alpha(k_\gamma - k_\alpha)\log\left(1 - \coth^2\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right)\right) \\ &+ 2k_\alpha \sum_{\substack{i=1 \\ i\neq\alpha,\gamma}}^{n} k_i \log\left(\frac{\coth^2(K\hat{x}^i/k_\gamma) - \coth^2(K\hat{x}^\alpha/k_\gamma)}{1 - \coth^2(K\hat{x}^i/k_\gamma)}\right).\end{aligned}$$

Since $\coth(u) = \frac{e^{2u}-1}{e^{2u}+1}$, we note that

$$1 - \coth^2(u) = \frac{4e^{2u}}{(e^{2u}+1)^2}$$

and

$$\coth^2(u) - \coth^2(v) = \frac{4e^{2v}(e^{2(u-v)}-1)(e^{2(u+v)}-1)}{(e^{2u}+1)^2(e^{2v}+1)^2}.$$

This allows us to expand and simplify $\widehat{F}_{\alpha\alpha}$ as follows:

$$\begin{aligned}\widehat{F}_{\alpha\alpha} \doteq 4k_\alpha K\hat{x}^\gamma &+ \frac{4k_\alpha K}{k_\gamma}(K - k_0 - 2k_\alpha)\hat{x}^\alpha - \frac{4k_\alpha K}{k_\gamma}\sum_{\substack{i=1 \\ i\neq\alpha,\gamma}}^{n} k_i\hat{x}^i \\ &+ 4k_\alpha(k_\alpha + k_0)\log\left(e^{2K\hat{x}^\alpha/k_\gamma} - 1\right) + 4k_\alpha(k_\alpha - K)\log\left(e^{2K\hat{x}^\alpha/k_\gamma} + 1\right) \\ &+ 2k_\alpha \sum_{\substack{i=1 \\ i\neq\alpha,\gamma}}^{n} k_i\left[\log\left(e^{2K(\hat{x}^i - \hat{x}^\alpha)/k_\gamma} - 1\right) + \log\left(e^{2K(\hat{x}^i + \hat{x}^\alpha)/k_\gamma} - 1\right)\right].\end{aligned}$$

We may rewrite this expression using the following identity:

$$\log(1+e^u) = \log(1-e^{2u}) - \log(1-e^u).$$

This leads to the required statement.

Case 3: α, β, γ distinct.

By direct calculation, we have

$$\widehat{F}_{\alpha\beta} = F_{\alpha\beta} \doteq 2k_\alpha k_\beta \log\left(x^\alpha + x^\beta\right) - 2k_\alpha k_\beta \log\left(x^\alpha - x^\beta\right),$$

and we can make substitutions using Lemma 6.1.1 to find

$$\widehat{F}_{\alpha\beta} \doteq 2k_\alpha k_\beta \log\left(\coth\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right) + \coth\left(\frac{K}{k_\gamma}\hat{x}^\beta\right)\right) - 2k_\alpha k_\beta \log\left(\coth\left(\frac{K}{k_\gamma}\hat{x}^\alpha\right) - \coth\left(\frac{K}{k_\gamma}\hat{x}^\beta\right)\right). \quad (6.1.5)$$

We note the following identities:

$$\coth(u) + \coth(v) = \frac{2(e^{2(u+v)}-1)}{(e^{2u}+1)(e^{2v}+1)}$$

and

$$\coth(u) - \coth(v) = \frac{2e^{2v}(e^{2(u-v)}-1)}{(e^{2u}+1)(e^{2v}+1)}.$$

Using these, we can reformulate (6.1.5) to obtain the required expression. □

Theorem 6.1.3. *The Legendre transform S_γ, for arbitrary $\gamma \in \{1, \ldots, n\}$, applied to the function F as defined in (6.0.1) has the form*

$$\widehat{F} = \frac{\eta_{\gamma\gamma}}{6}\left(\hat{x}^\gamma\right)^3 + \sum_{\substack{i=1 \\ i\neq\gamma}}^{n} \frac{\eta_{ii}}{2}\hat{x}^\gamma\left(\hat{x}^i\right)^2 + \sum_{\substack{i<j \\ i,j\neq\gamma}}^{n} \frac{2k_i k_j k_\gamma^2}{K^2} f\left(-\frac{K}{k_\gamma}\left(\hat{x}^i \pm \hat{x}^j\right)\right) + \sum_{\substack{i=1 \\ i\neq\gamma}}^{n}\left[\frac{4k_i k_\gamma^2(k_0+K)}{K^2} f\left(-\frac{K}{k_\gamma}\hat{x}^i\right) + \frac{k_i k_\gamma^2\left(k_i - K\right)}{K^2} f\left(-\frac{2K}{k_\gamma}\hat{x}^i\right)\right], \quad (6.1.6)$$

up to quadratic terms and coordinate shifts. The function $f(z)$ is as defined in (2.5.7).

Proof. Taking (6.1.6) as an ansatz, we can compare its second-order derivatives with the expressions given in Lemma 6.1.2. Note that we use identity (2.5.8) to express logarithmic terms via the function $f(z)$. □

6.2 Relating families of trigonometric solutions

The function $\widehat{F} = S_\gamma\left(F^{\text{rat}}_{B_n(k)}\right)$, given in equality (6.1.6), has a similar form to a known trigonometric solution corresponding to a trigonometric $\vee$-system of type BC_{n-1}. Following the results of Alkadhem and Feigin in [2], specifically Theorem 5.5, the trigonometric BC_{n-1}-type WDVV solutions have the form

$$\widetilde{F} = \frac{1}{3}\xi_0^3 + h\xi_0 \sum_{i=1}^{n-1} m_i \xi_i^2 + \lambda r \sum_{i=1}^{n-1} m_i \tilde{f}(\xi_i) + \lambda \sum_{i=1}^{n-1}\left(s m_i + \frac{1}{2} q m_i(m_i - 1)\right)\tilde{f}(2\xi_i) + \lambda q \sum_{1\le i<j\le n-1} m_i m_j \tilde{f}(\xi_i \pm \xi_j),$$

with coordinates $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^n$, and independent parameters $q, r, s \in \mathbb{C}$, such that $q \neq 0$, and $m = (m_1, \ldots, m_{n-1}) \in (\mathbb{C}^\times)^{n-1}$. The constants h, λ, $M \in \mathbb{C}^\times$ are defined as follows:

$$h = r + 4s + 2q(M-1); \tag{6.2.1}$$

$$\lambda = \left(\frac{2h^3}{q(r + 8s + 2q(M-2))}\right)^{1/2}; \tag{6.2.2}$$

$$M = \sum_{i=1}^{n-1} m_i; \tag{6.2.3}$$

with the requirement that

$$q\left(r + 8s + 2q(M-2)\right) \neq 0. \tag{6.2.4}$$

Finally, the function $\tilde{f}(z)$ is given by

$$\tilde{f}(z) = -f(-iz), \tag{6.2.5}$$

where $f(z)$ is defined in (2.5.7).

First, we show that an arbitrary solution of the form $\widehat{F}$ is equivalent to a solution of the form $\widetilde{F}$.

Theorem 6.2.1. *Suppose* $K=\sum_{i=0}^{n}k_i\neq k_\gamma$. *Define* q,r,s,m *as*

$$q=-\frac{2k_\gamma^2}{RK^2}; \tag{6.2.6}$$
$$r=-\frac{4k_\gamma^2(k_0+K)}{RK^2}; \tag{6.2.7}$$
$$s=\frac{k_\gamma^2(K-1)}{RK^2}; \tag{6.2.8}$$
$$m_\alpha=\begin{cases}k_\alpha & 1\leq\alpha<\gamma,\\ k_{\alpha+1} & \gamma\leq\alpha\leq n-1,\end{cases} \tag{6.2.9}$$

where $R\in\mathbb{C}^\times$ *is an independent scalar. Then functions* $\widetilde{F}$ *and* $\widehat{F}$ *satisfy*

$$\frac{R}{\lambda}\widetilde{F}(\xi)=\widehat{F}(\hat{x}) \tag{6.2.10}$$

with coordinate transformation

$$\xi_0=\left(\frac{4k_\gamma(K-k_\gamma)}{R}\right)^{1/2}\hat{x}^\gamma; \tag{6.2.11}$$
$$\xi_\alpha=\begin{cases}\frac{iK}{k_\gamma}\hat{x}^\alpha & 1\leq\alpha<\gamma,\\ \frac{iK}{k_\gamma}\hat{x}^{\alpha+1} & \gamma\leq\alpha\leq n-1.\end{cases} \tag{6.2.12}$$

Condition (6.2.4) is satisfied by the choice of parameters (6.2.6)–(6.2.9), and we also have $\lambda,\ h\neq 0$.

Proof. We start by inverting expressions (6.2.11) and (6.2.12). The resulting coordinate transformation $\hat{x}\to\xi$ in $\widehat{F}$ gives us

$$\widehat{F}=\sum_{i=1}^{n-1}\left[-\frac{4m_ik_\gamma^2(k_0+K)}{K^2}\tilde{f}(\xi_i)+\frac{m_ik_\gamma^2(K-m_i)}{K^2}\tilde{f}(2\xi_i)\right]-\sum_{i<j}^{n-1}\frac{2m_im_jk_\gamma^2}{K^2}\tilde{f}(\xi_i\pm\xi_j)+\text{cubic terms},$$

where we have also used (6.2.5) and (6.2.9). This expression matches the corresponding terms in $R\widetilde{F}/\lambda$.

Note that $M=K-k_0-k_\gamma$ by (6.2.3). Since we have q,r,s,M in terms of the k_α, we may use (6.2.1) and (6.2.2) to derive

$$h=\frac{4k_\gamma^2(k_\gamma-K)}{RK^2} \tag{6.2.13}$$

and

$$\lambda^2 = \frac{16k_\gamma(K-k_\gamma)^3}{RK^2}. \tag{6.2.14}$$

We can now consider the cubic terms in $\widehat{F}(\xi)$, which are

$$\frac{KR^{3/2}}{12k_\gamma^{1/2}(K-k_\gamma)^{3/2}}\xi_0^3 - \frac{k_\gamma^{3/2}R^{1/2}}{K(K-k_\gamma)^{1/2}}\xi_0\sum_{i=1}^{n-1} m_i\xi_i^2 = \frac{R}{3\lambda}\xi_0^3 + \frac{hR}{\lambda}\xi_0\sum_{i=1}^{n-1} m_i\xi_i^2.$$

This matches the cubic terms in $R\widetilde{F}/\lambda$: hence, relation (6.2.10) holds.

Substitution of the expressions for q, r, s, M in terms of the k_α into condition (6.2.4) gives us

$$q\left(r + 8s + 2q(M-2)\right) = -\frac{8k_\gamma^5}{R^2K^4} \neq 0.$$

Also, expressions (6.2.13), (6.2.14) are non-zero. □

The family of solutions $\widehat{F}$ depends on $n+1$ parameters $k_0, \dots, k_n$ while the family $\widetilde{F}$ depends on $n+2$ parameters q, r, s, $m_1, \dots, m_{n-1}$. As we introduced an extra scale factor R in Theorem 6.2.1, we can now relate a general solution of the family $\widetilde{F}$ to a solution of the family $\widehat{F}$. More precisely, the following statement holds.

Theorem 6.2.2. *Given a solution $\widetilde{F}$, such that $q \neq 2s$, we define:*

$$k_\alpha = \begin{cases} \frac{r-2q+4s}{2q} & \alpha = 0, \\ m_\alpha & 0 < \alpha < \gamma, \\ \frac{2q(2-M)-r-8s}{2q} & \alpha = \gamma, \\ m_{\alpha-1} & \gamma < \alpha \leq n; \end{cases} \tag{6.2.15}$$

$$R = -\frac{(r+8s+2q(M-2))^2}{2q(q-2s)^2}. \tag{6.2.16}$$

Then $\widehat{F}$ and $\widetilde{F}$ are related by $\frac{\lambda}{R}\widehat{F}(\hat{x}) = \widetilde{F}(\xi)$ with coordinate transformation

$$\hat{x}^\alpha = \begin{cases} \frac{i(r+8s+2q(M-2))}{2(q-2s)}\xi_\alpha & 1 \leq \alpha < \gamma, \\ \frac{h}{\lambda(q-2s)}\xi_0 & \alpha = \gamma, \\ \frac{i(r+8s+2q(M-2))}{2(q-2s)}\xi_{\alpha-1} & \gamma < \alpha \leq n. \end{cases}$$

The proof follows by inverting parameter relations and the change of coordinates from

Theorem 6.2.1. In particular, it is useful to note that

$$K = k_0 + k_\gamma + M = \frac{q - 2s}{q}.$$

Chapter 7

Concluding remarks

In this thesis, we built on work by Strachan and Stedman [59], which expands Dubrovin's definition of a Legendre transformation to include transformations generated by non-flat fields. We analysed the conditions for which an arbitrary vector field on a two-dimensional Frobenius manifold defines a Legendre transformation. This enabled us to describe all homogeneous Legendre fields for these manifolds.

In the second part of the thesis, we explored specific Legendre transformations applied to families of rational solutions of the generalised WDVV equations found by Chalykh and Veselov [11]. These transformations produced a mapping between rational solutions and certain trigonometric solutions. We found that some Legendre transformations of type A_n rational solutions produce type A_{n-1} trigonometric solutions, which are included in a multi-parameter family of A-type trigonometric solutions. This family generalises solutions previously found by Hoevenaars and Martini [35], among others. We also found that some Legendre transformations of type B_n rational solutions result in type BC_{n-1} trigonometric solutions, which belong to a family discovered by Alkadhem and Feigin [2].

We now discuss some possibilities for future research based on this work.

7.1 Legendre fields

In Chapter 3, we studied the Legendre field condition for the two-dimensional Frobenius manifolds. When restricting the Legendre field condition to homogeneous fields, we found that the hypergeometric differential equation controls all cases with a non-constant multiplication.

In [40], Liu, Qu, and Zhang showed that homogeneous Legendre fields map Frobenius manifolds to (quasi-homogeneous) generalised Frobenius manifolds, and that generalised Frobenius manifolds are mapped to Frobenius manifolds by flat Legendre fields of the form ∂_i. Since we have now described all homogeneous Legendre fields for the two-dimensional Frobenius manifolds, it is worth exploring whether these can be used to classify all generalised Frobenius manifolds in two dimensions.

A further step might be to explore the Legendre field condition for Frobenius manifolds in higher dimensions, or for rational solutions of the WDVV equations. In the latter case, one could use Coxeter root systems in two dimensions as toy examples. It would be interesting to see whether the hypergeometric differential equation arises in other cases when considering homogeneous Legendre fields in two dimensions. We note also that general hypergeometric systems related to root systems have been studied by, for example, Gelfand et al [28]. Whether there is a connection between this literature and Legendre fields remains to be seen.

As discussed in Proposition 2.4.15, the series expansion of a flat section of the Dubrovin connection provides an infinite family of Legendre fields indexed by the natural numbers. Studying integral representations of these flat sections could provide a way to find a larger family of Legendre fields.

7.2 Legendre transformations with non-flat fields

In § 3.4.2, we showed that flat twisted Legendre fields for the A_2 almost dual Frobenius manifold are given by the twisting of non-flat Legendre fields for the A_2 Frobenius manifold. From the results in Chapter 5, certain choices of flat Legendre fields for the A_2 almost dual produce a trigonometric A_1-type solution. A natural question is to ask what happens to the A_2 Frobenius manifold under the corresponding non-flat Legendre transformation, and whether any direct connection can be made between the resulting solutions and the trigonometric solutions obtained in Chapter 5.

It seems that Legendre transformations with non-flat fields have only been carried out in low-dimensional examples in [52, 59]. In both cases, the extended affine A_1 or A_2 Frobenius manifold, with a flat Legendre field δ, is taken as a starting point. The twisted (non-flat) Legendre field is then applied to the almost dual, a trigonometric A-type solution, which is mapped to a rational solution. Here, the original Legendre field δ is chosen to be a specific distinguished direction associated with the extended affine Weyl group.

7.3 Natural choices of Legendre fields

Recall from Remark 2.4.14 that the inversion of the coordinate relations $\hat{t}(t)$ presents the main obstacle in explicitly calculating the new solution $\widehat{F}$ produced by a Legendre transformation. Since it is not always possible to complete this inversion step even for a flat field, it is remarkable that it is sometimes possible for a non-flat field as in [52, 59]. It is also remarkable that it was possible to achieve this for the Legendre transformations explored in Chapters 5 and 6, where there is no distinguished direction in the starting system. It is not completely clear what makes Legendre transformations computationally achievable for some fields and not others. It seems that Legendre fields which are in some sense natural or distinguished directions produce "nicer" transformations, which are possible to compute. More general geometric analysis on this question would be interesting.

Chapters 5 and 6 provide examples of Legendre transformations of A_n- and B_n-type rational solutions for, arguably, natural choices of Legendre field. Although the A_n and B_n root systems do not have one single distinguished direction, unlike their extended affine counterparts, simple roots and fundamental weights encode the fundamental geometric properties of these systems. We therefore take certain simple roots as Legendre fields for both A_n and B_n, as well as the n^{th} fundamental weight for A_n. The resulting Legendre transformations in all cases are both possible to compute explicitly and map the original, rational, solution to a trigonometric solution. Future research here could attempt to set other simple roots or fundamental weights as a Legendre field, to see whether such a transformation is possible to compute and whether it produces a trigonometric solution. It would also be interesting to take this approach for other root systems.

7.4 Legendre transforms of trigonometric solutions

In Chapter 4, we introduced a new family of A-type trigonometric solutions. This class of solutions contains, as smaller sub-classes, the trigonometric solutions obtained in Chapter 5 from Legendre transformations of a multi-parameter family of A-type rational solutions. Since Legendre transformations are invertible, there exist Legendre fields such that the corresponding Legendre transformations map (a sub-class of) the trigonometric solutions to rational solutions. Finding these fields could be an entry-point to exploring Legendre transformations of this larger class of trigonometric solutions in general. One may investigate whether there exists a larger family of rational A-type solutions than that considered in this thesis, which could be obtained through Legendre transformations of the new trigonometric family.

In [59, Example 5.5], different Legendre transformations were applied to the extended affine A_2 almost dual prepotential, which is a trigonometric solution of the form considered in Chapter 5. The Legendre transformations in this example produced distinct rational solutions associated with certain extensions of the A_2 root systems. Such configurations were studied as extended $\vee$-systems by Stedman and Strachan in [58]. The trigonometric solutions given in Chapter 4, which contain the almost dual prepotentials of extended affine A-type Frobenius manifolds, therefore provide a connection via Legendre transformations between completely distinct families of $\vee$-systems and extended $\vee$-systems, thus connecting different rational solutions. More work is needed to understand the full picture of how these different objects all fit together.

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