Bricks and Stones

in the Homological Minimal Model Programme

by Parth Shimpi

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Abstract

Bricks and stones, morphisms between them and their cones, often build and control the symmetries of many categories arising in algebra and geometry. This thesis classifies such objects and related structures within categories arising from mild singularities in dimensions two and three.

More precisely, we classify *t-structures* on the local derived category of a 3-fold flopping contraction that are intermediate with respect to the heart of perverse coherent sheaves. Equivalently, this describes the complete lattice of *torsion classes* for the associated modification algebra. The intermediate hearts are all given as

(1) categories of coherent sheaves on birational models and tilts thereof in skyscrapers,

(2) algebraic t-structures described in the homological minimal model programme, or

(3) combinations of the above over appropriate open covers.

An analogous classification is also proved for (minimal as well as partial) resolutions of Kleinian singularities, thus providing a description of all torsion pairs in the module categories of (contracted) affine preprojective algebras.

The results have immediate applications to the classification of spherical modules and (semi)bricks, and are first steps towards describing *all* t-structures and spherical objects in the derived categories.

Such a programme is executed for flopping contractions with irreducible exceptional fibres. We show that *any* complex of coherent sheaves which admit no negative self-extensions is, up to flops and mutation equivalences, either

(1) a module over a derived–equivalent algebra, or

(2) a two–term extension of a coherent sheaf by skyscraper sheaves, or

(3) a direct sum of shifts of skyscrapers.

Consequently classifications of bricks, spherical objects, stability conditions, and algebraic t-structures in the local derived category are obtained; the lists populated by the homological minimal programme turn out exhaustive. We deduce that the Bridgeland stability manifold is connected, and that all basic tilting complexes on the variety (equivalently on the associated g–tame algebras) are related by shifts and iterated mutation.

Contents

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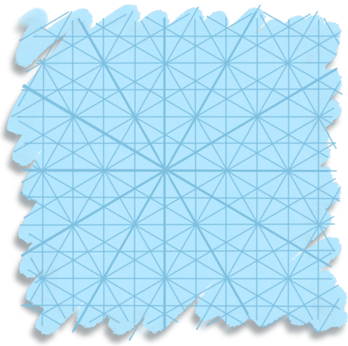
Declaration

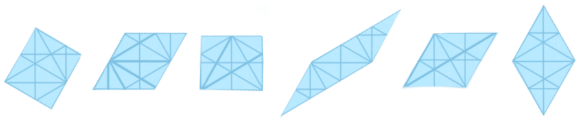
I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

A gist for everyone

In the process of renovating your university's mathematics department, you find yourself – as one does every so often – describing to the architect the pattern that will tile the central floor.



Of course, describing the entire infinite pattern is unnecessary. The architect simply needs the description of a single tile, the "heart" of the pattern, which they may repeat in every direction. For instance, any of the tiles below will suffice, and there are many more such.



The mathematically curious reader may wonder how much freedom they have in choosing this "heart", and in the process of cataloguing such tiles, convince themself that any two such can be linked by a sequence of "cut-and-paste" operations as shown.

What is the complete list of hearts associated to this pattern? Which of these tiles are linked by a single cut-and-paste? What is the list of "bricks" – smallest triangles – that makes up any heart?

The present work addresses questions of the above fashion. Tools for probing geometric and algebraic structures (such as those occurring naturally in some physical theories that describe the shape of the universe) often leave us with a kaleidoscopic version of the data we seek. That is to say, we only see an infinitely tiled mosaic in which the true geometry forms one repeating unit.

But, as above, there are many possible geometries from which the same infinite tiling we see could have originated. This thesis studies the structure and symmetries of the "tiled" data – presented in algebraic form as a *triangulated category* – by classifying *hearts of t-structures* and *brick objects* inside it.

Introduction

(Chapter 1)

Analyses of surfaces and 3-folds, that were definitive of algebraic geometry in the early and late twentieth century, motivated and demanded rapid advances in our understanding of embedded curves and the singularities that arise from contracting them. In the wake of such advancements a vortex of ideas has assumed a distinctly non-commutative flavour, adopting methods from representation theory and homological algebra to address various birational [Bri02; Wem18], enumerative [Gar22; NW23; Zha25] and deformation-theoretic [DW19; BW23] aspects of the geometry. The thesis at hand stems from this homological aspect of the minimal model programme.

Fix a curve C in a complex surface or a 3-fold X, and a proper birational morphism $\pi\colon X \to Z$ that non-discrepantly contracts C (and only C) to a point. The map π, a *partial resolution* of singularities, has been of interest to birational and symplectic geometers alike in both dimensions two [Cra00; IU05] and three [Asp03; Tod08; HW18]. Here homological algebra is governed by the *local derived category*

$$\mathbf{D}^0 X = \left\{ x \in \mathbf{D}^b X \;\middle|\; \operatorname{Supp} x \subseteq C \right\}$$

containing chain complexes of coherent $\mathcal{O}_X$-modules with support in the π-exceptional fibre; therefore an understanding of the structure and symmetries of this category is desirable.

Autoequivalences of derived categories are sometimes controlled by *spherical* objects, that is, complexes whose self-extensions resemble the singular cohomology of a sphere. Structural properties are often determined by collections of *stable objects*, from which any other object in the category can be reconstructed in an essentially unique way. Classifications of spherical objects and stability conditions have thence occupied the attentions of many [Bri09; IUU10; BDL23; HW23; KS24]. The two properties often appear intertwined though in theory the definitions share little other than the fact that an object x that is spherical or stable necessarily satisfies the condition

$$\operatorname{Hom}(x,x) = \mathbb{C}, \quad \text{and} \quad \operatorname{Hom}^i(x,x) = 0 \text{ whenever } i < 0. \tag{1.1}$$

This thesis is, then, about the unexpectedly thorough and fruitful classification that can stem from just the rather mild constraint (1.1).

Theorem 1.0.1 (= 7.3.2). *Suppose the curve C is integral, and the 3-fold Z is complete-local about the point $\pi(C)$ with at worst Gorenstein terminal singularities. Then, up to shifts, the Grothendieck duality functor*[1]*, and the action of a finitely generated group of Fourier–Mukai autoequivalences, the only complexes $x \in \mathbf{D}^0 X$ satisfying the condition* (1.1) *are*

- *(b1) the structure sheaves $\mathcal{O}_C, \mathcal{O}_{2C}, \ldots, \mathcal{O}_{\ell C}$ of the $\ell \leqslant 6$ thickenings that C may admit, or the unique non-trivial extension $\mathcal{Z}$ of $\mathcal{O}_{2C}$ by $\mathcal{O}_{3C}$ that exists when $\ell \geqslant 5$, or*
- *(b2) a skyscraper sheaf $\mathcal{O}_p$ at a point $p \in X$, or the image, under the Bridgeland–Chen flop equivalence, of a skyscraper sheaf $\mathcal{O}_q$ on the flop of X in C.*

Objects x satisfying the constraint (1.1) are called *bricks*. The above classification of bricks, and more generally, of orthogonal collections of bricks (*semibricks*) yields classifications of algebraic t-structures, tilting complexes, and Bridgeland stability conditions on the category (see § 7.3).

[1] Defined as $\mathbb{D}(-) = \mathbf{R}\mathcal{H}om(-, \varpi^!\mathbb{C})$, where $\varpi\colon X \to \operatorname{Spec}\mathbb{C}$ is the structure morphism defining X as a complex scheme

Essentially identical results hold when Z is a Gorenstein canonical surface (see § 1.3). The rest of the thesis carries these assumptions on the singularities of Z (and hence also of X). That is to say, we work in the setup below.

Setup 1.0.2. Let $\pi\colon X \to Z$ be a proper birational morphism between normal Gorenstein varieties that is *crepant*, i.e. $\pi^*\omega_Z = \omega_X$, and such that $Z = \operatorname{Spec}(R, \mathfrak{m})$ is complete local with a singularity at its closed point that is at worst canonical when $\dim Z = 2$, or at worst terminal when $\dim Z = 3$.

That is to say, we work with partial crepant resolutions of isolated singularities that are complete local and of (compound) du Val type [see Rei].

§ 1.1 Classifications in an ideal world

Some of earliest evidence that classifications of the kind this thesis addresses are not entirely quixotic to hope for came from the work of Smith and Wemyss [SW23], and of Hara and Wemyss [HW24].

Looking at a contraction π of two infinitesimally rigid[2] projective lines $C_1 \cup C_2 = C$ meeting transversely in a smooth 3-fold X, Smith–Wemyss found that any spherical object x satisfying $\mathbf{R}\pi_*(x) = 0$ can be algorithmically reduced to one of the sheaves $\mathcal{O}_{C_1}(-1)$, $\mathcal{O}_{C_2}(-1)$, or the complex $[\mathcal{O}_{C_1}(-1)[-1] \to \mathcal{O}_{C_2}(-1)]$. Hara–Wemyss weakened the assumptions on π to the generality considered in this thesis (setup 1.0.2), and showed that the category

$$\mathcal{C} = \left\{ x \in \mathbf{D}^0 X \mid \mathbf{R}\pi_*(x) = 0 \right\}$$

admits classifications of bricks, semibricks, t-structures, stability conditions, and more generally, of complexes x satisfying the (very mild!) hypothesis $\operatorname{Hom}^{<0}(x, x) = 0$.

Hara–Wemyss [HW25] trace this dreamy behaviour of their *Iyama dream category* to certain quirks in the representation theory of a contraction algebra. To set up a starting point for this thesis, to motivate the tools and constructions developed herein, and to appreciate the kinks that make $\mathbf{D}^0 X$ less dreamy than its counterpart $\mathcal{C}$, we present a distillation of Hara–Wemyss's argument.

The reduction algorithm. The full subcategory $H = \mathcal{coh}(X) \cap \mathcal{C}$ forms the *heart of a bounded t-structure* in $\mathcal{C}$, that is to say, every object in $x \in \mathcal{C}$ can be expressed as a bounded complex of objects in H. We write $x \in H[i, j]$ to mean the complex x is concentrated in degrees between i and j, and $H[\![i, j]\!]$ to indicate that this range cannot be made any smaller.

Now H is equivalent to the category of (right) modules over a finite dimensional algebra $\Lambda_{0,\mathrm{con}}$. When $\dim Z = 2$ and X is smooth this is a preprojective algebra of Dynkin type, in general it is the *contraction algebra* à la Donovan–Wemyss [DW19]. For such an *algebraic* heart, the constraint $\operatorname{Hom}^{(-n)}(x, x) = 0$ on an object $x \in H[\![0, n]\!]$ $(n > 0)$ guarantees the existence of another heart $K \subset H[-1, 0]$ (called a *tilt* of H) such that x lies in $K[1, n]$, see lemma 7.2.4.

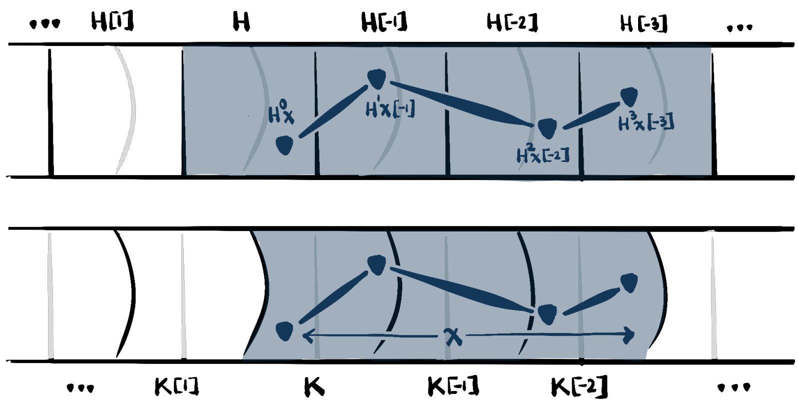


Figure 1.1. A complex $x \in H[\![-3, 0]\!]$, which has a 'spread' of 4 with respect to H, lies in only 3 degrees with respect to some tilted heart $K \subset H[-1, 0]$.

[2]This means their normal bundles are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

Now the algebra $\Lambda_{0,\mathrm{con}}$ turns out to be ***2-tilting finite*** and its derived equivalence class only contains other contraction algebras [Aug20a]. That is to say, there are finitely many hearts of bounded t-structures K contained in the subcategory $H[-1,0]$, and each such K is itself equivalent to the module category of some contraction algebra $\Lambda_{i,\mathrm{con}}$ $(i = 0, 1, \ldots, N)$. More precisely there is a finite system of categories and functorial equivalences

$$\Psi_{ij} : \mathcal{C}_i \longrightarrow \mathcal{C}_j \quad (0 \leqslant i, j \leqslant N) \tag{1.2}$$

with $\mathcal{C} = \mathcal{C}_0$, such that each $\mathcal{C}_i$ contains a heart H_i equivalent to $\mathcal{mod}\,\Lambda_{i,\mathrm{con}}$. This H_i has finitely many tilts each of which is the image of some H_j under some composite of the functors (1.2) and their inverses, and some shift.

Thus the complex $x \in H_0[\![0, n]\!]$ lies in $\Psi H_i[0, n-1]$ for some i and some composite functor Ψ, equivalently, $\Psi^{-1}(x)$ lies in $H_i[0, n-1]$. Rinsing and repeating shows that x must then sit in some H_i up to shifts and the system of functors (1.2).

If x happens to satisfy additional constraints then it can be reduced further:

if x is a semibrick then it reduces to a direct sum of distinct simples in $H_i \simeq \mathcal{mod}\,\Lambda_{i,\mathrm{con}}$ for some i,

if x is a brick (in particular if it is spherical, or stable under some stability condition) then it reduces to one of the finitely many simple $\Lambda_{i,\mathrm{con}}$-modules,

if x is a ***basic tilting object*** (i.e. if x is a classical generator of $\mathcal{C}$, satisfies $\mathrm{Hom}^i(x, x) = 0$ for all $i \neq 0$, and has pairwise non-isomorphic indecomposable summands) then x reduces to some $\Lambda_{i,\mathrm{con}}$ (viewed as a right module over itself); in particular all basic tilting objects can be obtained from $\Lambda_{0,\mathrm{con}}$ via shifts and iterated mutation in indecomposable summands.

As it turns out, the above classifications are enough to show that in fact *any* bounded heart in $\mathcal{C}$ must be equal to one of the H_is up to shifts and (1.2).

§ 1.2 Simple complexes on a flopped curve

This thesis is about establishing the above inductive machinery for the category $\mathbf{D}^0X$.

This category, too, has algebraic hearts. For $\pi\colon X \to Z \simeq \operatorname{Spec} R$ as in setup 1.0.2, Van den Bergh [Van04] observed that there is a module-finite R-algebra Λ and an equivalence $\mathrm{VdB}\colon \mathbf{D}^0X \to \mathbf{D}^{\mathrm{fl}}\Lambda$ onto the derived category of finite length Λ-modules. This identifies the natural heart $\mathrm{fl}\mathcal{mod}\,\Lambda$ with the full subcategory of ***perverse sheaves*** $\mathcal{per}(\frac{X}{Z}) \subset \mathbf{D}^0X$. This category serves as our reference heart H.

The homological minimal model programme [Wem18] extends this to a family of module-finite R-algebras $\{{}_\nu\Lambda_\nu \mid \nu \text{ a sequence of atomic mutations from } \Lambda\}$, called ***modification algebras***, that are equipped with derived equivalences

$$\Psi, \Phi\colon \mathbf{D}^{\mathrm{fl}}\,{}_\nu\Lambda_\nu \rightrightarrows \mathbf{D}^{\mathrm{fl}}\,{}_\upsilon\Lambda_\upsilon. \tag{1.3}$$

Tracking the natural hearts $\mathrm{fl}\mathcal{mod}\,{}_\nu\Lambda_\nu$ across these equivalences then produces more algebraic t-structures on $\mathbf{D}^0X$ which we say are ***mutations of*** $\mathcal{per}(\frac{X}{Z})$. These constructions are fleshed out in §§ 4.1 and 4.2.

However it is evident that there can be no autoequivalence which identifies the above algebraic hearts with the geometric (and non-Artinian) t-structure $\mathcal{coh}\,X$. Other geometric hearts can be constructed by iteratively flopping exceptional curves in X to obtain new flopping contractions $\pi\colon \nu X \to Z$, we say the 3-fold νX thus obtained is a ***birational model*** of X. Indeed flops give derived equivalences [Bri02; Che02], so tracking $\mathcal{coh}(\nu X)$ across the (inverse) Bridgeland–Chen functor $\mathbf{D}^b(\nu X) \to \mathbf{D}^bX$ gives a geometric t-structure on $\mathbf{D}^0X$. This construction is discussed in § 4.3.

The set of modification algebras has only finitely many isomorphism classes, and the functors (1.3) are compatible across preferred isomorphisms between modification algebras (see § 6.2), so in effect we are again reduced to a finite system of categories and equivalences up to which we may hope to classify.

Irreducible contractions. Chapter 7 addresses flopping contractions with irreducible exceptional fibres. This is the situation to which the Hara–Wemyss algorithm translates most readily.

The combinatorics of modification algebras are also especially simple – for any natural number $n \geqslant 1$ there are precisely two sequences of atomic mutations from Λ that have length n. We denote the resulting algebras Λ_{-n} and Λ_n respectively, and declare $\Lambda_0 = \Lambda$. The system of functors (1.2) then reduces to a diagram

$$\begin{tikzcd} \cdots \arrow[r, bend left] & \mathbf{D}^{\mathrm{fl}}\Lambda_{-2} \arrow[l, bend left] \arrow[r, bend left] & \mathbf{D}^{\mathrm{fl}}\Lambda_{-1} \arrow[l, bend left] \arrow[r, bend left] & \mathbf{D}^{\mathrm{fl}}\Lambda_{0} \arrow[l, bend left] \arrow[r, bend left] & \mathbf{D}^{\mathrm{fl}}\Lambda_{1} \arrow[l, bend left] \arrow[r, bend left] & \mathbf{D}^{\mathrm{fl}}\Lambda_{2} \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

where e.g. the two equivalences $\mathbf{D}^{\mathrm{fl}}\Lambda_n \rightrightarrows \mathbf{D}^{\mathrm{fl}}\Lambda_0$ in (1.2) arise as the composite of the bottom chain $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \leftarrow \mathbf{D}^{\mathrm{fl}}\Lambda_1 \leftarrow \cdots \leftarrow \mathbf{D}^{\mathrm{fl}}\Lambda_n$, and inverse of the top chain $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \to \mathbf{D}^{\mathrm{fl}}\Lambda_1 \to \cdots \to \mathbf{D}^{\mathrm{fl}}\Lambda_n$.

Van den Bergh gives an equivalence $\mathrm{VdB}\colon \mathbf{D}^0X \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$. There is only one other birational model W, with a preferred equivalence $\mathrm{VdB}\colon \mathbf{D}^0W \to \mathbf{D}^{\mathrm{fl}}\Lambda_{-1}$ such that the (inverse) Bridgeland–Chen flop functor $\mathbf{D}^0W \to \mathbf{D}^0X$ corresponds to the mutation functor $\mathbf{D}^{\mathrm{fl}}\Lambda_{-1} \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ after conjugation by VdB.

Example 1.2.1. Let $R = \mathbb{C}[\![u,v,x,y]\!]/(uv - xy)$ be the cA_1 singularity, i.e. the base of the Atiyah flop $X \dashrightarrow W$ where X is the total space of $\mathcal{O}_C(-1)^{\oplus 2}$ for the curve $C = \mathbb{P}^1$. The two flopping contractions $X, W \to Z = \operatorname{Spec} R$ are obtained by blowing up the ideals (u,x) and (u,y) respectively.

Accordingly the modification algebra associated to X is $\Lambda = \operatorname{End}(R \oplus (u,x))$, the remaining modification algebras are related by derived equivalences as below.

$$\begin{tikzcd} \cdots \arrow[r, bend left, "\Psi_1"] & \mathbf{D}^{\mathrm{fl}}\operatorname{End}\begin{pmatrix} L \\ \oplus \\ (u,y)\end{pmatrix} \arrow[l, bend left, "\Psi_1"] \arrow[r, bend left, "\Psi_0"] & \mathbf{D}^{\mathrm{fl}}\operatorname{End}\begin{pmatrix} R \\ \oplus \\ (u,y)\end{pmatrix} \arrow[l, bend left, "\Psi_0"] \arrow[r, bend left, "\Psi_1"] & \mathbf{D}^{\mathrm{fl}}\operatorname{End}\begin{pmatrix} R \\ \oplus \\ (u,x)\end{pmatrix} \arrow[l, bend left, "\Psi_1"] \arrow[r, bend left, "\Psi_0"] & \mathbf{D}^{\mathrm{fl}}\operatorname{End}\begin{pmatrix} K \\ \oplus \\ (u,x)\end{pmatrix} \arrow[l, bend left, "\Psi_0"] \arrow[r, bend left, "\Psi_1"] & \cdots \arrow[l, bend left, "\Psi_1"] \\ & & \mathbf{D}^0W \arrow[u, "\mathrm{VdB}"] & \mathbf{D}^0X \arrow[u, "\mathrm{VdB}"] & & \end{tikzcd} \tag{1.4}$$

Here the functors $\Psi_i\colon \mathbf{D}^{\mathrm{fl}}(\operatorname{End} N) \to \mathbf{D}^{\mathrm{fl}}(\operatorname{End} M)$ are given as $\mathbf{R}\mathrm{Hom}(\mathrm{Hom}(N,M), -)$, and the index records which summand changes between N and M. This changed summand is computed via mutation, thus for instance (u,x) and (u,y) are (first) syzygies of each other while $L \subset (u,y)^{\oplus 2}$ is the kernel of the natural map $((u,y) \hookrightarrow R) \oplus ((u,y) \xrightarrow{\sim} (v,x) \hookrightarrow R)$.

The action of the class group $\mathrm{Cl}(R) \simeq \mathbb{Z}$ yields natural R-algebra isomorphisms between all modification algebras for the Atiyah flop. Using these isomorphisms to identify their derived categories reduces the above system of functors to the diagram

$$\begin{tikzcd} \mathbf{D}^0X \arrow[r, "\mathrm{VdB}"] & \mathbf{D}^{\mathrm{fl}}\Lambda \arrow[loop above, "\Psi_0"] \arrow[loop below, "\Psi_1"] & \mathbf{D}^0W \arrow[l, "\mathrm{VdB}"'] \end{tikzcd}.$$

More generally there is a natural number $\wp \geqslant 1$, called the *helix period* of π, such that each Λ_i is isomorphic to $\Lambda_{i+\wp}$. The system of functors is then reduced to the following diagram[3], which the reader is invited to wrap around a cylinder after identifying the two copies of $\mathbf{D}^{\mathrm{fl}}\Lambda_0$ as in fig. 7.1.

$$\begin{tikzcd} \mathbf{D}^{\mathrm{fl}}\Lambda_0 \arrow[r, bend left, "\Psi_0"] & \mathbf{D}^{\mathrm{fl}}\Lambda_1 \arrow[l, bend left, "\Psi_0"] \arrow[r, bend left, "\Psi_1"] & \cdots \arrow[l, bend left, "\Psi_1"] \arrow[r, bend left, "\Psi_0"] & \mathbf{D}^{\mathrm{fl}}\Lambda_{\wp-1} \arrow[l, bend left, "\Psi_0"] \arrow[r, bend left, "\Psi_1"] & \mathbf{D}^{\mathrm{fl}}\Lambda_0 \arrow[l, bend left, "\Psi_1"] \\ \mathbf{D}^0X \arrow[u, "\mathrm{VdB}"] & & & \mathbf{D}^0W \arrow[u, "\mathrm{VdB}"] & \end{tikzcd} \tag{1.5}$$

[3]The reader is justified in worrying about how to interpret this diagram when $\wp$ is odd. When $\wp = 1$ it is the diagram for the Atiyah flop. Fortunately all other values of $\wp$ that can arise, namely 2,4,6,10, and 12, turn out to be even.

Our first significant advancement in establishing the reduction algorithm is a complete classification (theorem 7.2.5) of hearts that can be obtained by tilting $\mathrm{fl}\,\mathcal{mod}\,\Lambda_0$. In fact the bulk of this thesis (chapters 5 and 6) is concerned with establishing an analogue of the above result for a general flopping contraction with not-necessarily-irreducible exceptional fibre; thus we save the discussion for later in the introduction (see § 1.4). A discussion pertinent to the present single-curve setting appears as algorithm 1.4.3.

When all is said and done, all hearts $K \subset H[-1,0]$ are equal to some natural heart in the diagram (1.5) up to shifts and the equivalences shown. That is, for some composition Ψ of the functors appearing in (1.5) and their inverses and some shift, the category $\Psi^{-1}K$ either coincides with $\mathrm{fl}\,\mathcal{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}}\Lambda_i$, or with $\mathcal{coh}\,X \subset \mathbf{D}^0X$ or the minor modification

$$\left\langle \left\{ \mathcal{F} \in \mathcal{coh}\,X \;\middle|\; \begin{array}{l} \mathrm{Hom}(\mathcal{G},\mathcal{F}) = 0 \text{ whenever} \\ \mathcal{G} \in \mathcal{coh}\,X, \quad \mathrm{Supp}\,\mathcal{G} \subseteq U \end{array} \right\} \cup \left\{ \mathcal{G}[-1] \in \mathcal{coh}\,X[-1] \;\middle|\; \mathrm{Supp}\,\mathcal{G} \subseteq U \right\} \right\rangle \tag{1.6}$$

thereof defined by some subset $U \subseteq C$, or with the analogous hearts in $\mathbf{D}^0W$.

Given a complex $x \in \mathbf{D}^{\mathrm{fl}}\Lambda_0$ satisfying $\mathrm{Hom}^{<0}(x,x) = 0$, lying in $H[\![0,n]\!]$ without loss of generality, tilting to one of the above hearts K improves the spread of x to $[1,n]$. If some such K is algebraic, we may continue the induction much like we did for the category $\mathcal{C}$.

Much unlike the category $\mathcal{C}$ however, it is possible that none of the hearts that improve x are reachable by mutation. Indeed the complexes $\mathcal{O}_p \oplus \mathcal{O}_q[57]$ or $\mathrm{Cone}(\mathcal{O}_p \oplus \mathcal{O}_C \to \mathcal{O}_q[2])$ (for distinct points $p, q \in X$) can only be improved by tilting to a heart K of the form (1.6) with $q \in U \subset C \setminus \{p\}$.

Prima facie it is unclear how one continues the induction in such a situation, though for the examples shown it is evident that the latter is contained in K and admits a fairly explicit description while the former is degenerate enough to not cause alarm. In lemma 7.2.10 we obtain precise cohomological information about objects which do not improve by mutation to show that the two examples considered above are representative of the worst case, accounting for which yields a complete classification.

Theorem 1.2.2 (= 7.2.1). *Suppose* $\pi\colon X \to Z$ *as in setup 1.0.2 has irreducible fibres. Any complex* $x \in \mathbf{D}^0X$ *that satisfies the condition* $\mathrm{Hom}(x, x[<0]) = 0$*, up to a shift and iterated applications of the functors* (1.5)*, is given by*

- *(c1) a* Λ_i*-module for some modification algebra* Λ_i*, or*
- *(c2) a direct sum of shifts of coherent sheaves with zero–dimensional support on* X *(or on* W*), or*
- *(c3) a coherent sheaf on* X *(or on* W*), or a two-term complex thereof determined as a Yoneda-ext* $x \in \mathrm{Ext}^2(\mathcal{G},\mathcal{F})$ *between coherent sheaves* $\mathcal{F},\mathcal{G}$ *such that* $\mathrm{Hom}(\mathcal{G},\mathcal{F}) = 0$ *and* $\dim(\mathrm{Supp}\,\mathcal{G}) = 0$.

It is the irreducibility of C that guarantees the collapse into zero-dimensional sheaves of a complex that does not improve by mutation. This crucial role played by the very specific underlying geometry places our analysis of $\mathbf{D}^0X$ in stark contrast with Hara–Wemyss's of $\mathcal{C}$ – the latter arguments are very general and apply to any silting discrete triangulated category.

Tilting complexes and algebraic t-structures. Traditionally the questions of identifying and classifying Morita equivalences such as (1.5) are addressed by considering tilting objects. These are defined as classical generators $t \in \mathbf{D}^bX$ which satisfy $\mathrm{Hom}^i(t,t) = 0$ whenever $i \neq 0$, thus inducing a derived equivalence between X and $\mathrm{End}(t)$ [Ric89].

Indeed Van den Bergh [Van04] constructs the equivalence $\mathrm{VdB}\colon \mathbf{D}^bX \to \mathbf{D}^b\Lambda_0$ in (1.5) by identifying a tilting vector bundle $\mathcal{N}$ with two indecomposable summands and a globally generated dual. Replacing either summand with its left or right approximation by the other (i.e. *mutating*) constructs four more tilting objects, and hence four equivalences out of $\mathbf{D}^b\Lambda_0$. These are the functors $\Psi_0^{\pm1}, \Psi_{-1}^{\pm1}$ in (1.5). The remaining equivalences are likewise obtained by systematically propagating the mutation as in example 1.2.1, thus enumerating a class of basic tilting objects as a tetravalent exchange graph.

We show that this list of tilting complexes is complete.

Theorem 1.2.3 (= 7.3.4). *Suppose* $\pi\colon X \to Z$ *as in setup 1.0.2 has irreducible fibres. Every basic tilting complex in the category* $\mathbf{D}^bX$ *can be obtained from a shift of Van den Bergh's tilting bundle* $\mathcal{N}$ *by finitely many left or right mutations in indecomposable summands.*

In equivalent terms, every basic tilting complex of Λ_i-modules ($i \in \mathbb{Z}/\wp\mathbb{Z}$) can be obtained by iteratively mutating indecomposable summands of $\Lambda_i[n]$ for some $n \in \mathbb{Z}$. This classification applies, in particular, to all two-vertex idempotent contractions $(e_0 + e_i)\Pi(e_0 + e_i)$ of affine preprojective algebras Π.

The classical tilting theory toolkit excels at populating a list of tilting objects out of a single such instance. It however generally falls short of identifying when such a list is exhaustive (notable exceptions are the cases of Dynkin preprojective algebras and Donovan–Wemyss' contraction algebras, where Aihara–Mizuno [AM17] and August [Aug20b] leverage the finiteness of involved g-fans to obtain a complete classification.) Theorem 1.2.3 exhibits the first class of g-tame (in particular, silting-indiscrete) algebras for which the issue can be settled.

The proof is a *consequence* of a classification of Morita equivalences. More precisely we identify collections that could be the images of simple modules under the inclusion of a bounded heart $\mathit{mod}\,\Gamma \hookrightarrow \mathbf{D}^b\Gamma \xrightarrow{\sim} \mathbf{D}^b X$ for some module-finite R-algebra Γ. Any such R-linear equivalence restricts to the full subcategories supported on the singular point $[\mathfrak{m}] \in Z$, i.e. the subcategories $\mathbf{D}^0X$ and $\mathbf{D}^{fl}\Gamma$. The natural heart $\mathrm{fl}\,\mathit{mod}\,\Gamma$ is the extension–closure of its finitely many simples, whose images in $\mathbf{D}^0X$ form a *simple-minded collection* which can be tracked and identified using theorem 7.2.1.

Theorem 1.2.4 (= 7.3.1). *Suppose* $\pi\colon X \to Z$ *as in setup 1.0.2 has irreducible fibres. Then given any bounded t-structure with algebraic (i.e. Artinian and Noetherian) heart* $K \subseteq \mathbf{D}^0X$*, there is an index* $i \in \mathbb{Z}/\wp\mathbb{Z}$ *and equivalence* $\mathbf{D}^0X \to \mathbf{D}^{fl}\Lambda_i$ *composed of shifts and the functors* (7.1) *and their inverses that restricts to an equivalence between* K *and* $\mathrm{fl}\,\mathit{mod}\,\Lambda_i$.

Bricks and stones. A complex x satisfying $\mathrm{Hom}^{<0}(x, x) = 0$ is a *brick* if its endomorphism ring is isomorphic to $\mathbb{C}$, it is a *stone* if it furthermore satisfies the rigidity constraint $\mathrm{Hom}^1(x, x) = 0$. When X is smooth it makes sense to talk of spherical objects in $\mathbf{D}^0X$, and by Serre duality this class of spherical objects coincides with the class of stones.

In theorem 6.4.1 we argue that a brick $x \in \mathbf{D}^0X$, once reduced to a Λ_i-module or a complex of sheaves as in theorem 7.2.1, can be further reduced to a simple Λ_i-module or a simple sheaf (i.e. skyscraper at a point) with one more tilt. The classification theorem 1.0.1 then follows from Donovan–Wemyss' analysis of simple Λ_i-modules [DW25, §1.2].

The group of Fourier–Mukai autoequivalences appearing in the statement is simply the group of all functors $\mathbf{D}^0X \to \mathbf{D}^0X$ that can be written as composites of the functors (1.5) and their inverses; this is (a quotient of) the fundamental group of a $\wp$-punctured cylinder (fig. 7.1).

The question of which bricks are stones and/or spherical can then be addressed by explicit computation, for instance skyscraper sheaves are never rigid. When $\dim X = 2$ and X is smooth, i.e. X is (a completion of) the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$, all simples (and hence all non-skyscraper bricks) are spherical. The question for more general surfaces reduces to a combinatorial calculation of the Gabriel quivers of contracted preprojective algebras.

Example 1.2.5. Take $R = \operatorname{Spec}\mathbb{C}[\![x, y, z]\!]/(x^2 + y^2z + z^3)$ the D_4 surface singularity, and $\pi\colon X \to Z$ the blow-up of the ideal (x, y, z). The exceptional curve C contains three points that are singular in X, blowing these up further produces the minimal resolution of Z. It is classical [KV00] that the minimal resolution is derived equivalent to the preprojective algebra Π presented as

$$\begin{tikzcd} & 2 \arrow[d, bend left, "b^*"] & \\ 0 \arrow[r, bend left, "a^*"] & 1 \arrow[l, bend left, "a"] \arrow[u, bend left, "b"] \arrow[r, bend left, "c"] \arrow[d, bend left, "d"] & 3 \arrow[l, bend left, "c^*"] \\ & 4 \arrow[u, bend left, "d^*"] & \end{tikzcd}, \qquad \begin{aligned} &aa^* + bb^* + cc^* + dd^* = 0, \\ &a^*a = 0, \qquad b^*b = 0, \\ &c^*c = 0, \qquad d^*d = 0. \end{aligned}$$

Writing $e_i \in \Pi$ for the primitive idempotent at vertex i in the Dynkin quiver, Kalck, Iyama, Wemyss, and Yang [KIWY15, theorem 4.6] show that all "modification" algebras here isomorphic to the (non-unital) subalgebra $(e_0 + e_1)\Pi(e_0 + e_1)$ of Π.

Writing $f = bb^*$ and $g = cc^*$, this contracted preprojective algebra can be explicitly presented as the completed path algebra of the quiver below [see Kar17a, example 4.10]. In particular, the vertex simple S_0 ($= \mathrm{VdB}\,\mathbb{D}\mathcal{O}_{2C}$) is a stone while S_1 ($= \mathrm{VdB}\,\mathcal{O}_C(-1)$) deforms.

$$\begin{tikzcd} 0 \arrow[r, bend left, "a^*"] & 1 \arrow[l, bend left, "a"] \arrow[loop above, "f"] \arrow[loop below, "g"] \end{tikzcd}, \qquad \begin{array}{c} a^*a = 0, \quad f^2 = 0, \quad g^2 = 0, \\ (aa^* + f + g)^2 = 0. \end{array}$$

The question when $\dim X = 3$ is more subtle, even when X is smooth. There is an obvious combinatorial obstruction – a generic hyperplane section $\overline{X} = X \times_Z (\mathrm{Spec}\, R/(t))$ is a partial resolution of a Kleinian singularity so the above methods can be used to compute if the simple S_i deforms after restriction to $\overline{X}$. If it does then there is no hope of rigidity in X. With these considerations alone, Donovan–Wemyss [DW25, corollary 7.5] narrow down the list of possible spheres to $\{\mathcal{O}_{\ell C}, \mathbb{D}\mathcal{O}_{\ell C}, \mathcal{Z}, \mathbb{D}\mathcal{Z}\}$.

But even if the restriction of the simple to the surface is rigid, the threefold may allow deformations.

Example 1.2.6. The modification algebra for the Atiyah flop of example 1.2.1, i.e. the endomorphism algebra of the R-module $R \oplus (u, x)$, can be presented as

$$\begin{tikzcd} 0 \arrow[r, bend left=40, "a"] \arrow[r, bend left=10, "b"] & 1 \arrow[l, bend left=10, "c"] \arrow[l, bend left=40, "d"] \end{tikzcd}, \qquad \begin{array}{cc} acb = bca, & adb = bda, \\ cad = dac, & cbd = dbc, \end{array}$$

where the maps $a, b \colon R \to (u, x)$ and $c, d \colon (u, x) \to R$ are multiplication by $u, x, 1$, and $\frac{y}{u}$ respectively. The quiver has no loops, both simples are rigid and hence spherical. It follows that every spherical object in $\mathbf{D}^0X$ is, up to shifts and mutations, the structure sheaf $\mathcal{O}_C$ of the exceptional curve. This classification of spherical objects on the Atiyah flop was proved by Keating and Smith [KS24, theorem 1.3] via symplectic dynamics on the mirror.

The Pagoda flop is a close cousin, where the flopping contractions involved are the blow-ups of the ideals $(u, x \pm y^2)$ in $S = \mathbb{C}[\![u, v, x, y]\!]/(uv - (x + y^2)(x - y^2))$. The modification algebras are all isomorphic to the endomorphism algebra of the S-module $S \oplus (u, x + y^2)$, and can be presented as

$$\begin{tikzcd} 0 \arrow[loop left, "f"] \arrow[r, bend left=40, "a"] \arrow[r, bend left=10, "b"] & 1 \arrow[l, bend left=10, "c"] \arrow[l, bend left=40, "d"] \arrow[loop right, "g"] \end{tikzcd}, \qquad \begin{array}{ccc} fa = ag, & fb = bg, & 2f^2 = bc - ad, \\ cf = gc, & df = gd, & cb - da = 2g^2, \end{array}$$

with the maps a, b, c, d, f, g being respectively given by multiplication by $u, x + y^2, 1, \frac{x - y^2}{u}, y$, and y. Neither of the simples is rigid, and there are no spherical objects.

Both singularities of example 1.2.6 are of type cA_1 (that is, they have generic hyperplane sections isomorphic to $\mathbb{C}[\![x, y, z]\!]/(xy - z^2)$) and are therefore indistinguishable as far as the methods of this thesis are concerned.

Stability conditions. Another consequence of theorem 1.2.4 is the connectedness of the manifold parametrising Bridgeland stability conditions on $\mathbf{D}^0X$, an object of interest not least because it allows for an analysis of the autoequivalence group of $\mathbf{D}^bX$. Such analyses were carried out by Bridgeland [Bri09], Toda [Tod08] and Hirano–Wemyss [HW23], where they found that all stability conditions arising from the standard hearts $\mathrm{fl}\,\mathit{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}}\Lambda_i$ ($i \in \mathbb{Z}/\wp\mathbb{Z}$) lie in a single connected component which is preserved by autoequivalences composed of the mutation functors.

Theorem 1.2.4 implies that a stability condition outside this component, if it exists, must only correspond to t-structures with non-algebraic hearts, in particular admitting no gaps in its phase distribution. By using theorem 1.0.1 to explicitly control which phases can be occupied by any collection of stable objects, we preclude the possibility of such a stability condition existing and deduce the following.

Theorem 1.2.7 (= 7.3.6). *Suppose* $\pi \colon X \to Z$ *as in setup 1.0.2 has irreducible fibres. Then the manifold of Bridgeland stability conditions* $\mathrm{Stab}(\mathbf{D}^0X)$ *is connected.*

Stringy monodromy. The stability manifold is a way to get hold of the autoequivalence group of $\mathbf{D}^0X$, or at least of a large intrinsically defined subgroup thereof. Indeed, viewing $\mathbf{D}^bX$ as the topological B-model of a superconformal field theory [see Bri06], one expects to recover a class of symmetries $G \subseteq \mathrm{Aut}(\mathbf{D}^bX)$ via variation of the complexified Kähler structure [Asp03, §4] on a conjectured moduli space modelled as a submanifold of $G\backslash\,\mathrm{Stab}(\mathbf{D}^0X)/\mathbb{C}$.

This proposal was brought to fruition [Bri09; Tod08; HW23; DW25] by taking G to be the subgroup of R-linear Fourier–Mukai equivalences (that preserve the connected component $\mathrm{Stab}^\circ(\mathbf{D}^0X)$ containing algebraic points), and computing that the double quotient $G\backslash\,\mathrm{Stab}^\circ(\mathbf{D}^0X)/\mathbb{C}$ is a $(\wp{+}2)$-punctured sphere – a space reasonably declared to be the *stringy Kähler moduli space* $\mathcal{M}_{\mathrm{SK}}$.

Happily, all clauses to do with connected components of $\mathrm{Stab}(\mathbf{D}^0X)$ in the above description can now be dropped in light of theorem 1.2.7, so that G can be simply taken to be the group of all R-linear Fourier–Mukai autoequivalences of $\mathbf{D}^0X$.

Hirano–Wemyss [HW18, theorems 1.3 and 1.5] show that the quotient $\mathrm{Stab}^\circ(\mathbf{D}^0X)/\mathbb{C} \twoheadrightarrow \mathcal{M}_{\mathrm{SK}}$ is a regular covering map whose deck group, equal to G, is generated by composites of mutation functors (7.1). The question of relations between these generators (i.e. of the faithfulness of the $\pi_1(\mathcal{M}_{\mathrm{SK}})$-action shown in fig. 7.1) is intertwined with simply-connectedness of $\mathrm{Stab}(\mathbf{D}^0X)$ and hence with long-standing questions on contractibility of stability manifolds.

Both questions – of faithfulness and of contractibility – can then be settled using Roberts–Woolf's work [RW+], where they claim that each connected component of $\mathrm{Stab}(\mathfrak{T})$ admits a contracting flow whenever $\mathfrak{T}$ is a triangulated category with Grothendieck group $\mathbb{Z}^{\oplus 2}$. Theorem 1.2.7, combined with Roberts–Woolf's claim, thus proves the folklore conjecture below.

Conjecture 1.2.8. *Suppose $\pi\colon X \to Z$ as in setup 1.0.2 has irreducible fibres. The stability manifold of $\mathbf{D}^0X$ is homotopy equivalent to a point, and the quotient $\mathrm{Stab}(\mathbf{D}^0X)/\mathbb{C}$ is the universal cover of $\mathcal{M}_{\mathrm{SK}}$. Consequently the group G of R-linear Fourier–Mukai autoequivalences of $\mathbf{D}^bX$ is isomorphic to a product of $\mathbb{Z}$ (generated by the shift functor) and the fundamental group of $\mathcal{M}_{\mathrm{SK}}$ (i.e. a free group on $\wp+1$ generators).*

§1.3 Elephants and 3-folds

Before proceeding, we justify the cavalier attitude towards $\dim(Z)$, despite all discussions seeming to favour the language and terminology (e.g. flops, modification algebras) reserved for 3-folds.

Suppose $\overline{X}$ is a Gorenstein canonical surface that admits a crepant birational morphism $\pi\colon \overline{X} \to \overline{Z}$ to a complete local base $\overline{Z} = \mathrm{Spec}(\overline{R}, \mathfrak{p})$. All results of this thesis apply verbatim to the derived category of $\overline{X}$ and its π-perverse heart provided one suitably reinterprets the notions of modification algebras and birational models.

Such a map π is necessarily a crepant (partial) resolution of the Kleinian singularity $\overline{Z}$, i.e. $\overline{X}$ is obtained by contracting some exceptional curves in the minimal resolution $\widetilde{X} \to \overline{Z}$. It is well-known that $\widetilde{X}$ is derived equivalent to an affine preprojective algebra Π; the contraction $\widetilde{X} \to \overline{X}$ determines an idempotent $e \in \Pi$ such that $\overline{X}$ is derived equivalent to the *contracted preprojective algebra* $e\Pi e$ [KIWY15]. This equivalence $\mathbf{D}^{\mathrm{fl}}(e\Pi e) \to \mathbf{D}^0(\overline{X})$ is also recovered by Van den Bergh's construction [Van04], and algebraic mutations of the category of π-perverse sheaves can be read off from the tilting theory of $e\Pi e$ [IW, §7.4].

The discussion of birational models is only slightly more subtle, as curves in surfaces do not flop. Birational transformations (including flops in dimension 3), on the other hand, are naturally induced via geometric invariant theory. Indeed $\widetilde{X}$ appears as a moduli space $\mathcal{M}(\theta, \delta)$ of stable Π-modules, for some generic stability parameter θ relative to a $\mathbf{K}$-theory class δ [CS98]. Variation of GIT parameter $\theta_1 \rightsquigarrow \theta_2$ produces a birational map $\mathcal{M}(\theta_1, \delta) \dashrightarrow \mathcal{M}(\theta_2, \delta)$ of $\overline{Z}$-schemes. In particular if θ_1 and θ_2 lie in adjacent chambers then this birational map is defined away from a single exceptional curve, and simple wall-crossing of the GIT parameter thus gives an analogue of flops in dimension 2.

The role of Bridgeland–Chen equivalences is played by *reflection functors* [SY13], and the results are extended to the contracted setting by Iyama–Wemyss [IW].

Given this, all arguments in chapters 4 to 6 regarding the poset of algebraic hearts and its interactions with convex geometry and line bundles on birational models hold verbatim. Thus the reader indifferent to dimensions can safely read the remainder of this thesis as if it were written for surfaces.

To the reader interested in multiple dimensions, we remark that the correspondence between our results for 3-folds and surfaces is natural. Indeed, the morphism $\overline{X} \to \overline{Z}$ can be embedded as a generic hyperplane section (called a *general elephant*[4]) of some 3-fold flopping contraction $X \to \operatorname{Spec}(R, \mathfrak{m})$, and the corresponding non-commutative algebras are related by the reduction $e\Pi e = \Lambda \otimes_R \overline{R}$.

By reducing both to the fibre $R/\mathfrak{m} = \overline{R}/\mathfrak{p}$, Kimura [Kim24, theorem 5.4] shows that the corresponding functor $\operatorname{fl}\mathcal{mod}(e\Pi e) \hookrightarrow \operatorname{fl}\mathcal{mod}\,\Lambda$ induces a bijection of torsion classes. Likewise lemma 5.1 *ibid.* gives the correspondence between bricks.

Example 1.3.1. Setting $R = \mathbb{C}[\![u,v,x,y]\!]/(u^2+v^2x+x^3+y(v^2+x^2y+xy^2))$, one side of the Brown–Wemyss flop [BW18, example 2.1] is the blow-up $\pi\colon X \to Z = \operatorname{Spec} R$ of the ideal $(vx-uy, xy^2+v^2, xy^2+uv) \subset R$. The associated modification algebra is generated by the following R-linear maps, where we write $t = x+y$ for convenience.

$$\begin{tikzcd} R \arrow[r, bend left, "a^*"] & \operatorname{coker}\begin{pmatrix} u & x & -v & -y \\ x^2 & -u & xy & -v \\ vt & yt & u & x \\ xyt & -vt & -x^2 & u \end{pmatrix} \arrow[l, bend left, "a"] \arrow[loop above, "f"] \arrow[loop below, "g"] \end{tikzcd},$$

$$a = \begin{pmatrix} -u & x & v & y \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 1 & -i & 0 \\ -x & 0 & 0 & i \\ it & 0 & 0 & 1 \\ 0 & -it & -x & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 & i & 0 \\ -x & 0 & 0 & -i \\ -it & 0 & 0 & 1 \\ 0 & it & -x & 0 \end{pmatrix}.$$

Upon passing to the elephant $\{y = 0\}$, this reduces to example 1.2.5. An identical statement is true for the Laufer flop over $\mathbb{C}[\![u,v,x,y]\!]/(u^2+v^2y+x^3+xy^3)$, where the D_4 singularity can be most readily seen e.g. as the elephant $\{y = x\}$.

Brown and Wemyss studied the above examples to show that they have distinct contraction algebras (and are therefore non-isomorphic) but have identical Gopakumar–Vafa invariants. The lattice-theoretic invariants (i.e. the poset of t-sructures, heart fans, and the set of bricks) studied in this thesis are likewise insensitive to any specifics of the 3-fold beyond its generic hyperplane section.

§ 1.4 Torsion pairs and 3-fold flops

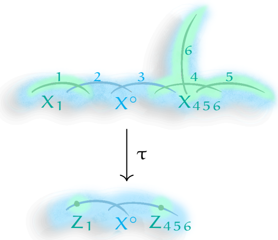
To set up the programme for a general flopping contraction $\pi\colon X \to Y$, an understanding of the mutation classes of modification algebras and birational classes of flops is necessary.

Matters are further complicated over non-irreducible fibers, there are hearts which are 'algebro-geometric' in quite the literal sense. The construction of the perverse heart is local, so any crepant morphism $\tau\colon X \to Y$ contracting some (but not all) of the π-exceptional curves gives a sheaf $\mathscr{A}$ of coherent $\mathcal{O}_Y$-algebras and a derived equivalence $\mathbf{D}^b X \to \mathbf{D}^b \mathscr{A}$. This identifies a category of suitable $\mathscr{A}$-modules with a heart $\mathcal{per}(\frac{X}{Y}) \subset \mathbf{D}^0 X$ whose objects are precisely complexes that look like finite length Λ-modules on the τ-exceptional locus and like $\mathcal{O}_X$-modules elsewhere (theorem 4.4.5). The category is again not equivalent to any heart examined previously, and it is hence apparent that any classification must also account for these (semi-)geometric hearts.

We show that at least for sufficiently small cohomological spread, the above possibilities are in fact exhaustive. This involves building semi-geometric hearts locally using algebraic and geometric categories for 'smaller' flopping contractions (§ 4.4), thus obtaining an inductive description of t-structures.

[4] As opposed to, say, a commander giraffe

Indeed any partial contraction $\tau : X \to Y$ is an isomorphism on the open locus X° of its non-exceptional fibers, while in any sufficiently small neighbourhood of a τ-exceptional fiber C_I, the map τ restricts to a flopping contraction $X_I \to Z_I$ with complete local base Z_I in Y. This gives a (flat) cover of X. Qualitatively, we show any tilt of $\mathcal{per}(\frac{X}{Z})$ decomposes into purely algebraic or purely geometric hearts with respect to some such cover.



Theorem 1.4.1 (= 1.4.2, 5.3.7). *Let K be the heart of a t-structure on $\mathbf{D}^0X$ that is contained in $\mathcal{per}(\frac{X}{Z})[-1,0]$. Then there is a birational model W of X, and a partial contraction $\tau\colon W \to Y$, such that K satisfies the following after being pulled back across the (inverse) Bridgeland–Chen equivalence $\Psi : \mathbf{D}^0W \to \mathbf{D}^0X$.*

(1) On the locus $W^\circ \subset W$ where τ is an isomorphism, $\Psi^{-1}K$ restricts to the category of coherent sheaves $\mathcal{coh}\,W^\circ$, possibly tilted in skyscraper sheaves $\langle \mathcal{O}_p \mid p \in Q\rangle$ for some subset of closed points $Q \subset W^\circ$.

(2) For each τ-exceptional fibre C_I in W, for sufficiently small neighbourhoods $W_I \supset C_I$ and $Z_I \ni \tau(C_I)$, the restriction of $\Psi^{-1}K$ to W_I is an (algebraic) mutation of the category $\mathcal{per}\big(\frac{W_I}{Z_I}\big) \subset \mathbf{D}^0W_I$.

Moreover, the above data uniquely determines K.

In particular if τ above contracts all exceptional curves, then K is an algebraic mutation of $\mathcal{per}(\frac{W}{Z})$ (hence also of $\mathcal{per}(\frac{X}{Z})$) and thus K is the category of finite length modules over a modification algebra. On the other extreme if τ contracts no curves then K is a geometric t-structure on the birational model W.

Mutating v/s tilting. The g-*fan* is a combinatorial manifestation of the set of algebraic tilts, allowing simple mutation to be recast as simple wall-crossing and hence for mutation to be tracked via paths in a hyperplane arrangement. Finiteness of the g-fan for silting-discrete algebras underpins Hara–Wemyss's argument as sketched in § 1.1, guaranteeing that any poset considered has maximal elements.

In contrast the category $\mathbf{D}^0X$ exhibits 'affine-type' behaviour, ensuring we don't have such privileges. The g-fan of the modification algebra Λ is infinite and incomplete. So while there still is a meaningful assignment of algebraic tilts to chambers, one is forced to reckon with infinite paths in the g-fan which culminate outside its support.

We thus replace mutation and the g-fan with the more general framework of *tilting in torsion pairs* and the *heart fan* respectively. Given any Abelian category H, Happel–Reiten–Smalø [HRS96] show that the set $\mathrm{tilt}(H)$ of intermediate t-structures (i.e. t-structures with hearts contained in $H[-1,0]$) can be recovered by tilting H in its torsion subcategories, and the containment order of torsion classes enhances $\mathrm{tilt}(H)$ to a *complete lattice* (i.e. a poset in which every subset has an infimum and a supremum). Broomhead–Pauksztello–Ploog–Woolf [BPPW24] assign each heart $K \in \mathrm{tilt}(H)$ to a cone $\mathbf{C}K$ and show that the ensemble $\mathrm{HFan}(H)$ of heart cones is a fan.

Giving a complete description of the lattice of torsion classes is hard, even for finite dimensional algebras [Tho21; DIRRT23]. The heart fan gives some insight into tilts satisfying numerical criteria (§ 2.4), but there are examples of tilts of algebraic hearts which have trivial (i.e. $\mathbf{0}$) heart cone [see BPPW24, example 6.7]. In such situations the numerical criteria lead to tautologies, and we are left to our own devices. We show that such situations do not arise when studying tilts of $\mathcal{per}(\frac{X}{Z})$, consequently furnishing a complete description of the lattice of torsion classes.

Theorem 1.4.2 (= 5.3.3). *For $\pi\colon X \to Z$ as in setup 1.0.2, the heart fan of $H = \mathcal{per}(\frac{X}{Z})$ is given by an intersection arrangement associated to the restriction of an affine Dynkin root system, i.e. by one of the arrangements described in* [IW]. *The heart cones are described as follows.*

(1) (= 6.3.1) *The trivial cone $\mathbf{0}$ is* not *the heart cone of any intermediate heart.*

(2) (= 5.2.3, 5.2.2) *A cone outside the imaginary-root hyperplane $\{\delta = 0\}$ is a heart cone if and only if it is full-dimensional, and in this case it is the heart cone of a unique algebraic heart in $\mathrm{tilt}(H)$. The positive and negative orthants C^+, C^- are the heart cones of H and $H[-1]$ respectively, and the heart associated to any other full-dimensional cone can be expressed as a mutation of $\mathcal{per}(\frac{X}{Z})$ by wall-crossing from $C^\pm$.*

(3) (= 5.1.10, 5.3.1) *The induced finite hyperplane arrangement on* $\{\delta = 0\}$ *is naturally identified with the movable fan of* X*. Every maximal cone* σ *in this subfan thus parametrises nef divisors on a unique birational model* W *of* X*, and such* σ *is then the heart cone of* $\mathcal{coh}\,W$ *tracked under the composite Bridgeland–Chen functor* $\mathbf{D}^0 W \to \mathbf{D}^0 X$.

(4) (= 5.3.9, 5.3.7) *More generally, every non-trivial cone* $\sigma \subset \{\delta = 0\}$ *can be assigned a unique partial contraction* $W \to Y$ *such that* $\mathcal{per}(\frac{W}{Y})$ *(tracked under flop equivalences) is the maximal heart in* $\mathrm{tilt}(H)$ *with heart cone* σ*. Every other heart with heart cone* σ *can be obtained by arbitrarily mutating the algebraic components of* $\mathcal{per}(\frac{X}{Y})$ *and tilting in skyscrapers in the geometric components.*

The hearts H *and* $H[-1]$ *are the maximal and minimal elements in* $\mathrm{tilt}(H)$ *respectively, and the partial order on remaining algebraic hearts respects (atomic) wall-crossing distance from these. An algebraic intermediate heart* K *then satisfies* $K > \mathcal{per}(\frac{W}{Y})$ *if and only if there is an atomic path* $C^+ \rightsquigarrow \mathbf{C}K$ *which can be atomically extended to an infinite sequence of wall-crossings* $C^+ \rightsquigarrow \mathbf{C}K \rightsquigarrow \mathbf{C}K_1 \rightsquigarrow \mathbf{C}K_2 \rightsquigarrow \cdots$ *with generic limit in* σ*. In this case* $\mathcal{per}(\frac{X}{Y})$ *is the infimum of the decreasing chain* $K > K_1 > K_2 > \cdots$.

In chapter 5 we establish the numerical story, enumerating for each cone in the intersection arrangement all the intermediate hearts and King-semistable objects associated to it. This involves establishing the fact that coherent sheaves on any birational model W (which are a priori only intermediate with respect to the perverse heart on W) are in fact intermediate with respect to the perverse heart on X. For this a careful analysis of the partial order is necessary, and convex geometric tools are needed not only to supply a systematic enumeration scheme but also to establish crucial limiting results as in theorem 1.4.2 (4).

Once the heart fan is filled in, the question of whether there are any hearts hidden away in the $\mathbf{0}$–cone remains. Here the affine-type behaviour of $\mathbf{D}^0 X$ ensures that the non-algebraic locus in the heart fan is codimension 1, so every heart in $\mathrm{tilt}(H)$ has tight algebraic bounds.

The following example is illustrative.

Single-curve flops. Take again the Atiyah flop over $R = \mathbb{C}[\![u, v, x, y]\!]/(uv - xy)$ as in example 1.2.1. All modification algebras are of the form $\mathrm{End}_R(M)$ for *modifying* R-modules M, and we have the equivalences Ψ_0, Ψ_1 and their composites between different modification algebras.

Any category $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}\,M)$ has a natural heart $H_M = \mathrm{fl}\mathcal{mod}(\mathrm{End}\,M)$, and tracking this across equivalences produces new t-structures in $\mathbf{D}^0 X$. Such a t-structure is intermediate with respect to the perverse heart when the chain of functors $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}\,M) \rightsquigarrow \mathbf{D}^0 X$ in (1.4) is chosen as short as possible, i.e. the tracks of H_M that lie in $\mathcal{per}(\frac{X}{Z})[-1, 0]$ are of the form

$$\mathrm{VdB}^{-1} \circ (\cdots \circ \Psi_0 \circ \Psi_1 \circ \Psi_0 \circ \cdots) H_M \quad \text{or} \quad \mathrm{VdB}^{-1} \circ (\cdots \circ \Psi_0 \circ \Psi_1 \circ \Psi_0 \circ \cdots)^{-1} H_M[-1].$$

Note the specified chain completely determines the domain of the functor, so we may drop the subscript for brevity. We also drop VdB from the notation, so for instance $\Psi_1 H$ is the image of $\mathrm{fl}\mathcal{mod}(R \oplus (u, y))$ under the functor $\mathrm{VdB}^{-1} \circ \Psi_1$. The algebraic hearts in $\mathbf{D}^0 X$ thus obtained are in natural bijection with full-dimensional cones in the $\widetilde{A}_1$ (○━●) intersection arrangement, see fig. 1.2.

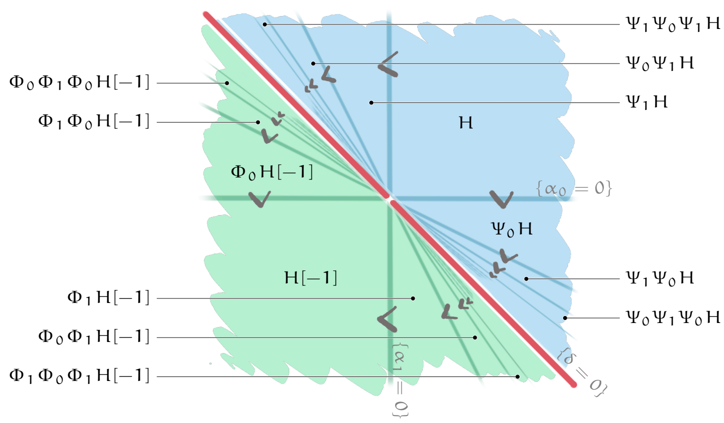


Figure 1.2. The heart fan for a minimal resolution of a cA_1 singularity.

Hyperplanes are induced by the affine A_1 root system, where the simple real roots α_0, α_1 are identified with the $\mathbf{K}$-theory classes of $\omega_C[1]$ and $\mathcal{O}_C(-1)$ respectively.

The ray $\{\delta = 0, \alpha_0 \geqslant 0\}$, geometrically the 'limit' of the path $\mathbf{C}(H) \to \mathbf{C}(\Psi_0 H) \to \mathbf{C}(\Psi_1\Psi_0 H) \to \ldots$, is the heart cone of $\mathit{coh}\, X = \inf\{H, \Psi_0 H, \Psi_1\Psi_0 H, \ldots\}$. Likewise it is also the heart cone of the *reversed geometric heart* $\overline{\mathit{coh}}\, X = \sup\{H[-1], \Phi_1 H[-1], \Phi_0\Phi_1 H[-1], \ldots\}$ which can be obtained by tilting $\mathit{coh}\, X$ in the torsion class generated by all skyscrapers.

The ray $\{\delta = 0, \alpha_0 \leqslant 0\}$ is similarly seen to be the heart cone of $\mathrm{flop}(\mathit{coh}\, W)$ and other geometric hearts that live on W, where $\mathrm{flop} : \mathbf{D}^0 W \to \mathbf{D}^0 X$ is the Bridgeland–Chen flop functor (in this case isomorphic to $\mathrm{VdB}^{-1} \circ \Psi_1 \circ \mathrm{VdB}$).

Now the partial order on algebraic hearts (indicated in fig. 1.2) is such that wall–crossings correspond to simple tilts (§ 6.1), thus $\Psi_0 H$ is obtained by tilting H in the torsion class $\langle \omega_C[1] \rangle$ while $\Phi_1 H[-1]$ and $H[-1]$ are related by a tilt in $\langle \mathcal{O}_C(-1) \rangle$. Considering how simples sit in relation to torsion pairs then allows us to 'push' non-algebraic intermediate hearts $K \in \mathrm{tilt}(H)$ towards the geometric hyperplane.

Algorithm 1.4.3 (Simple–tilting). Write $H = T * F$ for the torsion pair corresponding to $K(= F * T[-1])$.

1. Since $K \neq H$, one of the simples of H (say $S_0 = \omega_C[1]$) lies in T and this forces the inequality $K < \Psi_0 H$.
2. The inequality $\Psi_0 H > K$ shows that T intersects $\Psi_0 H$ nontrivially, hence must contain some simple $\{\Psi_0 S_0, \Psi_0 S_1\}$ of $\Psi_0 H$. Since $\Psi_0 S_0 = \omega_C$ lies outside H, we must have $\Psi_0 S_1 \in T$ and thus $\Psi_1\Psi_0 H > K$. Iterating, we have $K < \ldots\Psi_0\Psi_1\Psi_0 H$ for arbitrarily long chains, and hence $K \leqslant \mathit{coh}\, X$.
3. On the other hand $K \neq H[-1]$ so F contains some simple of H, which in this case must be $S_1 = \mathcal{O}_C(-1)$. This shows $K > \Phi_1 H[-1]$, and iterating as above gives $K \geqslant \overline{\mathit{coh}}\, X$.
4. Since $\overline{\mathit{coh}}\, X$ and $\mathit{coh}\, X$ both share the heart cone $\{\delta = 0, \alpha_0 \geqslant 0\}$, K must do so too i.e. K is numerical. In fact K can be expressed as a tilt of $\mathit{coh}\, X$ in $\langle \mathcal{O}_p \mid p \in Q \rangle$ for some $Q \subset C$.

Every heart in $\mathrm{tilt}(\mathit{per}(\frac{X}{Z}))$ is either algebraic, or can be shown to be geometric via the above recipe.

Multi-curve flops. After accounting for perverse hearts for partial contractions, the above description of the heart fan and the partial order carries over to higher rank cases. Thus for a crepant resolution $X \to Z$ of the cA_2 singularity $Z = \operatorname{Spec} \mathbb{C}[\![u, v, x, y]\!]/(uv - xy(x + y))$, the heart fan of $\mathit{per}(\frac{X}{Z})$ is described by an $\widetilde{A}_2$ ( ) root system as in fig. 1.3.

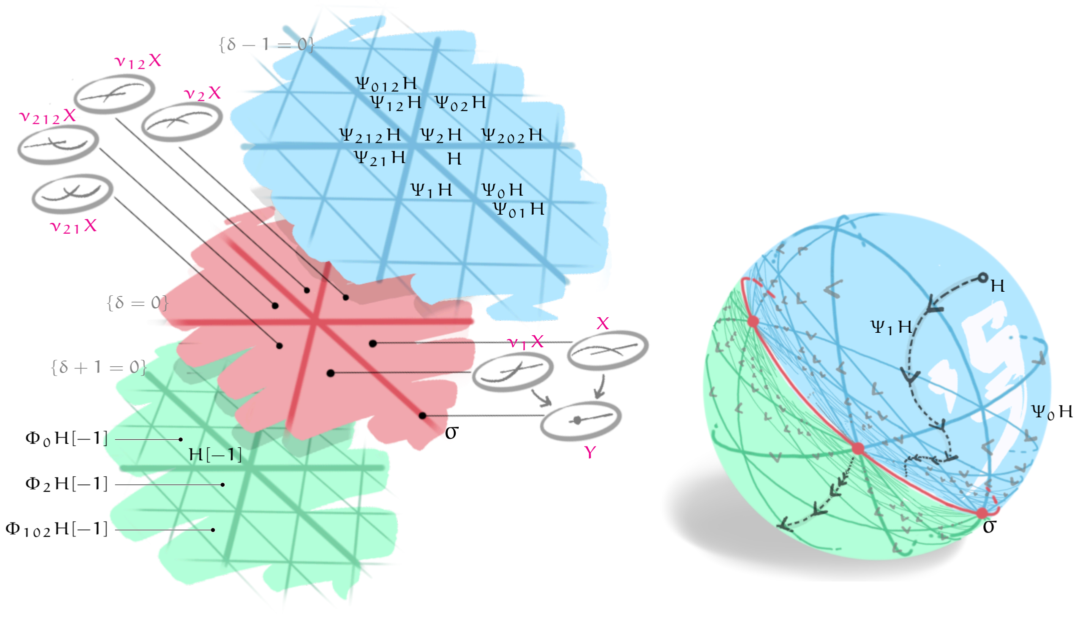


Figure 1.3. The 3-dimensional heart fan for a cA_2 resolution, sliced along affine hyperplanes (left) and the unit sphere (right). The mutation functors are abbreviated, e.g. by writing Ψ_{01} for $\Psi_1 \circ \Psi_0$. Cones in the hyperplane $\{\delta = 0\}$ have been labelled by the 3-fold whose geometric hearts they are associated to, where e.g. $\nu_{212}X$ is the flop of the second curve in $\nu_{12}X$.

Given a non-algebraic heart $\mathsf{K} \in \operatorname{tilt}(\mathcal{per}(\frac{X}{Z}))$, we can follow algorithm 1.4.3 to produce bounds

$$\dots \Psi_{i_2}\Psi_{i_1}\mathsf{H} \quad > \quad \mathsf{K} \quad > \quad \dots \Phi_{j_2}\Phi_{j_1}\mathsf{H}[-1]$$

by iterated simple tilting. However this does not suffice for 'pushing K to the geometric hyperplane', and it is possible that the corresponding paths in the heart fan do not converge to the same cone (illustrated in fig. 1.3). In this regard the single-curve case is deceptively simple.

Establishing $\mathbf{C}\mathsf{K} \neq 0$ thus demands more finesse, and our strategy is to utilise line bundles on birational models for this purpose. Under the identification of $\operatorname{Pic}_{\mathbb{R}} X$ with the hyperplane $\{\delta = 0\}$, (the proper transform of) any line bundle $\mathcal{L} \in \operatorname{Pic} W$ corresponds to a vector θ in the heart fan, so for any cone σ we can consider the submonoid $\operatorname{Pic} W \cap \sigma$ (e.g. if $\sigma = \mathbf{C}(\operatorname{flop} \mathcal{coh}\, W)$ then this is the monoid of nef bundles). On the other hand, $\operatorname{Pic} W$ has a natural action on $\mathbf{D}^0 W$ which induces an action $\operatorname{Pic} W \circlearrowright \mathbf{D}^0 X$ across the composite flop equivalence. Using the abbreviated notation $\mathcal{L} \otimes \mathsf{H} := \operatorname{flop}(\mathcal{L} \otimes_W (\operatorname{flop}^{-1} \mathsf{H}))$, we establish the following in § 6.2.

Theorem 1.4.4 (= 6.2.3, 6.2.4, 6.2.5). *Writing $\mathsf{H} = \mathcal{per}(\frac{X}{Z})$, the following statements hold.*

(1) Given any birational model W of X and a line bundle $\mathcal{L} \in \operatorname{Pic} W$, any heart in $\mathbf{D}^0 X$ of the form $\mathcal{L}^\vee \otimes \mathsf{H}$ or $\mathcal{L} \otimes \mathsf{H}[-1]$ lies in $\operatorname{tilt}(\mathsf{H})$ if and only if $\mathcal{L}$ is nef.

(2) If W' is another birational model such that $\mathcal{L} \in \operatorname{Pic} W$ and its proper transform $\mathcal{L}' \in \operatorname{Pic} W'$ are both nef, then there are equalities of t-structures $\mathcal{L}^\vee \otimes \mathsf{H} = \mathcal{L}'^\vee \otimes \mathsf{H}$, $\mathcal{L} \otimes \mathsf{H}[-1] = \mathcal{L}' \otimes \mathsf{H}[-1]$.

(3) For any cone $\sigma \subset \mathbf{C}(\operatorname{flop} \mathcal{coh}\, W)$ in $\operatorname{HFan}(\mathsf{H})$, the induced actions of monoid $\operatorname{Pic} W \cap \sigma$ on the subsets

$$\sigma\text{-}\operatorname{tilt}^+(\mathsf{H}) = \bigcup_{\mathcal{L} \in \operatorname{Pic} W \cap \sigma} [\mathcal{L}^\vee \otimes \mathsf{H}, \mathsf{H}], \qquad \sigma\text{-}\operatorname{tilt}^-(\mathsf{H}) = \bigcup_{\mathcal{L} \in \operatorname{Pic} W \cap \sigma} [\mathsf{H}[-1], \mathcal{L} \otimes \mathsf{H}[-1]] \qquad \subseteq \operatorname{tilt}(\mathsf{H})$$

respect the partial order inherited from $\operatorname{tilt}(\mathsf{H})$ and the monoid-order on $\operatorname{Pic} W \cap \sigma$.

(4) Each heart in the posets above is algebraic, and conversely every algebraic heart in $\operatorname{tilt}(\mathsf{H})$ lies in some poset of the above form for suitable W, σ.

(5) The infimum of σ-$\operatorname{tilt}^+(\mathsf{H})$ is the maximal element of $\operatorname{tilt}(\mathsf{H})$ with heart cone σ, likewise the supremum of σ-$\operatorname{tilt}^-(\mathsf{H})$ is the minimal element of $\operatorname{tilt}(\mathsf{H})$ with heart cone σ.

The above result can be seen as a 'fixed-point theorem' for $\operatorname{tilt}(\mathsf{H})$. Indeed if $\mathcal{L} \in \operatorname{Pic} X$ is an ample line bundle, then the orbit of H under iterative applications of $\mathcal{L}^\vee \otimes (-)$ limits to $\mathcal{coh}\, X$ which is fixed as a heart by $\operatorname{Pic} X$.

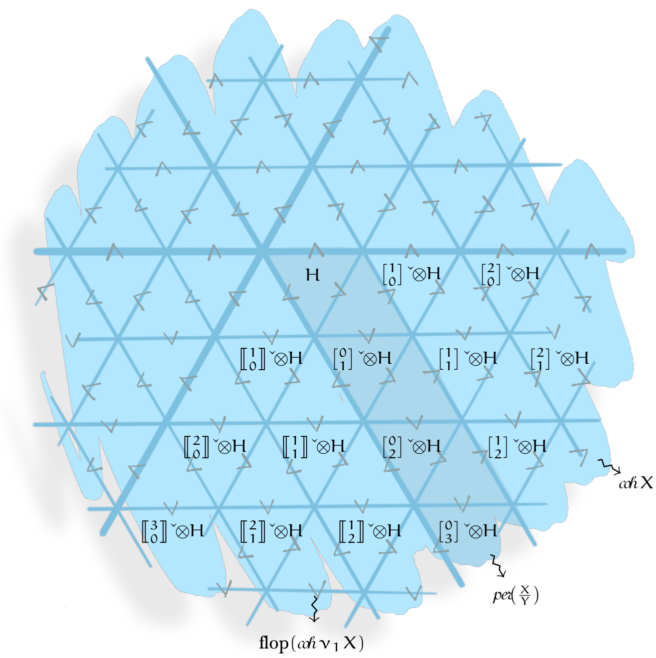


Figure 1.4. Continuing from fig. 1.3, the actions of $\operatorname{Pic} X$ and $\operatorname{Pic}(\nu_1 X)$ on H are shown. Here $\left[\begin{smallmatrix}i\\j\end{smallmatrix}\right]$ denotes the line bundle on X which has degrees i and j on the two exceptional curves respectively, and we use double brackets ($[\![\begin{smallmatrix}i\\j\end{smallmatrix}]\!]$) for bundles on $\nu_1 X$.

Note that since $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and its proper transform $[\![\begin{smallmatrix}0\\1\end{smallmatrix}]\!]$ are both trivial on the flopped curve, their actions coincide i.e. $[\![\begin{smallmatrix}0\\1\end{smallmatrix}]\!]^\vee \otimes \mathsf{H} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^\vee \otimes \mathsf{H}$.

The shaded region represents σ-$\operatorname{tilt}^+(\mathsf{H})$ for σ shown in fig. 1.3.

We then have the following recipe, which works generally to show every heart cone is non-zero.

Algorithm 1.4.5 (Skyscraper-hunting). Suppose $\mathsf{K} \subset \mathcal{per}(\frac{X}{Z})[-1,0]$ is a non-algebraic t-structure corresponding to a torsion pair $\mathcal{per}(\frac{X}{Z}) = \mathsf{T} * \mathsf{F}$.

1. By lemma 6.3.6 there is a unique birational model $W \in \mathrm{Bir}(\frac{X}{Z})$ such that $\mathrm{flop}^{-1}(\mathsf{K})$ is intermediate with respect to $\mathcal{per}(\frac{W}{Z})$ and also contains the full subcategory $\{w \in \mathcal{coh}\, W \mid \mathbf{R}\pi_* w = 0\}$ (we say K *lives* on W). Thus replacing K by $\mathrm{flop}^{-1}(\mathsf{K})$ if necessary, we may assume K lives on X.

   With this hypothesis satisfied, every skyscraper sheaf $\mathcal{O}_p \in \mathcal{per}(\frac{X}{Z}) \cap \mathcal{coh}\, X$ is either torsion or torsion-free with respect to the torsion pair $\mathsf{T} * \mathsf{F}$ (lemma 6.3.8).

2. Suppose $C_i \subset X$ is an exceptional curve and $\mathcal{L}_i \in \mathrm{Pic}\, X$ is the line bundle with degree 1 on C_i and degree 0 elsewhere. In particular, the maximal and minimal elements of $\mathrm{tilt}(\mathsf{H})$ whose heart cone is generated by $\mathcal{L}_i$ are the categories $\mathcal{per}\left(\frac{X}{X_i}\right)$ and $\overline{\mathcal{per}}\left(\frac{X}{X_i}\right)$ respectively, where $X \to X_i$ is the partial contraction of C_i.

   Now if some $p \in C_i$ satisfies $\mathcal{O}_p \in \mathsf{T}$, then lemma 6.3.2 shows $\mathcal{L}_i^{\otimes(-n)} \otimes \mathsf{H} > \mathsf{K}$ for arbitrarily large n, and thus $\mathcal{per}\left(\frac{X}{X_i}\right) \geqslant \mathsf{K}$. Likewise if some $p \in C_i$ satisfies $\mathcal{O}_p \in \mathsf{F}$, then we have the bound $\mathsf{K} \geqslant \overline{\mathcal{per}}\left(\frac{X}{X_i}\right)$.

3. Thus if there is some curve C_i with two points $p, q \in C_i$ such that $\mathcal{O}_p \in \mathsf{T}$ and $\mathcal{O}_q \in \mathsf{F}$, then K lies in the interval $\left[\mathcal{per}\left(\frac{X}{X_i}\right), \overline{\mathcal{per}}\left(\frac{X}{X_i}\right)\right]$ and in particular has non-zero heart cone.

4. Otherwise by connectivity of the exceptional curves in X, either T or F contains *all* the skyscrapers. By a similar argument, this forces one of the bounds $\mathsf{K} \geqslant \mathcal{coh}\, X$ or $\mathsf{K} \leqslant \overline{\mathcal{coh}}\, X$. The bound on the other side can then be found by chasing simple tilts from H or $\mathsf{H}[-1]$ as in algorithm 1.4.3, and this suffices to prove that $\mathbf{C}\mathsf{K}$ is non-zero (lemma 6.3.4).

Heart cones of semi-geometric hearts. Continuing to work with the cA_2 crepant resolution above, consider a flop $\rho : X \dashrightarrow \nu_1 X$, and the partial contraction $\tau : X \to Y$ of the flopped curve. The heart cone of the associated semi-geometric heart $\mathcal{per}(\frac{X}{Y})$ is given by the ray $\sigma = \mathbf{C}(\mathcal{coh}\, X) \cap \mathbf{C}(\mathrm{flop}\, \mathcal{coh}(\nu_1 X))$.

To examine other hearts that lie on σ, note that Y has a unique singular point (the image of the flopped curve), and ρ is simply the Atiyah flop over a neighbourhood Z_1 of this point. Computing the category of σ-semistable objects then recovers the category of perverse sheaves associated to the cA_1 flopping contraction $\tau^{-1} Z_1 \to Z_1$, and thus (up to tilting in skyscrapers in the smooth locus) every other intermediate heart with heart cone σ is obtained from tilts of this smaller category $\mathcal{per}\left(\frac{\tau^{-1} Z_1}{Z_1}\right)$.

Convex geometrically this manifests as the fact that the cA_2 intersection arrangement 'looks like' the cA_1 arrangement locally around σ (as is apparent from figs. 1.2 and 1.3). This phenomenon is more general, see remark 2.4.8.

Bricks. As a corollary of theorem 1.4.2 we get a succinct description of bricks in $\mathcal{per}(\frac{X}{Z})$. This extends to modification algebras a result of Crawley-Boevey [Craoo, lemma 1], who showed that the dimension vector of any brick-module over an affine preprojective algebra is a root.

Theorem 1.4.6 (= 6.4.1). *Let* $\pi \colon X \to Z$ *be as in setup 1.0.2, and let* $b \in \mathcal{per}(\frac{X}{Z})$ *be a brick. Then* b *is either*

(1) a simple module over some modification algebra ${}_{\nu}\Lambda_{\nu}$*, or*

(2) a skyscraper sheaf on some birational model W *of* X*,*

tracked under appropriate equivalences induced by mutations or flops. In particular, the $\mathbf{K}$*-theory class of* b *is a primitive restricted root of the (affine) Dynkin data associated to* X*.*

In sufficiently restricted settings (e.g. under the assumptions imposed in [Todo8]), every algebraic mutation of $\mathcal{per}(\frac{X}{Z})$ is equivalent to the category of perverse sheaves on some birational model W. Then the above result says that, up to applying mutation functors, every brick in $\mathcal{per}(\frac{X}{Z})$ is either a point sheaf $\mathcal{O}_p$, or the twisted structure sheaf $\mathcal{O}_{C_i}(-1)$ on some integral curve, or the suspended canonical sheaf $\omega_{\underline{C}}[1]$ of the scheme-theoretic exceptional fibre on some birational model W.

Woffle. Given theorem 1.4.1, it is only natural to attempt for a general flopping contraction $\pi\colon X \to Z$ the induction sketched in § 1.1. That is, given a complex $x \in \mathbf{D}^0 X$ we can write it as an extension of some objects $x_0, x_1[-1], \dots, x_n[-n]$ with $x_i \in H = \mathit{per}(\frac{X}{Z})$, and attempt to inductively improve its spread by tilting the reference heart.

If $\mathrm{Hom}^{<0}(x, x) = 0$ then the spread can be improved at least once. There is at least one torsion pair $H = T * F$ such that $x_0 \in F$ and $x_n \in T$ so that $x \in K[1, n]$ for $K = F * T[-1]$. If this K (classified by theorem 1.4.1) is algebraic then the induction continues, so the question really is whether there are enough obstructions to prevent this interval $\{K \in \mathrm{tilt}(H) \mid x \in K[1, n]\}$ from being too narrow.

Sadly the assumption $\mathrm{Hom}^{<0}(x, x) = 0$ alone isn't enough, here is a minimal criminal. Take $\pi\colon X \to Z$ to be the minimal resolution of the A_3 singularity $\mathbb{C}[\![x, y, z]\!]/(xy - z^4)$. The exceptional curve C has three integral components C_1, C_2, C_3, with $C_1 \cap C_2 = \{p\}$, $C_2 \cap C_3 = \{q\}$, and $C_1 \cap C_3 = \emptyset$.

So we can consider a complex such as $x = \mathcal{O}_{C_1}[-57] \oplus \mathcal{O}_{C_3}(-3)[1]$, satisfying the constraint on endomorphisms simply because its cohomologies $x_0 = \mathcal{O}_{C_3}(-3)[1]$ and $x_{57} = \mathcal{O}_{C_1}$ have disjoint support.

If tilting in a torsion pair $H = T * F$ improves the spread of x, then the constraint $x_0 \in F$ and the exact sequence $\mathcal{O}_q \hookrightarrow \mathcal{O}_{C_3}(-3)[1] \twoheadrightarrow \mathcal{O}_{C_3}(-2)[1]$ in H shows we must have $\mathcal{O}_q \in F$. An analogous reasoning shows $\mathcal{O}_p \in T$. Thus the curve C_2 has two points p, q such that $\mathcal{O}_p[1]$ and $\mathcal{O}_q$ lie in $K = F * T[-1]$. It follows (see the reasoning in lemma 6.3.5) that K is necessarily geometric on the curve C_2.

Thus understanding tilts of geometric and semi-geometric hearts seems imperative to continue the induction. Alternatively, more restrictions on the object being classified might prevent cohomologies such as above from arising entirely; indeed for minimal resolutions of A_n surface singularities it is known [IU05; IUU10] that spherical objects and stability conditions admit classifications analogous to those in § 1.2.

§ 1.5 Notation and Conventions

We work over the ground field $\mathbb{C}$, and fix once and for all a complete local isolated compound du Val singularity $Z = \operatorname{Spec} R$ with singular point corresponding to the maximal ideal $\mathfrak{m} \subset R$.

All algebras we consider are finitely generated R-algebras. Given such an algebra Λ, we write $\mathbf{D}^b\Lambda$ for the bounded derived category of right Λ-modules and $\mathbf{D}^{\mathrm{fl}}\Lambda$ for the full subcategory of complexes whose cohomology (with respect to $\mathit{mod}\,\Lambda$) has finite length. This forms a triangulated subcategory of $\mathbf{D}^b\Lambda$, and we write $\mathbf{K}\Lambda$ for the $\mathbf{K}$-theory (i.e. Grothendieck group) of $\mathbf{D}^{\mathrm{fl}}\Lambda$.

Likewise, all 3-folds we consider are Z-schemes. Given such a scheme $\pi : X \to Z$, we write $\mathbf{D}^b X$ for the bounded derived category of coherent sheaves on X and $\mathbf{D}^0 X$ for the full subcategory of $\mathbf{D}^b X$ containing objects supported on $\pi^{-1}[\mathfrak{m}]$. Again, we write $\mathbf{K}X$ for the Grothendieck group of $\mathbf{D}^0 X$.

When the map $\pi\colon X \to Z$ is a flopping contraction, the associated category $\mathscr{P}\!\mathit{er}(\frac{X}{Z})$ of perverse sheaves is central to our exposition. The definition depends on a choice of "perversity" $p \in \mathbb{Z}$, and we always set $p = 0$. Thus in Van den Bergh's [Van04] or Bridgeland's [Bri02] notation, the category $\mathscr{P}\!\mathit{er}(\frac{X}{Z})$ would be denoted ${}^0\mathrm{Per}(X/Z)$.

Torsion classes in an algebraic category

(Chapter 2)

Let H be an Abelian category. We say H is *algebraic* if it is Artinian and Noetherian, and has finitely many simple objects. In particular the Jordan–Hölder theorem holds and each object of H admits a finite filtration by simple objects, and the associated graded object (a direct sum of simples) is independent of the choice of filtration. Consequently the Grothendieck group $\mathbf{K}\mathsf{H}$ is a free Abelian group of finite rank with basis given by the classes of simples.

Given subcategories $\mathsf{U}, \mathsf{V} \subset \mathsf{H}$ we write $\mathsf{U} * \mathsf{V}$ for the full subcategory of objects $h \in \mathsf{H}$ which sit in some exact sequence $0 \to u \to h \to v \to 0$ with $u \in \mathsf{U}$, $v \in \mathsf{V}$. The operation $(*)$ is associative, so writing $\mathsf{U} * \mathsf{V} * \mathsf{W}$ for $\mathsf{U}, \mathsf{V}, \mathsf{W} \subset \mathsf{H}$ is unambiguous. Then we can define the *extension closure* of $\mathsf{U} \subset \mathsf{H}$ as

$$\langle \mathsf{U} \rangle = \bigcup_{n \geqslant 0} \underbrace{\mathsf{U} * \ldots * \mathsf{U}}_{n \text{ factors}}, \tag{2.1}$$

and say U is closed under extensions (or *extension-closed*) if $\langle \mathsf{U} \rangle = \mathsf{U}$. Equivalently, $\langle \mathsf{U} \rangle$ is the full subcategory of objects which admit a filtration by objects of U.

For a subcategory $\mathsf{U} \subset \mathsf{H}$, we define its left orthogonal complement ${}^{\perp}\mathsf{U}$ as the full subcategory of objects $h \in \mathsf{H}$ such that $\mathrm{Hom}(h, u) = 0$ for all $u \in \mathsf{U}$. The right orthogonal complement $\mathsf{U}^{\perp}$ is defined analogously. It can be shown that both $\mathsf{U}^{\perp}$ and ${}^{\perp}\mathsf{U}$ are always extension-closed.

Definition 2.0.1. We say a subcategory $\mathsf{T} \subset \mathsf{H}$ is a *torsion class* if $\mathsf{H} = \mathsf{T} * (\mathsf{T}^{\perp})$. Dually we say $\mathsf{F} \subset \mathsf{H}$ is a *torsion-free class* if $\mathsf{H} = ({}^{\perp}\mathsf{F}) * \mathsf{F}$. We write $\mathrm{tors}(\mathsf{H})$ for the collection of all torsion classes in H, and we write $\mathrm{torf}(\mathsf{H})$ for the collection of all torsion-free classes in H.

It is straightforward to check that if T is a torsion class then $\mathsf{T} = {}^{\perp}(\mathsf{T}^{\perp})$, and hence torsion classes are closed under factors (i.e. if T is a torsion class and we have $h \in \mathsf{T}$ then we also have $h' \in \mathsf{T}$ whenever there is a surjection $h \twoheadrightarrow h'$) and extensions. Dually, each torsion-free class F satisfies $\mathsf{F} = ({}^{\perp}\mathsf{F})^{\perp}$ and is closed under sub-objects and extensions.

It follows that the assignment $\mathsf{F} \mapsto {}^{\perp}\mathsf{F}$ is a bijection $\mathrm{torf}(\mathsf{H}) \to \mathrm{tors}(\mathsf{H})$ with the inverse map given by $\mathsf{T} \mapsto \mathsf{T}^{\perp}$. We say '$\mathsf{H} = \mathsf{T} * \mathsf{F}$ *is a torsion pair*' to mean T is a torsion class in H with corresponding torsion-free class F.

When H satisfies additional hypotheses, torsion and torsion-free classes can in fact be characterised by the above closure properties – if H is Noetherian then a subcategory $\mathsf{T} \subset \mathsf{H}$ that is closed under extensions and factors (for example, the subcategory ${}^{\perp}\mathsf{U}$ associated to any $\mathsf{U} \subset \mathsf{H}$) is a torsion class. Dually if H is Artinian then any subcategory closed under extensions and sub-objects (for example, the subcategory $\mathsf{U}^{\perp}$ associated to any $\mathsf{U} \subset \mathsf{H}$) is a torsion-free class.

§ 2.1 Recollections on t-structures

Let $\mathcal{T}$ be a triangulated category with shift functor $(-)[1]$. For subcategories $\mathsf{U}, \mathsf{V} \subset \mathcal{T}$, we define the subcategory $\mathsf{U} * \mathsf{V}$ analogously with exact triangles instead of short exact sequences. The operation $(*)$ is associative on subcategories [BBD82, lemma 1.3.10], so the extension closure is defined just as in (2.1). Likewise, the orthogonal complements ${}^{\perp}\mathsf{U}$ and $\mathsf{U}^{\perp}$ are defined as full subcategories of objects with no morphisms into (resp. from) any object of U.

Given a finite interval $[i,j] \subset \mathbb{Z}$ and a subcategory $\mathsf{U} \subset \mathfrak{T}$, we write $\mathsf{U}[i,j] = \mathsf{U}[j] * \mathsf{U}[j-1] * \cdots * \mathsf{U}[i]$. For infinite intervals $I \subset \mathbb{Z}$, we declare $\mathsf{U}[I] = \bigcup_{[i,j]\subset I} \mathsf{U}[i,j]$ and use obvious notational choices whenever convenient, so $\mathsf{U}[\leqslant 0] = \mathsf{U}[(-\infty, 0]]$.

Definition 2.1.1. A full additive subcategory $\mathsf{H} \subset \mathfrak{T}$ is the *heart of a (bounded) t-structure* if it is closed under extensions, $\mathsf{H}[\mathbb{Z}] = \mathfrak{T}$, and $\mathsf{H}[\geqslant 0] = {}^{\perp}(\mathsf{H}[< 0])$. We write t-str$(\mathfrak{T})$ for the collection of all hearts of bounded t-structures in $\mathfrak{T}$.

We remark that each of the three subcategories $\mathsf{H}[\geqslant 0]$, $\mathsf{H}[< 0]$, H (called the *aisle*, *coaisle*, and the *heart* respectively) determines the other two and hence saying "the t-structure $\mathsf{H}[\geqslant 0]$" is unambiguous.

The heart H of a t-structure is always Abelian, with $0 \to h' \to h \to h'' \to 0$ exact if and only if the corresponding triangle $h' \to h \to h'' \to h'[1]$ is distinguished in $\mathfrak{T}$. Further, the natural map of Grothendieck groups $\mathbf{K}\mathsf{H} \to \mathbf{K}\mathfrak{T}$ is an isomorphism. We say a t-structure is *algebraic* if its heart is algebraic as an Abelian category.

Further, if $\mathsf{H} \subset \mathfrak{T}$ is the heart of a t-structure then there are additive functors $\mathrm{H}^i \colon \mathfrak{T} \to \mathsf{H}$ $(i \in \mathbb{Z})$, called the *cohomology functors with respect to* H, such that any $t \in \mathfrak{T}$ is filtered by the objects $\{\mathrm{H}^i(t)[-i] \mid i \in \mathbb{Z}\}$ finitely many of which are non-zero.

In particular, for an interval $I \subset \mathbb{Z}$ we see that $\mathsf{H}[I]$ contains precisely those objects t for which $\mathrm{H}^{-i}(t) = 0$ whenever $i \notin I$. We say such objects t are *concentrated in cohomological degrees in* I.

Objects concentrated in cohomological degrees -1 and 0 with respect to H, the so-called *two-term complexes*, are of special interest since their properties depend only on H as an Abelian category. We say a t-structure containing only such objects is *intermediate with respect to* H, and write tilt$(\mathsf{H}) \subset$ t-str$(\mathfrak{T})$ for the collection of intermediate t-structures, namely

$$\mathrm{tilt}(\mathsf{H}) = \{\mathsf{K} \in \text{t-str}(\mathfrak{T}) \mid \mathsf{K} \subset \mathsf{H}[-1,0]\}.$$

The notation is explained by the fact that any $\mathsf{K} \in$ tilt(H) can be obtained from H by the process of *tilting* as described in [HRS96], who show that in this case there is a torsion pair $\mathsf{H} = \mathsf{T} * \mathsf{F}$ on H such that $\mathsf{K} = \mathsf{F} * \mathsf{T}[-1]$. This torsion pair is obtained as $\mathsf{T} = \mathsf{K}[1] \cap \mathsf{H}$, $\mathsf{F} = \mathsf{K} \cap \mathsf{H}$, and the correspondence is bijective, and we say that K is the *(negative) tilt of* H *along the given torsion pair.* Given this, it follows that tilt(H) depends only on H as an Abelian category and not on the ambient triangulated category.

We write alg-tilt(H) for the set of algebraic t-structures that are intermediate with respect to H, and alg-tors(H), alg-torf(H) for the corresponding torsion (resp. torsion-free) classes. When H is the module category of a finite dimensional algebra, the torsion theories corresponding to algebraic tilts are called *functorially finite* and can be characterised in multiple ways (see for example [Asa20].)

§ 2.2 The lattice theory of torsion classes

Again, let $\mathfrak{T}$ be a triangulated category with a t-structure $\mathsf{H} \subset \mathfrak{T}$. The set of (bounded) t-structures t-str$(\mathfrak{T})$ can be assigned a partial order given by inclusion of coaisles, i.e. we write $\mathsf{K}' \leqslant \mathsf{K}$ whenever we have $\mathsf{K}'[< 0] \subseteq \mathsf{K}[< 0]$. The subset tilt$(\mathsf{H})$ is then precisely the interval $[\mathsf{H}[-1], \mathsf{H}]$ and in particular inherits this partial order. If we equip tors(H), torf(H) with their natural (inclusion) orders then the correspondences between these sets can be upgraded to a commutative diagram of poset isomorphisms

$$\begin{tikzcd}
(\mathrm{tors}(\mathsf{H}),\subseteq)^{\mathrm{op}} \arrow[rr, shift left, "(\text{-})^{\perp}"] \arrow[dr, shift right, "(\text{-})^{\perp} * (\text{-})[-1]"'] & & (\mathrm{torf}(\mathsf{H}),\subseteq) \arrow[ll, shift left, "{}^{\perp}(\text{-})"] \arrow[dl, shift left, "(\text{-}) * {}^{\perp}(\text{-})[-1]"] \\
& (\mathrm{tilt}(\mathsf{H}),\leqslant) \arrow[ul, shift right, "(\text{-})[1]\cap \mathsf{H}"'] \arrow[ur, shift left, "(\text{-})\cap \mathsf{H}"] &
\end{tikzcd} \tag{2.2}$$

and these restrict to isomorphisms between the sub-posets alg-tors$(\mathsf{H})^{\mathrm{op}}$, alg-torf$(\mathsf{H})$, and alg-tilt$(\mathsf{H})$.

When H is algebraic, each poset appearing in (2.2) is a *complete lattice* – i.e. it admits arbitrary infima (greatest lower bounds) and suprema (least upper bounds) – indeed the intersection of torsion classes remains a torsion class and this gives the infima, while the existence of suprema follows from the poset isomorphism with $(\operatorname{torf} H)^{\mathrm{op}}$ and the analogous observation that the intersection of torsion-free classes remains torsion-free. Explicitly, given a collection of torsion classes $\{T_i \mid i \in I\} \subset \operatorname{tors}(H)$ we have

$$\inf_{i\in I} T_i = \bigcap_{i\in I} T_i, \qquad \sup_{i\in I} T_i = \left\langle \left\{ h \in H \mid \text{there is a surjection } t \twoheadrightarrow h \text{ for some } t \in \bigcup_{i\in I} T_i \right\} \right\rangle.$$

We assume H is algebraic for the rest of this section.

Recall that we say a *covers* b (written $a \gtrdot b$) in a partially ordered set if $a > b$ and there is no element c satisfying $a > c > b$. In this case we say the relations $a > b$ and $b < a$ are *covering*. The *Hasse quiver* of a poset has vertices given by the elements, and an arrow $a \to b$ for each covering relation $a > b$. The join and meet operations on lattices of torsion classes are semidistributive so their Hasse quivers are naturally labelled by certain indecomposable objects [see BCZ19]. We now describe the construction.

Definition 2.2.1. An object $b \in H$ is a *brick* if all non-zero endomorphisms of b are invertible. A *semibrick* $S \subset H$ is a set of bricks that are pairwise orthogonal, i.e. each $b \in S$ is a brick, and whenever $b_1, b_2 \in S$ are non-isomorphic bricks, we have $\operatorname{Hom}(b_1, b_2) = 0$.

By Schur's lemma, every simple object in H is a brick and the collection of all simples forms a semibrick. It is not hard to see that if $s \in H$ is simple, then $T = \langle s \rangle$ is a minimal non-zero torsion class in H (i.e. the relation $0 \subset T$ is covering in $\operatorname{tors}(H)$), and any torsion class covering 0 must be of this form. Likewise every maximal proper torsion class (i.e. a torsion class U such that the relation $U \subset H$ is covering) is precisely of the form ${}^{\perp}s$ for some simple object $s \in H$. Intermediate hearts arising from such torsion theories are called *simple tilts*.

In fact all covering relations arise in this way from simple tilts.

Theorem 2.2.2. *If* $K' \lessdot K$ *is a covering relation in* $\operatorname{tilt}(H)$*, then* $(K \cap H) \setminus K'$ *contains a unique brick* b *which we call the* brick-label *of the covering relation. This brick is a simple object of* K*, and the corresponding simple tilt in the torsion class* $\langle b \rangle \in \operatorname{tors}(K)$ *is* K'*.*

More generally if $K' < K$ *are intermediate t-structures with respect to* H *then the set of bricks in* $(K \cap H) \setminus K'$ *is non-empty and equals the set of brick-labels of covering relations in the interval* $[K', K] \subset \operatorname{tilt}(H)$*.*

Proof. If H is the module category of a finite dimensional algebra, the result is [DIRRT23, theorems 3.3–3.4]. The proof is readily adapted to the generality required here, by using the Jordan–Hölder length as a substitute for the dimension. □

Thus each arrow in the Hasse quiver of $\operatorname{tilt}(H)$ is labelled by a unique brick in H, and each brick in H arises as the brick-label of at least one such arrow.

The posets $\operatorname{tors}(H)$ and $\operatorname{torf}(H)$ inherit this labelling across the isomorphisms (2.2). In particular if the covering relation $T' \supset T$ in $\operatorname{tors}(H)$ has brick-label b, then b is the unique brick in $T' \setminus T$ and we have $T' \cap T^{\perp} = \langle b \rangle$, $T = {}^{\perp}b \cap T'$, and $T' = T * \langle b \rangle$. The analogous statement holds for covering relations of torsion-free classes.

§ 2.3 Widely generated torsion theories

Given an algebraic Abelian category H, we review how the lattice of torsion classes can be employed to study the set $\operatorname{sbrick}(H)$ of semibricks in H.

Note that if $K \subset H[-1, 0]$ is an intermediate heart, then every simple object of K is either torsion or torsion-free with respect to the torsion pair $K = (H \cap K) * (H[-1] \cap K)$. Thus writing $\operatorname{simp}(K)$ for the set of simple objects of K, we see that $\operatorname{simp}(K)[1] \cap H$ is a semibrick in H. This defines a map $\operatorname{simp}(-)[1] \cap H \colon \operatorname{tilt}(H) \to \operatorname{sbrick}(H)$, equivalently a map $\operatorname{tors}(H) \to \operatorname{sbrick}(H)$ which maps any torsion class to a semibrick inside it.

On the other hand each $S \in \mathrm{sbrick}(H)$ defines a torsion-free class $F = S^{\perp}$, with corresponding torsion class $T = {}^{\perp}(S^{\perp})$ characterised by the property of being minimal among all torsion classes containing S. We say the torsion class T in this case is *generated by* S. This defines a map $\mathrm{sbrick}(H) \to \mathrm{tors}(H)$, equivalently a map $\mathrm{sbrick}(H) \to \mathrm{tilt}(H)$ which we show is a section of $\mathrm{simp}(-)[1] \cap H$.

Lemma 2.3.1. *Suppose* $H = T * F$ *is a torsion pair such that the torsion class* T *is generated by a semibrick* $S \subseteq H$. *Writing* $K = F * T[-1]$ *for the tilt, an object* $k \in K \cap H[-1]$ *is simple in* K *if and only if it lies in* $S[-1]$.

Proof. Given $k \in S[-1]$, suppose there is an injection $k' \hookrightarrow k$ in K, with $k' \neq 0$. Since $T[-1] \subset K$ is a torsion-free class, k' also lies in $T[-1]$. Thus the object $k'[1] \in T$ is filtered by objects in S and their factors in H [see for example MS17, lemma 3.1].

In particular, there is some $s \in S$ with a non-zero morphism $s \to k'[1]$. By injectivity of $k' \hookrightarrow k$, the composite map $s[-1] \to k' \to k$ is non-zero. Since S is a semibrick, it follows that $s[-1] \cong k$ and this is a splitting of the injection $k' \hookrightarrow k$. But k is indecomposable (since $k[1]$ is a brick), so the map $k' \to k$ is an isomorphism. Thus k has no non-trivial sub-objects in K, i.e. k is simple as required.

Conversely suppose $k \in K \cap H[-1]$ is simple in K. It follows that $k[1]$ is the quotient (in H) of some $s \in S$. In other words, there is an $h \in H$ and an exact triangle $h \to s \to k[1] \to h[1]$. Claim $h = 0$, so that $k[1] = s$ lies in S as required.

To prove the claim, first note that the triangle $s[-1] \to k \to h \to s$ shows that h cannot be a non-zero object in K. Thus if h is non-zero, then considering the torsion part of h shows that there is a non-zero composite map $s' \to h \to s$ for some $s' \in S$. But S is a semibrick, so a similar argument as above shows that the map $h \to s$ is an isomorphism and $k = 0$, which is a contradiction. □

It follows that assigning a semibrick $S \subseteq H$ to the minimal torsion class it generates gives an injective map $\mathrm{sbrick}(H) \to \mathrm{tors}(H)$. As an application, we have the following lemma about torsion classes generated by a single brick.

Lemma 2.3.2. *Let* $b \in H$ *be a brick, and* $T = {}^{\perp}(b^{\perp})$ *the torsion class generated by* b. *For any brick* $b' \in T$, *we either have* $b' = b$ *or* $\mathrm{Hom}(b', b) = 0$.

Proof. When T is a torsion class generated by a semibrick S, Asai [Asa20, lemma 2.7] observed that a morphism from any $t \in T$ to any $s \in S$ is either zero or surjective. In the given situation, this means if $\mathrm{Hom}(b', b) \neq 0$ then there is a surjection $b' \twoheadrightarrow b$. In particular, b lies in the minimal torsion class T' containing b', and we must have the equality $T = T'$. The injectivity of the assignment $\mathrm{sbrick}(H) \to \mathrm{tors}(H)$ then gives us $b = b'$ as required. □

The following proposition provides lattice-theoretic and homological characterisations of the image of the map $\mathrm{sbrick}(H) \to \mathrm{tors}(H)$.

Proposition 2.3.3. *Given a torsion pair* $H = T * F$, *the following are equivalent.*

(1) The torsion class T *is generated by a semibrick* $S \subseteq H$.

(2) The interval $[0, T] \subset \mathrm{tors}(H)$ *is* coatomic, *i.e. for every* $U \in [0, T)$ *there is a* $U' \in [0, T)$ *with* $U \subseteq U' \lessdot T$.

(3) In the tilted heart $K = F * T[-1]$, *every non-zero object has a simple sub-object.*

If the above statements hold, then the semibrick S *which generates* T *is unique and is determined as*

$$\begin{aligned} S &= \{b \in H \mid b \textit{ is the brick-label of a relation } U \lessdot T \textit{ in } \mathrm{tors}(H)\} \\ &= \{b \in H \mid b[-1] \textit{ is a simple object of } K\}. \end{aligned}$$

We have seen (lemma 2.3.1) that if (1) holds, then the semibrick S which generates T is unique and is given by $\mathrm{simp}(K)[1] \cap H$. The equivalence (1) $\iff$ (2) is the content of [AP22, theorem 7.2], which also shows that in this case T is generated by the set $\{b \in H \mid b \text{ is the brick-label of a relation } U \lessdot T \text{ in } \mathrm{tors}(H)\}$. Since this set is a semibrick (see theorem 4.2 *ibid.*), it must coincide with S.

It thus remains to show the equivalence (1) $\iff$ (3), which we now do.

Proof of proposition 2.3.3 (1)⇒(3). Suppose T is generated by a semibrick S, and $k \in \mathsf{K}$ is a non-zero object with no simple sub-object. In particular any sub-object $k' \hookrightarrow k$ shares this property (i.e. k' has no simple sub-object), and there is at least one such proper non-zero sub-object k'.

Now no object of $\mathsf{S}[-1] \subset \operatorname{simp}(\mathsf{K})$ can map to k, so k lies in $\mathsf{T}[-1]^{\perp}$. Thus neither k nor k' lie in $\mathsf{T}[-1]$, so passing to sub-objects if necessary, we may in fact assume k' and k lie in F. But $\mathsf{F} \subset \mathsf{K}$ is a torsion class, so the cokernel k/k' of this morphism also lies in F. It follows that the map $k' \to k$ is also a proper injection in H.

But repeating the argument, this produces a chain of proper injections $\cdots \hookrightarrow k'' \hookrightarrow k' \hookrightarrow k$ in H, which is a contradiction since H is Artinian. Thus every non-zero $k \in \mathsf{K}$ necessarily has a simple sub-object. □

Proof of proposition 2.3.3 (3)⇒(1). Suppose (3) holds. We show that any torsion-free class that is larger than F must contain some element of the semibrick $\mathsf{S} = \operatorname{simp}(\mathsf{K})[1] \cap \mathsf{H}$. It follows that T is the minimal torsion class containing S, i.e. T is generated by S as required.

Thus consider any torsion pair $\mathsf{H} = \mathsf{T}' * \mathsf{F}'$ such that $\mathsf{F} \subsetneq \mathsf{F}'$. Thus $\mathsf{T} \cap \mathsf{F}'$ is a non-zero torsion-free class in the Abelian category $\mathsf{K}[1] = \mathsf{F}[1] * (\mathsf{T} \cap \mathsf{T}') * (\mathsf{T} \cap \mathsf{F}')$, in particular $\mathsf{T} \cap \mathsf{F}'$ is closed under taking sub-objects in this category. But by hypothesis on K (equivalently $\mathsf{K}[1]$), any non-zero object in $\mathsf{T} \cap \mathsf{F}'$ has a simple sub-object which therefore also lies in $\mathsf{T} \cap \mathsf{F}'$. Thus F' has non-trivial intersection with the set $\operatorname{simp}(\mathsf{K}[1]) \cap \mathsf{H} = \mathsf{S}$, as required. □

The obvious dual statements to lemma 2.3.1 and proposition 2.3.3 hold. In particular if $\mathsf{H} = \mathsf{T} * \mathsf{F}$ is a torsion pair, then F is generated by a semibrick S if and only if every non-zero object in the tilt $\mathsf{K} = \mathsf{F} * \mathsf{T}[-1]$ has a simple factor, and in this case the semibrick generating F is determined as $\mathsf{S} = \operatorname{simp}(\mathsf{K}) \cap \mathsf{H}$.

Remark 2.3.4. If $W \subseteq \mathsf{H}$ is a ***wide*** subcategory (i.e. W is closed under extensions, kernels, and cokernels, and is therefore Abelian), then the set of simples of W is evidently a semibrick of H. Ringel [Rin76, §1.2] shows that every wide subcategory of H is in fact the extension-closure of its simples, and conversely the extension closure of any semibrick is a wide subcategory. Thus torsion(-free) classes in H that are generated by a semibrick are called ***widely generated*** [see Asa20; AP22; BCZ19; MS17].

By proposition 2.3.3 and its dual, any torsion (torsion-free) class in H associated to an Artinian (resp. Noetherian) tilted heart is widely generated. However, chain conditions on an Abelian category are in general strictly stronger than the requirement that every non-zero object admit a simple sub-object or factor. Indeed in the setting of a 3-fold flopping contraction $\pi\colon X \to Z$, we show that the algebraic heart $\mathsf{H} = \mathit{per}(\frac{X}{Z})$ has tilts which exhibit each of the following behaviours (see § 4.3 and remark 5.3.5 for relevant constructions).

Tilted heart	Chain conditions Artin.?	Noeth.?	Widely generated tors.?	torf.?
Any algebraic, e.g. H	✓	✓	✓	✓
Any geometric, e.g. $\mathit{coh}\, X$	✗	✓	✗	✓
Any reversed-geometric, e.g. $\overline{\mathit{coh}}\, X$	✓	✗	✓	✗

If we further assume that the exceptional fiber $\pi^{-1}[\mathfrak{m}]$ has $n \geqslant 3$ integral components $C_1, C_2, \ldots, C_n$ (indexed such that $C_1 \cap C_n = \emptyset$), then choosing closed points $p_i \in C_i$ ($i = 1, \ldots, n$) allows us to construct hearts in $\mathsf{H}[-1, 0]$ which exhibit the following additional behaviours.

Tilt of $\mathit{coh}\, X$ in...				
$\langle \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_n} \rangle$	✗	✗	✓	✓
$\langle \mathcal{O}_{p_1} \rangle$	✗	✗	✗	✓
$\langle \mathcal{O}_p \mid p \in C_1 \cup \{p_2, \ldots, p_n\} \rangle$	✗	✗	✓	✗
$\langle \mathcal{O}_p \mid p \in C_1 \rangle$	✗	✗	✗	✗

Every other tilt of H fits into one of the seven categories above. In fact ***any*** tilt of ***any*** algebraic Abelian category fits into one of the seven categories above; in particular if a torsion pair $\mathsf{H} = \mathsf{T} * \mathsf{F}$ is such that both T and F are widely generated then it is straightforward to show that the heart $\mathsf{K} = \mathsf{F} * \mathsf{T}[-1]$ is Artinian if and only if it is Noetherian.

§ 2.4 Heart fans and numerical hearts

Given a triangulated category $\mathcal{T}$ with a t-structure $H \subset \mathcal{T}$, the poset of intermediate hearts with respect to H can be encoded into convex-geometric data, possibly incurring the loss of some information.

To explain the construction, first fix a surjection $\mathbf{K}\mathcal{T} \twoheadrightarrow \mathfrak{h}$ of the Grothendieck group onto a free Abelian group $\mathfrak{h}$ of finite rank. Taking $\mathbb{R}$-linear duals, note that the finite dimensional Euclidean space $\Theta = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}, \mathbb{R})$ thus injects into $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{K}\mathcal{T}, \mathbb{R})$ and hence any vector $\theta \in \Theta$ can be regarded as an $\mathbb{R}$-linear functional on $\mathbf{K}\mathcal{T}$.

We say a subset of Θ is *almost rational* if it can be defined by inequalities with coefficients in $\mathfrak{h}$, and we say it is *rational* if it can be defined by finitely many such inequalities. For example, given any $\alpha \in \mathfrak{h}$ the hyperplane $\{\theta \in \Theta \mid \theta(\alpha) = 0\}$ and the closed half-space $\{\theta \in \Theta \mid \theta(\alpha) \geqslant 0\}$ determined by α are rational. We use obvious notational shorthands when convenient, for example $\{\alpha, \beta \geqslant 0\}$ denotes the rational subset $\{\theta \in \Theta \mid \theta(\alpha) \geqslant 0 \text{ and } \theta(\beta) \geqslant 0\}$.

A *cone* in Θ is a closed, convex, almost rational subset $\sigma \subset \Theta$ such that whenever we have vectors $\theta, \theta' \in \sigma$ and non-negative reals $a, a' \geqslant 0$, we also have $a\theta + a'\theta' \in \sigma$. If a cone σ lies in the half-space $\{\alpha \geqslant 0\}$, we say the subset $\sigma \cap \{\alpha = 0\}$ is a *face* of σ. We write $\mathrm{faces}(\sigma)$ for the collection of all faces of σ.

Definition 2.4.1. We say a collection of cones Σ is a *fan* in Θ if it is closed under taking faces (i.e. if $\sigma \in \Sigma$ then $\mathrm{faces}(\sigma) \subset \Sigma$) and any two cones in Σ intersect only in faces (i.e. if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma' \in \mathrm{faces}(\sigma)$). We say the fan Σ is is *complete* if every vector $\theta \in \Theta$ is in some cone $\sigma \in \Sigma$, and Σ is *simplicial* if each cone $\sigma \in \Sigma$ is simplicial (i.e. generated by linearly independent vectors).

For any subcategory $U \subset \mathcal{T}$, the *dual cone of* U (with respect to $\mathfrak{h}$) is defined to be

$$\mathbf{C}U = \{\theta \in \Theta \mid \theta[u] \geqslant 0 \text{ for all } u \in U\}.$$

The following lemma is immediate from the constructions, and is useful when relating heart cones of intermediate hearts to torsion theories.

Lemma 2.4.2. *If K is the tilt of H in a torsion pair $H = T * F$, then $\mathbf{C}K = \mathbf{C}F \cap (-\mathbf{C}T)$ as subsets of Θ.*

We also make the useful observation that dual cones transform via an ‘inverse–transpose’ rule under functorial equivalences.

Lemma 2.4.3. *Suppose $\Phi \colon \mathcal{T}_1 \to \mathcal{T}_2$ is an exact equivalence of triangulated categories and the finite-rank free Abelian group quotients $\mathbf{K}\mathcal{T}_i \twoheadrightarrow \mathfrak{h}_i$ ($i = 1, 2$) are such that Φ induces a linear isomorphism $\varphi \colon \mathfrak{h}_1 \to \mathfrak{h}_2$. Then the corresponding isomorphism $(\varphi^\vee)^{-1} \colon \mathrm{Hom}(\mathfrak{h}_1, \mathbb{R}) \to \mathrm{Hom}(\mathfrak{h}_2, \mathbb{R})$ governs the transformation of dual cones, i.e. for any $U \subset \mathcal{T}_1$ we have $\mathbf{C}(\Phi U) = (\varphi^\vee)^{-1}\mathbf{C}U$.*

Now if $K \subset \mathcal{T}$ is the heart of a t-structure, the dual cone $\mathbf{C}K$ is also called the *heart cone.* We say K is a *numerical* heart if $\mathbf{C}K$ contains a non-zero vector.

Broomhead, Pauksztello, Ploog, and Woolf [BPPW24] show that the heart cones coming from $\mathrm{tilt}(H)$ form a fan in Θ, which generalises various well-known constructions of fans coming from silting theory and the theory of stability conditions.

Theorem 2.4.4 [BPPW24, theorem A]. *For H, $\mathcal{T}$, $\mathfrak{h}$, and Θ as above, the collection of cones given by*

$$\mathrm{HFan}(H) = \bigcup_{K \in \mathrm{tilt}(H)} \mathrm{faces}\,(\mathbf{C}K)$$

is a fan in Θ, called the heart fan *of H (with respect to $\mathfrak{h}$). If H is algebraic, then this fan is complete and simplicial. If in addition the map $\mathbf{K}\mathcal{T} \twoheadrightarrow \mathfrak{h}$ is an isomorphism, then an intermediate heart K is algebraic if and only if $\mathbf{C}K$ is a full-dimensional cone, and in this case K is the unique intermediate heart with heart cone $\mathbf{C}K$.*

For the rest of this subsection, assume H is algebraic and that the map $\mathbf{K}\mathcal{T} \twoheadrightarrow \mathfrak{h}$ is an isomorphism. Under these assumptions, we have plenty of control over the natural map

$$\mathbf{C}\colon \operatorname{tilt}(\mathsf{H}) \to \mathrm{HFan}(\mathsf{H}). \tag{2.3}$$

For instance, by the last part of theorem 2.4.4, the fiber of this map over a full dimensional cone contains a single element. However, in general, one can have multiple torsion classes giving the same heart cone. It is possible to describe the complete fibre of $\mathbf{C}$ over any non-zero cone (and hence all numerical hearts) by considering the *numerical torsion theories* defined in [BKT14]. To define these, given $\theta \in \Theta$ we consider the subcategories

$$\begin{aligned}
\mathsf{H}^{\mathrm{tr}}(\theta) &= \{h \in \mathsf{H} \mid \theta[f] \leqslant 0 \text{ for all factors } h \twoheadrightarrow f\},\\
\mathsf{H}_{\mathrm{tr}}(\theta) &= \{h \in \mathsf{H} \mid \theta[f] < 0 \text{ for all non-zero factors } h \twoheadrightarrow f \neq 0\},\\
\mathsf{H}^{\mathrm{tf}}(\theta) &= \{h \in \mathsf{H} \mid \theta[s] \geqslant 0 \text{ for all sub-objects } s \hookrightarrow h\},\\
\mathsf{H}_{\mathrm{tf}}(\theta) &= \{h \in \mathsf{H} \mid \theta[s] > 0 \text{ for all non-zero sub-objects } 0 \neq s \hookrightarrow h\}.
\end{aligned}$$

These define two torsion pairs $\mathsf{H} = \mathsf{H}^{\mathrm{tr}}(\theta) * \mathsf{H}_{\mathrm{tf}}(\theta) = \mathsf{H}_{\mathrm{tr}}(\theta) * \mathsf{H}^{\mathrm{tf}}(\theta)$, and we call the corresponding tilts $\mathsf{H}_{\mathrm{tt}}(\theta), \mathsf{H}^{\mathrm{tt}}(\theta)$ respectively. Evidently $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$, and we have an interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta] \subset \operatorname{tilt}(\mathsf{H})$. This interval contains precisely the intermediate hearts K which satisfy $\theta \in \mathbf{C}\mathsf{K}$.

Lemma 2.4.5. *Let* $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ *be an intermediate heart, and fix a vector* $\theta \in \Theta$. *Then we have* $\theta \in \mathbf{C}\mathsf{K}$ *if and only* $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{K} \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$ *in* $\operatorname{tilt}(\mathsf{H})$.

Proof. Write $\mathsf{H} = \mathsf{T} * \mathsf{F}$ for the torsion pair associated to K, and note that if $\theta \in \mathbf{C}\mathsf{K}$ then the definitions imply $\mathsf{T} \subseteq \mathsf{H}^{\mathrm{tr}}(\theta)$ and $\mathsf{F} \subseteq \mathsf{H}^{\mathrm{tf}}(\theta)$, which under the bijections (2.2) gives us $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{K} \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$. Conversely if K lies in the said interval, then we have $\mathsf{T} \subseteq \mathsf{H}^{\mathrm{tr}}(\theta)$ and $\mathsf{F} \subseteq \mathsf{H}^{\mathrm{tf}}(\theta)$, and hence $\mathbf{C}(\mathsf{H}^{\mathrm{tr}}\theta) \subseteq \mathbf{C}\mathsf{T}$ and $\mathbf{C}(\mathsf{H}^{\mathrm{tf}}\theta) \subseteq \mathbf{C}\mathsf{F}$. This gives us the desired conclusion

$$\theta \quad \in \quad \mathbf{C}(\mathsf{H}^{\mathrm{tf}}\theta) \cap \big(-\mathbf{C}(\mathsf{H}^{\mathrm{tr}}\theta)\big) \quad \subseteq \quad \mathbf{C}\mathsf{F} \cap (-\mathbf{C}\mathsf{T}) = \mathbf{C}\mathsf{K}. \qquad \square$$

Given the above lemma we can now describe the interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tr}}\theta]$ in $\operatorname{tilt}(\mathsf{H})$ using semistable objects following [AP22]. Recall that given an Abelian category K and a parameter $\theta \in \operatorname{Hom}(\mathbf{K}\mathsf{K}, \mathbb{R})$, King [King94] defines the full subcategory of θ-*semistable objects in* K as

$$\mathsf{K}_{\mathrm{ss}}(\theta) = \{k \in \mathsf{K} \mid \theta[k] = 0, \quad \theta[s] \geqslant 0 \text{ for all sub-objects } s \hookrightarrow k\}.$$

This is a wide (in particular, Abelian) subcategory of K. We say a θ-semistable object $k \in \mathsf{K}$ is *stable* if it is simple in $\mathsf{K}_{\mathrm{ss}}(\theta)$, equivalently if the inequality $\theta[s] > 0$ is strict for every sub-object $s \hookrightarrow k$ in K. If K is algebraic, then every θ-semistable object admits a filtration by θ-stable ones and thus $\mathsf{K}_{\mathrm{ss}}(\theta)$ is the extension closure of the set of θ-stable objects.

In the context of our algebraic Abelian category H with $\theta \in \Theta$, we clearly have $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{H}^{\mathrm{tr}}(\theta) \cap \mathsf{H}^{\mathrm{tf}}(\theta)$, and hence there is a decomposition

$$\mathsf{H} = \underbrace{\mathsf{H}_{\mathrm{tr}}(\theta) * \overbrace{\mathsf{H}_{\mathrm{ss}}(\theta)}^{}}_{\mathsf{H}^{\mathrm{tr}}(\theta)}\!\!\overbrace{\phantom{\mathsf{H}_{\mathrm{ss}}(\theta)}* \mathsf{H}_{\mathrm{tf}}(\theta)}^{\mathsf{H}^{\mathrm{tf}}(\theta)}.$$

In particular if $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{U} * \mathsf{V}$ is a torsion pair, then there is an induced torsion pair $\mathsf{H} = (\mathsf{H}_{\mathrm{tr}}\theta * \mathsf{U}) * (\mathsf{V} * \mathsf{H}_{\mathrm{tf}}\theta)$ and the corresponding tilt $\mathsf{K} = \mathsf{V} * \mathsf{H}_{\mathrm{tf}}\theta * \mathsf{H}_{\mathrm{tr}}\theta[-1] * \mathsf{U}[-1]$ evidently lies in the interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta]$. Further considering θ as a stability parameter on K (using the isomorphism $\mathbf{K}\mathsf{K} \cong \mathbf{K}\mathsf{H}$), lemma 2.4.5 shows θ is non-negative on every class of K so one computes

$$\mathsf{K}_{\mathrm{ss}}(\theta) = \{k \in \mathsf{K} \mid \theta[k] = 0\} \quad = \mathsf{V} * \mathsf{U}[-1]$$

and this clearly lies in $\operatorname{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$. The following theorem shows that considering semistable objects thus gives a poset isomorphism $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta] \to \operatorname{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$.

Lemma 2.4.6. *The maps below are well defined and give mutually inverse poset isomorphisms*

$$\operatorname{tilt}(\mathsf{H}_{ss}\theta) \underset{\langle\, (\,\text{-}\,), \mathsf{H}_{tf}\theta, \mathsf{H}_{tr}\theta[-1]\,\rangle}{\overset{(\,\text{-}\,)_{ss}\theta}{\rightleftarrows}} \left[\mathsf{H}_{tt}\theta, \mathsf{H}^{tt}\theta\right] \quad \subseteq \operatorname{tilt}(\mathsf{H})$$

Further the covering relations in $\operatorname{tilt}(\mathsf{H}_{ss}\theta)$ *are precisely those that remain covering in* $\operatorname{tilt}(\mathsf{H})$ *under the above correspondence, and the brick-labels coming from the two posets coincide.*

Proof. This is essentially [AP22, theorem 1.4], which is stated in terms of torsion classes and gives the poset isomorphisms

$$\operatorname{tors}(\mathsf{H}_{ss}\theta) \underset{\mathsf{H}_{tr}(\theta)*(\,\text{-}\,)}{\overset{(\,\text{-}\,)\cap \mathsf{H}_{ss}(\theta)}{\rightleftarrows}} \left[\mathsf{H}_{tt}\theta, \mathsf{H}^{tt}\theta\right] \quad \subseteq \operatorname{tors}(\mathsf{H}). \qquad \square$$

A consequence of lemmas 2.4.5 and 2.4.6 is that in addition to being a tilt of H, every intermediate heart K with $\theta \in \mathbf{C}(\mathsf{K})$ is also a tilt of $\mathsf{H}^{tt}(\theta)$. Indeed, we have $\mathsf{K} = (\mathsf{V} * \mathsf{H}_{tf}\theta) * (\mathsf{H}_{tr}\theta * \mathsf{U})[-1]$ for some torsion pair $\mathsf{H}_{ss}(\theta) = \mathsf{U} * \mathsf{V}$, and we can verify that U is a torsion class in $\mathsf{H}^{tt}(\theta)$.

Further, lemma 2.4.6 gives us the following characterisations of $\mathsf{H}^{tt}(\theta)$.

Corollary 2.4.7. *Given an intermediate heart* $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ *and a vector* $\theta \in \mathbf{C}\mathsf{K}$*, the following statements are equivalent.*

(1) The heart K *is maximal among tilts of* H *with* θ *in their heart cone, i.e.* $\mathsf{K} = \mathsf{H}^{tt}(\theta)$.

(2) The categories of θ*-semistable objects in* H *and* K *coincide, i.e.* $\mathsf{H}_{ss}(\theta) = \mathsf{K}_{ss}(\theta)$.

(3) The category $\mathsf{H}_{ss}(\theta)$ *lies in* K.

(4) The category $\mathsf{K}_{ss}(\theta)$ *lies in* H.

We say a vector θ is *generic* in a cone σ if it lies in σ but not in any proper face of σ, equivalently if it is contained in the interior of σ in the Euclidean space $\operatorname{span}(\sigma) \subset \Theta$. If θ, θ' are both generic in some cone $\sigma \in \operatorname{HFan}(\mathsf{H})$, it is easy to see that all their associated numerical categories coincide i.e. $\mathsf{H}_{tr}(\theta) = \mathsf{H}_{tr}(\theta')$, $\mathsf{H}_{tf}(\theta) = \mathsf{H}_{tf}(\theta')$, and so forth. Thus we use the notations $\mathsf{H}_{tr}(\sigma)$, $\mathsf{H}^{tr}(\sigma)$, $\mathsf{H}_{tf}(\sigma)$, $\mathsf{H}^{tf}(\sigma)$, $\mathsf{H}_{tt}(\sigma)$, $\mathsf{H}^{tt}(\sigma)$, and $\mathsf{H}_{ss}(\sigma)$ to denote the respective categories associated to any generic vector $\theta \in \sigma$.

Remark 2.4.8 (Zooming into the heart fan). Given a stability parameter $\theta \in \Theta$, the correspondence 2.4.6 can be used to essentially 'read off' the heart fan of $\mathsf{H}_{ss}(\theta)$ from that of H. To see this, note that the exact inclusion $\mathsf{H}_{ss}(\theta) \hookrightarrow \mathsf{H}$ induces a map of Grothendieck groups which can be factored through its image $\mathfrak{h}_{ss}$ as

$$\mathbf{K}\mathsf{H}_{ss} \twoheadrightarrow \mathfrak{h}_{ss} \hookrightarrow \mathfrak{h} \xrightarrow{\sim} \mathbf{K}\mathsf{H},$$

where $\mathfrak{h}_{ss}$ is again a free Abelian group of finite rank. The heart fan of $\mathsf{H}_{ss}(\theta)$ with respect to $\mathfrak{h}_{ss}$ thus lives in the vector space $\Theta_{ss} = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{h}_{ss}, \mathbb{R})$ which is naturally a quotient of Θ. It can be shown that the surjection $\Theta \twoheadrightarrow \Theta_{ss}$ is such that the image of (the sub-fan generated by)

$$\{\sigma \in \operatorname{HFan}(\mathsf{H}) \mid \theta \text{ lies in some face of } \sigma\}$$

is a sub-fan of $\operatorname{HFan}(\mathsf{H}_{ss}\theta)$. More precisely, given a heart $\mathsf{K} \in [\mathsf{H}_{tt}\theta, \mathsf{H}^{tt}\theta]$ and the corresponding category $\mathsf{K}_{ss}(\theta) \in \operatorname{tilt}(\mathsf{H}_{ss}\theta)$, it can be shown that the image of $\mathbf{C}\mathsf{K}$ is always a face of $\mathbf{C}(\mathsf{K}_{ss}\theta)$ and is in fact equal to $\mathbf{C}(\mathsf{K}_{ss}\theta)$ if K is algebraic.

To see an example where the image of $\mathbf{C}\mathsf{K}$ is a proper face of $\mathbf{C}(\mathsf{K}_{ss})$, one should examine the category $\mathsf{H} = \operatorname{Rep}(2 \rightrightarrows 1)$ associated to the 2-Kronecker quiver [BPPW24, example 4.6], and the stability parameter $\theta = (-1, 1)$ given in the basis dual to vertex simples.

§ 2.5 Arcs in the stability manifold

Heart fans of algebraic t-structures $\mathsf{H} \subset \mathcal{T}$ are combinatorial models of the so-called *walls-and-chambers for categories* encountered in appropriate regions of the stability manifold $\operatorname{Stab}(\mathcal{T})$. We briefly spell out this tautology, and record a recipe for reconstructing the heart fan by 'probing' $\operatorname{Stab}(\mathcal{T})$ with arcs.

Recall [see e.g. Bri07] that there is a complex manifold $\mathrm{Stab}(\mathfrak{T})$ parametrising *Bridgeland stability conditions* $(Z, \mathscr{P})$ on $\mathfrak{T}$ – here the *central charge* $Z\colon \mathbf{K}\mathfrak{T} \to \mathbb{C}$ is a $\mathbb{Z}$-linear map that factors through $\mathfrak{h}$, and the *slicing* $\mathscr{P}\colon \mathbb{R} \to 2^{\mathfrak{T}}$ is a sequence of full subcategories $\mathscr{P}(t) \subset \mathfrak{T}$ subject to constraints that we shall recall as necessary. The additive group $(\mathbb{R}, +)$ acts naturally on $\mathrm{Stab}(\mathcal{T})$ via a 'rotation'

$$u * (Z, \quad t \mapsto \mathscr{P}(t)) = (e^{-i\pi u} \cdot Z, \quad t \mapsto \mathscr{P}(t+u))$$

in a fashion that the biholomorphism $1{*}(-)\colon \mathrm{Stab}(\mathcal{T}) \to \mathrm{Stab}(\mathcal{T})$ is induced by the functor $(-)[1]\colon \mathcal{T} \to \mathcal{T}$.

Each subcategory $\mathscr{P}I = \langle \bigcup_{t \in I} \mathscr{P}(t) \rangle$ determined by a unit interval $I \subset \mathbb{R}$ is the heart of a t-structure. Choosing $(0, 1] \subset \mathbb{R}$ as the reference frame, each $K \in \text{t-str}(\mathfrak{T})$ thus determines a subset

$$\mathrm{Stab}(K) = \{(Z, \mathscr{P}) \in \mathrm{Stab}(\mathcal{T}) \mid \mathscr{P}(0, 1] = K\}$$

called the *chamber* associated to K. Identifying central charges (i.e. functions $\mathfrak{h} \simeq \mathbf{K}\mathfrak{T} \to \mathbb{C}$) with elements of the vector space $\Theta \otimes \mathbb{C} = \Theta \oplus i\Theta$, the forgetful map $\mathrm{Stab}(\mathfrak{T}) \to \Theta \otimes \mathbb{C}$ restricts to an injection on any chamber $\mathrm{Stab}(K)$. That is to say, stability conditions in any given chamber are determined by their central charge. Furthermore, the compatibility constraint

$$(Z, \mathscr{P}) \in \mathrm{Stab}(\mathcal{T}), \quad t \in \mathbb{R}, \quad x \in \mathscr{P}(t) \quad \Longrightarrow \quad e^{-i\pi t} \cdot Z[x] \in \mathbb{R}_{>0}$$

shows the image of $\mathrm{Stab}(K)$ must lie in the subset $\Theta \oplus i \cdot \mathbf{C}K$.

Given this, we show how the heart fan of an algebraic heart $H \subset \mathfrak{T}$ records the chambers encountered in the $(\mathbb{R}, +)$-orbit of $\mathrm{Stab}(H)$. By [Bri07, propositions 2.4 and 5.3], a sufficient condition for $Z \in \Theta \oplus i \cdot \mathbf{C}H$ to determine a stability condition in $\mathrm{Stab}(H)$ is for the imaginary component $\operatorname{Im} Z$ to lie in the interior of $\mathbf{C}H$. Pick such a central charge

$$Z = \theta_{1/2} + i \cdot \theta_0 \quad \in \quad \Theta \oplus i \cdot \mathrm{int}(\mathbf{C}H)$$

determining a point $(Z, \mathscr{P}) \in \mathrm{Stab}(H) \subset \mathrm{Stab}(\mathfrak{T})$.

Following [BPPW24, § 6.2], the slicing $\mathscr{P}$ can be explicitly described by extending the pair of vectors $\theta_0, \theta_{1/2}$ to an arc $[0, 1] \to \Theta$ given by

$$\theta_t = \cos(\pi t)\, \theta_0 + \sin(\pi t)\, \theta_{1/2}. \tag{2.4}$$

Lemma 2.5.1. *For* $(Z, \mathscr{P})$ *as above and* $t \in (0, 1]$, *we have* $\mathscr{P}(1-t) = H_{ss}(\theta_t)$.

Proof. Begin by observing that $\theta_t\colon \mathbf{K}\mathfrak{T} \to \mathbb{R}$ is the functional $w_t \circ Z$, where $w_t\colon \mathbb{C} \to \mathbb{R}$ is the $\mathbb{R}$-linear map given by

$$w_t(z) = \mathrm{Im}(e^{i\pi t} z).$$

Now given $x \in H$, the complex number $Z[x]$ has argument in the range $(0, 1]$ and hence

$$\begin{aligned} \arg Z[x] &\leqslant 1-t \quad \text{if and only if } w_t\,(Z[x]) \geqslant 0\ ,\ \text{i.e. } \theta_t[x] \geqslant 0\ , \\ \arg Z[x] &\geqslant 1-t \quad \text{if and only if } w_t\,(Z[x]) \leqslant 0\ ,\ \text{i.e. } \theta_t[x] \leqslant 0\ . \end{aligned}$$

The result follows, since x lies in $\mathscr{P}(1-t)$ if and only if $\arg Z[x] = 1-t$ and $\arg Z[s] \leqslant 1-t$ for all sub-objects $s \hookrightarrow x$ [Bri07, definition 2.2]. □

Proposition 2.5.2. *Let* $(Z, \mathscr{P}) \in \mathrm{Stab}(H)$ *be as above, and consider the stability condition* $(W, \mathscr{Q}) = -t * (Z, \mathscr{P})$ *determined by some* $t \in [0, 1]$. *Then we have* $\mathscr{Q}(0, 1] = H^{tt}(\theta_t)$ *and* $\mathscr{Q}[0, 1) = H_{tt}(\theta_t)$ *where* θ_t *is determined by* Z *as in* (2.4).

In particular, $-t * (Z, \mathscr{P})$ *lies in the chamber* $\mathrm{Stab}(H^{tt}\theta_t)$ *for all* $0 \leqslant t \leqslant 1$.

Proof. The heart $K = \mathscr{Q}(0, 1] = \mathscr{P}(-t, 1-t]$ is the extension-closure of $\mathscr{P}(0, 1-t]$ and $\mathscr{P}(1-t, 1][-1]$. Evidently, we have $\theta_t[x] \geqslant 0$ whenever $x \in \mathscr{P}(0, 1-t]$ and $\theta_t[x] < 0$ whenever $x \in \mathscr{P}(1-t, 1]$. It follows that θ_t lies in $\mathbf{C}K$. Since K also contains the subcategory $\mathscr{P}(1-t) = H_{ss}(\theta_t)$, we must have $K = H^{tt}(\theta_t)$ by corollary 2.4.7. The result follows. □

Dynkin combinatorics for flops

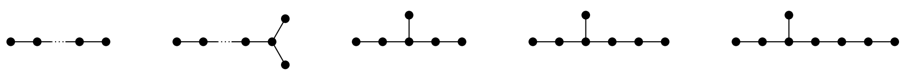
(Chapter 3)

A recurring motif in our treatment is that of mutation combinatorics, which allow us to enumerate sets over Dynkin data. We briefly explain the construction in an abstract setting, and recognise its various manifestations as they come up in the rest of the thesis.

Let G be a (simply laced) *Coxeter–Dynkin graph*, i.e. a finite set equipped with a map $m\colon G\times G \to \{1,2,3,\infty\}$ satisfying $m(i,i)=1$ and $m(i,j)=m(j,i)\geqslant 2$ whenever $i\neq j$ are distinct elements of G. From this data we can construct a *Coxeter–Weyl group* $W(G)$, presented by the generators $\{s_i \mid i\in G\}$ (called *simple reflections*) and relations $(s_i s_j)^{m(i,j)}=1$ for all pairs i,j satisfying $m(i,j)<\infty$.

The data of a Coxeter–Dynkin graph can and will be succinctly represented as a graph with vertex set G, with a single unlabelled edge $i \text{——} j$ whenever $m(i,j)=3$, and a double edge $i \Longleftrightarrow j$ whenever $m(i,j)=\infty$. An absence of edge thus indicates $m(i,j)=2$, in particular if (the graphical representation of) G is a disjoint union $G=G_1\sqcup G_2$ of Dynkin subgraphs then we have the decomposition $W(G)\simeq W(G_1)\times W(G_2)$. In this situation we say $W(G_1)$ and $W(G_2)$ are *parabolic factors* of G.

Proposition 3.0.1 [Hum90, chapter 2]. *The group* $W(G)$ *is finite if and only if* G *is a (possibly empty) disjoint union of the graphs below.*

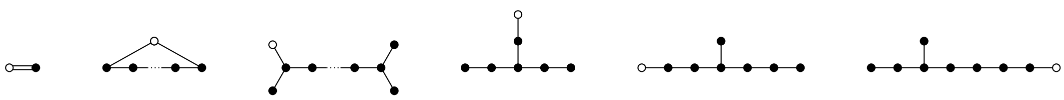
(3.1) 

When $W(G)$ is finite there is a unique element $w_G\in W(G)$ that is maximal in the weak (Bruhat) order, i.e. has longest minimum-length factorisation into simple reflections [Hum90, § 1.8]. More generally, we declare w_G to be the maximal word w_H in the largest finite parabolic factor $W(H)\subset W(G)$. In particular if none of the graphs above appear as connected components of G, we have $w_G=1\in W(G)$.

Lemma 3.0.2. *Any Coxeter–Dynkin graph* G *has a unique involution (i.e. a bijection* $\mathrm{inv}_G\colon G\to G$ *satisfying* $\mathrm{inv}_G\circ\mathrm{inv}_G=\mathrm{id}_G$ *and* $m(i,j)=m(\mathrm{inv}_G i,\mathrm{inv}_G j)$ *for all* $i,j\in G$*) such that* $w_G s_i w_G = s_{\mathrm{inv}_G i}$ *for all* $i\in G$.

Proof. Any such involution must preserve connected components. Evidently if $G=G_1\sqcup G_2$ then $w_G=w_{G_1}w_{G_2}$. Moreover by maximality and uniqueness of w_G, we must have $w_G w_G=1$. This allows us to reduce to the case when G is connected. If $W(G)$ is finite, the result then is [IW, lemma 1.2]. If $W(G)$ is infinite, the identity is clearly the only such involution. $\square$

Remark 3.0.3. Following convention, we shall say a Coxeter–Dynkin graph G is a *Dynkin graph* if each connected component of G is one of the *spherical graphs* (3.1), or one of the *affine graphs* shown below.

(3.2) 

We note that each connected affine graph $\underline{\Delta}$ has a preferred spherical subgraph Δ obtained by deleting the hollow vertex indicated in the above figure. This yields a bijection between connected spherical and affine graphs, and we say $\underline{\Delta}$ is the 'affine counterpart' of Δ.

When G is a Dynkin graph, we shall call $W(G)$ the associated *Weyl group*.

§ 3.1 Dynkin data and mutation

By *Dynkin data* we mean a Coxeter–Dynkin graph G with a subset of vertices $J \subset G$. Given such a pair, we may use lemma 3.0.2 to define a map $\iota_J \colon G \setminus J \to G$ given by $\iota_J(i) = \mathrm{inv}_{J\cup\{i\}}(i)$. Here $J \cup \{i\}$ is considered a (full) subgraph of G.

Definition 3.1.1. We say the *simple mutation of the Dynkin data* $J \subset G$ at $i \in G \setminus J$ is the new Dynkin data given by the collection of vertices

$$\nu_i J = J \cup \{i\} \setminus \{\iota_J(i)\} \quad \subset G.$$

When iterating simple mutations, we omit brackets thus writing $\nu_{i_n} \dots \nu_{i_1} J$ to mean $\nu_{i_n}(\nu_{i_{n-1}} \dots \nu_{i_1} J)$, noting that the sequence makes sense only if $i_1 \in G \setminus J$ and $i_{j+1} \in G \setminus (\nu_{i_j} \dots \nu_{i_1} J)$ for each $j = 1, \dots, n-1$. In this case we say the sequence of symbols $\nu = \nu_{i_n} \dots \nu_{i_1}$ is a J*-path of length* n. We treat the empty word $(\,)$ as a J-path of length 0.

Paths are composed in the obvious way, namely if ν is a J-path of length n and μ is a νJ-path of length m then the concatenation $\mu\nu$ is a J-path of length $m + n$.

Clearly, simple mutation is involutive in the sense that $\nu_{\iota_J(i)} \nu_i J = J$. Given a J-path $\nu = \nu_{i_n} \dots \nu_{i_1}$, we can thus define a *reversed* J*-path* to be the νJ-path given by $\overline{\nu} = \nu_{\iota(i_1)} \dots \nu_{\iota(i_n)}$, where the maps ι in order correspond to the subsets J, $\nu_{i_1} J$, ..., $\nu_{i_{n-1}} \dots \nu_{i_1} J$.

When the choice of J is clear as above, we simply write ι instead of ι_J.

Definition 3.1.2. A *set with* G*-mutation* is a set A equipped with a map $\mathbb{J} : A \to 2^G$ that assigns each element of A to some subgraph of G, and a collection of functions $(\nu_i \colon \{a \in A \mid i \notin \mathbb{J}(a)\} \to A)_{i \in G}$ (called *simple mutations*) such that for each $a \in A$, we have $\mathbb{J}(\nu_i a) = \nu_i \mathbb{J}(a)$ and $\nu_{\iota(i)}(\nu_i a) = a$.

If A, A' are two sets with G-mutation, we say a map $f \colon A \to A'$ *respects mutation* if for all $a \in A$ and $i \in G \setminus \mathbb{J}(a)$, we have $\mathbb{J}(fa) = \mathbb{J}(a)$ and $\nu_i(fa) = f(\nu_i a)$.

Example 3.1.3. Given a Coxeter–Dynkin graph G, the set of all its subgraphs 2^G is naturally a set with G-mutation. This naturally breaks up into smaller sets with G-mutation called *mutation classes*, where the mutation class of $J \subseteq G$ is the subset of 2^G containing subgraphs that can be obtained from J by iterated mutation. Thus $\{\emptyset\}$ and $\{G\}$ are mutation classes of the trivial subgraphs $\emptyset$ and G respectively.

Iyama–Wemyss [IW, § 4.2] compute many non-trivial mutation classes associated to affine Dynkin graphs. We draw one such mutation class (which Iyama–Wemyss label $\mathcal{E}_{7,4}$) below.

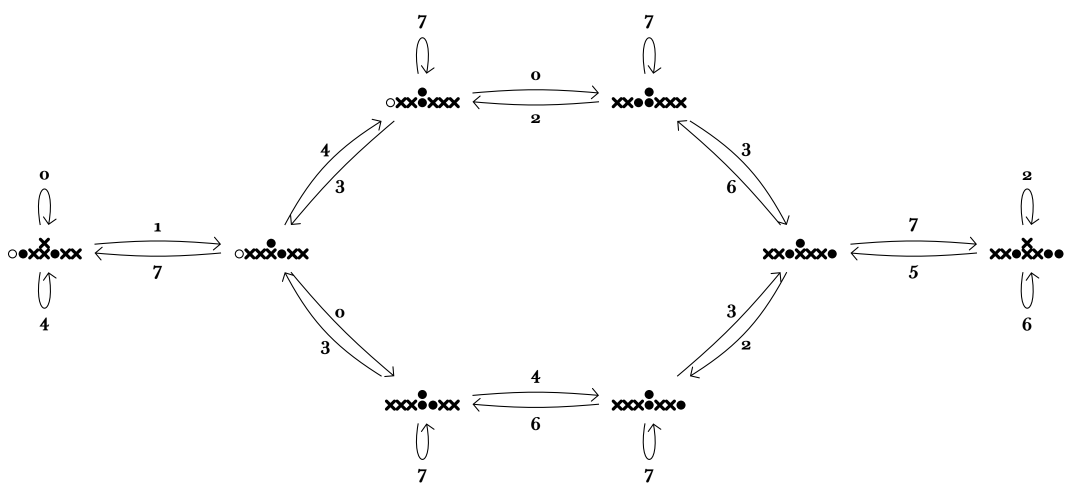


Figure 3.1. The mutation class of $J = \{2, 3, 5, 6, 7\}$ inside the $\widetilde{E}_7$ graph, where edges are not drawn and subgraphs are indicated by marking off their vertices with a cross (✕). Arrows indicate simple mutation, the symbol ν has been omitted for brevity.

Example 3.1.4. If A is a set with an action of the Coxeter–Weyl group $W(G)$, then there is a natural G-mutation structure given by $\mathbb{J}(a) = \emptyset$, $\nu_i(a) = a \cdot s_i$ for all $a \in A$, $i \in G$. Thus for instance the set of chambers $\mathrm{Cham}(G)$ in the Tits cone of $W(G)$ [see e.g. Hum90, § 5.13] is a set with G-mutation, where the (right)-action is by simple wall crossings.

More generally, a subset $J \subset G$ determines a hyperplane arrangement in Euclidean space [see IW, chapter 1], and any set with an action of the associated Deligne groupoid (also called the J-*cone groupoid*, see § 2.3.1 *ibid.*) is a set with G-mutation. This includes the prototypical example $\mathrm{Cham}(G, J)$, which is the set of chambers in the Tits cone of the hyperplane arrangement. When G is a connected Dynkin graph (i.e. one shown in (3.1) or (3.2)), we explain the construction of the hyperplane arrangement and the set $\mathrm{Cham}(G, J)$ in §§ 3.2 and 3.3.

We shall see (in § 4.1) that a 3-fold flopping contraction $\pi\colon X \to Z = \mathrm{Spec}(R)$ can naturally be assigned an affine Dynkin graph $\underline{\Delta}$ and a subset $\mathfrak{J}$ of the preferred spherical subgraph $\Delta \subset \underline{\Delta}$. In this situation the set of modifying R-modules (theorem 4.2.4), the set of torsion(-free) classes in $\mathcal{per}(\frac{X}{Z})$ (theorem 5.2.1), and the set of birational models of X (proposition 5.1.10) all also have natural mutation structures governed by $\mathfrak{J} \subset \Delta \subset \underline{\Delta}$.

Paths and exchange quivers. If A is a set with G-mutation, the *exchange quiver* $\mathrm{ExQuiv}(A)$ is a quiver with vertices A and a labelled arrow $i\colon a \to \nu_i a$ for each $a \in A$ and $i \in G \setminus \mathbb{J}(a)$. Paths in the exchange quiver can be described by a sequence of valid mutations from a specified starting vertex – it is easy to see that each $\mathbb{J}(a)$-path $\nu = \nu_{i_n} \dots \nu_{i_1}$ describes a unique path in the exchange quiver

$$a \xrightarrow{i_1} \nu_{i_1} a \xrightarrow{i_2} \cdots \xrightarrow{i_n} \nu_{i_n}\nu_{i_{n-1}} \dots \nu_{i_1} a \tag{3.3}$$

which we call the *positive path* ν from a, and further every path in the exchange quiver corresponds to a unique pair (a, ν) in this way. It is convenient to write νa for the end-point of this path.

The reversed path $\overline{\nu}$ is then a positive path from νa and we have $\overline{\nu}(\nu a) = a$.

If the positive path (3.3) has minimal length among all paths in $\mathrm{ExQuiv}(A)$ from a to νa, we say it is *minimal*. Clearly all minimal positive paths from a to νa have the same length, and a positive path ν from a is minimal if and only if the reversed path $\overline{\nu}$ from νa is so. Further, if two vertices a, b lie in the same connected component of $\mathrm{ExQuiv}(A)$ then it is always possible to find a minimal path ν from a such that $b = \nu a$.

Remark 3.1.5. In what follows, we will work with a spherical Dynkin graph Δ as in (3.1) and its affine counterpart $\underline{\Delta} = \Delta \cup \{0\}$ as in (3.2). Thus to avoid confusion when considering subsets $J \subset \Delta \subset \underline{\Delta}$, we use the term *spherical* J-*paths* for the paths corresponding to the ambient graph $G = \Delta$ and reserve the unqualified term J-path to mean paths for $G = \underline{\Delta}$. It can be seen that for such J, there is a natural bijective correspondence between spherical J-paths and J-paths in which the symbol ν_0 does not occur.

§ 3.2 Restricted root systems of affine type

We recap the unrestricted (Coxeter) setting first, see [Kac80, § 1] for a detailed introduction. Given an affine Dynkin diagram $\underline{\Delta}$ as in (3.2) of rank n (i.e. cardinality $n + 1$), the *root lattice* $\mathfrak{h} = \mathfrak{h}(\underline{\Delta})$ is a free $\mathbb{Z}$-module with basis given by the *simple roots* $\{\alpha_i \mid i \in \underline{\Delta}\}$. The Dynkin graph $\underline{\Delta}$ also gives a degenerate symmetric bilinear form $(-,-)$ on $\mathfrak{h}$ given by

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & , \quad m(i,j) = 1 \\ -2 & , \quad m(i,j) = \infty \\ 2 - m(i,j) & , \quad \text{otherwise} \end{cases}$$

and we use these to define the *simple reflections* $s_i\colon \mathfrak{h} \to \mathfrak{h}$ $(i \in \underline{\Delta})$ given by

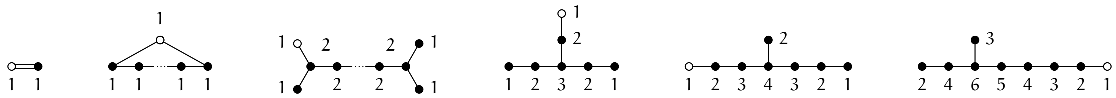
$$s_i(\alpha_j) = \alpha_j - (\alpha_i, \alpha_j)\alpha_i. \tag{3.4}$$

These define a faithful action of $W(\underline{\Delta})$ on $\mathfrak{h}$, i.e. an injection $W(\underline{\Delta}) \hookrightarrow \mathrm{GL}(\mathfrak{h})$.

Likewise each subset $J \subseteq \underline{\Delta}$ determines the ***parabolic subgroup*** $W(J) = \langle s_i \mid i \in J \rangle \subseteq W(\underline{\Delta})$, isomorphic to the Weyl group of the full subgraph spanned by J. Evidently if $J \neq \underline{\Delta}$ then the group $W(J)$ is finite.

The ***set of real roots*** $\mathrm{Root}(\underline{\Delta})$ is the union of $W(\underline{\Delta})$-orbits of the simple roots. We say a real root is ***positive*** if it can be expressed as a non-negative linear combination of the simple roots, and write $\mathrm{Root}^+(\underline{\Delta})$ for the set of positive real roots. The set $\mathrm{Root}^-(\underline{\Delta})$ of ***negative real roots*** is defined likewise. Every real root is either positive or negative, and the map $\alpha \mapsto -\alpha$ gives a bijection between $\mathrm{Root}^+(\underline{\Delta})$ and $\mathrm{Root}^-(\underline{\Delta})$. In fact for each real root α, there is a unique reflection s (i.e. an element of order 2) in $W(\underline{\Delta})$ such that $s(\alpha) = -\alpha$, and this gives a bijection between the set of positive real roots and the set of reflections in the Weyl group.

Vertices in the diagram $\underline{\Delta}$ can be Lie-theoretically assigned numerical labels $(\delta_i)_{i \in \underline{\Delta}}$ which are computed and given for each Dynkin type in [Kac80, table Z], stated below for convenience.



In particular (δ_i) is always a tuple of positive integers, and the extended (hollow) vertex $0 \in \underline{\Delta} \setminus \Delta$ is assigned the integer $\delta_0 = 1$. The vector $\delta = \sum_{i \in \underline{\Delta}} \delta_i \alpha_i$ is called the ***primitive positive imaginary root*** in $\mathfrak{h}$, and any non-zero multiple of δ is called an ***imaginary root***. The ***root system*** of $\underline{\Delta}$ is then the set of all (real and imaginary) roots in $\mathfrak{h}$, and is preserved by the action of the Weyl group. This action fixes δ.

Restricted roots. Consider now the Dynkin data $\mathfrak{J} \subset \underline{\Delta}$ determined by a subset $\mathfrak{J}$ such that $|\underline{\Delta} \setminus \mathfrak{J}| \geqslant 2$. This defines a decomposition $\mathfrak{h}(\underline{\Delta}) = \mathfrak{h}(\mathfrak{J}) \oplus \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, where we write $\mathfrak{h}(\mathfrak{J})$ for the span of the simple roots $\{\alpha_i \mid i \in \mathfrak{J}\}$ (and define $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ likewise). We say $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ is the ***restricted root lattice*** associated to $\mathfrak{J}$, and the notions of restricted (simple, positive, negative, imaginary) roots are defined as the non-zero images of the corresponding objects in $\mathfrak{h}$. We say any non-imaginary restricted root is a restricted real root, and the set of all restricted roots in $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ is the ***restricted root system***. In particular there are $|\underline{\Delta} \setminus \mathfrak{J}|$ restricted simple roots $\{\alpha_i \mid i \in \underline{\Delta} \setminus \mathfrak{J}\}$, and a ***primitive positive restricted imaginary root*** $\delta_{\mathfrak{J}} = \sum_{i \in \underline{\Delta} \setminus \mathfrak{J}} \delta_i \alpha_i$. Writing $\mathrm{Root}(\underline{\Delta}, \mathfrak{J})$ for the set of restricted real roots (and defining the subsets $\mathrm{Root}^{\pm}(\underline{\Delta}, \mathfrak{J})$ of positive and negative restricted real roots accordingly), we thus see that the restricted root system is given as

$$\underbrace{\mathrm{Root}^+(\underline{\Delta}, \mathfrak{J}) \ \sqcup \ \mathrm{Root}^-(\underline{\Delta}, \mathfrak{J})}_{\mathrm{Root}(\underline{\Delta}, \mathfrak{J})} \ \sqcup \ \{n\delta_{\mathfrak{J}} \mid n \in \mathbb{Z} \setminus \{0\}\}.$$

To obtain the analog of a Weyl-group action for restricted roots, it is necessary to consider simultaneously all restricted root lattices $\mathfrak{h}(\underline{\Delta} \setminus \nu\mathfrak{J})$ ranging over $\mathfrak{J}$-paths ν. In the Coxeter setting, a simple reflection $s_i \in W(\underline{\Delta})$ was characterised by the property of being an involution such that $s_i(\alpha_j) - \alpha_j$ is equal to $-2\alpha_i$ if $j = i$, and is a non-negative multiple of α_i otherwise. Iyama–Wemyss suggest the following generalisation for restricted root systems.

Proposition 3.2.1 [NW23, lemmas 5.1 and 5.2]. *For $\mathfrak{J} \subset \underline{\Delta}$ and $i \in \underline{\Delta} \setminus \mathfrak{J}$, the linear map $\mathfrak{h} \to \mathfrak{h}$ given by the action of $w_{\mathfrak{J}} w_{\mathfrak{J}+i} \in W(\underline{\Delta})$ maps the subset $\mathfrak{h}(\nu_i \mathfrak{J})$ isomorphically onto $\mathfrak{h}(\mathfrak{J})$, thus inducing an isomorphism*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "w_{\mathfrak{J}} w_{\mathfrak{J}+i}"] \arrow[d, two heads] & \mathfrak{h} \arrow[d, two heads] \\ \mathfrak{h}(\underline{\Delta} \setminus \nu_i \mathfrak{J}) \arrow[r, "\varphi_i", "\sim"'] & \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J}) \end{tikzcd} \tag{3.5}$$

This map preserves the restricted root systems, in particular acting on the simple roots and the primitive positive imaginary root as

$$\varphi_i(\delta_{\nu_i \mathfrak{J}}) = \delta_{\mathfrak{J}}, \qquad \varphi_i(\alpha_{\iota(i)}) = -\alpha_i, \qquad \varphi_i(\alpha_j) - \alpha_j \ \in \ \mathbb{Z}_{\geqslant 0} \cdot \alpha_i \quad (j \neq i).$$

Further, defining $\varphi_{\iota(i)} \colon \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J}) \to \mathfrak{h}(\underline{\Delta} \setminus \nu_i \mathfrak{J})$ analogously, we have $\varphi_{\iota(i)} = (\varphi_i)^{-1}$.

§ 3.3 Hyperplane arrangements

An important tool in the study of root systems and Weyl groups is the study of the dual representation. Let $\mathfrak{h}$ be the root lattice associated to $\underline{\Delta}$ as in the previous section, with the action of $W(\underline{\Delta})$ as described. The dual action on the Euclidean space $\Theta = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}, \mathbb{R})$ preserves each of the subsets $\{\delta > 0\}$, $\{\delta < 0\}$, and $\{\delta < 0\}$, and the fundamental domains for the respective actions have closures given by the cones

$$C^+ = \{\theta \in \Theta \mid \theta(\alpha_i) \geqslant 0 \text{ for all } i \in \underline{\Delta}\}, \qquad C^- = -C^+,$$
$$C^0 = \{\theta \in \Theta \mid \theta(\alpha_i) \geqslant 0 \text{ for all } i \in \Delta, \quad \theta(\delta) = 0\}.$$

Further the faces of wC^+ are all given by wC^+_J for some subset $J \subset \underline{\Delta}$, where

$$C^+_J = C^+ \cap \bigcap_{i \in J} \{\alpha_i = 0\}.$$

The faces $wC^-_J \subseteq wC^-$ $(J \subseteq \underline{\Delta})$ and $wC^0_J \subseteq wC^0$ $(J \subseteq \Delta)$ are given likewise, and as a consequence we have a complete simplicial fan in Θ called the *Weyl arrangement* given as

$$\mathrm{Arr}(\underline{\Delta}) = \underbrace{\left\{ \begin{array}{l} wC^+_J \\ : w \in W(\underline{\Delta}) \\ : J \subseteq \underline{\Delta} \end{array} \right\}}_{\mathrm{Arr}^+(\underline{\Delta})} \cup \underbrace{\left\{ \begin{array}{l} wC^0_J \\ : w \in W(\Delta) \\ : J \subseteq \Delta \end{array} \right\}}_{\mathrm{Arr}(\Delta)} \cup \underbrace{\left\{ \begin{array}{l} wC^-_J \\ : w \in W(\underline{\Delta}) \\ : J \subseteq \underline{\Delta} \end{array} \right\}}_{\mathrm{Arr}^-(\underline{\Delta})}. \tag{3.6}$$

The Weyl arrangement is induced by the root system for $\underline{\Delta}$, in the sense that every face can be described as the intersection of half-spaces defined by root hyperplanes $\{\alpha = 0\}$. Further each such hyperplane corresponds to a unique positive root in $\mathrm{Root}^+(\underline{\Delta}) \cup \{\delta\}$.

The subfan $\mathrm{Arr}^+(\underline{\Delta})$ is supported on the rational subset $\{\delta > 0\} \cup \{0\}$, and is called the *Tits cone*. The subfan $\mathrm{Arr}(\Delta)$ is supported on $\{\delta = 0\}$, and induces a complete simplicial fan on this hyperplane. Further it has finitely many cones which are indexed over the parabolic subgroup $W(\Delta)$. This is possible because $W(\underline{\Delta})$ decomposes as a semi-direct product of $W(\Delta)$ and the coroot lattice, and the action on $\{\delta = 0\}$ is such that $W(\Delta)$ acts faithfully while the coroot lattice fixes the hyperplane pointwise.

We write $\mathrm{Cham}(\underline{\Delta})$ for the set of maximal cones (*chambers*) in $\mathrm{Arr}^+(\underline{\Delta})$, noting that subfans $\mathrm{Arr}^+(\underline{\Delta})$ and $\mathrm{Arr}^-(\underline{\Delta})$ are isomorphic so it suffices to consider just one of them. Likewise, we write $\mathrm{Cham}(\Delta)$ for the set of chambers in $\mathrm{Arr}(\Delta)$.

Intersection arrangements. Now a subset $\mathfrak{I} \subset \underline{\Delta}$ with $|\underline{\Delta} \setminus \mathfrak{I}| \geqslant 2$ defines the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I})$, which we view as a split quotient of $\mathfrak{h}$. Accordingly, the dual space $\Theta(\underline{\Delta} \setminus \mathfrak{I}) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I}), \mathbb{R})$ can be identified with the subspace $\bigcap_{i \in \mathfrak{I}} \{\alpha_i = 0\} \subset \Theta$.

Since the subspace is defined by root hyperplanes, every cone $\sigma \in \mathrm{Arr}(\underline{\Delta})$ intersects $\Theta(\underline{\Delta} \setminus \mathfrak{I})$ in a face of σ, whence the Weyl arrangement induces a complete simplicial fan

$$\mathrm{Arr}(\underline{\Delta}, \mathfrak{I}) = \underbrace{\left\{ \begin{array}{l} \sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \\ : \sigma \in \mathrm{Arr}^+(\underline{\Delta}) \end{array} \right\}}_{\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{I})} \cup \underbrace{\left\{ \begin{array}{l} \sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \\ : \sigma \in \mathrm{Arr}(\Delta) \end{array} \right\}}_{\mathrm{Arr}(\Delta, \mathfrak{I})} \cup \underbrace{\left\{ \begin{array}{l} \sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \\ : \sigma \in \mathrm{Arr}^-(\underline{\Delta}) \end{array} \right\}}_{\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{I})} \tag{3.7}$$

called the (full) $\mathfrak{I}$*-cone arrangement*. This is precisely the subfan of $\mathrm{Arr}(\underline{\Delta})$ supported on $\Theta(\underline{\Delta} \setminus \mathfrak{I})$. It can be seen that the $\mathfrak{I}$-cone arrangement is induced by the restricted root system associated to $\mathfrak{I} \subset \underline{\Delta}$.

The subfan $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{I})$, called the *Tits cone*, contains infinitely many cones and is supported on the subset $\{\delta_{\mathfrak{I}} > 0\} \cup \{0\}$. The subfan $\mathrm{Arr}(\Delta, \mathfrak{I})$ on the other hand is a complete simplicial fan on the hyperplane $\{\delta_{\mathfrak{I}} = 0\}$, with finitely many cones whose combinatorics are controlled by the ADE Dynkin diagram Δ. We write $\mathrm{Cham}(\underline{\Delta}, \mathfrak{I})$ for the subset of maximal cones in $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{I})$, and $\mathrm{Cham}(\Delta, \mathfrak{I})$ for the subset of maximal cones in $\mathrm{Arr}(\Delta, \mathfrak{I})$.

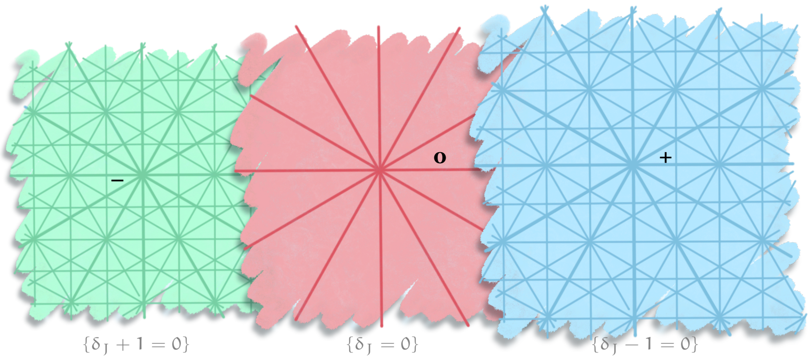


Figure 3.2. Representative affine slices of the $\mathcal{E}_{7,4}$ arrangement, associated to the Dynkin data ($J = \{2, 3, 5, 6, 7\}$) as in fig. 3.1. The principal chambers C_J^+, C_J^0, C_J^- are indicated by the symbols **+**,**o**,**–** respectively.

Simple wall crossings. Any chamber $wC^+ \in \mathrm{Cham}(\underline{\Delta})$ has $|\underline{\Delta}|$ codimension-one faces (*walls*), where the ith wall $wC^+_{\{i\}}$ spans the root hyperplane $\{w\alpha_i = 0\}$. The wall $wC^+_{\{i\}}$ is a face of precisely one other chamber, which lies on the 'other side' of this hyperplane. Examining (3.4) shows that this chamber is given by ws_iC^+, and we say ws_iC^+ is obtained from wC^+ by a *simple wall crossing.* Since every $w \in W(\underline{\Delta})$ is a product of simple reflections, it follows that any two chambers in $\mathrm{Cham}(\underline{\Delta})$ are connected by a finite sequence of simple wall crossings. The analogous statement holds for $\mathrm{Cham}(\Delta)$.

Iyama–Wemyss [IW, §1] show that while there may not be an underlying group action on the chambers of the $\mathfrak{J}$-cone arrangement, one can navigate between them by wall-crossing combinatorics.

To explain the construction, we first note that if $\sigma \in \mathrm{Arr}(\underline{\Delta})$ can be expressed as $wC_J^+ = \nu C_I^+$, then we must have $J = I$ and $wW(J) = \nu W(I)$ as cosets. Thus for each $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ there is a unique subset $\mathbb{J}(\sigma) \subset \underline{\Delta}$ such that σ can be expressed as $\sigma = wC^+_{\mathbb{J}(\sigma)}$. Further the codimension-one faces of σ are in bijection with $\underline{\Delta} \setminus \mathbb{J}(\sigma)$, where $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$ determines the face $\sigma_i = wC_{\mathbb{J}(\sigma)+i}$.

We then have the following.

Lemma 3.3.1 [IW, lemma 1.23]. *Given* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and a vertex* $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$, *there is a unique cone* $\nu_i\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *which satisfies* $\nu_i\sigma \cap \sigma = \sigma_i$, *which we call the* simple wall crossing of σ at i. *Explicitly, if* $\sigma = wC_J^+$ *then this cone is given by*

$$\nu_i \cdot wC_J^+ = w \cdot w_J w_{J+i} \cdot C^+_{\nu_i J}.$$

This gives $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *the structure of a set with* $\underline{\Delta}$*-mutation, i.e. we have* $\mathbb{J}(\nu_i\sigma) = \nu_i\mathbb{J}(\sigma)$ *and* $\nu_{\iota(i)}\nu_i\sigma = \sigma$. *Moreover, this set has a connected exchange quiver (i.e. any two chambers in* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *are connected by a finite sequence of simple wall crossings.)*

Analogously, $\mathrm{Cham}(\Delta, \mathfrak{J})$ *equipped with the data of simple wall crossings is a set with* Δ*-mutation, and this has a finite and connected exchange quiver.*

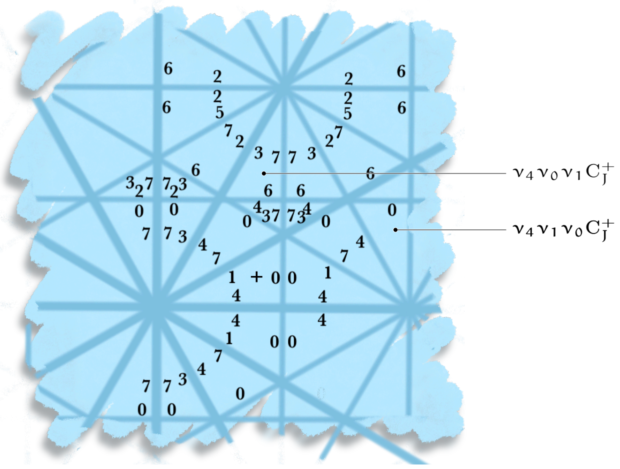
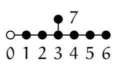


Figure 3.3. Continuing fig. 3.2, some wall crossings in the $\mathcal{E}_{7,4}$ Tits cone are indicated. Within each chamber σ, the walls are labelled by indices $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$ such that crossing the i^{th} wall lands in the adjacent chamber $\nu_i\sigma$.

$\underline{\Delta} =$ 

The cone $C^+_{\mathfrak{J}}$ lies in the $\mathfrak{J}$-cone arrangement, and it can be seen that the simple wall crossing $\nu_i C^+_{\mathfrak{J}}$ is precisely the image of $C^+_{\nu_i\mathfrak{J}} \in \mathrm{Cham}(\underline{\Delta}, \nu_i\mathfrak{J})$ under the map

$$(\varphi_i{}^\vee)^{-1} \colon \Theta(\underline{\Delta} \setminus \nu_i\mathfrak{J}) \to \Theta(\underline{\Delta} \setminus \mathfrak{J})$$

transpose to the map (3.5). More generally, if $\nu = \nu_{i_n} \dots \nu_{i_1}$ is an $\mathfrak{J}$-path then the cone $\nu C^+_{\mathfrak{J}}$ is the image of $C^+_{\nu\mathfrak{J}} \in \mathrm{Cham}(\underline{\Delta}, \nu\mathfrak{J})$ under the map $(\varphi_\nu{}^\vee)^{-1}$ where $\varphi_\nu = \varphi_{i_n} \circ \cdots \circ \varphi_{i_1}$. Every chamber in the Tits cone of the $\mathfrak{J}$-cone arrangement has this form.

Simple wall crossing from a maximal cone in $\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$ is defined likewise. This is such that if ν is an $\mathfrak{J}$-path then we have $\nu C^-_{\mathfrak{J}} = -\nu C^+_{\mathfrak{J}}$, and every maximal cone in $\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$ is of this form.

Similarly each $\sigma \in \mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ can be assigned a unique subset $\mathbb{J}(\sigma) \subset \Delta$ such that we have $\sigma = w C^0_{\mathbb{J}(\sigma)}$. The codimension-one faces of σ are thus in bijection with $\Delta \setminus \mathbb{J}(\sigma)$, and a statement analogous to lemma 3.3.1 holds showing that $\mathrm{Cham}(\Delta, \mathfrak{J})$ is a set with Δ-mutation and a connected exchange quiver.

In particular if we write $C^0_{\mathfrak{J}} = C^0 \cap \Theta(\underline{\Delta} \setminus \mathfrak{J})$ (even when $\mathfrak{J}$ does not lie in Δ), we see that every $\sigma \in \mathrm{Cham}(\Delta, \mathfrak{J})$ can be written as $\nu C^0_{\mathfrak{J}}$ for some spherical $\mathbb{J}\left(C^0_{\mathfrak{J}}\right)$-path ν. We remark that $\mathbb{J}\left(C^0_{\mathfrak{J}}\right) = \mathfrak{J}$ if and only if $\mathfrak{J} \subset \Delta$.

It follows that if $\mathfrak{J} \subset \Delta$, then the $\mathfrak{J}$-cone arrangement can be explicitly described as

$$\mathrm{Arr}(\underline{\Delta}, \mathfrak{J}) = \bigcup_{\nu \text{ an } \mathfrak{J}\text{-path}} \mathrm{faces}\left(\nu C^+_{\mathfrak{J}}\right) \;\cup \bigcup_{\substack{\nu \text{ a spherical} \\ \mathfrak{J}\text{-path}}} \mathrm{faces}\left(\nu C^0_{\mathfrak{J}}\right) \;\cup \bigcup_{\nu \text{ an } \mathfrak{J}\text{-path}} \mathrm{faces}\left(\nu C^-_{\mathfrak{J}}\right). \tag{3.8}$$

§ 3.4 The weak order

The set of chambers defined by hyperplane arrangements in Euclidean space, and more generally the set of topes of any oriented matroid [BLSWZ99], can be endowed with a partial order (called the *weak order*) by simply choosing a base element. For the chamber set of the Weyl arrangement associated to a (finite or affine) Dynkin graph, this order corresponds to the Bruhat order on the Weyl group.

We define this order and establish some of its properties in the more general setting of an $\mathfrak{J}$-cone arrangement corresponding to some Dynkin data $\mathfrak{J} \subset \underline{\Delta}$. The $\mathfrak{J}$-cone arrangement is induced by the restricted root system, hence every chamber $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is determined by the subset of positive roots which σ lies in the 'negative half-space' of, namely

$$[\sigma \leqslant 0] := \left\{\alpha \in \mathrm{Root}^+(\underline{\Delta}, \mathfrak{J}) \mid \sigma \subset \{\alpha \leqslant 0\}\right\}.$$

This assignment gives a partial order on $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ given by

$$\sigma \geqslant \sigma' \iff [\sigma \leqslant 0] \subseteq [\sigma' \leqslant 0].$$

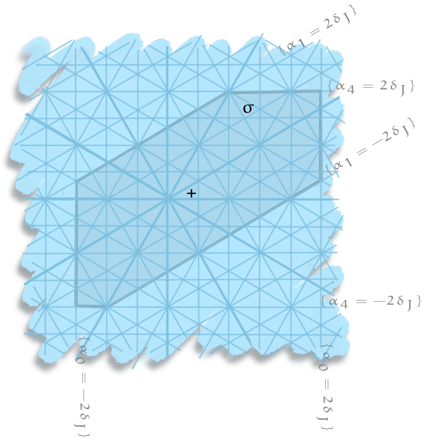
Clearly the chamber $C^+_{\mathfrak{J}}$ is maximal with respect to this order since we have $[C^+_{\mathfrak{J}} \leqslant 0] = \emptyset$, and hence for a general $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, the set $[\sigma \leqslant 0]$ determines the hyperplanes which 'separate' σ and $C^+_{\mathfrak{J}}$.

We now exhibit various properties of this partial order.

Lemma 3.4.1. *Bounded intervals in the poset* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *are finite, i.e. if* $\sigma, \sigma' \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *satisfy* $\sigma' \leqslant \sigma$, *then there are finitely many chambers in the interval* $[\sigma', \sigma]$.

Proof. It suffices to show the set $[\sigma \leqslant 0]$ is finite for any $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, i.e. σ and $C^+_{\mathfrak{J}}$ are separated by finitely many hyperplanes. Now we can write the Tits cone as the nested union of 'boxes'

$$\{\delta_{\mathfrak{J}} > 0\} \cup \{0\} = \bigcup_{N \geqslant 0} \left\{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \;\middle|\; -N\delta_{\mathfrak{J}}(\theta) \leqslant \alpha_i(\theta) \leqslant N\delta_{\mathfrak{J}}(\theta) \quad \text{for all } i \in \underline{\Delta} \setminus \mathfrak{J}\right\}.$$



In particular, both σ and $C_{\mathfrak{J}}^{+}$ lie in some sufficiently large box

$$\bigcap_{i\in\Delta\setminus\mathfrak{J}}\{\alpha_i + N\delta_{\mathfrak{J}} \geqslant 0\} \cap \{\alpha_i - N\delta_{\mathfrak{J}} \leqslant 0\}$$

for $N \gg 0$ (see figure beside for a representative box in the $\mathcal{E}_{7,4}$ Tits cone with $N = 2$). But each box is cut only by finitely many root hyperplanes, since the intersection of any box with the level set $\{\delta_{\mathfrak{J}} = 1\}$ is compact and the root hyperplanes form a locally finite arrangement away from the origin. The conclusion then follows since any hyperplane separating σ and $C_{\mathfrak{J}}^{+}$ must pass through this box. $\square$

The poset $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is also particularly amiable to the study of paths and covering relations.

Lemma 3.4.2. *For each* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and* $i \in \underline{\Delta} \setminus \mathfrak{J}(\sigma)$, *we either have* $\sigma \lessdot \nu_i\sigma$ *or* $\sigma \gtrdot \nu_i\sigma$.

Proof. By definition of simple wall crossing there is precisely one hyperplane that separates σ and $\nu_i\sigma$, i.e. σ and $\nu_i\sigma$ lie on the same side of all root hyperplanes except for one. Thus one of the sets $[\sigma \leqslant 0], [\nu_i\sigma \leqslant 0]$ is contained in the other, and their difference contains exactly one element. The result follows. $\square$

Lemma 3.4.3. *Given* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and a positive path* $\nu = \nu_{i_n} \dots \nu_{i_1}$ *from* σ *such that* $\sigma \geqslant \nu\sigma$, *the following are equivalent.*

(1) ν *is* minimal, *i.e.* ν *has minimal length among all positive paths from* σ *to* $\nu\sigma$.

(2) ν *is* reduced, *i.e.* ν *crosses each root hyperplane at most once.*

(3) the length of ν *is equal to the number of hyperplanes separating* σ *and* $\nu\sigma$, *i.e. the set* $[\nu\sigma \leqslant 0] \setminus [\sigma \leqslant 0]$ *contains* n *elements.*

(4) ν *is* atomic, *i.e. successive truncations of* ν *give us a sequence* $\sigma > \nu_{i_1}\sigma > \nu_{i_2}\nu_{i_1}\sigma > \dots > \nu\sigma$.

Proof. Since the arrangement of root hyperplanes is locally finite away from 0, we have the equivalence (1) $\iff$ (2) [see for example Sal87, lemma 2]. Further, the implications (4)⇒(3)⇒(1) are immediate where for the latter we note that any positive path must cross each separating hyperplane at least once. Lastly, suppose ν is reduced but not atomic, i.e. we have $\nu_{i_{j-1}} \dots \nu_{i_1}\sigma < \nu_{i_j} \dots \nu_{i_1}\sigma$ for some j. Thus there is a positive root α such that ν passes from $\{\alpha \leqslant 0\}$ into $\{\alpha \geqslant 0\}$, and since this hyperplane is crossed exactly once, we therefore have $\alpha \in [\sigma \leqslant 0]$ but $\alpha \notin [\nu\sigma \leqslant 0]$. This contradicts $\sigma \geqslant \nu\sigma$, thus showing (2)⇒(4). $\square$

Now any two chambers in a locally finite hyperplane arrangement can be connected by a reduced path. It follows that every relation in the poset $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is realised by the Hasse quiver, i.e. we have $\sigma < \sigma'$ if and only if there is a chain of covering relations $\sigma = \sigma^0 \lessdot \dots \lessdot \sigma^n = \sigma'$. This also shows that every covering relation in the poset must arise from simple wall crossing as in lemma 3.4.2, so that the Hasse quiver is a subquiver of the exchange quiver in the obvious way and a positive path is atomic if and only if it lies in the Hasse quiver.

Since the Hasse quiver of $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ realises all relations, we have the following consequence.

Proposition 3.4.4. *Suppose* $(P, \leqslant)$ *is a poset and* $f\colon \mathrm{Cham}(\underline{\Delta}, \mathfrak{J}) \to P$ *is a bijection that induces an isomorphism of Hasse quivers. Then* f *is an isomorphism of posets.*

Functorial equivalences

(Chapter 4)

Let $Z = \operatorname{Spec} R$ be a complete local isolated (compound) du Val singularity as in setup 1.0.2.

When $\dim Z = 3$, a ***flopping contraction*** over Z [KM98, definition 6.10] is a projective birational morphism $\pi\colon X \to Z$ such that X is normal, the exceptional locus is codimension 2, and the canonical divisor on X is numerically π-trivial (equivalently $\pi^*\omega_Z = \omega_X$, i.e. π is *crepant*). Following §1.3 we broaden our use of the term 'flopping contraction' to also include crepant partial resolutions $\pi\colon X \to Z$ when $\dim Z = 2$.

Fixing such a flopping contraction $\pi\colon X \to Z$ we look at the triangulated category $\mathbf{D}^0 X$ containing complexes of coherent sheaves supported within $\pi^{-1}[\mathfrak{m}]$, and seek to examine *hearts of (bounded) t-structures* on $\mathbf{D}^0 X$ – an example being $\mathit{coh}\, X = \mathcal{C}\!\mathit{oh}\, X \cap \mathbf{D}^0 X$. Noting that more generally any Abelian category H is naturally the heart of a t-structure inside its bounded derived category $\mathbf{D}^b H$, in this section we enumerate and examine derived equivalences of X with a multitude of algebras, varieties, and more generally, non-commutative schemes.

§ 4.1 Van den Bergh's equivalence

By Kawamata's vanishing theorem [KMM, theorem 1-2-5], the map π is acyclic i.e. $\mathbf{R}\pi_*\mathcal{O}_X = \mathcal{O}_Z$ and thus the exceptional fiber $\underline{C} = \pi^{-1}[\mathfrak{m}]$ is such that the reduced subscheme $C = \underline{C}_{\mathrm{red}}$ is a finite collection of rational curves [Van04, lemma 3.4.1].

In this context, for each integral exceptional curve $C_i \subset C$ Van den Bergh [Van04, §3.5] gives the construction of a distinguished indecomposable vector bundle $\mathcal{N}_i$ on X, such that the bundle $\mathcal{V}(\frac{X}{Z}) = \mathcal{O}_X \oplus \bigoplus_i \mathcal{N}_i$ is tilting. Consequently X is derived equivalent to the basic R-algebra $\Lambda = \operatorname{End}_X \mathcal{V}(\frac{X}{Z})$ via the functor

$$\mathrm{VdB}\colon \mathbf{D}^b\Lambda \xrightarrow{(-)\otimes^{\mathbf{L}}_{\Lambda}\mathcal{V}(\frac{X}{Z})} \mathbf{D}^b X. \tag{4.1}$$

This equivalence maps the natural heart $\mathit{mod}\,\Lambda \subset \mathbf{D}^b\Lambda$ to the category $\mathcal{P}\!\mathit{er}(\frac{X}{Z}) = \mathrm{VdB}(\mathit{mod}\,\Lambda)$ of ***perverse (coherent) sheaves***, which should be thought of as the 'standard' heart in $\mathbf{D}^b X$ owing to its amiable cohomological properties. By [Van04, corollary 3.2.8],

$$\mathcal{P}\!\mathit{er}(\tfrac{X}{Z}) = \left\{ x \in \mathcal{C}\!\mathit{oh}\, X[0,1] \;\middle|\; \begin{array}{l} \mathbf{R}^1\pi_*(H^0x) = 0, \quad \pi_*(H^{-1}x) = 0, \\ \operatorname{Hom}(c, H^{-1}x) = 0 \text{ for all } c \in \mathcal{C}\!\mathit{oh}\, X \\ \text{satisfying } \mathbf{R}\pi_* c = 0. \end{array} \right\}. \tag{4.2}$$

The equivalence (4.1), being R-linear, identifies the subcategory $\mathbf{D}^0 X \subset \mathbf{D}^b X$ of complexes with cohomology supported within C with the subcategory $\mathbf{D}^{\mathrm{fl}}\Lambda \subset \mathbf{D}^b\Lambda$ of complexes whose cohomology modules have finite length over Λ (equivalently, the thick subcategory generated by simple Λ-modules.)

We write $\mathit{per}(\frac{X}{Z})$ for the subcategory $\mathcal{P}\!\mathit{er}(\frac{X}{Z}) \cap \mathbf{D}^0 X$ of $\mathbf{D}^b X$, which under the equivalence (4.1) corresponds to the natural heart $\mathrm{fl}\,\mathit{mod}\,\Lambda \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ of finite length Λ-modules.

Dynkin data for flopping contractions. Returning to the contraction $\pi\colon X \to Z$, the integral components of the reduced exceptional fibre C can be naturally indexed over some subgraph of a Dynkin graph as we shall now explain.

When $\dim Z = 2$, the partial resolution X is obtained by contracting a subset of exceptional curves in the minimal resolution of Z. By the McKay correspondence [McK80], the exceptional curves in the minimal resolution are indexed over some Dynkin graph Δ in the list (3.1). Recording the indices of the contracted curves in a subset $\mathfrak{J} \subset \Delta$ shows that the integral components of C are naturally indexed over vertices of the subgraph $\Delta \setminus \mathfrak{J}$ and we can write the reduced exceptional fiber as $C = \bigcup_{i \in \Delta \setminus \mathfrak{J}} C_i$ where each component C_i is a $\mathbb{P}^1$.

The construction for 3-folds reduces to that for surfaces by considering a generic hyperplane section (*general elephant*) $\overline{Z} = \operatorname{Spec} \overline{R} \hookrightarrow \operatorname{Spec} R$ and the pullback $\overline{X} = X \times_Z \overline{Z}$. Indeed by assumptions on R, the variety $\overline{Z}$ is the germ of a canonical surface singularity and the map $\overline{X} \to \overline{Z}$ is a crepant partial resolution which is independent of the choice of general elephant [Rei, theorem 1.14]. The exceptional curves in X are naturally in bijection with those in $\overline{X}$, which are in turn indexed over some subset $\Delta \setminus \mathfrak{J}$ of a Dynkin graph by the preceding discussion.

The indexing naturally carries over to indecomposable summands of Van den Bergh's bundle $\mathcal{V}(\frac{X}{Z})$ in the following fashion.

Lemma 4.1.1. *The bundle $\mathcal{V}(\frac{X}{Z})$ has precisely one indecomposable summand isomorphic to $\mathcal{O}_X$. Further, for each irreducible exceptional curve $C_i \subset C$ ($i \in \Delta \setminus \mathfrak{J}$) in X, the bundle $\mathcal{V}(\frac{X}{Z})$ has an indecomposable summand $\mathcal{N}_i$ such that some representative Weil divisor $c_1(\mathcal{N}_i{}^\vee) \subset X$ transversely intersects C_i once, and is disjoint from C_j for $j \neq i$.*

This assignment $C_i \mapsto \mathcal{N}_i$ is a bijection between integral π-exceptional curves in X and non-free indecomposable summands of $\mathcal{V}(\frac{X}{Z})$.

Proof. This is implicit in the construction of $\mathcal{V}(\frac{X}{Z})$, see [Van04, §§ 3.4 and 3.5]. □

As in § 3.1, it is convenient to view Δ as sitting inside the associated extended (affine) Dynkin diagram $\underline{\Delta}$ with extended (hollow) vertex $0 \in \underline{\Delta} \setminus \Delta$. We reserve this index 0 for the trivial summand of $\mathcal{V}(\frac{X}{Z})$, writing $\mathcal{N}_0 = \mathcal{O}_X$.

Thus indecomposable summands $\mathcal{N}_i \subset \mathcal{V}(\frac{X}{Z})$, and hence also finitely generated indecomposable projective Λ-modules $P_i = \operatorname{Hom}_X\left(\mathcal{V}(\frac{X}{Z}), \mathcal{N}_i\right)$, are naturally indexed over $\underline{\Delta} \setminus \mathfrak{J}$. The indexing carries over to simple Λ modules via the standard simple-projective duality. By [Van04, proposition 3.5.7], the simple Λ-module S_i dual to P_i is given by

$$\operatorname{VdB}(\, S_i \,) = \begin{cases} \omega_{\underline{C}}[1] & , \quad i = 0 \\ \mathcal{O}_{C_i}(-1) & , \quad i \in \Delta \setminus \mathfrak{J}. \end{cases}$$

This indexing of simples also yields an identification of the Grothendieck group $\mathbf{K}\Lambda = \bigoplus_i \mathbb{Z} \cdot [S_i]$ with the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, where we map the class of $S_i \in \mathrm{fl}\mathcal{mod}\,\Lambda$ to the simple root α_i. Dually, the split-Grothendieck group of projective Λ-modules $\mathbf{K}_{\text{split}}(\mathcal{proj}\,\Lambda) \simeq \bigoplus_i \mathbb{Z} \cdot [P_i]$ is identified with the set $\operatorname{Hom}(\mathfrak{h}(\underline{\Delta}, \mathfrak{J}), \mathbb{Z})$ of integral points in $\Theta(\underline{\Delta}, \mathfrak{J})$, where the class $[P_i]$ of an indecomposable projective is mapped to the vector θ_i dual to the corresponding simple class $[S_i] = \alpha_i$.

§ 4.2 Tilting equivalences

When $\pi\colon X \to Z \simeq \operatorname{Spec} R$ is a flopping contraction of 3-folds, Iyama–Wemyss observed [IW; IW14] that the tilting theory of Λ, and hence also a large class of algebraic t-structures on $\mathbf{D}^{\mathrm{fl}}\Lambda \simeq \mathbf{D}^0 X$, is controlled by the combinatorics of certain reflexive R-modules. We describe the constructions below, emphasising that they are only pertinent to the $\dim Z = 3$ case of setup 1.0.2. This will be followed by a dictionary of appropriate translations to the $\dim Z = 2$ case.

Definition 4.2.1. An R-module N is said to be *modifying* if it is basic (i.e. has pairwise non-isomorphic indecomposable summands), reflexive, and its endomorphism algebra $\operatorname{End}_R(N)$ is Cohen–Macaulay. We say N is a *modifying generator* if in addition it contains R as a direct summand (this is equivalent to N itself being Cohen–Macaulay).

The set of (isomorphism classes of) modifying R-modules is naturally partitioned into *mutation classes* [IW, corollary 9.29], where two modifying modules M, N lie in the same mutation class if they have the same number of indecomposable summands and if M admits a two-term approximation by $\mathrm{add}(N)$.

We write $\mathrm{MM}^N(R)$ for the mutation class of N, and $\mathrm{MMG}^N(R) \subset \mathrm{MM}^N(R)$ for the subset of modifying generators in the mutation class. Each mutation class furnishes a family of derived-equivalent algebras via the following Auslander–McKay type correspondence.

Theorem 4.2.2 [IW, theorem 9.25]. *If* M *and* N *are modifying* R*-modules in the same mutation class, then* $\mathrm{Hom}_R(N, M)$ *is a reflexive tilting* $\mathrm{End}_R N$*-module, and further every reflexive tilting* $\mathrm{End}_R N$*-module arises in this way from a unique module* M *in the mutation class of* $\mathrm{MM}^N N$.

There is a natural isomorphism $\mathrm{End}_{\mathrm{End}_R N}(\mathrm{Hom}_R(N, M)) \cong \mathrm{End}_R M$ given by reflexive equivalence [IW14, lemma 2.5]. Then as is standard, the tilting module $\mathrm{Hom}_R(N, M)$ in the above setup induces quasi-inverse derived equivalences

$$\mathbf{D}^b(\mathrm{End}_R N) \underset{\mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(N,M),-)}{\overset{(-)\otimes^{\mathbf{L}}\mathrm{Hom}_R(N,M)}{\leftrightarrows}} \mathbf{D}^b(\mathrm{End}_R M)\,. \tag{4.3}$$

Being R-linear, the functors restrict to equivalences between the categories $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}_R N)$ and $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}_R M)$ containing complexes with finite length cohomology.

Modification for surfaces. When $Z = \mathrm{Spec}\, R$ as in setup 1.0.2 is 2-dimensional, modifying modules as in definition 4.2.1 coincide with basic Cohen–Macaulay modules [Yos90, (1.5.3)] and there are only finitely many such (see e.g. chapter 10 *ibid*). In particular there is a bijection $i \mapsto N_i$ between the vertex set of $\underline{\Delta}$ and the indecomposable modifying R-modules, with $N_0 \simeq R$.

As such, theorem 4.2.2 and related statements therefore do not hold when $\dim Z = 2$ if $\mathrm{MM}^N(R)$ is simply taken to be a set of modifying R-modules. This can be remedied by 'tagging' the modules.

Definition 4.2.3. A *tagged modifying module* for R is a pair (σ, N) consisting of a chamber $\sigma \in \mathrm{Cham}(\underline{\Delta}\backslash\mathfrak{J})$ in the $\mathfrak{J}$-cone arrangement (for some $\mathfrak{J} \subset \underline{\Delta}$) and the modifying R-module $N = \bigoplus_{i\in\underline{\Delta}\backslash\mathbb{J}(\sigma)} N_i$. We say (σ, N) is a *tagged modifying generator* if σ is connected to $C_{\mathfrak{J}}^+$ via a spherical $\mathfrak{J}$-path.

Mutation of tagged modifying modules can be defined in the obvious way, i.e. by declaring $\nu(\sigma, N)$ to be the tagged modifying module determined by $\nu\sigma$. Happily, the two modifying R-modules $\bigoplus_{i\in\underline{\Delta}\backslash J} N_i$ and $\bigoplus_{i\in\underline{\Delta}\backslash\nu J} N_i$ are also related by mutation in the conventional sense (i.e. both have the same number of summands and one admits a two-term approximation by summands of the other).

For example, any modifying R-module $N = \bigoplus_{i\in\underline{\Delta}\backslash\mathfrak{J}} N_i$ can be upgraded to the trivially-tagged modifying module $(C_{\mathfrak{J}}^+, N)$, and we will abuse notation to simply write N for this pair. It then makes sense to write $\mathrm{MM}^N R$ for the mutation class of $(C_{\mathfrak{J}}^+, N)$, and $\mathrm{MMG}^N R$ for the subset of tagged modifying generators in this mutation class.

Lastly, suppose we have tagged modifying modules (σ, N) and (σ', N') in the same mutation class, determined by two cones $\sigma, \sigma' \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$. Let $\nu = \nu_{i_n} \dots \nu_{i_1}$ be any sequence of wall-crossings from σ to $\sigma' = \nu\sigma$. For brevity write $\nu_k = \nu_{i_k} \dots \nu_{i_1}$ for the kth truncation of ν, and write $J = \mathbb{J}(\sigma)$ and $J' = \mathbb{J}(\sigma')$. Recalling that w_J denotes the longest element of $W(J)$, we define an element

$$w = (w_J \cdot w_{J+i_1}) \cdot (w_{\nu_1 J} \cdot w_{\nu_1 J + i_2}) \cdots (w_{\nu_{n-1} J} \cdot w_{\nu_{n-1} J + i_n}) \quad \in W(\underline{\Delta}).$$

This is defined precisely so that if $\sigma = \nu C_J^+$ then $\sigma' = \nu \cdot w C_{J'}^+$ (lemma 3.3.1), and is independent of the choice of path $\nu\colon \sigma \rightsquigarrow \sigma'$ by the results of [IW, §1.6]. The word w naturally defines an ideal $\mathcal{I}_w$ of the $\underline{\Delta}$-preprojective algebra Π [BIRS09, chapter 2]. Writing $e_J = \sum_{i\in\underline{\Delta}\backslash J} e_i \in \Pi$ for the idempotent determined by vertices $i \in \underline{\Delta} \setminus J$ (and likewise for $e_{J'}$), we *declare* the symbol $\mathrm{Hom}_R((\sigma, N), (\sigma', N'))$ to mean the subspace $e_J \mathcal{I}_w e_{J'} \subset \Pi$.

Happily again, we see that $\mathrm{End}_R((\sigma, N)) = \mathrm{Hom}_R((\sigma, N), (\sigma, N))$ coincides with the usual endomorphism algebra $\mathrm{End}_R N$, i.e. the *contracted preprojective algebra* $e_J \Pi e_J$.

Further, $T = \mathrm{Hom}_R((\sigma, N), (\sigma', N'))$ is clearly a left $\mathrm{End}_R((\sigma, N))$-module. Iyama–Wemyss [IW, chapter 5] show that T is in fact a tilting $\mathrm{End}_R((\sigma, N))$-module, and furthermore every tilting module arises in this way from a unique tagged module in the mutation class of (σ, N).

That is to say, theorem 4.2.2 holds verbatim for surfaces once modifying R-modules are replaced with their tagged counterparts. This is also true for the rest of the thesis.

Mutating modifying modules. The key example of a modifying generator is the module $\pi_*\mathcal{V}(\frac{X}{Z})$, which is Cohen–Macaulay with a Cohen–Macaulay endomorphism algebra by [Van04, proposition 3.2.10]. Given the setup of our problem, we fix once and for all this choice of tilting module and algebra

$$N = \pi_*\mathcal{V}(\tfrac{X}{Z}), \qquad \Lambda = \mathrm{End}_R N \cong \mathrm{End}_X \mathcal{V}(\tfrac{X}{Z}).$$

Note that the indecomposable summands of N are naturally indexed over $\underline{\Delta} \setminus \mathfrak{J}$ by lemma 4.1.1.

Iyama–Wemyss's correspondence theorem 4.2.2 is complemented by the following result which enumerates the entire mutation class of N.

Theorem 4.2.4. *There is a bijection* $\mathbf{C}\colon \mathrm{MM}^N R \to \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *such that the following statements hold.*

(1) Given $M \in \mathrm{MM}^N(R)$ *and* $i \in \underline{\Delta} \setminus \mathbb{J}(\mathbf{C}M)$*, let* $M' \in \mathrm{MM}^N(R)$ *be such that* $\mathbf{C}(M')$ *is the adjacent chamber* $\nu_i \mathbf{C}(M)$*. Then there is a unique indecomposable summand* $M_i \subset M$ *such that* M/M_i *is a summand of* M'*; this defines a bijection between* $\underline{\Delta} \setminus \mathbb{J}(\mathbf{C}M)$ *and the indecomposable summands of* M*.*

(2) If M *above is a modifying generator, then the extended vertex* $0 \in \underline{\Delta}$ *lies outside* $\mathbb{J}(\mathbf{C}M)$ *and corresponds to the summand* $M_0 = R$*.*

(3) For $N = \pi_*\mathcal{V}(\frac{X}{Z})$*, we have* $\mathbf{C}(N) = C^+_{\mathfrak{J}}$ *and the bijection in (1) coincides with the* $(\underline{\Delta} \setminus \mathfrak{J})$*-indexing of summands given by lemma 4.1.1.*

(4) Given $M, M' \in \mathrm{MM}^N(R)$ *corresponding to the tilting* Λ*-modules* $T = \mathrm{Hom}(N, M)$ *and* $T' = \mathrm{Hom}(N, M')$*, we have* $\mathbf{C}(M) \geqslant \mathbf{C}(M')$ *in the weak order on* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *if and only if* $\mathrm{Ext}^1(T, T') = 0$*.*

Proof. If X is a surface (i.e. $\mathrm{MM}^N R$ is the mutation class of the tagged modifying module $(C^+_{\mathfrak{J}}, N)$) then the assignment $\mathbf{C}$ is simply $(\sigma, N') \mapsto \sigma$. Iyama–Wemyss [IW, chapter 5] show that $\sigma = \mathbf{C}(\sigma, N')$ is precisely the g-cone (i.e. the cone spanned by the g-vectors) of the tilting Λ-module $\mathrm{Hom}_R((C^+_{\mathfrak{J}}, N), (\sigma, N'))$.

The first three statements then hold for straightforward reasons. As for the final statement, we note that the assignment $T \mapsto g\text{-cone}(T)$ for tilting modules respects partial orders since wall-crossing is compatible with tilting mutation and $g\text{-cone}(T) \geqslant g\text{-cone}(T')$ in the weak order if and only if $\mathrm{Ext}^1(T, T') = 0$ [IW18, theorem 5.2]. Thus $\mathbf{C}$ has all the required properties.

If X is a $\mathbb{Q}$-factorial threefold (i.e. a minimal model), the statement is [IW, corollary 9.8]. We sketch the argument for more general 3-folds, following corollary 9.29 *ibid.* Consider a general elephant $\mathrm{Spec}\,\overline{R} \hookrightarrow \mathrm{Spec}\,R$ and the corresponding reduction functor $\mathbb{F}(-) = (-) \otimes_R \overline{R}$ to construct the required bijection as a composite

$$\begin{tikzcd}[column sep=small]
\mathrm{MM}^N(R) \arrow[d, "\mathrm{Hom}_R(N,-)"] & & \mathrm{Cham}(\underline{\Delta}, \mathfrak{J}) \\
\{\text{basic reflexive tilting } \Lambda\text{-modules}\} \arrow[rr, "\mathbb{F}(-)"] & & \{\text{basic tilting } \mathbb{F}\Lambda\text{-modules}\} \arrow[u, "g\text{-cone}(-)"']
\end{tikzcd} \tag{4.4}$$

where $\mathbb{F}\Lambda$ is a contracted preprojective algebra.

We have noted that the final map $g\text{-cone}(-)$ has all the required properties. Since $\mathrm{Hom}_R(N, -)$ and $\mathbb{F}(-)$ both preserve indecomposability, and since $\mathbb{F}$ preserves the tilting order [Kim24, proposition 4.2], it follows that the composite $\mathbf{C}(-) = g\text{-cone}(\mathbb{F}\,\mathrm{Hom}(N, -))$ has the required properties too. □

The upshot is that $\mathrm{MM}^N(R)$ inherits both – a partial order (where $M \geqslant M'$ if and only if $\mathbf{C}(M) \geqslant \mathbf{C}(M')$ in the weak order), and the structure of a set with $\underline{\Delta}$-mutation (with exchange quiver isomorphic to that of $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$). In particular for $M, M' \in \mathrm{MM}^N(R)$ there exists a positive $\mathbb{J}(M)$-path ν such that

$M' = \nu M$, and further if M and M' are modifying generators then this path ν can be chosen to be spherical. Thus we have

$$\text{(4.5)} \qquad \mathrm{MM}^N(R) = \{\nu N \mid \nu \text{ a } \mathfrak{J}\text{-path}\}, \quad \mathrm{MMG}^N(R) = \{\nu N \mid \nu \text{ a spherical } \mathfrak{J}\text{-path}\}.$$

The enumeration (4.5) extends to an enumeration of a large class of algebras and their tilting modules. Given $\mathfrak{J}$-paths ν, μ we write

$$_{\mu}\Lambda_{\nu} = \mathrm{Hom}_R(\nu N, \mu N)$$

and note that this is a reflexive tilting $\mathrm{End}_R(\nu N)$-module by theorem 4.2.2. The empty path ( ) omits itself from the notation, so for example we have $\Lambda_\nu = \mathrm{Hom}_R(\nu N, N)$, $_{\nu}\Lambda = \mathrm{Hom}_R(N, \nu N)$. Happily, this is consistent with $\Lambda = \mathrm{Hom}_R(N, N)$.

Then each module $\nu N \in \mathrm{MM}^N(R)$ can be used to obtain two equivalences to $\mathbf{D}^{\mathrm{fl}}\Lambda$. Indeed the constructions in (4.3) are symmetric in the input data so we obtain a tilting Λ-module $_{\nu}\Lambda$ and a tilting $_{\nu}\Lambda_{\nu}$-module Λ_ν, which in turn give two *mutation functors* $\mathbf{D}^{\mathrm{fl}}\,{}_{\nu}\Lambda_{\nu} \rightrightarrows \mathbf{D}^{\mathrm{fl}}\Lambda$ defined as

$$\text{(4.6)} \qquad \Phi_\nu(-) = (-) \otimes^{\mathbf{L}} {}_{\nu}\Lambda, \qquad \Psi_\nu(-) = \mathbf{R}\mathrm{Hom}(\Lambda_\nu, -).$$

Remark 4.2.5. Given $M \in \mathrm{MM}^N(R)$ and a summand $M_i \subset M$ indexed by $i \in \underline{\Delta} \setminus \mathbb{J}(M)$, the simple mutation $M' = \nu_i M$, whose existence (and uniqueness) is asserted by theorem 4.2.4, can be computed explicitly via approximation sequences.

We say a map of R-modules $f\colon V \to M_i$ is a *minimal right* $\mathrm{add}(M/M_i)$*-approximation* if $V \in \mathrm{add}(M/M_i)$, the induced map $\mathrm{Hom}(M/M_i, V) \to \mathrm{Hom}(M/M_i, M_i)$ is surjective, and for any non-invertible map $h \in \mathrm{Hom}(V, V)$ we have $f \circ (\mathrm{id} - h) \neq 0$. The map f is surjective if M/M_i contains R as a summand.

We can dually define what it means for $g\colon M_i \to U$ to be a *minimal left* $\mathrm{add}(M/M_i)$*-approximation*. Since R is assumed to be a complete local commutative Noetherian ring, minimal (left and right) approximations exist. With the additional assumption that R is isolated, we then have

$$\nu_i M = (M/M_i) \oplus \ker(f) = (M/M_i) \oplus \ker(g^{\vee})^{\vee}$$

where $(-)^{\vee} = \mathrm{Hom}_R(-, R)$ [see IW, corollaries 9.28 and 9.31]. The same results also show mutation is involutive in our context, so writing $M'_{\iota(i)} = \ker(f) = \ker(g^{\vee})^{\vee}$ we see that M_i can be recovered via an exchange sequence

$$0 \to M_i \xrightarrow{g''} V' \xrightarrow{f'} M'_{\iota(i)}$$

where f' is a minimal right approximation by $\mathrm{add}(M'/M'_{\iota(i)}) = \mathrm{add}(M/M_i)$. As it turns out, the map g'' is a minimal left approximation – this follows from arguments in [IW14] and is used extensively in [Wem18] and subsequent literature, nonetheless we spell out Wemyss's proof below. Since minimal approximations are unique, we thus must have $g'' = g$. In particular g as above is always injective.

Proposition 4.2.6. *Let* $L \oplus M$ *be a modifying* R*-module, and* $f\colon V \to L$ *a minimal right* $\mathrm{add}(M)$*-approximation. Then the kernel morphism* $g\colon K \to V$ *of* f *is a minimal left* $\mathrm{add}(M)$*-approximation.*

Proof. By [IW14, proposition 6.4], applying $\mathrm{Hom}(-, M)$ to the sequence $0 \to K \to V \to L$ yields a short exact sequence of left $\Gamma = \mathrm{End}(M)$-modules

$$0 \to \mathrm{Hom}(L, M) \xrightarrow{f^*} \mathrm{Hom}(V, M) \xrightarrow{g^*} \mathrm{Hom}(K, M) \to 0.$$

That the map g is a left $\mathrm{add}(M)$-approximation is precisely the surjectivity of g^* above, and it remains to prove its (left) minimality. For this, Krause–Saorín's characterisation [KS98, corollary 1.4] of minimal morphisms in a Krull–Schmidt Abelian category is handy: for A, B finite coproducts of indecomposable objects, a map $h\colon A \to B$ is left minimal if and only if $\mathrm{im}(h)$ intersects every indecomposable summand of B, and dually it is right minimal if and only if no indecomposable summand of A lies in $\ker(h)$.

Since R is a complete local ring, this characterisation applies in the category $\mathit{mod}\, R$, and more generally in the module category of the module-finite R-algebra Γ.

Now $L \notin \operatorname{add}(M)$ since $L \oplus M$ is basic, hence the Γ-module $\operatorname{Hom}(L, M)$ has no projective summands, that is to say it cannot surject onto any projective Γ-module. On the other hand $V \in \operatorname{add}(M)$ by assumption, so each summand of $\operatorname{Hom}(V, M)$ is projective. It follows that $\operatorname{im}(f^*) = \ker(g^*)$ does not contain any summand of $\operatorname{Hom}(V, M)$, so that g^* is right minimal. The left minimality of g follows. □

Composing mutation functors. Of course for the purpose of defining mutation functors there is nothing special about the choice of N and $\Lambda = \operatorname{End}_R N$, and we can define mutation functors starting at $\mathbf{D}^{\mathrm{fl}}(\operatorname{End}_R M)$ for any modifying module M.

That is to say if $M = \nu N$ for some positive $\mathfrak{J}$-path ν, and if μ is a positive $\nu\mathfrak{J}$-path, then the reflexive tilting modules ${}_{\mu\nu}\Lambda_{\nu} \in \mathit{mod}({}_{\nu}\Lambda_{\nu})$ and ${}_{\nu}\Lambda_{\mu\nu} \in \mathit{mod}({}_{\mu\nu}\Lambda_{\mu\nu})$ define two functors

$$(-) \otimes^{\mathbf{L}} {}_{\mu\nu}\Lambda_{\nu}, \quad \mathbf{R}\mathrm{Hom}({}_{\nu}\Lambda_{\mu\nu}, -) \colon \mathbf{D}^{\mathrm{fl}}\, {}_{\mu\nu}\Lambda_{\mu\nu} \rightrightarrows \mathbf{D}^{\mathrm{fl}}\, {}_{\nu}\Lambda_{\nu}$$

which we again denote by $\Phi_{\mu}(-)$ and $\Psi_{\mu}(-)$ respectively.

The following lemmas record composition relations between such functors. It will suffice to consider functors corresponding to paths of length 1, and for simplicity of notation we write Φ_i (resp. Ψ_i) for the functor Φ_{ν_i} (resp. Ψ_{ν_i}).

Lemma 4.2.7. *Let $M = \nu N$ be a modifying module determined by $\mathfrak{J}$-path ν, and fix a vertex $i \in \underline{\Delta} \setminus \mathsf{J}(M)$.*

(1) We have natural isomorphisms $\Phi_i \circ \Psi_{\iota(i)} \simeq \Psi_i \circ \Phi_{\iota(i)} \simeq \mathrm{id}$ of functors $\mathbf{D}^{\mathrm{fl}}\, {}_{\nu}\Lambda_{\nu} \to \mathbf{D}^{\mathrm{fl}}\, {}_{\nu}\Lambda_{\nu}$.

(2) If $M > \nu_i M$, we have natural isomorphisms $\Phi_{\nu_i\nu} \simeq \Phi_{\nu} \circ \Phi_i$ and $\Psi_{\nu_i\nu} \simeq \Psi_{\nu} \circ \Psi_i$ of functors $\mathbf{D}^{\mathrm{fl}}\, {}_{\nu_i\nu}\Lambda_{\nu_i\nu} \to \mathbf{D}^{\mathrm{fl}}\Lambda$.

Proof. The first statement is immediate from definitions.

For the second, if $\nu N > \nu_i \nu N$ in $\mathrm{MM}^N(R)$, then claim there is an isomorphism of ${}_{\nu_i\nu}\Lambda_{\nu_i\nu}$-$\Lambda$ bimodules ${}_{\nu_i\nu}\Lambda_{\nu} \otimes^{\mathbf{L}} {}_{\nu}\Lambda \simeq {}_{\nu_i\nu}\Lambda$ (where the tensor product is over ${}_{\nu}\Lambda_{\nu}$). The claim is essentially [HW18, theorem 4.6], we spell out the argument there for convenience.

Indeed if we have $\nu N > \nu_i \nu N$, then by theorem 4.2.4 we have $\operatorname{Ext}^1({}_{\nu}\Lambda, {}_{\nu_i\nu}\Lambda) = 0$ over Λ. Then applying [HW18, proposition B.1] with $T = {}_{\nu}\Lambda$ and $\Gamma = {}_{\nu}\Lambda_{\nu}$, we see that there is a Λ-module isomorphism ${}_{\nu_i\nu}\Lambda_{\nu} \otimes^{\mathbf{L}} {}_{\nu}\Lambda \cong {}_{\nu_i\nu}\Lambda$. To show this is a bimodule isomorphism, first note that ${}_{\nu_i\nu}\Lambda_{\nu} \otimes^{\mathbf{L}} {}_{\nu}\Lambda$ is concentrated in degree zero so we can replace the tensor product with its non-derived version. Then the given isomorphism of Λ-modules factors as

$${}_{\nu_i\nu}\Lambda_{\nu} \otimes {}_{\nu}\Lambda \xrightarrow{\sim} \operatorname{Hom}_{\Lambda}({}_{\nu}\Lambda, {}_{\nu_i\nu}\Lambda) \otimes {}_{\nu}\Lambda \xrightarrow{\sim} {}_{\nu_i\nu}\Lambda,$$

where the first is reflexive equivalence $\operatorname{Hom}_{\Lambda}(\operatorname{Hom}_R(N, \nu N), \operatorname{Hom}_R(N, {}_{\nu_i\nu}N)) \cong \operatorname{Hom}_R(\nu N, {}_{\nu_i\nu}N)$ and the second is the evaluation map. Both of these preserve the $({}_{\nu_i\nu}\Lambda_{\nu_i\nu})^{\mathrm{op}}$-module structure, hence we have an isomorphism of bimodules as required.

The isomorphism $\Psi_{\nu_i\nu} \simeq \Psi_{\nu} \circ \Psi_i$ is then immediate, while the isomorphism $\Phi_{\nu_i\nu} \simeq \Phi_{\nu} \circ \Phi_i$ follows from the $\otimes^{\mathbf{L}}$–$\mathbf{R}\mathrm{Hom}$ adjunction [IR08, lemma 2.10]. □

Lemma 4.2.8. *Given a positive $\mathfrak{J}$-path $\nu = \nu_{i_n} \ldots \nu_{i_1}$, we can write $\Phi_{\nu} = \Upsilon_{i_1} \circ \Upsilon_{i_2} \circ \ldots \circ \Upsilon_{i_n}$ for some sequence of simple mutations $\Upsilon_i \in \{\Phi_i, \Psi_i\}$. Further if the path ν is atomic (i.e. $N > \nu_{i_1} N > \nu_{i_2}\nu_{i_1} N > \ldots > \nu N$), then each Υ_i above is of the form Φ_i and we have $\Phi_{\nu} = \Phi_{i_1} \circ \Phi_{i_2} \circ \ldots \circ \Phi_{i_n}$.*

The analogous statement holds for Ψ_{ν}.

Proof. For $n \geqslant 1$ we may write $\nu = \nu_i \nu'$ where $i = i_n$ and $\nu' = \nu_{i_{n-1}} \ldots \nu_{i_1}$ is a shorter path. By lemma 3.4.3 we have either $\nu' N > \nu N$ or $\nu N > \nu' N$. In the first case lemma 4.2.7 gives us $\Phi_{\nu} = \Phi_{\nu'} \circ \Phi_i$, while in the latter case we write $\nu' N = \nu_{\iota(i)} \nu N$ to get $\Phi_{\nu} = \Phi_{\nu'} \circ (\Phi_{\iota(i)})^{-1} = \Phi_{\nu'} \circ \Psi_i$. The result follows by induction. □

§ 4.3 Birational models

Geometric t-structures naturally arise from birational modifications of X, where surgery operations described below replace exceptional curves in a way that preserves the derived category. Again, the discussion that follows is pertinent to the $\dim Z = 3$ case of setup 1.0.2, and will be followed by a dictionary of appropriate translations to the $\dim Z = 2$ case.

Theorem 4.3.1 [Che02, theorem 1.1]. *Suppose Y is a normal Gorenstein 3-fold with at worst terminal singularities, and the maps $\tau\colon X \to Y$, $\tau'\colon W \to Y$ are flopping contractions over Y such that for every τ-nef Cartier divisor D on X, the proper transform of $-D$ across the birational map $X \dashrightarrow W$ is $\mathbb{Q}$-Cartier and τ'-nef. Then there is a Fourier–Mukai type equivalence $\mathbf{D}^b X \to \mathbf{D}^b W$ induced by universal perverse point sheaves on X.*

Given one of the maps $\tau\colon X \to Y$ as above, its counterpart $\tau'\colon W \to Y$ satisfying the given properties exists, is unique, and is called the *flop* of τ (see remark below). We say the equivalence $\mathbf{D}^b X \to \mathbf{D}^b W$ given above is the *Bridgeland–Chen flop functor.*

Remark 4.3.2. There are various equivalent definitions of a flop in the literature, see for example [Kol90]. Given a flopping contraction $\tau\colon X \to Y$ between normal 3-folds, one typically needs to choose a $\mathbb{Q}$-cartier $\mathbb{Q}$-divisor D on X such that $-D$ is τ-ample in order to define the D-flop (i.e. the $(K_X + D)$-flip) of τ, which exists [KM98, theorem 6.14] and is unique and independent of the choice of D by corollary 6.4 *ibid.*

Now the flopping contraction $\pi\colon X \to Z$ of interest to us is over a complete local base, and therefore we can freely contract exceptional curves [Sch01, §2]. That is to say, given any collection of exceptional curves $C_I = \bigcup_{i\in I} C_i \subset X$, the map π admits a unique factorisation

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "\pi"'] & Y \arrow[r, "\varpi"] & Z \end{tikzcd} \tag{4.7}$$

such that τ contracts C_i to a point if and only if $i \in I$, and is an isomorphism away from $\bigcup_{i\in I} C_i$. We say τ is a *partial contraction over* Z. Since the singularity on Z is assumed to be isolated, it follows that both τ and ϖ are flopping contractions. In particular we can consider the flop $\tau'\colon W \to Y$ of τ to obtain a derived equivalence $\mathbf{D}^b W \to \mathbf{D}^b X$; further the map $\varpi \circ \tau'\colon W \to Z$ is itself a flopping contraction over Z so we can iterate.

Wemyss [Wem18] addresses the question of classifying *birational models* of X, i.e. varieties W obtained by iteratively flopping subsets of π-exceptional curves. Abusing notation to write $\pi\colon W \to Z$ for the corresponding flopping contraction, we can again consider the associated vector bundle $\mathcal{V}(\frac{W}{Z})$, the modifying R-module generator $M = \pi_*\mathcal{V}(\frac{W}{Z})$, and the equivalence

$$\mathrm{VdB}\colon\ \mathbf{D}^b(\operatorname{End} M) \to \mathbf{D}^b W$$

constructed analogously to (4.1). In this context, Wemyss shows that flops obey mutation-combinatorics following the modifying generators.

Theorem 4.3.3 [Wem18]. *If $\pi\colon W \to Z$ is a birational model of $\pi\colon X \to Z$, then the associated modifying R-module generators $M = \pi_*\mathcal{V}(\frac{W}{Z})$ and $N = \pi_*\mathcal{V}(\frac{X}{Z})$ lie in the same mutation class, and every module in $\mathrm{MMG}^N(R)$ arises in this way from a unique birational model of X.*

Thus the set $\mathrm{Bir}(\frac{X}{Z})$ of birational models of X is in bijection with $\mathrm{MMG}^N(R)$ via the map $W \mapsto \pi_*\mathcal{V}(\frac{W}{Z})$. Since theorem 4.2.4 shows $\mathrm{MMG}^N(R)$ is a set with Δ-mutation, $\mathrm{Bir}(\frac{X}{Z})$ inherits this structure. That is to say for any $W \in \mathrm{Bir}(\frac{X}{Z})$ with associated modifying generator $\pi_*\mathcal{V}(\frac{W}{Z}) = M$ we have $\mathbb{J}(W) = \mathbb{J}(M)$, and for any $\mathbb{J}(W)$-path ν the birational model νW is the unique element of $\mathrm{Bir}(\frac{X}{Z})$ satisfying $\pi_*\mathcal{V}(\frac{\nu W}{Z}) = \nu M$.

Flops as mutation. The above mutation structure on $\mathrm{Bir}(\frac{X}{Z})$ can be realised intrinsically as arising from flops – the indecomposable summands of M are indexed over $\underline{\Delta} \setminus \mathbb{J}(M)$ in a way that applying ν_i corresponds to mutating the ith summand, and the only free summand is $M_0 = R$. It follows that the non-free indecomposable summands of $\mathcal{V}(\frac{W}{Z})$ are indexed over $\Delta \setminus \mathbb{J}(W)$. By lemma 4.1.1 these summands are naturally in bijection with the integral exceptional curves of W. Therefore we see that the reduced exceptional fiber of $\pi\colon W \to Z$ can be written as a union of integral curves $\bigcup_{i\in\Delta\setminus\mathbb{J}(W)} C_i$. The following result of Wemyss realises the operation ν_i on $\mathrm{Bir}\left(\frac{X}{Z}\right)$ as a flop of the ith curve.

Theorem 4.3.4 [Wem18, theorem 4.2]. *Fix* $i \in \Delta\setminus\mathfrak{J}$, *and suppose* $\nu_i X = W \to Z$ *is the birational model of* X *corresponding to the modifying* R*-module generator* $M = \nu_i N$. *With indexing as above, write* $\bigcup_{i\in\Delta\setminus\nu_i\mathfrak{J}} C_i$ *for the reduced exceptional fibre in* $\nu_i X$.

Then W *is precisely the flop of* X *in the curve* C_i, *and* $C_{\iota(i)} \subset W$ *is the proper transform of* $C_i \subset X$. *Further, in this case the inverse Bridgeland–Chen flop functor* $\mathbf{D}^b W \to \mathbf{D}^b X$ *coincides with the composite*

$$\mathbf{D}^b W \xrightarrow{\mathrm{VdB}^{-1}} \mathbf{D}^b(\operatorname{End} M) \xrightarrow{\Psi_i(-)} \mathbf{D}^b(\operatorname{End} N) \xrightarrow{\mathrm{VdB}} \mathbf{D}^b X,$$

and restricts to an equivalence $\mathbf{D}^0 W \to \mathbf{D}^0 X$ *between the full subcategories supported on exceptional curves.*

In what follows, we omit VdB from notation whenever convenient, thus for example writing Ψ_i for both the mutation equivalence and the flop functor in the above theorem.

By examining the connectedness of $\mathrm{ExQuiv}(\mathrm{MMG}^N R)$ we also conclude that any two birational models of X are connected by a chain of single-curve flops, i.e. given birational models $W, W' \in \mathrm{Bir}\left(\frac{X}{Z}\right)$ there is a $\mathbb{J}(W)$-path $\nu = \nu_{i_n}\dots\nu_{i_1}$ and a *sequence of flops*

$$W \dashrightarrow \nu_{i_1} W \dashrightarrow \nu_{i_2}\nu_{i_1} W \dashrightarrow \cdots \dashrightarrow \nu_{i_n}\dots\nu_{i_1} W = W'.$$

Extending the notion of Bridgeland–Chen functors to chains of flops, in this case we say the *inverse flop functor* associated to ν is the equivalence

$$\mathbf{D}^b \nu W \xrightarrow{\mathrm{VdB}^{-1}} \mathbf{D}^b(\operatorname{End} \nu M) \xrightarrow{\Psi_\nu(-)} \mathbf{D}^b(\operatorname{End} M) \xrightarrow{\mathrm{VdB}} \mathbf{D}^b W$$

which is evidently independent of the choice of ν.

Birational models of surfaces. When $Z = \operatorname{Spec} R$ as in setup 1.0.2 is 2-dimensional, there is a unique crepant resolution $\widetilde{X} \to Z$ with reduced exceptional fibre $\bigcup_{i\in\underline{\Delta}} C_i$ and all partial crepant resolutions are given by contracting some subset of these exceptional curves. As such, there is no definition of a flop. To remedy this, we again 'tag' our partial crepant resolutions with some combinatorial data.

Definition 4.3.5. A *tagged partial resolution* of Z is a pair (σ, W) consisting of a chamber $\sigma \in \mathrm{Cham}(\Delta\setminus\mathfrak{J})$ in the spherical $\mathfrak{J}$-cone arrangement (for some $\mathfrak{J} \subset \Delta$) and the partial crepant resolution W of Z obtained by contracting precisely the curves $\bigcup_{i\in\mathbb{J}(\sigma)} C_i$ in $\widetilde{X}$.

Mutation of tagged partial resolutions can be defined in the obvious way, i.e. by declaring $\nu(\sigma, W)$ to be the tagged partial resolution determined by $\nu\sigma$. We shall call ν a *sequence of flops* in this setting. We say (σ', W') is a *birational model* of (σ, W) the two are related by some sequence of flops.

Happily if (σ, W) is a tagged partial resolution then the exceptional curves of $\pi\colon W \to Z$ are naturally indexed over $\Delta\setminus\mathbb{J}(\sigma)$, and for a birational model (σ', W') of (σ, W) we can construct the 'flop diagram'

$$\begin{tikzcd} W \arrow[rr, dashed, "\rho"] \arrow[dr, "\tau"'] & & W' \arrow[dl, "\tau'"] \\ & Y & \end{tikzcd}$$

where τ contracts precisely the curves in W with indices in $\mathbb{J}(\sigma)\setminus\mathbb{J}(\sigma')$ and likewise for τ', and ρ is the birational map given by $\tau\circ\tau'^{-1}$ on the locus where τ, τ' are isomorphisms.

Note $\sigma = \upsilon C^0_{\mathfrak{J}}$ for some spherical $\mathfrak{J}$-path υ. Recalling that indecomposable Cohen–Macaulay R-modules in the 2-dimensional setting are indexed over vertices of $\underline{\Delta}$, the modifying R-module $M = \pi_*\mathcal{V}\left(\frac{W}{Z}\right)$ corresponds to the index set $\underline{\Delta}\setminus\upsilon\mathfrak{J}$. We associate (σ, W) to the tagged modifying generator $(\upsilon C^+_{\mathfrak{J}}, M)$.

Now X as in setup 1.0.2 can be upgraded to a trivially-tagged partial resolution $(C^0_{\mathfrak{J}}, X)$, and we will abuse notation to simply write X for this pair. It then makes sense to write $\mathrm{Bir}\left(\frac{X}{Z}\right)$ for the mutation class of X, this is the set of all tagged partial resolutions $\nu(C^0_{\mathfrak{J}}, X)$ obtained from a spherical $\mathfrak{J}$-path ν. The assignment to tagged modifying generators gives a natural isomorphism $\mathrm{Bir}\left(\frac{X}{Z}\right) \to \mathrm{MMG}^{\,N}(R)$.

Lastly, suppose (σ, W) and $(\sigma', W') = \nu(\sigma, W)$ are two tagged partial resolutions related by a sequence of flops ν. Writing $M = \pi_* \mathcal{V}\left(\frac{W}{Z}\right)$ and $M' = \pi_* \mathcal{V}\left(\frac{W'}{Z}\right)$, we declare the *inverse Bridgeland Chen flop functor* associated to ν to be the composite equivalence

$$\mathbf{D}^b W' \xrightarrow{\mathrm{VdB}^{-1}} \mathbf{D}^b(\mathrm{End}_R M') \xrightarrow{\Psi_\nu(-)} \mathbf{D}^b(\mathrm{End}_R M) \xrightarrow{\mathrm{VdB}} \mathbf{D}^b W.$$

Once the notions of partial resolutions, birational models, and modifying generators are correctly interpreted as their tagged versions, theorems 4.3.3 and 4.3.4 hold verbatim for surfaces. This is also true for the rest of the thesis.

§ 4.4 Semi-geometric resolutions

When the flopping contraction $\pi\colon X \to Z$ is not irreducible (i.e. the reduced exceptional fibre has multiple components), there are intermediate partial contractions $X \to Y \to Z$ to be considered. In this subsection we establish basic structural results about perverse sheaves arising from such morphisms.

Thus fix a subset $I \subset \Delta \setminus \mathfrak{J}$, equivalently a collection of exceptional curves $C_I = \bigcup_{i \in I} C_i$ in X which can be contracted via the map $\tau\colon X \to Y$ as in (4.7). As noted previously, τ is a flopping contraction and in particular [Van04, theorem A] applies. Thus there is a vector bundle $\mathcal{V}\left(\frac{X}{Y}\right)$ on X, furnishing a sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = \tau_* \mathcal{E}nd\, \mathcal{V}\left(\frac{X}{Y}\right)$ such that there is a derived equivalence

$$\mathrm{VdB}\colon \mathbf{D}^b \mathcal{A} \xrightarrow{\tau^{-1}(-) \otimes^{\mathbf{L}} \mathcal{V}\left(\frac{X}{Y}\right)} \mathbf{D}^b X \tag{4.8}$$

where the tensor product is over $\tau^{-1}\mathcal{A}$. We write $\mathcal{P}er\left(\frac{X}{Y}\right)$ for the image of $\mathcal{C}oh\, \mathcal{A}$ under this equivalence. By [Van04, proposition 3.3.1], this category admits the description

$$\mathcal{P}er\left(\tfrac{X}{Y}\right) = \left\{ x \in \mathcal{C}oh\, X[0,1] \;\middle|\; \begin{array}{l} \mathbf{R}^1\tau_*(H^0 x) = 0, \quad \tau_*(H^{-1}x) = 0, \\ \mathrm{Hom}(c, H^{-1}x) = 0 \text{ for all } c \in \mathcal{C}oh\, X \\ \text{satisfying } \mathbf{R}\tau_* c = 0. \end{array} \right\}. \tag{4.9}$$

Since we are concerned with complexes supported on the exceptional fibres of the maps in (4.7), we also define the corresponding full subcategories

$$\mathbf{D}^0\mathcal{A} = \left\{ y \in \mathbf{D}^b\mathcal{A} \mid \mathrm{Supp}\, y \subseteq \varpi^{-1}[\mathfrak{m}] \right\},$$
$$coh\, \mathcal{A} = \mathcal{C}oh\, \mathcal{A} \,\cap\, \mathbf{D}^0\mathcal{A}, \qquad per\left(\tfrac{X}{Y}\right) = \mathcal{P}er\left(\tfrac{X}{Y}\right) \,\cap\, \mathbf{D}^0 X.$$

The functor VdB clearly restricts to an equivalence $\mathbf{D}^0\mathcal{A} \to \mathbf{D}^0 X$, thus identifying $coh\, \mathcal{A}$ with $per\left(\frac{X}{Y}\right)$. The truncation functors associated to the heart $\mathcal{C}oh\, \mathcal{A} \subset \mathbf{D}^b\mathcal{A}$ restrict to $\mathbf{D}^0\mathcal{A}$, showing that $coh\, \mathcal{A} \subset \mathbf{D}^0\mathcal{A}$ (and hence $per\left(\frac{X}{Y}\right) \subset \mathbf{D}^0 X$) is the heart of a bounded t-structure.

Remark 4.4.1. The following calibration is helpful – when $I = \emptyset$ the map τ is an isomorphism and we have $\mathcal{V}\left(\frac{X}{Y}\right) = \mathcal{A} = \mathcal{O}_X$, so that $\mathcal{P}er\left(\frac{X}{Y}\right) = \mathcal{C}oh\, \mathcal{A} = \mathcal{C}oh\, X$. In this case we have $\mathbf{D}^0\mathcal{A} = \mathbf{D}^0 X$ by definition. At the other extreme, when $I = \Delta \setminus \mathfrak{J}$ the map ϖ is an isomorphism and the sheaf $\mathcal{A}$ on $Y = \mathrm{Spec}\, R$ corresponds to the R-algebra Λ. Since $(R, \mathfrak{m})$ is complete local, a complex of R-modules has support in $[\mathfrak{m}]$ if and only if its total cohomology has finite length, i.e. $\mathbf{D}^0\mathcal{A} = \mathbf{D}^{\mathrm{fl}}\Lambda$.

Characterising perversity. We now show that the 'algebraic part' of $per\left(\frac{X}{Y}\right) \subset \mathbf{D}^0 X$ is generated by finitely many simple perverse sheaves which depend only on a neighbourhood of the contracted curves, and further can be read off as a subcategory of $per\left(\frac{X}{Z}\right)$. For simplicity we first consider the case where $C_I \subset X$ is connected, i.e. τ is an isomorphism away from a point $p = \tau(C_I) \in Y$. Write $\underline{C}_I = \tau^{-1}(p)$ for the scheme theoretic exceptional fibre of τ, this has underlying reduced subscheme C_I.

Then we have the following.

Theorem 4.4.2. *Let $C_I \subset X$ be a connected component of the reduced exceptional fiber in X, and $\tau : X \to Y$ the crepant contraction of C_I. For any complex of coherent sheaves $x \in \mathbf{D}^0 X$ supported within C_I, the following are equivalent.*

(1) The complex x lies in $\mathit{per}(\frac{X}{Z})$, i.e. x is perverse with respect to the contraction $\pi\colon X \to Z$.

(2) The complex x lies in $\mathit{per}(\frac{X}{Y})$, i.e. x is perverse with respect to the contraction $\tau\colon X \to Y$.

(3) The complex x is filtered by the sheaves $\mathcal{O}_{C_i}(-1)$ for $i \in I$ and the complex $\omega_{\underline{C}_I}[1]$ (the suspended canonical sheaf of scheme-theoretic exceptional fiber of τ).

To prove this, we first establish the following lemma which reduces conditions involving the null-category $\{c \in \mathcal{Coh}\, X \mid \mathbf{R}\tau_* c = 0\}$ (such as those appearing in the descriptions (4.2), (4.9)) to checks on a finite collection of objects.

Lemma 4.4.3. *Let $\tau : X \to Y$ be the crepant contraction of C_I and $c \in \mathcal{Coh}\, X$ be such that $\mathbf{R}\tau_* c = 0$. Then c is filtered by the sheaves $\mathcal{O}_{C_i}(-1)$ for $i \in I$.*

Proof. Note if c is as given, then we have $\mathbf{R}\pi_* c = 0$ and hence $c \in \mathcal{P}\mathit{er}(\frac{X}{Z})$ from the description (4.2). Further c is supported within $\pi^{-1}[\mathfrak{m}]$. Thus across the equivalence (4.1), we see that c corresponds to some finite length Λ-module (which we again denote by c). In particular c is filtered by the simples of $\mathcal{H} = \mathit{per}(\frac{X}{Z})$ and the problem is reduced to showing that the only simples which can occur in a composition series for c are the S_i for $i \in I$.

But by [Wem18, theorem 2.15], Λ can be expressed as the quotient of the path algebra of a quiver with vertex set $\underline{\Delta} \setminus \mathfrak{J}$, and the simple representation supported on vertex i is precisely the module S_i. Further writing $e_i \in \Lambda$ for the vertex idempotent at $i \in \underline{\Delta} \setminus \mathfrak{J}$, [Wem18, proposition 2.14] shows that a Λ-module x is annihilated by $\sum_{i \notin I} e_i$ if and only if the corresponding complex $x \in \mathcal{P}\mathit{er}(\frac{X}{Z})$ satisfies $\mathbf{R}\tau_* x = 0$. In particular, the Λ-module c is annihilated by $\sum_{i \notin I} e_i$, and hence is filtered only by the vertex simples S_i for $i \in I$ as required. □

We then have the following.

Proof of theorem 4.4.2, (1) $\Longleftrightarrow$ (2). Let x be a complex of coherent sheaves with support in C_I, and for brevity write $x_i = H^{-i}(x)$ for its cohomology sheaves. Since the objects of both $\mathit{per}(\frac{X}{Y})$ and $\mathit{per}(\frac{X}{Z})$ can be described as two-term complexes of coherent sheaves on X, we can assume $x_i = 0$ unless $i = 0, 1$.

If $x \in \mathit{per}(\frac{X}{Y})$, then by the description (4.9) we see that $\tau_*(x_1) = 0$ and hence $\pi_*(x_1) = 0$.

Likewise we have $\mathrm{Hom}(\mathcal{O}_{C_j}(-1), x_1) = 0$ whenever C_j is contracted by τ, because $\mathbf{R}\tau_*\mathcal{O}_{C_j}(-1) = 0$. On the other hand if C_j is not contracted by τ and we have a non-zero morphism $f : \mathcal{O}_{C_j}(-1) \to x_1$, then the image $\mathrm{im}\, f$ is a non-zero subsheaf of x_1 supported within the finite collection of points $C_j \cap (\bigcup_{i \in I} C_i)$. But in that case, $\pi_*(\mathrm{im}\, f)$ is a non-zero subsheaf of $\pi_* x_1 = 0$, a contradiction. Thus $\mathrm{Hom}(\mathcal{O}_{C_j}(-1), x_1) = 0$ for all exceptional curves C_j, and hence by lemma 4.4.3 we have $\mathrm{Hom}(c, x_1) = 0$ for all $c \in \mathcal{Coh}\, X$ satisfying $\mathbf{R}\pi_*(c) = 0$.

Lastly, examining the Leray spectral sequence

$$\mathbf{R}^p \varpi_* \circ \mathbf{R}^q \tau_*(x_0) \Rightarrow \mathbf{R}^{p+q}\pi_*(x_0),$$

which degenerates since all maps have fiber dimension $\leqslant 1$, shows that $\mathbf{R}^1\pi_*(x_0)$ is filtered by $\varpi_* \circ \mathbf{R}^1\tau_*(x_0)$ ($= 0$ since $\mathbf{R}^1\tau_*(x_0)$ vanishes) and $\mathbf{R}^1\varpi_* \circ \tau_*(x_0)$ ($= 0$ since $\tau_*(x_0)$ is supported on a zero-dimensional subset of Y). Thus we have $\mathbf{R}^1\pi_*(x_0) = 0$ as well, and hence $x \in \mathit{per}(\frac{X}{Z})$.

Conversely if $x \in \mathit{per}(\frac{X}{Z})$, we see that $\tau_*(x_1)$ is a coherent sheaf on Y with zero-dimensional support such that $\varpi_* \circ \tau_*(x_1) = \pi_*(x_1) = 0$. This is possible only if $\tau_*(x_1) = 0$. Likewise since $\mathbf{R}^1\pi_*(x_0) = 0$, the Leray sequence shows we have $\varpi_* \circ \mathbf{R}^1\tau_*(x_0) = 0$ and hence $\mathbf{R}^1\tau_*(x_0) = 0$. Lastly if $c \in \mathcal{Coh}\, X$ is such that $\mathbf{R}\tau_* c = 0$ then we have $\mathbf{R}\pi_* c = \mathbf{R}\varpi_* \circ \mathbf{R}\tau_* c = 0$, and hence $\mathrm{Hom}(c, x_1) = 0$. This shows $x \in \mathit{per}(\frac{X}{Y})$. □

To prove (2) $\iff$ (3), we reduce the problem to a formal neighbourhood $Z_I \hookrightarrow Y$ of p, i.e. the spectrum of the complete local ring $\widehat{\mathcal{O}}_{Y,p}$. The scheme Z_I has an isolated cDV singularity at its closed point, and the restriction of τ is a flopping contraction $X_I \to Z_I$ with reduced exceptional fiber $C_I \subset X_I$. We can therefore consider the full subcategory $\mathbf{D}^0 X_I \subset \mathbf{D}^b X_I$ of complexes supported on C_I.

Lemma 4.4.4. *The restriction (i.e. pullback) functor* $\mathbf{D}^0 X \to \mathbf{D}^0 X_I$ *gives an equivalence of* $\mathbf{D}^0 X_I$ *with the full subcategory of complexes in* $\mathbf{D}^0 X$ *supported within* C_I.

Proof. The schemes X and X_I have isomorphic completions along C_I, and we write $\mathfrak{X}$ for the associated Noetherian formal scheme. Now [Orl11, lemma 2.1] shows that the category $\mathbf{D}^0 X_I$ is the bounded derived category of the Abelian category

$$\mathcal{coh}(X_I) = \{x \in \mathcal{Coh}\, X_I \mid \operatorname{Supp}(x) \subset C_I\},$$

whilst the subcategory of $\mathbf{D}^0 X$ containing complexes supported within C_I is the derived category of

$$\mathcal{Coh}_{C_I}(X) = \{x \in \mathcal{Coh}\, X \mid \operatorname{Supp}(x) \subset C_I\}.$$

But proposition 2.8 *ibid.* shows that both categories above are equivalent (via pullback along the canonical map associated to completion) to the category of torsion coherent sheaves on $\mathfrak{X}$. In particular, the restriction functor $\mathcal{Coh}_{C_I}(X) \to \mathcal{coh}(X_I)$ is an exact equivalence and passing to bounded derived categories gives the result. □

Thus we can identify $\mathbf{D}^0 X_I$ and its subcategories $\mathcal{coh}\, X_I$, $\mathcal{per}\big(\frac{X_I}{Z_I}\big)$ with their images in $\mathbf{D}^0 X$. In particular, $\mathcal{per}\big(\frac{X_I}{Z_I}\big)$ (as a subcategory of $\mathbf{D}^0 X$) is an algebraic Abelian category equal to the extension-closure of its simple objects $\omega_{\underline{C}_I}[1]$, $\mathcal{O}_{C_i}(-1)$ for $i \in I$. The following is then immediate.

Proof of theorem 4.4.2, (2) $\iff$ (3). By [Van04, proposition 3.1.4], the membership of any complex in $\mathcal{per}(\frac{X}{Y})$ can be checked locally with respect to the flat topology on Y, in particular with respect to the flat cover $Y = (Y \setminus \{p\}) \cup Z_I$. Now if x is supported within contracted curves, the restriction of x to $Y \setminus \{p\}$ vanishes and thus x lies in $\mathcal{per}(\frac{X}{Y})$ if and only if its restriction to X_I (i.e. x viewed as an object of $\mathbf{D}^0 X_I$) lies in $\mathcal{per}\big(\frac{X_I}{Z_I}\big)$, equivalently if x is filtered by the simples of $\mathcal{per}\big(\frac{X_I}{Z_I}\big)$. □

It is straightforward to generalise this to partial contractions $\tau\colon X \to Y$ associated to possibly disconnected collections of exceptional curves $C_I \subset X$, since any complex in $\mathbf{D}^0 X$ decomposes into a direct sum of complexes with connected support. In particular we can define the category

$$\mathcal{per}_I(\tfrac{X}{Z}) = \left\{ x \in \mathcal{per}(\tfrac{X}{Z}) \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i \in I} C_i \right\}$$

and note that if C_I has connected components $C_{I_1}, \dots, C_{I_k}$, then this decomposes as

$$\mathcal{per}_I(\tfrac{X}{Z}) = \bigoplus_{J \in \{I_1, \dots, I_k\}} \mathcal{per}_J(\tfrac{X}{Z}).$$

But following theorem 4.4.2, each $\mathcal{per}_J(\frac{X}{Z})$ is equivalent to the category $\mathcal{per}\big(\frac{X_J}{Z_J}\big)$ associated to a formal neighbourhood Z_J of the point $p_J = \tau(C_J) \in Y$, and in particular is generated by finitely many complexes which depend only on the scheme theoretic exceptional fiber $\underline{C}_J$ overlying C_J. Further, considering the flat cover $Y = Z_{I_1} \cup \cdots \cup Z_{I_k} \cup (Y \setminus \{p_{I_1}, \dots, p_{I_k}\})$ shows that a complex x supported within C_I lies in $\mathcal{per}(\frac{X}{Y})$ if and only if its restriction to each X_J lies in $\mathcal{per}\big(\frac{X_J}{Z_J}\big)$, and this gives us the description

$$\mathcal{per}_I(\tfrac{X}{Z}) = \Big\langle \{\mathcal{O}_{C_i}(-1) \mid i \in I\} \cup \{\omega_{\underline{C}_J}[1] \mid J = I_1, \dots, I_k\} \Big\rangle.$$

A consequence of the equivalence is that the property of a two-term complex x of coherent sheaves being 'perverse' can be checked with respect to any partial contraction which maps $\operatorname{Supp}(x)$ to a (finite collection of) points. That is to say, if $\tau\colon X \to Y$ and $\tau'\colon X \to Y'$ are two crepant partial contractions which contract all the curves containing $\operatorname{Supp}(x)$, then we have $x \in \mathcal{per}(\frac{X}{Y})$ if and only if $x \in \mathcal{per}(\frac{X}{Y'})$.

Since $\mathcal{per}(\frac{X}{Y})$ away from the contracted curves should mimic $\mathcal{per}(\frac{X}{X}) = \mathcal{coh}\,X$, we then have the following result which reduces the construction of $\mathcal{per}(\frac{X}{Y})$ to a binary choice between $\mathcal{coh}\,X$ and $\mathcal{per}(\frac{X}{Z})$ on each C_i, effectively eliminating the need to consider the geometry of the non-commutative scheme $(Y, \mathcal{A}_0)$.

Theorem 4.4.5. *If* $\tau\colon X \to Y$ *is the crepant contraction of the exceptional subset* $C_I \subset X$*, then the associated heart* $\mathcal{per}(\frac{X}{Y})$ *is the smallest extension-closed subcategory of* $\mathbf{D}^0X$ *containing the full subcategory* $\mathcal{per}_I(\frac{X}{Z})$ *and all coherent sheaves on* X *that are supported within the uncontracted curves* $\bigcup_{i\notin I} C_i$*. In other words,*

$$\mathcal{per}(\tfrac{X}{Y}) = \left\langle \underbrace{\left\{ x \in \mathcal{per}(\tfrac{X}{Z}) \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i\in I} C_i \right\}}_{\mathcal{per}_I(\frac{X}{Z})} \cup \left\{ x \in \mathcal{coh}\,X \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i\notin I} C_i \right\} \right\rangle.$$

Proof. By theorem 4.4.2 the category $\mathcal{per}_I(\frac{X}{Z})$ lies in $\mathcal{per}(\frac{X}{Y})$, while if $x \in \mathcal{coh}\,X$ is supported within $\bigcup_{i\notin I} C_i$ then $\mathbf{R}\tau_* x$ is a sheaf on Y and hence $x \in \mathcal{per}(\frac{X}{Y})$. To furnish the required description of $\mathcal{per}(\frac{X}{Y})$, it thus suffices to show that every object in the category is an extension of complexes of the two given forms.

Now an arbitrary complex $x \in \mathcal{per}(\frac{X}{Y})$ can be written as an extension of its cohomology objects with respect to $\mathcal{coh}\,X$, namely $x_0 = H^0(x)$ and $x_1[1] = H^{-1}(x)[1]$. It is clear from (4.9) that x_0, $x_1[1]$ are themselves contained in $\mathcal{per}(\frac{X}{Y})$. Further we have $\tau_*(x_1) = 0$ and hence x_1 must be supported within the contracted curve $\bigcup_{i\in I} C_i$, from which it follows that $x_1[1] \in \mathcal{per}_I(\frac{X}{Z})$.

On the other hand, writing $\mathcal{I} \subset \mathcal{O}_X$ for the ideal sheaf of the closed subscheme $\bigcup_{i\in I} C_i$, note that $y = \mathcal{I}^n x_0$ (i.e. the image of the natural map $\mathcal{I}^n \otimes x_0 \to x_0$) is a subsheaf of x_0 (in particular, an object of $\mathcal{coh}\,X$) supported within $\bigcup_{i\notin I} C_i$ for $n \gg 0$. The quotient x_0/y is clearly supported within $\bigcup_{i\in I} C_i$. Further, applying τ_* to this quotient and examining the long exact sequence shows $\mathbf{R}^1\tau_*(x_0/y)$ vanishes since $\mathbf{R}^1\tau_*(x_0)$ does, i.e. x_0/y lies in $\mathcal{per}(\frac{X}{Y})$ (and hence in $\mathcal{per}_I(\frac{X}{Z})$).

This conclude our proof that x is an extension of objects of the required form. $\square$

It is immediate from theorem 4.4.5 that an object of $\mathcal{per}(\frac{X}{Y})$ is simple if and only if it is simple in some summand $\mathcal{per}_J(\frac{X}{Z}) \subset \mathcal{per}_I(\frac{X}{Z})$ or a simple coherent sheaf supported on the uncontracted locus, i.e. we have the following.

Corollary 4.4.6. *For* $\tau\colon X \to Y$ *the crepant contraction of* $C_I \subset X$*, the simple objects of* $\mathcal{per}(\frac{X}{Y})$ *are the skyscrapers* $\mathcal{O}_p$ *at closed points* $p \notin \bigcup_{i\in I} C_i$*, the sheaves* $\mathcal{O}_{C_j}(-1)$ *for* $j \in I$*, and suspended canonical sheaves* $\omega_{\underline{C}_J}[1]$ *for each scheme-theoretic exceptional fiber* $\underline{C}_J$ *of* τ.

Enumerating numerical hearts

(Chapter 5)

Synthesising chapters 2 to 4, we now compute the complete heart fan of the algebraic heart $\mathsf{H} = \mathit{per}(\frac{X}{Z})$ and subsequently describe every intermediate t-structure $\mathsf{K} \in \mathrm{tilt}(\mathsf{H})$ that is *numerical*, in the sense that its heart cone $\mathbf{C}\mathsf{K}$ is non-zero.

§ 5.1 Flops and mutation via wall-crossing

We show that the $\mathbf{K}$-theoretic maps induced by the mutation functors Φ_ν, Ψ_ν obey the same combinatorial rules which govern mutations of root lattices. For this, recall that we make the identification $\mathbf{K}X \simeq \mathbf{K}\Lambda \simeq \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ as in § 4.1, and more generally, we identify $\mathbf{K}_\nu\Lambda_\nu$ with $\mathfrak{h}(\underline{\Delta} \setminus \nu\mathfrak{J})$ by declaring the ith simple root α_i to be the class of the simple module S_i dual to the indecomposable projective $\mathrm{Hom}(\nu N, (\nu N)_i) \in \mathit{proj}({}_\nu\Lambda_\nu)$.

The following lemmas allow us to track simple objects under mutation functors.

Lemma 5.1.1. *For $i \in \underline{\Delta} \setminus \mathfrak{J}$ and $S_i \in \mathrm{fl}\,\mathit{mod}\,\Lambda$, we have $\Psi_{\iota(i)}(S_i) = \Phi_i^{-1}(S_i) = S_{\iota(i)}[-1] \in \mathbf{D}^{\mathrm{fl}}{}_{\nu_i}\Lambda_{\nu_i}$.*

Proof. This is [HW18, lemma 5.3], the only change is how we are indexing the simples. □

Lemma 5.1.2. *For $i \in \underline{\Delta} \setminus \mathfrak{J}$, let the non-negative integers $(b_j \mid j \in \underline{\Delta} \setminus (\mathfrak{J} + i))$ be such that N_i has minimal left $\mathrm{add}(N/N_i)$-approximation*

$$N_i \to \bigoplus_{j \in \underline{\Delta} \setminus (\mathfrak{J}+i)} N_j^{\oplus b_j}. \tag{5.1}$$

Then for a simple $S_j \in \mathrm{fl}\,\mathit{mod}\,{}_{\nu_i}\Lambda_{\nu_i}$ with $j \neq \iota(i)$, we have that $\Phi_i(S_j)$ and $\Psi_i(S_j)$ lie in the subcategory $\langle S_i, S_j \rangle \subset \mathrm{fl}\,\mathit{mod}\,\Lambda$. Further, in any Jordan–Hölder filtration of $\Phi_i(S_j)$ (resp. $\Psi_i(S_j)$), the simple S_j occurs exactly once while the simple S_i occurs b_j times.

Proof. Claim both $\Phi_i(S_j)$ and $\Psi_i(S_j)$ (which a priori are two-term complexes of Λ-modules) actually lie in $\mathrm{fl}\,\mathit{mod}\,\Lambda$, that is to say $\mathrm{Tor}_{>0}(S_j\,,\,{}_{\nu_i}\Lambda) = 0 = \mathrm{Ext}^{>0}(\Lambda_{\nu_i}\,,\,S_j)$ over ${}_{\nu_i}\Lambda_{\nu_i}$. This claim is essentially the content of [Wem18, corollary 5.10], where it is proved via identities on singular Calabi–Yau algebras and the Auslander–Buchsbaum formula. Notwithstanding, we present an elementary alternative proof.

To see the Ext-vanishing, consider the projective cover of right modules $P_j \to S_j \to 0$. The tilting module Λ_{ν_i} has projective dimension $\leqslant 1$, so we have an exact sequence $\mathrm{Ext}^1(\Lambda_{\nu_i}, P_j) \to \mathrm{Ext}^1(\Lambda_{\nu_i}, S_j) \to 0$. But P_j is a summand of Λ_{ν_i}, and the latter has no self-extensions whence every term in the sequence vanishes. Higher ext-groups vanish for projective dimension reasons.

To see the Tor-vanishing, we first note the decomposition of left modules

$${}_{\nu_i}\Lambda = \mathrm{Hom}(N_i, \nu_i N) \;\oplus \bigoplus_{k \in \underline{\Delta} \setminus (\mathfrak{J}+i)} Q_k$$

where each $Q_k = \mathrm{Hom}(N_k, \nu_i N)$ appearing above is an indecomposable projective. By flatness of projectives, we deduce $\mathrm{Tor}_{>0}(S_j\,,\,{}_{\nu_i}\Lambda) = \mathrm{Tor}_{>0}(S_j\,,\,\mathrm{Hom}(N_i, \nu_i N))$.

We compute this Tor by writing a projective resolution of $\mathrm{Hom}(N_i, \nu_i N) \in {}_{\nu_i}\Lambda_{\nu_i}\text{-}\mathit{mod}$, for which we follow the arguments of remark 4.2.5 to extend (5.1) to the exact sequence

$$0 \to N_i \to \bigoplus_{j \in \Delta \setminus (\mathfrak{J}+i)} N_j^{\oplus b_j} \to N'_{\iota(i)} \tag{5.2}$$

where $N'_{\iota(i)} \subset \nu_i N$ is the summand corresponding to the projective $Q_{\iota(i)} = \mathrm{Hom}(N'_{\iota(i)}, \nu_i N)$. Applying $\mathrm{Hom}(-, \nu_i N)$ as in [IW14, proposition 6.4] then gives the required projective resolution

$$0 \to Q_{\iota(i)} \to \bigoplus_{j \in \Delta \setminus (\mathfrak{J}+i)} Q_j^{\oplus b_j} \to \mathrm{Hom}(N_i, \nu_i N) \to 0.$$

Thus the groups $\mathrm{Tor}_{>1}(S_j, \mathrm{Hom}(N_i, \nu_i N))$ again vanish for projective dimension reasons, while the group $\mathrm{Tor}_1(S_j, \mathrm{Hom}(N_i, \nu_i N))$ vanishes since it injects into $S_j \otimes Q_{\iota(i)}$ $(= 0$ since S_j is annihilated by the $\iota(i)$th vertex idempotent of ${}_{\nu_i}\Lambda_{\nu_i})$.

Given the claim, the required result follows immediately from [Wem18, lemma 5.7]. □

Proposition 5.1.3. *For $i \in \Delta \setminus \mathfrak{J}$, the simple mutation functors $\Phi_i, \Psi_i \colon \mathbf{D}^{\mathrm{fl}}\,{}_{\nu_i}\Lambda_{\nu_i} \to \mathbf{D}^{\mathrm{fl}}\Lambda$ both induce the same map on $\mathbf{K}$-theory and this agrees with the map $\varphi_i \colon \mathfrak{h}(\Delta \setminus \nu_i\mathfrak{J}) \to \mathfrak{h}(\Delta \setminus \mathfrak{J})$ given as in (3.5).*

Proof. This assertion is due to Nabijou–Wemyss [NW23, remark 5.2], we spell out the details for clarity. Evidently from lemmas 5.1.1 and 5.1.2, both Φ_i and Ψ_i induce the same map on $\mathbf{K}$-theory which we momentarily call $\psi_i \colon \mathfrak{h}(\Delta \setminus \nu_i\mathfrak{J}) \to \mathfrak{h}(\Delta \setminus \mathfrak{J})$.

As in the proof of theorem 4.2.4, the connection to mutation-combinatorics comes by reducing to surfaces. Thus we consider a general elephant $\mathrm{Spec}\,\overline{R} \hookrightarrow \mathrm{Spec}\,R$ and the reduction functor $\mathbb{F}(-) = (-) \otimes_R \overline{R}$.

In the natural identifications of vector spaces

$$\mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\mathbb{F}\Lambda) \otimes \mathbb{R} \quad \simeq \quad \mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\Lambda) \otimes \mathbb{R} \quad \simeq \quad \Theta(\Delta \setminus \mathfrak{J}),$$

the first map (induced by $\mathbb{F}$) is an identification of g-fans [Gar23, proposition B]. Passing to $\Theta(\Delta \setminus \mathfrak{J})$, Iyama–Wemyss [IW, theorem 5.2] show that this g-fan of the contracted preprojective algebra $\mathbb{F}\Lambda$, and hence also that of the modification algebra Λ, coincides with the fan $\mathrm{Arr}^+(\Delta, \mathfrak{J}) \cup \mathrm{Arr}^-(\Delta, \mathfrak{J})$.

Analogously, the g-fan of ${}_{\nu_i}\Lambda_{\nu_i}$ sits in the $\nu_i\mathfrak{J}$-cone arrangement.

Being induced by a derived equivalence, the map

$$(\psi_i{}^{\vee})^{-1} \colon \mathbf{K}_{\mathrm{split}}(\mathit{proj}\;{}_{\nu_i}\Lambda_{\nu_i}) \otimes \mathbb{R} \to \mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\Lambda) \otimes \mathbb{R},$$

respects silting theory and is a map of g-fans. In particular, the image of $C^+_{\nu_i\mathfrak{J}}$ under $(\psi_i{}^{\vee})^{-1}$ is a cone in the intersection arrangement of $\Theta(\Delta \setminus \mathfrak{J})$. Reading off the matrix $(\psi_i{}^{\vee})^{-1}$ from lemmas 5.1.1 and 5.1.2, we see that $(\psi_i{}^{\vee})^{-1} C^+_{\nu_i\mathfrak{J}}$ intersects (and hence is equal to) the cone $\nu_i C^+_{\mathfrak{J}}$, and the images of primitive integral generators are primitive integral.

By proposition 3.2.1 the map $(\varphi_i{}^{\vee})^{-1}$ admits an identical description, so that $\psi_i = \varphi_i$ as required. □

The above proposition does not depend on the specific choice of modifying module, and if ν is any $\mathfrak{J}$-path and $i \in \Delta \setminus \nu\mathfrak{J}$, the functors $\Psi_i, \Phi_i \colon \mathbf{D}^{\mathrm{fl}}\,{}_{\nu_i\nu}\Lambda_{\nu_i\nu} \to \mathbf{D}^{\mathrm{fl}}\,{}_{\nu}\Lambda_{\nu}$ analogously induce the map φ_i on $\mathbf{K}$-theory. Combined with lemma 4.2.8, we therefore see that if $\nu = \nu_{i_n} \ldots \nu_{i_1}$ is a $\mathfrak{J}$-path then the maps on $\mathbf{K}$-theory induced by Φ_ν and Ψ_ν both agree with the composite $\varphi_\nu = \varphi_{i_n} \circ \cdots \circ \varphi_{i_1}$.

We register the following corollary.

Corollary 5.1.4. *If ν is a $\mathfrak{J}$-path and s is a simple ${}_{\nu}\Lambda_{\nu}$-module, then the object $\Phi_\nu(s)$ (resp. $\Psi_\nu(s)$) in $\mathbf{D}^{\mathrm{fl}}\Lambda$ has $\mathbf{K}$-theory class given by a primitive positive restricted root in $\mathfrak{h}(\Delta \setminus \mathfrak{J})$.*

Cones via geometric invariant theory. We have seen that the set with Δ-mutation $\mathrm{MMG}^{\mathrm{N}}\mathrm{R}$ is naturally isomorphic to both $\mathrm{Bir}(\frac{X}{Z})$ and $\mathrm{Cham}(\Delta,\mathfrak{J})$. Karmazyn [Kar17b, §5.2] and Wemyss [Wem18, §5] use moduli constructions to provide another natural interpretation of the bijection $\mathrm{Bir}(\frac{X}{Z}) \simeq \mathrm{Cham}(\Delta,\mathfrak{J})$.

The category $\mathcal{P}er(\frac{X}{Z}) \cong \mathit{mod}\,\Lambda$ can be identified with a subcategory of representations of a quiver with vertices $\underline{\Delta}\setminus\mathfrak{J}$ – this is immediate when $\dim Z = 2$ since Λ is a contracted preprojective algebra, and it is proven by Wemyss [Wem18, theorem 2.15] when $\dim Z = 3$. Viewing elements of $\mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J}) = \bigoplus_{i\in\underline{\Delta}\setminus\mathfrak{J}} \mathbb{Z}\cdot\alpha_i$ as dimension vectors for this quiver, the identification is such that the dimension vector of the representation corresponding to $x \in \mathit{per}(\frac{X}{Z})$ is precisely the $\mathbf{K}$-theory class $[x]$.

We are particularly interested in representations of dimension $\delta_{\mathfrak{J}}$, since they correspond to point sheaves.

Lemma 5.1.5. *Let $p \in X$ be a closed point on the exceptional fibre of $\pi\colon X \to Z$. Identifying $\mathbf{K}X \simeq \mathbf{K}\Lambda$ with $\mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J})$, any skyscraper sheaf $\mathcal{O}_p \in \mathit{coh}\,X$ has $\mathbf{K}$-theory class equal to $\delta_{\mathfrak{J}}$. Likewise for any birational model $\nu X \to Z$ containing a closed point q in its exceptional fibre, the perverse point sheaf $\Psi_\nu\mathcal{O}_q \in \mathit{per}(\frac{X}{Z})$ has $\mathbf{K}$-theory class $\delta_{\mathfrak{J}}$.*

Proof. By [Kar17b, theorem 5.2.4], the $\mathbf{K}$-theory class of $\mathcal{O}_p$ is determined by the indecomposable summands of $\mathcal{N}$ as $[\mathcal{O}_p] = \sum \mathrm{rk}(\mathcal{N}_i)\cdot\alpha_i$. By [IW, proposition 9.4], this vector is precisely $\delta_{\mathfrak{J}}$. The statement for $\Psi_\nu\mathcal{O}_q$ follows since the map $\mathbf{K}(\nu X) \to \mathbf{K}X$ induced by Ψ_ν sends $\delta_{\nu\mathfrak{J}}$ to $\delta_{\mathfrak{J}}$. □

Now elements of the dual vector space $\Theta(\underline{\Delta}\setminus\mathfrak{J})$ naturally give stability parameters (in the sense of [King94]) on $\mathcal{P}er(\frac{X}{Z})$ and thus for any stability parameter $\theta \in \{\delta_{\mathfrak{J}} = 0\}$ we can construct the coarse moduli space $\mathcal{M}_\Lambda(\theta,\delta_{\mathfrak{J}})$ of θ-semistable perverse sheaves of dimension $\delta_{\mathfrak{J}}$. When θ is generic in a chamber $\nu C^0_{\mathfrak{J}}$, there are no strictly semistable objects and $\mathcal{M}_\Lambda(\theta,\delta_{\mathfrak{J}})$ is a fine moduli space.

Proposition 5.1.6. *There is a natural isomorphism $\mathcal{M}_\Lambda(\theta,\delta_{\mathfrak{J}}) \cong \nu X$ under which closed points $p \in \nu X$ correspond to perverse point sheaves $\Psi_\nu\mathcal{O}_p \in \mathcal{P}er(\frac{X}{Z})$.*

Proof. If θ lies in $C^+_{\mathfrak{J}}$ (i.e. if ν is the empty path) then this is [Kar17b, corollary 5.24].

For more general ν, we can first consider the vector $\varphi_\nu^\vee\theta \in C^+_{\nu\mathfrak{J}}$ and construct the fine moduli space $\mathcal{M}_{{}_\nu\Lambda_\nu}(\varphi_\nu^\vee\theta,\delta_{\nu\mathfrak{J}})$ of ${}_\nu\Lambda_\nu$-representations. We have seen this is isomorphic to νX and its closed points parametrise point sheaves in $\mathcal{P}er(\frac{\nu X}{Z})$. Then [Wem18, corollary 4.13] shows that there is an isomorphism of schemes $\mathcal{M}_{{}_\nu\Lambda_\nu}(\varphi_\nu^\vee\theta,\delta_{\nu\mathfrak{J}}) \simeq \mathcal{M}_\Lambda(\theta,\delta_{\mathfrak{J}})$ for which the bijection between points is induced by Ψ_ν. □

The above moduli construction yields the bijection $\mathrm{Cham}(\Delta,\mathfrak{J}) \to \mathrm{Bir}(\frac{X}{Z})$. Another upshot for us is that this gives excellent control over the subcategories of semistable objects.

Proposition 5.1.7. *For a chamber $\nu C^0_{\mathfrak{J}} \in \mathrm{Cham}(\Delta,\mathfrak{J})$ and a generic vector $\theta \in \nu C^0_{\mathfrak{J}}$, an object of $\mathcal{H} = \mathit{per}(\frac{X}{Z})$ is θ-stable if and only if it is equal to a perverse point sheaf $\Psi_\nu\mathcal{O}_p$ for some closed point $p \in \nu X$. In particular, the subcategory of semistable objects associated to $\nu C^0_{\mathfrak{J}}$ is given by the extension closure of such objects.*

Proof. Any object in $\mathcal{H}_{ss}(\nu C^0_{\mathfrak{J}})$ admits a finite filtration by θ-stable objects so it suffices to characterise the latter set. Furthermore, by [Wem18, theorem 5.12] an object of $\mathit{per}(\frac{X}{Z})$ is θ-semistable if and only if it is equal to $\Psi_\nu(x)$ for some $\varphi_\nu^\vee\theta$-semistable object $x \in \mathit{per}(\frac{\nu X}{Z})$. Thus it suffices to show that for a generic $\theta \in C^0_{\mathfrak{J}}$, the θ-stable objects in $\mathit{per}(\frac{X}{Z})$ are precisely all the skyscraper sheaves on the exceptional fibre.

Now if an object is θ-stable, it remains so upon small perturbations of θ within the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$ and thus it must have dimension vector $n\delta_{\mathfrak{J}}$ for some integer $n > 0$. The objects with $n = 1$ can be read off from the closed points in $\mathcal{M}_\Lambda(\theta,\delta_{\mathfrak{J}}) \cong X$, thus the collection of θ-stable objects in $\mathit{per}(\frac{X}{Z})$ that have $\mathbf{K}$-theory class $\delta_{\mathfrak{J}}$ is precisely $\{\mathcal{O}_p \mid p \in X(\mathbb{C})\}$.

We show the case $n \geqslant 2$ does not occur, by replicating the argument of [Gar22, lemma 4.12]. Indeed suppose for the sake of contradiction that $x \in \mathit{per}(\frac{X}{Z})$ is θ-stable and $[x] = n\delta_{\mathfrak{J}}$ for $n \geqslant 2$. Since any skyscraper sheaf $\mathcal{O}_p \in \mathcal{P}er(\frac{X}{Z})$ is θ-stable and necessarily distinct from x, we have $\mathrm{Hom}(x,\mathcal{O}_p) = 0$ for all $p \in X$. Combining this with the fact that x is a two-term complex of coherent sheaves, we see that the sheaf $H^0_{\mathit{coh}\,X}(x)$ has empty support and hence $x[-1]$ is a sheaf on X. Choosing a sufficiently ample bundle $\mathcal{L}$ on X then gives us $\chi(\mathcal{L}\otimes x[-1]) \geqslant 0$, but this is absurd since x is numerically equivalent to some skyscraper sheaf $\mathcal{O}_p^{\oplus n}$. □

Combinatorics of the movable fan. The correspondence between the $\mathrm{Cham}(\Delta, \mathfrak{J})$ and $\mathrm{Bir}\left(\frac{X}{Z}\right)$ can also be described directly via the classical birational geometry of movable divisors.

Suppose W is a birational model of X with associated Dynkin data $\mathfrak{J} = \mathbb{J}(W)$. Given the flopping contraction $\pi\colon W \to Z$ with reduced exceptional fibre $\bigcup_{i\in\Delta\setminus\mathfrak{J}} C_i$, it is well known [see for example Van04, lemma 3.4.3] that the Picard group $\operatorname{Pic} W$ is naturally dual to the group of π-relative 1-cycles $Z_1(W) = \bigoplus_{i\in\Delta\setminus\mathfrak{J}} \mathbb{Z}\cdot C_i$ via the intersection pairing $(\mathscr{L}\cdot C_i) = \deg(\mathscr{L}|_{C_i})$.

If $\mathcal{V}\left(\frac{W}{Z}\right)$ has indecomposable summands $(\mathcal{N}_i')_{i\in\Delta\setminus\mathfrak{J}}$ indexed as in lemma 4.1.1, then the line bundles $\mathscr{L}_i' = (\det \mathcal{N}_i')^\vee$ $(i \in \Delta\setminus\mathfrak{J})$ form a basis of $\operatorname{Pic} W$ dual to the standard basis of $Z_1(W)$.

In the $\dim Z = 3$ case of setup 1.0.2, there is a naturally induced isomorphism between the Picard groups of any two birational models.

Lemma 5.1.8. *The threefold flop* $\rho\colon W \dashrightarrow X$ *induces an isomorphism of Picard groups* $\rho_*\colon \operatorname{Pic} W \to \operatorname{Pic} X$.

Proof. Since ρ is an isomorphism away from a codimension two locus, taking proper transforms gives an isomorphism of class groups $\rho_*\colon \mathrm{Cl}(W) \to \mathrm{Cl}(X)$. We show that the proper transform of a Cartier divisor remains Cartier. For this it suffices to consider the case when $W = \nu_i X$ is the flop of X in a single curve C_i, and show that if $D_j \subset W$ is a Cartier divisor representing a basis element $\mathscr{L}_j' \in \operatorname{Pic} W$ then $\rho_* D_j \subset X$ is Cartier too.

Thus writing $C_{\iota(i)}$ for the flopped curve in W, if $j \neq \iota(i)$ then D_j may be chosen to lie entirely in $W \setminus C_{\iota(i)} \simeq X \setminus C_i$ so that its proper transform is Cartier. On the other hand if $j = \iota(i)$, then $\rho_*(-D_j)$ (and hence also $\rho_* D_j$) is shown to be Cartier in [Wem18, proof of theorem 4.17 (1)]. □

The isomorphism above can be computed explicitly in terms of the natural bases $\{\mathscr{L}_i' \mid i \in \Delta\setminus\mathfrak{J}\} \subset \operatorname{Pic} W$ and (analogously defined) $\{\mathscr{L}_i \mid i \in \Delta\setminus\mathfrak{J}\} \subset \operatorname{Pic} X$. Suppose for simplicity W is the birational model $\nu_i X$, obtained by flopping C_i. It is evident from the above proof that $\rho_*\mathscr{L}_j' = \mathscr{L}_j$ if $j \neq \iota(i)$. To compute the proper transform $\rho_*\mathscr{L}_{\iota(i)}'$, we use the isomorphisms of class groups $\mathrm{Cl}(X) \simeq \mathrm{Cl}(Z)$ and $\mathrm{Cl}(W) \simeq \mathrm{Cl}(Z)$ to calculate in the category of reflexive R-modules. Here we know that $\pi_*\mathcal{N}_i = N_i$ is the kernel of a minimal right $\mathrm{add}(\nu_i N/N_{\iota(i)})$ approximation f of $\pi_*\mathcal{N}_{\iota(i)}' = N_{\iota(i)}$, therefore sitting in the exact sequence (5.2). Further, $\nu_i N/N_{\iota(i)}$ contains $N_0 = R$ as a summand, whence the approximation f is surjective. Considering determinants gives us the identity $[\det N_{\iota(i)}] + [\det N_i] = \sum_{j\in\underline{\Delta}\setminus(\mathfrak{J}+i)} b_j\,[\det N_j]$ in $\mathrm{Cl}(Z)$, so that the map ρ_* can be read off as

$$\rho_*\mathscr{L}_j' = \mathscr{L}_j \quad (j \neq \iota(i)), \qquad \rho_*\mathscr{L}_{\iota(i)}' = \bigotimes_{j\in\underline{\Delta}\setminus(\mathfrak{J}+i)} \mathscr{L}_j^{\otimes b_j} \otimes \mathscr{L}_i^{\vee}. \tag{5.3}$$

Remark 5.1.9. In the $\dim Z = 2$ case of setup 1.0.2, given the tagged partial resolution $X = (C^0_{\mathfrak{J}}, X)$ and its birational model $\nu_i X = (\nu_i C^0_{\mathfrak{J}}, W)$, the birational map $\rho\colon W \dashrightarrow X$ of interest is defined away from the codimension one locus $C_{\iota(i)} \subset W$ and there is no induced isomorphism of class groups as such. In this case we *declare* the isomorphism of Picard groups $\rho_*\colon \operatorname{Pic} W \to \operatorname{Pic} X$ to be given by the formula (5.3).

For any $W \in \mathrm{Bir}\left(\frac{X}{Z}\right)$, write $\operatorname{Pic}^+ W$ for the submonoid of $\operatorname{Pic} W$ generated by the basis elements $(\mathscr{L}_i')$. This is the monoid of ***nef*** line bundles (i.e. line bundles which intersect every exceptional curve non-negatively). The birational map $\rho\colon W \dashrightarrow X$ allows us to consider the submonoid $\rho_*(\operatorname{Pic}^+ W) \subset \operatorname{Pic} X$ of line bundles on X which are nef on W, and we now argue that considering all such submonoids gives a decomposition

$$\operatorname{Pic} X = \bigcup_{W\in\mathrm{Bir}\left(\frac{X}{Z}\right)} \rho_*(\operatorname{Pic}^+ W) \tag{5.4}$$

whose combinatorics are given by the wall–crossing rules of Iyama–Wemyss.

For this it is convenient to consider the space of $\mathbb{R}$-divisors $\operatorname{Pic}_{\mathbb{R}} X = \operatorname{Pic} X \otimes \mathbb{R}$. The nef cone $\operatorname{Pic}^+_{\mathbb{R}} W$ is then an orthant generated by the basis vectors $\mathscr{L}_i$, and the isomorphisms of Picard groups given by proper transforms extend linearly to give isomorphisms $\rho_*\ \operatorname{Pic}_{\mathbb{R}} W \to \operatorname{Pic}_{\mathbb{R}} X$ so that the nef cone $\rho_*(\operatorname{Pic}^+_{\mathbb{R}} W)$ is a rational polyhedral cone in $\operatorname{Pic}_{\mathbb{R}} X$. We then show the following.

Proposition 5.1.10. *Given a flopping contraction $\pi\colon X \to Z$, the collection of rational polyhedral cones given by*

$$\operatorname{Mov}(X) = \bigcup_{W \in \operatorname{Bir}(\frac{X}{Z})} \operatorname{faces}(\rho_* \operatorname{Pic}^+_{\mathbb{R}} W)$$

forms a complete simplicial fan in $\operatorname{Pic}_{\mathbb{R}} X$ in which each nef cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ is full-dimensional.

Further, a natural linear isomorphism of $\operatorname{Pic}_{\mathbb{R}} X$ with the hyperplane $\{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \mid \theta(\delta_{\mathfrak{J}}) = 0\}$ induces an isomorphism of fans $\operatorname{Mov}(X) \to \operatorname{Arr}(\Delta, \mathfrak{J})$ such that for each sequence of flops ν from X, the cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}}(\nu X)$ gets identified with the chamber $\nu C^0_{\mathfrak{J}}$.

Note the decomposition (5.4) can be immediately deduced by noting that each nef monoid is the set of integral points in the corresponding nef cone.

In order to prove proposition 5.1.10, we first explain the construction of the natural map $\operatorname{Pic}_{\mathbb{R}} X \to \Theta(\underline{\Delta}, \mathfrak{J})$. Now the objects in $\mathcal{coh}\, X$ have proper support of dimension $\leqslant 1$, so for any $\mathscr{L} \in \operatorname{Pic} X$ and $\mathscr{F} \in \mathcal{coh}\, X$ we have a well-defined Snapper–Kleiman intersection number

$$(\mathscr{L} \cdot \mathscr{F}) = \chi(\mathscr{F}) - \chi(\mathscr{L}^\vee \otimes \mathscr{F}).$$

By standard properties of the intersection product [see for example Kol96, §VI] this defines a $\mathbb{Z}$-bilinear map $(- \cdot -) : \operatorname{Pic} X \otimes \mathbf{K}X \to \mathbb{Z}$, and therefore a linear map $\operatorname{Pic}_{\mathbb{R}} X \to \operatorname{Hom}(\mathbf{K}X, \mathbb{R})$. But recall that we identify $\mathbf{K}X$ (across the equivalence (4.1)) with $\mathbf{K}\Lambda$, and hence with the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$. This identifies $\operatorname{Hom}(\mathbf{K}X, \mathbb{R})$ with the vector space $\Theta(\underline{\Delta}, \mathfrak{J})$ furnishing the desired map.

We can likewise construct maps $\operatorname{Pic}_{\mathbb{R}} W \to \Theta(\underline{\Delta}, \mathbb{J}(W))$ for each birational model. These maps induced by intersection pairings are compatible with various identifications induced by flops.

Lemma 5.1.11. *Given a birational model $W = \nu X$, the two intersection pairings*

$$\operatorname{Pic} X \otimes \mathbf{K}X \to \mathbb{Z} \quad \textit{and} \quad \operatorname{Pic} W \otimes \mathbf{K}W \to \mathbb{Z}$$

are compatible with the isomorphisms $\operatorname{Pic} W \cong \operatorname{Pic} X$ and $\mathbf{K}W \cong \mathbf{K}X$, where the Picard groups are identified by taking proper transforms across $\rho : W \dashrightarrow X$, and the Grothendieck groups are identified via the flop equivalence.

Proof. It suffices to prove this for a single-curve flop $W = \nu_i X$, $i \in \Delta \setminus \mathfrak{J}$. Let $\{\mathscr{L}_i \mid i \in \underline{\Delta} \setminus \mathfrak{J}\} \subset \operatorname{Pic} X$ and $\{\mathscr{L}'_i \mid i \in \underline{\Delta} \setminus \nu_i\mathfrak{J}\} \subset \operatorname{Pic} W$ be the natural bases, these are related by the formula (5.3).

The Grothendieck groups too have preferred choices of bases in which the intersection product is readily computed – for $\mathbf{K}W$ we choose the class of a skyscraper $\mathscr{O}_p$ ($p \in W$) and the classes of simple perverse sheaves $S_k = \mathscr{O}_{C_k}(-1)$ ($k \in \Delta \setminus \nu_i\mathfrak{J}$), and likewise for $\mathbf{K}X$. By lemmas 5.1.1 and 5.1.2, the flop functor Ψ_i relates these bases as

$$[\Psi_i \mathscr{O}_p] = [\mathscr{O}_q], \qquad [\Psi_i S_{\iota(i)}] = -[S_i], \qquad [\Psi_i S_k] = [S_k] + b_k[S_i] \quad (k \in \Delta \setminus (\mathfrak{J} + i)).$$

To prove the result it suffices to show the equalities of intersection pairings

$$(\rho_*\mathscr{L}'_j,\ \Psi_i\mathscr{O}_p) = (\mathscr{L}'_j,\ \mathscr{O}_p) \quad \text{and} \qquad (\rho_*\mathscr{L}'_j,\ \Psi_i S_k) = (\mathscr{L}'_j,\ S_k) \qquad \text{for all } j, k \in \Delta \setminus \nu_i\mathfrak{J}$$

which are straightforward to verify explicitly. □

We can now prove the proposition.

Proof of proposition 5.1.10. The intersection of any line bundle with a sheaf that has zero-dimensional support is trivial, thus considering $\mathscr{F} = \mathscr{O}_p$ (the skyscraper sheaf at a closed point) and using lemma 5.1.5 shows that the map $\operatorname{Pic}_{\mathbb{R}} X \to \Theta(\underline{\Delta} \setminus \mathfrak{J})$ constructed above has image in $\{\delta_{\mathfrak{J}} = 0\}$. On the other hand considering the sheaves $\mathscr{F} = \mathscr{O}_{C_i}(-1)$ shows the map is injective, and hence an isomorphism onto the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$ by comparing dimensions.

Now the nef cone $\operatorname{Pic}^+_{\mathbb{R}} X$ is spanned by divisors which pair with each $\alpha_i = [\mathscr{O}_{C_i}(-1)]$ non-negatively, and hence gets identified with $C^0_{\mathfrak{J}}$ by the above injection. If $W = \nu X$ is another birational model, its nef cone likewise gets identified with the chamber $C^0_{\nu\mathfrak{J}} \subset \Theta(\underline{\Delta}, \nu\mathfrak{J})$. By lemma 5.1.11 these identifications are compatible with flops and mutation, i.e. there is a commutative square

$$\begin{tikzcd} \operatorname{Pic}_{\mathbb{R}} W \arrow[r, "\rho_*"] \arrow[d, hook] & \operatorname{Pic}_{\mathbb{R}} X \arrow[d, hook] \\ \Theta(\underline{\Delta} \setminus \nu\mathfrak{J}) & \Theta(\underline{\Delta} \setminus \mathfrak{J}). \arrow[l, "\varphi_{\check{\nu}}"] \end{tikzcd}$$

Thus the nef cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ in $\operatorname{Pic}_{\mathbb{R}} X$ gets mapped to the cone $(\varphi_{\check{\nu}})^{-1} C^0_{\nu\mathfrak{J}} = \nu C^0_{\mathfrak{J}}$ in $\Theta(\underline{\Delta}, \mathfrak{J})$, in particular each nef cone is full dimensional and all maximal cones in $\operatorname{Mov}(X)$ arise as nef cones. It follows that there is an isomorphism of fans $\operatorname{Mov}(X) \to \operatorname{Arr}(\Delta, \mathfrak{J})$ of the required form. The remaining properties (completeness, simpliciality) of $\operatorname{Mov}(X)$ follow from the corresponding properties of $\operatorname{Arr}(\Delta, \mathfrak{J})$. □

We examine some geometric consequences for 3-folds. The completeness of the fan $\operatorname{Mov}(X)$ implies that any line bundle $\mathcal{L} \in \operatorname{Pic} X$, after some appropriate sequence of flops, becomes nef and hence globally generated [see for example Van04, lemma 3.4.5]. In particular the base locus of the linear system associated to $\mathcal{L}$ is contained entirely within the exceptional curves in X, which has co-dimension 2. Recalling that an $\mathbb{R}$-divisor is *movable* if and only if it is a non-negative linear combinations of such line bundles, we have the following.

Corollary 5.1.12. *If $\pi : X \to Z$ is a threefold flopping contraction, every divisor on X is movable.*

This explains the notation $\operatorname{Mov}(X)$ for the fan, since its support can be identified with the cone of movable divisors on X. When X is a minimal model of Z, work of Kawamata [Kaw88], Kollár [Kol89], Mori [Mor82], and Reid [Rei] shows that the (closure of the) cone of movable divisors can be decomposed into a union of nef cones of minimal models in a way that that the nef cones of non-isomorphic models have disjoint interiors [the result is summarised in Mat02, theorem 12.2.7]. Via mutation combinatorics of birational models, modifying generators, and chambers in the intersection arrangement, this decomposition theorem generalises to arbitrary flopping contractions over isolated cDV singularities.

Corollary 5.1.13. *Given a flopping contraction $\pi : X \to Z$, the nef cones $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ for $W \in \operatorname{Bir}(\frac{X}{Z})$ cover the cone of movable divisors and their interiors (i.e. the corresponding ample cones) are pairwise disjoint.*

§ 5.2 Algebraic intermediate hearts

Turning to the poset $\operatorname{tilt}(H)$, we now show all algebraic hearts that are intermediate with respect to H must arise via the mutation functors, and show that the corresponding heart cones fill out the subfan $\operatorname{Arr}^+(\underline{\Delta}, \mathfrak{J}) \cup \operatorname{Arr}^-(\underline{\Delta}, \mathfrak{J})$ in the intersection arrangement.

Each modifying module $\nu N \in \operatorname{MM}^N(R)$ yields a tilting Λ-module ${}_\nu\Lambda$ and a tilting ${}_\nu\Lambda_\nu$-module Λ_ν, defining the functors $\mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \rightrightarrows \mathbf{D}^{\mathrm{fl}} \Lambda$ given as in (4.6). Applying the Brenner–Butler theorem [AHK, chapter 4, remark 3.10] to these equivalences gives two torsion pairs

$$\mathrm{fl}\,\mathit{mod}\,\Lambda = T_\nu * F_\nu = U_\nu * V_\nu, \tag{5.5}$$

$$\begin{aligned} T_\nu &= \{x \in \mathrm{fl}\,\mathit{mod}\,\Lambda \mid \operatorname{Ext}^1({}_\nu\Lambda, x) = 0\} & U_\nu &= \{x \in \mathrm{fl}\,\mathit{mod}\,\Lambda \mid x \otimes \Lambda_\nu = 0\} \\ F_\nu &= \{x \in \mathrm{fl}\,\mathit{mod}\,\Lambda \mid \operatorname{Hom}({}_\nu\Lambda, x) = 0\} & V_\nu &= \{x \in \mathrm{fl}\,\mathit{mod}\,\Lambda \mid \operatorname{Tor}_1(x, \Lambda_\nu) = 0\} \end{aligned}$$

such that the squares below commute.

$$\begin{tikzcd} \mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \arrow[r, "\Phi_\nu", "\sim"'] & \mathbf{D}^{\mathrm{fl}} \Lambda \\ \mathrm{fl}\,\mathit{mod}\, {}_\nu\Lambda_\nu \arrow[u, hook] \arrow[r, "\sim"] & F_\nu[1] * T_\nu \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \arrow[r, "\Psi_\nu", "\sim"'] & \mathbf{D}^{\mathrm{fl}} \Lambda \\ \mathrm{fl}\,\mathit{mod}\, {}_\nu\Lambda_\nu \arrow[u, hook] \arrow[r, "\sim"] & V_\nu * U_\nu[-1] \arrow[u, hook] \end{tikzcd}$$

Abusing notation to write H for $\mathrm{fl}\,\mathit{mod}\,\Lambda \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ as well as $\mathrm{fl}\,\mathit{mod}\, {}_\nu\Lambda_\nu \subset \mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu$, each $\mathfrak{J}$-path ν thus gives us two hearts in $\mathbf{D}^{\mathrm{fl}}\Lambda$ that are intermediate with respect to $\mathrm{fl}\,\mathit{mod}\,\Lambda$, namely

$$\Phi_\nu H[-1] = F_\nu * T_\nu[-1], \qquad \Psi_\nu H = V_\nu * U_\nu[-1]. \tag{5.6}$$

Evidently the hearts (5.6) are algebraic, allowing us to define the following sub-posets of alg-tors(H), alg-torf(H), and alg-tilt(H).

$$\begin{aligned} &\mathrm{tors}^-(H) = \{T_\nu \mid \nu \text{ a } \mathfrak{J}\text{-path}\} && \mathrm{torf}^-(H) = \{F_\nu \mid \nu \text{ a } \mathfrak{J}\text{-path}\} && \mathrm{tilt}^-(H) = \{\Phi_\nu H[-1] \mid \nu \text{ is a } \mathfrak{J}\text{-path}\} \\ &\mathrm{tors}^+(H) = \{U_\nu \mid \nu \text{ a } \mathfrak{J}\text{-path}\} && \mathrm{torf}^+(H) = \{V_\nu \mid \nu \text{ a } \mathfrak{J}\text{-path}\} && \mathrm{tilt}^+(H) = \{\Psi_\nu H \mid \nu \text{ a } \mathfrak{J}\text{-path}\} \end{aligned}$$

Each of the above sets is naturally in bijection with $\mathrm{MM}^N(R)$, where the bijection simply identifies representative $\mathfrak{J}$-paths. This is well-defined because the constructions of § 4.2 depend only on the end points of positive paths in $\mathrm{ExQuiv}(\mathrm{MM}^N R)$. In other words, if ν, ν' are $\mathfrak{J}$-paths such that $\nu N = \nu' N$ in $\mathrm{MM}^N(R)$, then we also have $T_\nu = T_{\nu'}$, $U_\nu = U_{\nu'}$, and so on. It follows that each of the above subposets carries the structure of a set with $\underline{\Delta}$-mutation.

In chapter 6 we will show that the bijections respect partial orders, too.

Algebraic heart cones. Now $\mathrm{MM}^N(R)$ is naturally in bijection with both $\mathrm{tilt}^+(H)$ and $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$. From the computation proposition 5.1.3 we deduce the induced bijection $\mathrm{tilt}^+(H) \to \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is nothing but the assignment of heart cones $K \mapsto \mathbf{C}K$. The heart cones of hearts in $\mathrm{tilt}^-(H)$ are computed analogously, and the following theorem summarises the situation.

Theorem 5.2.1. *For each $\mathfrak{J}$-path ν, the cones $\nu C_{\mathfrak{J}}^{\pm} \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$ are heart cones in* $\mathrm{HFan}(H)$. *In particular, these are described as*

$$\mathbf{C}(\Psi_\nu H) = \nu C_{\mathfrak{J}}^+, \qquad \mathbf{C}(\Phi_\nu H[-1]) = \nu C_{\mathfrak{J}}^-$$

and therefore the associated numerical torsion theories and intermediate hearts are

$$\begin{aligned} &H_{\mathrm{tr}}(\nu C_{\mathfrak{J}}^+) = H^{\mathrm{tr}}(\nu C_{\mathfrak{J}}^+) = U_\nu, && H_{\mathrm{tf}}(\nu C_{\mathfrak{J}}^+) = H^{\mathrm{tf}}(\nu C_{\mathfrak{J}}^+) = V_\nu, && H_{\mathrm{tt}}(\nu C_{\mathfrak{J}}^+) = H^{\mathrm{tt}}(\nu C_{\mathfrak{J}}^+) = \Psi_\nu H, \\ &H_{\mathrm{tr}}(\nu C_{\mathfrak{J}}^-) = H^{\mathrm{tr}}(\nu C_{\mathfrak{J}}^-) = T_\nu, && H_{\mathrm{tf}}(\nu C_{\mathfrak{J}}^-) = H^{\mathrm{tf}}(\nu C_{\mathfrak{J}}^-) = F_\nu, && H_{\mathrm{tt}}(\nu C_{\mathfrak{J}}^-) = H^{\mathrm{tt}}(\nu C_{\mathfrak{J}}^-) = \Phi_\nu H[-1]. \end{aligned}$$

Corollary 5.2.2. *Every algebraic heart in* $\mathrm{tilt}(H)$ *lies in exactly one of the subsets* $\mathrm{tilt}^+(H)$ *or* $\mathrm{tilt}^-(H)$, *i.e. is either equal to $\Psi_\nu H$ or to $\Phi_\nu H[-1]$ for some $\mathfrak{J}$-path ν. Thus we have disjoint-union decompositions*

$$\text{alg-tors}(H) = \mathrm{tors}^+(H) \sqcup \mathrm{tors}^-(H), \qquad \text{alg-torf}(H) = \mathrm{torf}^+(H) \sqcup \mathrm{torf}^-(H),$$
$$\text{alg-tilt}(H) = \mathrm{tilt}^+(H) \sqcup \mathrm{tilt}^-(H).$$

Proof. If $K \in \mathrm{tilt}(H)$ is algebraic, then by theorem 2.4.4 we have that $\mathbf{C}(K)$ is a full dimensional simplicial cone in $\mathrm{HFan}(H)$ and further K the unique intermediate heart with this heart cone. But from theorem 5.2.1 we see that the union of Tits cones

$$\bigcup \mathrm{faces}(\nu C_{\mathfrak{J}}^+) \quad \cup \quad \bigcup \mathrm{faces}(\nu C_{\mathfrak{J}}^-)$$

defines a subfan of $\mathrm{HFan}(H)$ supported on the complement of the hyperplane $\Theta^0(\underline{\Delta} \setminus \mathfrak{J})$. The cone $\mathbf{C}(K)$, being full-dimensional, lies in this subfan and therefore must be of the form $\nu C_{\mathfrak{J}}^{\pm}$, giving the result. □

Faces of algebraic heart cones. We show that non-maximal cones in $\mathrm{Arr}^{\pm}(\underline{\Delta} \setminus \mathfrak{J})$ cannot arise as heart cones. We remark this can be accomplished by observing that the corresponding subcategories of semistable modules are module categories over τ-tilting finite algebras [Gar23], but we take a more direct approach below.

Theorem 5.2.3. *A non-zero cone $\sigma \in \mathrm{Arr}^{\pm}(\underline{\Delta}, \mathfrak{J})$ is the heart cone of some $K \in \mathrm{tilt}(H)$ if and only if σ is full-dimensional. In this case, (K, σ) are of the form $(\Psi_\nu H, \nu C_{\mathfrak{J}}^+)$ or $(\Phi_\nu H[-1], \nu C_{\mathfrak{J}}^-)$ for some positive $\mathfrak{J}$-path ν.*

Proof. If σ is full dimensional it must be of the form $\nu C_{\mathfrak{J}}^{\pm}$, so it is a heart cone of some $K \in \mathrm{tilt}(H)$ determined uniquely by theorem 5.2.1.

To show the converse, suppose for the sake of contradiction that there is a heart K in $\mathrm{tilt}(H)$ whose heart cone $\sigma = \mathbf{C}(K)$ is non-zero but non-maximal in (without loss of generality) $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J})$. Consider the sets

$$\{H' \in \mathrm{tilt}(H) \mid H' \text{ algebraic}, \quad \sigma \subset \mathbf{C}(H'), \quad H' \geqslant K\} \tag{5.7}$$
$$\{H' \in \mathrm{tilt}(H) \mid H' \text{ algebraic}, \quad \sigma \subset \mathbf{C}(H'), \quad H' \leqslant K\} \tag{5.8}$$

of algebraic hearts which are comparable with K and have σ in their heart cone. Since the $\mathfrak{J}$-cone arrangement is locally finite away from the origin, these are finite.

Replacing (K, σ) by another pair if necessary, we may assume one of the sets (5.7) or (5.8), without loss of generality the former, is non-empty. Indeed if not, then in particular the heart $K' = H_{tt}(\sigma)$ is non-algebraic so that $\sigma' = \mathbf{C}(K')$ is a non-maximal cone containing σ, and for each chamber $\nu C^+_{\mathfrak{J}} = \mathbf{C}(\Psi_\nu H)$ surrounding σ' we evidently have $K' = H_{tt}(\sigma) \leqslant \Psi_\nu H$.

Thus there is a minimal algebraic heart H' satisfying $H' \geqslant K$ and $\sigma \subset \mathbf{C}(H')$. Furthermore equality can't hold, so by atomicity of the interval $[H[-1], H']$ (proposition 2.3.3) there is a heart $H'' \in \mathrm{tilt}(H)$ satisfying $H' \gtrdot H'' \geqslant K$ where the first relation is covering. But covering relations arise from simple tilts (theorem 2.2.2), and by [HW23, lemma 5.5], the simple tilt H'' of $H' \in \mathrm{tilt}^+(H)$ is algebraic – thus contradicting the minimality of H'. $\square$

We remark that the key ingredient in the above proof, namely covering relations arising from simple tilts, are addressed in greater detail in theorem 6.1.1 below and in particular we identify explicitly all simple tilts of $\Psi_\nu H$ and $\Phi_\nu H[-1]$ as arising from simple mutation.

§ 5.3 (Semi-)geometric intermediate hearts

It is evident from the description (4.2) of $H = \mathcal{per}(\frac{X}{Z})$ that we have $\mathcal{coh}\, X \subset H[-1, 0]$, and from theorem 4.4.5 we deduce that the same can be said of all the hearts $\mathcal{per}(\frac{X}{Y})$ arising from crepant contractions $X \to Y$. We will now show that for any birational model $W = \nu X$ and any crepant contraction $W \to Y$, the heart $\Psi_\nu \mathcal{per}(\frac{W}{Y})$ is also intermediate with respect to $\mathcal{per}(\frac{X}{Z})$.

Proposition 5.3.1. *For any sequence of flops ν from X and any crepant contraction $W \to Y$ involving the birational model $W = \nu X$, the heart $\Psi_\nu \mathcal{per}(\frac{W}{Y}) \subset \mathbf{D}^0 X$ is intermediate with respect to $\mathcal{per}(\frac{X}{Z})$.*

Proof. Again, theorem 4.4.5 allows us to reduce to the case of $\Psi_\nu\, \mathcal{coh}(W)$. Since $\mathcal{per}(\frac{X}{Z}) \geqslant \Psi_\nu \mathcal{per}(\frac{W}{Z})$ (see § 5.2) and $\mathcal{per}(\frac{W}{Z}) \geqslant \mathcal{coh}(W)$, we have $\mathcal{per}(\frac{X}{Z}) \geqslant \Psi_\nu\, \mathcal{coh}(W)$ and it thus suffices to exhibit the lower bound

$$\Psi_\nu\, \mathcal{coh}(W) \geqslant \mathcal{per}(\tfrac{X}{Z})[-1].$$

Applying the equivalence $(\Psi_\nu)^{-1}$ and swapping X and W for simplicity, we equivalently show

$$\mathcal{coh}(X) \geqslant \Phi_\nu \mathcal{per}(\tfrac{\nu X}{Z})[-1]$$

for any spherical and (without loss of generality) atomic $\mathfrak{J}$-path ν.

Note there are only finitely many atomic spherical $\mathfrak{J}$-paths, since the set $\mathrm{Cham}(\Delta, \mathfrak{J})$ is finite. We use this to further reduce to the case where ν is a maximal spherical atomic $\mathfrak{J}$-path. Indeed if ν could be extended to a longer spherical atomic $\mathfrak{J}$-path $\nu_i \nu$, then we have $\Phi_i \mathcal{per}(\frac{\nu_i \nu X}{Z}) \geqslant \mathcal{per}(\frac{\nu X}{Z})$, and hence (by lemma 4.2.7) $\Phi_{\nu_i\, \nu} \mathcal{per}(\frac{\nu_i \nu X}{Z}) \geqslant \Phi_\nu \mathcal{per}(\frac{\nu X}{Z})$, so that we could replace ν with $\nu_i \nu$ and proceed.

Noting $\mathrm{Cham}(\Delta, \mathfrak{J})$ is the face poset of a finite hyperplane arrangement, any longest spherical $\mathfrak{J}$-path ν defines the chamber $\nu C^0_{\mathfrak{J}} = -C^0_{\mathfrak{J}}$ that is furthest from the principal chamber $C^0_{\mathfrak{J}}$. Likewise $\nu C^-_{\mathfrak{J}}$ contains the ray C^-_Δ in its boundary and is the furthest such chamber from $C^-_{\mathfrak{J}}$.

It follows that any vector θ in the interior of $C^-_{\mathfrak{J}}$ satisfies $\theta(\alpha_i) > 0$ for all $i \in \Delta \setminus \mathfrak{J}$, and thus (up to rescaling) can be written as $\theta = \theta_0 - \alpha_0^*$ for some $\theta_0 \in C^0_{\mathfrak{J}}$ and α_0 defined as

$$\alpha_0^*(\alpha_i) = \begin{cases} 1, & i = 0, \\ 0, & i \in \Delta \setminus \mathfrak{J} \end{cases}. \tag{5.9}$$

Since θ lies in the heart cone of $\Phi_\nu \mathit{per}(\frac{W}{Z})[-1]$ (theorem 5.2.1), by lemma 2.4.6 we see that the corresponding torsion-free class satisfies

$$\begin{aligned}
\Phi_\nu \mathit{per}(\tfrac{W}{Z})[-1] \cap H &= H_{tf}(\theta) \\
&= \{h \in H \mid \theta[s] > 0 \text{ for all non-zero sub-objects } s \hookrightarrow h\} \\
&= \{h \in H \mid \theta_0[s] > \alpha_0^*[s] \text{ for all non-zero sub-objects } s \hookrightarrow h\} \\
&\subseteq \{h \in H \mid \theta_0[s] > 0 \text{ for all non-zero sub-objects } s \hookrightarrow h\} \\
&= H_{tf}(\theta_0),
\end{aligned}$$

but $H_{tf}(\theta_0)$ is evidently contained in the torsion-free class $\mathit{coh}(X) \cap H$ since θ_0 lies in $\mathbf{C}(\mathit{coh}\, X)$ – to see this, note that the functional $\theta_0 \colon \mathbf{K}X \to \mathbb{R}$ is precisely the intersection product with a nef $\mathbb{R}$-divisor on X by proposition 5.1.10. The desired inequality of t-structures can then be read off from the above containment of torsion-free classes. □

Having established the intermediacy of the geometric and semi-geometric hearts of §§ 4.3 and 4.4, we now show that each non-zero cone in the finite subfan $\mathrm{Arr}(\Delta, \mathfrak{J}) \subset \mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ is the heart cone of some $\Psi_\nu \mathit{per}(\frac{\nu X}{Y})$ arising from a sequence of flops $X \dashrightarrow \nu X$ and crepant contraction $\nu X \to Y$.

For this, we use the identification $\mathrm{Mov}(X) \simeq \mathrm{Arr}(\Delta, \mathfrak{J})$ of proposition 5.1.10 that furnishes an explicit description of functionals via intersection products – an argument that has already appeared in the proof of proposition 5.3.1 above.

Proposition 5.3.2. *Let* $\tau \colon X \to Y$ *be the crepant contraction of the collection of exceptional curves* $\{C_i \mid i \in I\}$ *defined by* $I \subset \Delta \setminus \mathfrak{J}$, *and suppose* $Y \not\simeq Z$ *(i.e. not all* π*-exceptional curves are contracted). Then the heart* $\mathit{per}(\frac{X}{Y}) \subset \mathbf{D}^0 X$ *of* τ*-perverse sheaves has heart cone* $C^0_{\mathfrak{J}\cup I} \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$.

Proof. If θ lies in the heart cone of $K = \mathit{per}(\frac{X}{Y})$, then considering the sheaves $\mathcal{O}_{C_i}(n)$ $(n \in \mathbb{Z})$ on uncontracted curves and the sheaves $\mathcal{O}_{C_i}(-1)$ on contracted curves shows θ lies in the chamber $C^0_{\mathfrak{J}}$. In particular, θ necessarily vanishes on the class $\delta_{\mathfrak{J}}$ of skyscraper sheaves. But for a connected component $C_J \subset C_I$ and a closed point $p \in C_J$, we see from lemma 5.1.5 that $\mathcal{O}_p$ admits a filtration by the simples of $\mathit{per}_J(\frac{X}{Z})$ in which every simple appears at least once. Thus in $\mathbf{K}$-theory we have

$$[\mathcal{O}_p] = \big[\omega_{\underline{C}_J}[1]\big] + \sum_{i \in J} m_i\, [\mathcal{O}_{C_i}(-1)] \quad = \big[\omega_{\underline{C}_J}[1]\big] + \sum_{i \in J} m_i\, \alpha_i \tag{5.10}$$

for positive integers m_i. Since θ is non-negative on each class appearing in the above expression and in particular vanishes on $[\mathcal{O}_p]$, it must vanish on every class on the right-hand side. Repeating the argument on all connected components of C_I shows $\theta(\alpha_i) = 0$ whenever $i \in I$, and thus the heart cone $\mathbf{C}K$ is contained in $C^0_{\mathfrak{J}\cup I}$.

Conversely if θ lies in $C^0_{\mathfrak{J}\cup I}$, then in particular θ lies in the image of the nef cone in $\mathrm{Pic}(X)$ under the map of lemma 5.1.5, and thus $\theta[x] \geqslant 0$ whenever x is a coherent sheaf. On the other hand if $x \in \mathit{per}(\frac{X}{Y})$ is supported in C_I, then we may assume x is supported in some connected component $C_J \subset C_I$ so that x lies in $\mathit{per}_J(\frac{X}{Z})$. But then theorem 4.4.2 and the expression (5.10) show that the $\mathbf{K}$-theory class $[x]$ can be expressed as some integral linear combination

$$[x] = n_\delta\, [\mathcal{O}_p] + \sum_{i \in J} n_i\, [\mathcal{O}_{C_i}(-1)] \quad = n_\delta\, \delta_{\mathfrak{J}} + \sum_{i \in J} n_i\, \alpha_i,$$

so that $\theta[x] = 0$. Since an arbitrary complex $x \in \mathit{per}(\frac{X}{Y})$ is an extension of objects of the above forms (theorem 4.4.5), it follows that $\theta[x] \geqslant 0$ and thus $C^0_{\mathfrak{J}\cup I}$ is equal to the heart cone of $\mathit{per}(\frac{X}{Y})$. □

Likewise if $W = \nu X$ is a birational model of X, we have an injection of $\mathrm{Pic}_{\mathbb{R}}(W)$ into $\Theta(\underline{\Delta} \setminus \nu\mathfrak{J})$, and hence into $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ via the wall-crossing isomorphisms in proposition 3.2.1. It follows that for $I \subsetneq \Delta \setminus \nu\mathfrak{J}$ and the crepant contraction $\tau \colon W \to Y$ of the curves $\{C_i \mid i \in I\}$ in W, the heart $\Psi_\nu \mathit{per}(\frac{W}{Y}) \subset \mathbf{D}^0 X$ has heart cone

$$(\varphi_\nu^\vee)^{-1} C^0_{\nu\mathfrak{J}\cup I} \quad = \nu C^0_{\mathfrak{J}} \cap \bigcap_{i \in I} \{\varphi_\nu \alpha_i = 0\}.$$

The complete heart fan. An important consequence of theorem 5.2.1 and the above discussion is that we can now write a complete description of the heart fan of $H = \mathcal{per}(\frac{X}{Z})$, and identify precisely which non-zero cones are heart cones.

Theorem 5.3.3. *Under the identification* $\mathrm{Hom}(\mathbf{K}X, \mathbb{R}) \cong \Theta(\underline{\Delta} \setminus \mathfrak{J})$, *the heart fan of* $H = \mathcal{per}(\frac{X}{Z})$ *is given by the* $\mathfrak{J}$*-cone arrangement* $\mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$. *The non-zero heart cones in the heart fan are precisely the chambers in* $\mathrm{Arr}^{\pm}(\underline{\Delta}, \mathfrak{J})$, *and all non-zero cones in* $\mathrm{Arr}^{0}(\underline{\Delta}, \mathfrak{J})$.

The sub-poset of numerical tilts of H is by lemma 2.4.6 the union $\bigcup_\sigma [H_{tt}\sigma, H^{tt}\sigma]$ of intervals defined by non-zero heart cones σ in the heart fan of H. We thus see that all numerical tilts of $H = \mathcal{per}(\frac{X}{Z})$ can be enumerated as

$$\{\Psi_\nu H \mid \nu \text{ a } \mathfrak{J}\text{-path}\} \cup \{\Phi_\nu H[-1] \mid \nu \text{ a } \mathfrak{J}\text{-path}\} \cup \bigcup_{\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J}) \smallsetminus 0} [H_{tt}\sigma, H^{tt}\sigma]$$

In particular it remains to address the question of when (semi-)geometric hearts are maximal occupants of their heart cones, and we shall do that now.

Mixed and reversed geometric hearts. Considering the heart cones $\sigma = \nu C^0_{\mathfrak{J}}$ associated to geometric hearts $\Psi_\nu\, \mathcal{coh}(\nu X)$, proposition 5.1.7 is precisely the computation that

$$H^{ss}(\sigma) = \langle \mathcal{O}_p \mid p \in \nu X(\mathbb{C}) \rangle$$

where we note that each closed point of νX is automatically supported on the exceptional fiber of $\pi\colon \nu X \to Z$ in our complete local setting. From corollary 2.4.7 and lemma 2.4.6, it is thus immediate that the interval $[H_{tt}\sigma, H^{tt}\sigma]$ is a Boolean lattice (i.e. a lattice isomorphic to the inclusion-poset of all subsets of some set) in which the geometric heart is maximal.

Proposition 5.3.4. *Let* ν *be a spherical* $\mathfrak{J}$*-path, defining a birational model* $W = \nu X$ *and a cone* $\sigma = \nu C^0_{\mathfrak{J}}$. *An intermediate heart* $K \in \mathrm{tilt}(\mathcal{per}(\frac{X}{Z}))$ *has heart cone* σ *for some spherical* $\mathfrak{J}$*-path* ν *if and only if* K *is a tilt of* $\Psi_\nu\, \mathcal{coh}(W)$ *in a torsion class* $\langle \Psi_\nu \mathcal{O}_p \mid p \in U \rangle$ *associated to some subset of closed points* $U \subset W(\mathbb{C})$. *That is,*

$$\Psi_\nu^{-1} K = \langle \mathcal{O}_p \mid p \in W(\mathbb{C}) \setminus U \rangle \ast \{x \in \mathcal{coh}\, W \mid x \text{ pure of dimension } 1\} \ast \langle \mathcal{O}_p[-1] \mid p \in U \rangle. \tag{5.11}$$

This correspondence $\left\{K \in \mathrm{tilt}(\mathcal{per}(\frac{X}{Z})) \mid \mathbf{C}K = \nu C^0_{\mathfrak{J}}\right\} \longrightarrow \{U \subset \nu X(\mathbb{C})\}$ *is an anti-isomorphism of posets.*

In particular $\mathcal{coh}(X)$, corresponding to $U = \varnothing$, is maximal with heart cone $C^0_{\mathfrak{J}}$, while the minimal such heart $\overline{\mathcal{coh}}(X)$, corresponding to $U = X(\mathbb{C})$, is the category of two-term complexes

$$\{x \in \mathcal{coh}\, X \mid x \text{ pure of dimension } 1\} \ast \{y[-1] \in \mathcal{coh}\, X[-1] \mid \dim \mathrm{Supp}(y) = 0\}.$$

Analogous categories considered in [BPPW24, example 2.6] were dubbed *reversed geometric hearts.*

Remark 5.3.5. Geometric hearts and their associated torsion theories furnish a rich bank of examples when studying chain conditions on Abelian categories and wide generation of torsion theories. In particular, the table in § 2.3 is constructed using the following observations.

(When does a geometric heart arise from semibricks?) Suppose the torsion pair $H = T \ast F$ is such that the tilt $K = F \ast T[-1]$ lies in $[\overline{\mathcal{coh}}\, X, \mathcal{coh}\, X]$, and is associated to $U \subseteq X(\mathbb{C})$ as in proposition 5.3.4. Evidently each $\mathcal{O}_p$ $(p \in X(\mathbb{C}) \setminus U)$ and each $\mathcal{O}_p[-1]$ $(p \in U)$ is simple in K. Considering the decomposition (5.11) shows that any simple object of K must be of this form; in particular a purely one-dimensional sheaf $k \in \mathcal{coh}\, X$ cannot be simple in K since the sets $\mathrm{Hom}(\mathcal{O}_p[-1], k)$ and $\mathrm{Hom}(k, \mathcal{O}_p)$ are both non-zero for any closed point $p \in \mathrm{Supp}(k)$.

By a similar reasoning, we see that every non-zero object in K has a simple sub-object (i.e. T is widely generated, see proposition 2.3.3) if and only if U contains at least one point in each exceptional curve $C_i \subset X$. Likewise, non-zero object in K has a simple quotient (i.e. F is widely generated) if and only if the complement $X(\mathbb{C}) \setminus U$ contains at least one point in every exceptional curve C_i.

(What chain conditions do geometric hearts satisfy?) It is a classical fact that $\mathcal{coh}\,X$ is a Noetherian category. On the other hand, any geometric heart $K \in [\overline{\mathcal{coh}}\,X, \mathcal{coh}\,X)$ is necessarily non-Noetherian, since K then contains some $\mathcal{O}_p[-1]$ ($p \in C_i \subset X$) and hence the sequence of morphisms

$$\mathcal{O}_{C_i} \to \mathcal{O}_{C_i}(1) \to \mathcal{O}_{C_i}(2) \to \cdots,$$

each of which has cone $\mathcal{O}_p \in K[1]$, gives an infinite chain of proper surjections in K.

Likewise, $\overline{\mathcal{coh}}\,X$ is the only Artinian heart in the interval $[\overline{\mathcal{coh}}\,X, \mathcal{coh}\,X]$. The argument for why any $K \in (\overline{\mathcal{coh}}\,X, \mathcal{coh}\,X]$ is non-Artinian is dual to the above discussion. To see why $\overline{\mathcal{coh}}\,X$ is Artinian, note that if a morphism $x \to y$ is injective in $\overline{\mathcal{coh}}\,X$ with non-zero cokernel y/x, then considering cohomologies with respect to $\mathcal{coh}\,X$ produces a long exact sequence of sheaves

$$0 \to \underbrace{H^0(x) \to H^0(y) \to H^0(y/x)}_{\text{pure sheaves with one-dimensional support}} \longrightarrow \underbrace{H^1(x) \to H^1(y) \to H^1(y/x)}_{\text{sheaves with zero-dimensional support}} \to 0.$$

In particular, either H^0x has lower rank than H^0y on some exceptional curve $C_i \subset X$, or y/x has zero-dimensional support (i.e. $H^0(y/x) = 0$ and the length of H^1x is lower than that of H^1y. Thus $\overline{\mathcal{coh}}\,X$ has no infinite descending chains of inclusions ... $\hookrightarrow x_2 \hookrightarrow x_1 \hookrightarrow x$.

The zoos of semi-geometric hearts. We now compute the semistable categories associated to non-zero non-maximal cones $\sigma \in \mathrm{Arr}^0(\Delta, \mathfrak{J})$, which will subsequently furnish a description of the interval $[H_{tt}\sigma, H^{tt}\sigma]$ in $\mathrm{tilt}(H)$.

Lemma 5.3.6. *Let $W = \nu X$ be a birational model of X, and $\tau\colon W \to Y$ a crepant contraction of some (but not all) exceptional curves in W. Writing $K = \Psi_\nu \mathcal{per}(\frac{W}{Y})$ for the corresponding heart in $\mathbf{D}^0X$ and picking a generic vector $\theta \in \mathbf{C}(K)$, an object $k \in K$ satisfies $\theta[k] = 0$ if and only if it is filtered by the simple objects of K.*

Proof. It suffices to address the case $W = X$ since the rest are analogous. Thus supposing τ contracts the locus $C_I = \bigcup_{i\in I} C_i$ defined by $I \subset \Delta \setminus \mathfrak{J}$, we must show $K = \mathcal{per}(\frac{X}{Y})$ satisfies

$$K_{ss}(\theta) = \mathcal{per}_I(\tfrac{X}{Z}) \quad \oplus \bigoplus_{p\in C\setminus C_I} \langle \mathcal{O}_p \rangle \tag{5.12}$$

for θ generic in $\mathbf{C}(K) = C^0_{\mathfrak{J}\cup I}$. It is evident that θ vanishes on every simple of K, so we address the converse containment.

Thus suppose $k \in K$ satisfies $\theta[k] = 0$. By theorem 4.4.5, $k \in \mathcal{per}(\frac{X}{Z})$ is an extension of the simples of $\mathcal{per}_I(\frac{X}{Z})$ by some coherent sheaf supported on the τ-uncontracted curves, so we can reduce to the case where k lies in $\mathcal{coh}(X)$ and is supported within the closure of $C \setminus C_I$.

We show such k continues to satisfy $\theta[k] = 0$ if θ perturbed from the wall $C^0_{\mathfrak{J}\cup I}$ into the interior of $C^0_{\mathfrak{J}}$, from whence the result follows from the observation

$$\{x \in \mathcal{coh}(X) \mid \dim \mathrm{Supp}(x) = 0\} \overset{(5.1.7)}{=} H_{ss}(C^0_{\mathfrak{J}}) \overset{(2.4.6)}{=} \left\{x \in H^{tt}(C^0_{\mathfrak{J}}) \;\middle|\; \theta'[x] = 0 \text{ for generic } \theta' \in C^0_{\mathfrak{J}}\right\}.$$

Thus consider the vector $\theta' = \theta + \sum_{i\in I} \alpha_i^*$, where the vectors $(\alpha_i^*)_{i\in I}$ are defined as

$$\alpha_i^*(\delta_{\mathfrak{J}}) = 0, \qquad \alpha_i^*(\alpha_j) = \begin{cases} 1, & j = i \\ 0, & j \in \Delta \setminus (\mathfrak{J} \cup \{i\}) \end{cases}.$$

Evidently θ' lies generically in $C^0_{\mathfrak{J}}$. Writing $I^c = \Delta \setminus (\mathfrak{J} \cup I)$ so that k is supported on $C_{I^c} = \bigcup_{i\in I^c} C_i$, note that k lies in the thick subcategory generated by $\mathcal{per}_J(\frac{X}{Z})$. It follows that the class $[k] \in \mathbf{K}X$ can be written as a linear combination of the classes $\delta_{\mathfrak{J}}$ and α_i ($i \in I^c$), and consequently $\theta'[k] = 0$ as required. □

Consider the crepant contraction $\tau\colon X \to Y$ of exceptional curves $C_I = \bigcup_{i\in I} C_i$, defining the intermediate heart $K = \mathcal{per}(\frac{X}{Y})$ and the category $K_{ss}(\theta)$ as in (5.12) for generic $\theta \in \mathbf{C}(K)$. The category $K_{ss}(\theta)$ entirely lies within $H = \mathcal{per}(\frac{X}{Z})$, and thus proposition 5.1.7 shows $H^{tt}(C^0_{\mathfrak{J}\cup I}) = K$.

The minimal tilt with heart cone $C^0_{\mathfrak{J}\cup I}$, which we call the *reversed semi-geometric heart* $\overline{\mathit{per}}(\frac{X}{Y})$, is the Artinian heart obtained by tilting K in the torsion class $K_{ss}(\theta)$. To spell this out, writing $I^c = \Delta \setminus (\mathfrak{J} \cup I)$ and $C_J = \bigcup_{i\in I^c} C_i$ we have

$$\overline{\mathit{per}}(\tfrac{X}{Y}) = \left\{ x \in \mathit{coh}\, X \;\middle|\; \begin{array}{l} \mathrm{Supp}(x) \subset C_{I^c} \\ x \text{ pure of dim. } 1 \end{array} \right\} * \bigoplus_{p\in C\setminus C_I} \langle \mathcal{O}_p[-1] \rangle \, * \, \mathit{per}_I(\tfrac{X}{Z})[-1]$$
$$= \big\langle \{x \in \overline{\mathit{coh}}(X) \mid \mathrm{Supp}(x) \subset C_{I^c}\} \cup \big\{x \in \mathit{per}(\tfrac{X}{Z})[-1] \mid \mathrm{Supp}(x) \subset C_I\big\} \big\rangle.$$

The poset $\{K \in \mathrm{tilt}(H) \mid \mathbf{C}(K) \supset C^0_{\mathfrak{J}\cup I}\}$ is thus equal to the interval $[\overline{\mathit{per}}(\frac{X}{Y}), \overline{\mathit{per}}(\frac{X}{Y})]$. The structure can be described explicitly – decomposing $I = I_1 \sqcup \cdots \sqcup I_k$ such that each $C_J = \bigcup_{i\in J} C_i$ $(J = I_1, \ldots, I_k)$ is a connected component of C_I, we have the following.

Lemma 5.3.7. *Let* $\tau \colon X \to Y$ *be the crepant contraction of* $C_I = C_{I_1} \sqcup \cdots \sqcup C_{I_k}$ *as above. Given* $K \in \big[\overline{\mathit{per}}(\frac{X}{Y}), \mathit{per}(\frac{X}{Y})\big]$, *each restricted heart*

$$K \cap \mathbf{D}^0 X_J = \{k \in K \mid \mathrm{Supp}(k) \subset C_J\} \qquad (J = I_1, \ldots, I_k)$$

is a tilt of $\mathit{per}_J(\frac{X}{Z})$. *Further, these restrictions and the set* $\{p \in C \setminus C_I \mid \mathcal{O}_p[-1] \in K\}$ *uniquely determine* K, *and this determines a poset isomorphism*

$$\big[\overline{\mathit{per}}(\tfrac{X}{Y}), \mathit{per}(\tfrac{X}{Y})\big] \quad \simeq \quad \prod_{J=I_1,\ldots,I_k} \mathrm{tilt}\left(\mathit{per}_J(\tfrac{X}{Z})\right) \times \mathrm{Bool}\,(C \setminus C_I)\,.$$

Here $\mathrm{Bool}(-)$ denotes the boolean lattice on closed points, and the poset on the right hand side is given the product order i.e. $(a_i) \leqslant (b_i)$ if and only if $a_i \leqslant b_i$ for every i.

Proof. By lemma 2.4.6 and the discussion following it, hearts $K \in \big[\overline{\mathit{per}}(\frac{X}{Y}), \mathit{per}(\frac{X}{Y})\big]$ are in bijection with torsion classes $T \subset \mathit{per}(\frac{X}{Y})$ that satisfy $T \subseteq K_{ss}\left(C^0_{\mathfrak{J}\cup I}\right)$. For such a T, lemma 5.3.6 induces the decomposition

$$T = \bigoplus_{J=I_1,\ldots,I_k} T \cap \mathit{per}_J(\tfrac{X}{Z}) \;\oplus\; \bigoplus_{p\in C\setminus C_I} T \cap \langle \mathcal{O}_p \rangle,$$

and each summand in the above decomposition is closed under extensions and factors (in $K_{ss}\left(C^0_{\mathfrak{J}\cup I}\right)$) since T is so. It follows that each $T_J = T \cap \mathit{per}_J(\frac{X}{Z})$ is a torsion class in $\mathit{per}_J(\frac{X}{Z})$, and $K \cap \mathbf{D}^0 X_J$ is the tilt of $\mathit{per}_J(\frac{X}{Z}) = \mathit{per}(\frac{X}{Y}) \cap \mathbf{D}^0 X_J$ in $T_J = T \cap \mathbf{D}^0 X_J$. Further T is clearly determined the summands, i.e. the desired bijection follows. $\square$

Note when C has two or more irreducible components, each non-zero non-maximal cone $\sigma \in \mathrm{Arr}(\Delta \setminus \mathfrak{J})$ is the heart-cone of multiple hearts of the form $\Psi_\nu \mathit{per}(\frac{\nu X}{Y})$ – to be precise, one for each birational model νX such that $\sigma \subset \nu C^0_{\mathfrak{J}}$. Indeed each chamber $\nu C^0_{\mathfrak{J}}$ containing σ uniquely determines a subset $I \subset \Delta \setminus \nu\mathfrak{J}$ such that $\varphi_\nu^\vee \sigma = C^0_{\nu\mathfrak{J}\cup I}$, and the data (ν, I) together determine a partial contraction $\nu X \to Y$ such that $\mathbf{C}\left(\Psi_\nu \mathit{per}(\frac{\nu X}{Y})\right) = \sigma$.

Definition 5.3.8. Given a non-zero cone $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$, we say a birational model $\nu X \in \mathrm{Bir}\,(\frac{X}{Z})$ is σ-*positive* if σ is a face of $\nu C^0_{\mathfrak{J}} = \mathbf{C}(\Psi_\nu\, \mathit{coh}(\nu X))$.

Fixing the reference heart $H = \mathit{per}(\frac{X}{Z})$, and in particular the model X, endows the set of birational models $\mathrm{Bir}\,(\frac{X}{Z})$ with a partial order where $W \geqslant W'$ if and only if there is a minimal sequence of simple flops from X to W' passing through W. This order, which under the natural bijections $\mathrm{Bir}\,(\frac{X}{Z}) \simeq \mathrm{Cham}(\Delta, \mathfrak{J}) \simeq \mathrm{MMG}^N(R)$ corresponds to the weak order on chambers or the mutation-order on modifying modules, helps discern when a semi-geometric heart $\Psi_\nu \mathit{per}(\frac{\nu X}{W})$ is the maximal occupant of its heart cone.

Theorem 5.3.9. *Consider a birational model* $W = \nu X$ *of* X *and let* $\tau : W \to Y$ *be the crepant contraction of a collection of exceptional curves* $C_I \subset W$. *Writing* $\sigma \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$ *for the heart cone of* $\Psi_\nu \mathit{per}(\frac{W}{Y})$, *the following statements are equivalent.*

(1) The birational model W *is maximal among all* σ*-positive elements of* $\mathrm{Bir}\,(\frac{X}{Z})$.

(2a) The heart $\Psi_\nu \mathit{per}(\frac{W}{Y})$ *is the supremum (in* $\mathrm{tilt}(H)$*) of* $\left\{\Psi_\upsilon\, \mathit{coh}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\right\}$.

(2b) The heart $\Psi_\nu \mathit{per}(\frac{W}{Y})$ is equal to the numerically defined tilt $H^{tt}(\sigma)$.

(3) The reversed semi-geometric heart $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y})$ is intermediate with respect to H.

(3a) The heart $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y})$ is the infimum of $\{\Psi_\upsilon\, \overline{\mathit{coh}}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\}$.

(3b) The heart $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y})$ is equal to the numerically defined tilt $H_{tt}(\sigma)$.

(4) For generic $\theta \in \sigma$, an object of H is θ-stable if and only if it is a simple of $\Psi_\nu \mathit{per}(\frac{W}{Y})$.

Further for any non-zero cone $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$, there is a unique birational model $W = \nu X$ and a unique partial contraction $W \to Y$ for which $\sigma = \mathbf{C}(\Psi_\nu \mathit{per}(\frac{W}{Y}))$ and the above holds.

Before proving the various equivalences we sketch a proof of the final claim i.e. we show that for any $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ there is a unique birational model and partial contraction satisfying the said conditions. By the preceding discussion, each σ-positive birational $W = \nu X$ admits precisely one crepant contraction $W \to Y$ such that $\mathbf{C}(\Psi_\nu \mathit{per}(\frac{W}{Y})) = \sigma$ and hence it suffices to show that exactly one σ-positive $W \in \mathrm{Bir}(\frac{X}{Z})$ can be maximal. But the corresponding statement is evident for the weak order of cones – perturbing a generic vector in σ 'in the direction of' $C^0_{\mathfrak{J}}$ picks out the maximal chamber surrounding σ, see for example [BLSWZ99, lemma 4.2.12].

Returning to the setup of theorem 5.3.9, we now establish the equivalence of statements (1)–(4). The following equivalences arise from the interplay between numerical tilts and semistable categories.

Proof of (2b) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (3b). Considering the heart $K = \Psi_\nu \mathit{per}(\frac{W}{Y})$ and a generic parameter $\theta \in \sigma$, note that the statement (4) is equivalent to $H_{ss}\theta = K_{ss}\theta$ by lemma 5.3.6. But we know K is intermediate with respect to H and has heart cone σ, so corollary 2.4.7 shows that we have $H_{ss}\theta = K_{ss}\theta$ if and only if K is equal to $H^{tt}\theta$, thus showing (2b) $\Longleftrightarrow$ (4).

If (2b) (and hence also (4)) is true, then it is immediate that (3b) holds, since $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y})$ (i.e. the tilt of K in $K_{ss}\theta$) must coincide with $H_{tt}\theta$ (i.e. the tilt of $H^{tt}\theta$ in $H_{ss}\theta$). Conversely suppose $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y}) = H^{tt}\theta$. Since K lies in $[H_{tt}\theta, H^{tt}\theta]$, we see that $K \cap H_{tt}\theta[1]$ is a torsion-free class in $H_{ss}\theta$ and in particular is contained in H. But we have

$$K \cap H_{tt}\theta[1] = \Psi_\nu \mathit{per}(\tfrac{W}{Y}) \cap \Psi_\nu \overline{\mathit{per}}(\tfrac{W}{Y})[1]) = K_{ss}\theta,$$

so in particular $K_{ss}\theta \subset H$ and hence corollary 2.4.7 shows that both (2b) and (4) hold. □

Likewise, the equivalences (2a) $\Longleftrightarrow$ (2b) and (3a) $\Longleftrightarrow$ (3b) only rely on the geometry of heart cones. We show how to prove the former, the latter being similar.

Proof of (2a) $\Longleftrightarrow$ (2b). It suffices to show that $H^{tt}(\sigma)$ is the supremum of the given collection of geometric hearts. If σ is a face of $\upsilon C^0_{\mathfrak{J}} = \mathbf{C}(\Psi_\upsilon\, \mathit{coh}(\upsilon X))$, then we clearly have $\Psi_\upsilon\, \mathit{coh}(\upsilon X) \leqslant H^{tt}(\sigma)$.

Conversely, choose generic vectors $\theta_\upsilon \in \upsilon C^0_{\mathfrak{J}}$ for each $\upsilon C^0_{\mathfrak{J}}$ containing σ as a face. Since the fan $\mathrm{Arr}(\Delta, \mathfrak{J})$ is induced from a simplicial hyperplane arrangement, there is a tuple of positive reals (λ_υ) such that the weighted average $\theta = \sum \lambda_\upsilon \theta_\upsilon$ sits generically in the face σ. By the construction of numerical torsion theories, we see that if h lies in each torsion class $H_{tr}(\theta_\upsilon) = \Psi_\upsilon\, \mathit{coh}(\upsilon X)[1] \cap H$, then we also have $h \in H_{tr}(\theta) = H^{tt}(\sigma)[1] \cap H$. It follows that we have $H^{tt}(\sigma) \leqslant \sup\{\Psi_\upsilon\, \mathit{coh}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\}$, and hence equality holds. □

Note that the implication (3b)⇒(3) is immediate from definitions, so the following establishes that the statements (2a)–(4) are all equivalent.

Proof of (3)⇒(4). Suppose we have $\Psi_\nu \overline{\mathit{per}}(\frac{W}{Y}) \subset H[-1, 0]$. Since $K = \Psi_\nu \mathit{per}(\frac{W}{Y})$ also lies in $H[-1, 0]$, we see that for generic $\theta \in \sigma$ the category

$$K_{ss}(\theta) = \Psi_\nu \mathit{per}(\tfrac{W}{Y}) \cap \Psi_\nu \overline{\mathit{per}}(\tfrac{W}{Y})[1]$$

lies in H, and hence by corollary 2.4.7 and corollary 4.4.6 we conclude that every object in $H_{ss}(\theta)$ is filtered by the simples of K i.e. (4) holds. □

It remains to show these are equivalent to (1), which asserts that the shortest $\mathfrak{J}$-path μ satisfying $\sigma \subset \mu C^0_{\mathfrak{J}}$ gives a sequence of flops from X to W (i.e. $W = \mu X$). Unsurprisingly, we leverage our control over compositions of atomic mutations.

Proof of (1)⇒(2a). First consider the case when the shortest such path μ is given by the empty word $\emptyset$, i.e. $W = X$ and $\sigma = C^0_{\mathfrak{J}\cup I}$ is a face of $C^0_{\mathfrak{J}}$. Then for $K = \mathcal{per}(\frac{X}{Y})$ and $\theta \in \sigma$ generic, the category $K_{ss}(\theta)$ (as computed in lemma 5.3.6) clearly lies in H and thus corollary 2.4.7 shows we have $K = H^{tt}(\theta)$, i.e. statement (2b) (and hence also (2a)) holds.

Now suppose W, Y, σ are as in the general situation and μ is the shortest spherical $\mathfrak{J}$-path satisfying $\sigma \subset \mu C^0_{\mathfrak{J}}$, i.e. $\varphi_\nu^\vee \sigma = C_{\mu\mathfrak{J}\cup I}$ for some $I \subset \mu\mathfrak{J}$. If (1) holds, then we have $W = \mu X$ and thus $W \to Y$ is the crepant contraction of $C_I \subset W$. The reasoning above then shows

$$\mathcal{per}(\tfrac{W}{Y}) = \sup \left\{ \Psi_\upsilon \operatorname{coh}(\upsilon W) \mid C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}} \right\},$$

and it suffices to restrict the above expression to spherical $\mu\mathfrak{J}$-path υ that are atomic. But if υ is atomic then so is $\upsilon\mu$, since every hyperplane crossed by υ must contain σ while the choice of μ ensures that no hyperplane crossed by it contains σ. Thus have

$$\begin{aligned}\Psi_\mu \mathcal{per}(\tfrac{W}{Y}) &= \sup \left\{ \Psi_\mu \circ \Psi_\upsilon \operatorname{coh}(\upsilon W) \mid C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}} \right\} \\ &= \sup \left\{ \Psi_{\upsilon\mu} \operatorname{coh}(\upsilon\mu X) \mid \sigma \subset \upsilon\mu C^0_{\mathfrak{J}} \right\}\end{aligned}$$

To see this is equivalent to the required statement (2b), observe that for ny spherical $\mu\mathfrak{J}$-path υ we have $C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}}$ if and only if $\sigma \subset \upsilon\mu C^0_{\mathfrak{J}}$, and every maximal chamber containing σ as a face can be realised in this way since $\varphi_\mu^\vee \colon \operatorname{Arr}(\underline{\Delta}, \mathfrak{J}) \to \operatorname{Arr}(\underline{\Delta}, \mu\mathfrak{J})$ is an isomorphism of fans. □

Proof of (4)⇒(1). Let $W' = \mu X$ be the maximal σ-positive birational model, admitting the partial contraction $W' \to Y'$ such that $\mathbf{C}\left(\Psi_\mu \mathcal{per}\left(\frac{W'}{Y'}\right)\right) = \sigma$. The statement (1) (and hence (4)) holds for this partial contraction by construction, so we see that an object of H is θ-stable if and only if it is a simple object of $\Psi_\mu \mathcal{per}\left(\frac{W'}{Y'}\right)$.

Choosing an atomic sequence of flops υ from W' to W, we note that the path υ from $\mu C^0_{\mathfrak{J}}$ to $\upsilon\mu C^0_{\mathfrak{J}} = \nu C^0_{\mathfrak{J}}$ only crosses hyperplanes containing σ while the path μ by definition never passes through a chamber containing σ. Thus the composite path $\upsilon\mu$ is also atomic and lemma 4.2.8 allows us to write $\Psi_\nu = \Psi_\mu \circ \Psi_\upsilon$.

Suppose (4) also holds for $W \to Y$, i.e. an object of H is θ-stable if and only if it is a simple of $\Psi_\nu \mathcal{per}(\frac{W}{Y})$. Thus we have the equality of sets

$$\left\{ \Psi_\mu^{-1} h \mid h \in H \ \theta\text{-stable} \right\} = \left\{ s \mid s \in \mathcal{per}\left(\tfrac{W'}{Y'}\right) \text{ simple} \right\} = \left\{ \Psi_\upsilon t \mid t \in \mathcal{per}(\tfrac{W}{Y}) \text{ simple} \right\}.$$

We need to show $W = W'$, or equivalently that υ is the empty path. If not, it has length $\geqslant 1$ so can be written as $\upsilon = \nu_i \cdot \upsilon'$ for some $\mu\mathfrak{J}$-path υ' and $i \in \Delta \setminus \upsilon'\mu\mathfrak{J}$. In other words, the final curve flopped by υ is $C_i \subset \upsilon' W'$, which has proper transform $C_{\iota(i)} \subset W$.

Further the wall $\upsilon' C^0_{\mathfrak{J}} \cap \nu_i \upsilon' C^0_{\mathfrak{J}}$ contains the cone σ, so we must have $\iota(i) \in I$. Thus the curve $C_{\iota(i)} \subset W$ is contracted by $\tau : W \to Y$.

In particular the object $t = \mathcal{O}_{C_{\iota(i)}}(-1)$ is simple in $\mathcal{per}(\frac{W}{Y})$, and tracking it across the flop functors using [Wem18, lemma 4.15] we see that

$$\begin{aligned}\Psi_\upsilon(t) &= \Psi_{\upsilon'} \circ \Psi_i(\mathcal{O}_{C_{\iota(i)}}(-1)) \\ &= \Psi_{\upsilon'}\left(\mathcal{O}_{C_i}(-1)\right)[-1] \\ &\in \mathcal{per}\left(\tfrac{W'}{Z}\right)[-2, -1].\end{aligned}$$

But all simples of $\mathcal{per}\left(\frac{W'}{Y'}\right)$ must lie in $\mathcal{per}\left(\frac{W'}{Z}\right)$, this furnishes the desired contradiction. □

This concludes our analysis of the non-maximal heart cones in $\operatorname{HFan}(\mathcal{per}(\frac{X}{Z}))$, and thus of all numerical tilts of $\mathcal{per}(\frac{X}{Z})$. In the next section we examine local and global properties of the partial order to conclude that all tilts of $\mathcal{per}(\frac{X}{Z})$ are numerical.

Covering and limiting relations

(Chapter 6)

With the goal of showing that every heart in $\mathrm{tilt}(\mathcal{per}(\frac{X}{Z}))$ can be detected via $\mathbf{K}$-theory – thus proving theorem 1.4.2 (1) – we now examine covering relations in the poset of algebraic t-structures and the dynamics of the actions of $\mathrm{Pic}(\nu X)$ on the poset. A happy consequence of the study of covering relations is the classification of bricks in $\mathcal{per}(\frac{X}{Z})$, following theorem 2.2.2.

§ 6.1 Simple tilts of length hearts

Following [HW18, §5], we will show that simple tilts of algebraic hearts in $\mathrm{tilt}(\mathcal{per}(\frac{X}{Z}))$ arise precisely from simple mutation functors. This will yield a complete description of the poset of algebraic tilts $\mathrm{alg\text{-}tilt}(\mathcal{per}(\frac{X}{Z}))$, and also give insight into the embedding $\mathrm{alg\text{-}tilt}(\mathcal{per}(\frac{X}{Z})) \subset \mathrm{tilt}(\mathcal{per}(\frac{X}{Z}))$.

Recall that for any $M = \nu N \in \mathbf{MM}^N(R)$, we write H for the natural heart $\mathrm{fl}\mathcal{mod}\, {}_\nu\Lambda_\nu$ in $\mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu$. Each indecomposable summand $M_i \subset M$ ($i \in \underline{\Delta} \setminus \nu\mathfrak{J}$) then gives an indecomposable projective ${}_\nu\Lambda_\nu$-module $\mathrm{Hom}_R(M, M_i)$, and hence a dual simple module $S_i \in H$.

The following theorem, which extends lemma 4.2.7, is then a complete record of the behaviour of simple mutations with respect to various partial orders.

Theorem 6.1.1. *Let ν be a positive path starting at N, and $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$. Then the following are equivalent:*

(1) $\nu N > \nu_i\nu N$ *in* $\mathbf{MM}^N(R)$.

(2a) $T_\nu \supset T_{\nu_i\nu}$, *equivalently* $F_\nu \subset F_{\nu_i\nu}$.

(2b) $F_{\nu_i\nu} = \Phi_\nu F_i * F_\nu$, *equivalently* $T_{\nu_i\nu} = \Phi_\nu T_i \cap T_\nu$.

(2c) In $\mathrm{fl}\mathcal{mod}\, {}_\nu\Lambda_\nu$ *we have* $S_i \in \Phi_\nu^{-1} T_\nu$.

(2d) The heart $\Phi_\nu(\Phi_i H)[-1] \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ *is intermediate with respect to* H, *i.e. lies in* $H[-1, 0]$.

(2e) There is a natural isomorphism of functors $\Phi_{\nu_i\nu} \cong \Phi_\nu \circ \Phi_i$.

(3a) $U_\nu \subset U_{\nu_i\nu}$, *equivalently* $V_\nu \supset V_{\nu_i\nu}$.

(3b) $U_{\nu_i\nu} = U_\nu * \Psi_\nu U_i$, *equivalently* $V_{\nu_i\nu} = V_\nu \cap \Psi_\nu V_i$

(3c) In $\mathrm{fl}\mathcal{mod}\, {}_\nu\Lambda_\nu$ *we have* $S_i \in \Psi_\nu^{-1} V_\nu$.

(3d) The heart $\Psi_\nu(\Psi_i H) \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ *is intermediate with respect to* H.

(3e) There is a natural isomorphism of functors $\Psi_{\nu_i\nu} \cong \Psi_\nu \circ \Psi_i$.

(4) There is an isomorphism of bimodules ${}_{\nu_i\nu}\Lambda \cong {}_{\nu_i\nu}\Lambda_\nu \otimes^{\mathbf{L}} {}_\nu\Lambda$, *where the tensor product is over* ${}_\nu\Lambda_\nu$.

We now prove theorem 6.1.1 in parts, showing (1) is equivalent to (2a)–(2e) and to (4). Proving the equivalence of (1) with (3a)–(3e) is entirely analogous. Note also that in (2a), (2b), (3a), and (3b), the equivalence of the two involved clauses follows from formal properties of torsion theories– for instance whenever $H = T * F = T' * F'$ are two torsion pairs with $T \subset T'$, then $T' \cap F$ is a torsion class in the Abelian category $F * T[-1]$ with corresponding torsion-free class $F' * T[-1]$.

The implication (1)⇒(2e) is precisely lemma 4.2.7, and the proof given establishes the implications (1)⇒(4)⇒(2e). The statements (2e)⇒(2d) and (2b)⇒(2a) are immediate. Likewise the following are straightforward from the definitions of the involved torsion theories.

Proof of (2a)⇒(1). If (1) does not hold, then necessarily $\nu_i\nu N > \nu N$ by lemma 3.4.2 and theorem 4.2.4 and hence there is a $y \in \operatorname{add}{}_\nu\Lambda$ and an exact sequence

$$0 \to {}_{\nu_i\nu}\Lambda \to y \to {}_\nu\Lambda \to 0.$$

In particular for $x \in T_\nu$, we have $\mathrm{Ext}^1({}_\nu\Lambda, x) = 0$ and hence $\mathrm{Ext}^1(y, x) = 0$. Since $\mathrm{Ext}^2({}_\nu\Lambda, x) = 0$ for projective dimension reasons, the long exact sequence associated to $\mathrm{Hom}(-, x)$ shows $\mathrm{Ext}^1({}_{\nu_i\nu}\Lambda, x) = 0$ i.e. $x \in T_{\nu_i\nu}$. Thus we have $T_{\nu_i\nu} \supsetneq T_\nu$, i.e. (2a) does not hold. □

Proof of (2e)⇒(2b). Using $\Phi_{\nu_i\nu} = \Phi_\nu \circ \Phi_i$, we have $T_{\nu_i\nu} = H \cap \Phi_{\nu_i\nu} H = H \cap \Phi_\nu(\Phi_i H) = H \cap (\Phi_\nu F_i[1] * \Phi_\nu T_i)$. But since $\Phi_\nu F_i \subset H[0,1]$, we have $\Phi_\nu F_i[1] \cap H = \{0\}$ and hence $T_{\nu_i\nu} = \Phi_\nu T_i \cap H$. Likewise writing $H = T_\nu * F_\nu$, we observe that $\Phi_\nu T_i \subset \Phi_\nu H = F_\nu[1] * T_\nu$, and hence $\Phi_\nu T_i \cap F_\nu = \{0\}$. Thus we have $T_{\nu_i\nu} = \Phi_\nu T_i \cap T_\nu$ as required. □

Proof of (2d)⇒(2c). If (2c) does not hold, then noting that $H = \Phi_\nu^{-1} F_\nu[1] * \Phi_\nu^{-1} T_\nu$ is a torsion theory on $\mathbf{D}^{\mathrm{fl}}{}_\nu\Lambda_\nu$, we have that $S_i \in \Phi_\nu^{-1} F_\nu[1]$ and hence $\Phi_\nu S_i \in F_\nu[1] \subset H[1]$. But then we see that $\Phi_\nu(\Phi_i H)[-1] = \Phi_\nu F_i * \Phi_\nu T_i[-1]$ contains the object $\Phi_\nu S_i$ (since $S_i \in F_i$) and hence cannot be intermediate. □

Thus we have proven the chain of equivalences (1)⇒ (4)⇒ (2e)⇒ (2b)⇒ (2a)⇒ (1). Since we also have (2e)⇒(2d)⇒(2c), showing (2c)⇒(1) will finish the proof of theorem 6.1.1.

Proof of (2c)⇒(1). This is essentially [HW18, lemma 5.4]. If (1) does not hold, then by lemma 3.4.2 and theorem 4.2.4 we must have $\nu_i\nu N > \nu N = \nu_{\iota(i)}\nu_i\nu N$. Thus applying (1)⇒(2e) to this chain of mutations, we have $\Phi_\nu = \Phi_{\nu_i\nu} \circ \Phi_{\iota(i)}$ and hence by lemma 5.1.1, $\Phi_\nu(S_i) = \Phi_{\nu_i\nu}(S_{\iota(i)})[1]$. But noting $\Phi_{\nu_i\nu}(S_{\iota(i)}) \in \Phi_{\nu_i\nu} H \subset H[0,1]$, we immediately have that $\Phi_\nu(S_i) \notin H$ and hence $S_i \notin \Phi_\nu^{-1} T_\nu$. □

We consequently have a complete description of covering relations involving algebraic hearts.

Corollary 6.1.2. *If ν is a $\mathfrak{J}$-path and $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$ is such that $\nu N > \nu_i\nu N$, then* $\mathrm{tilt}(H)$ *has covering relations*

$$\Phi_\nu H[-1] \lessdot \Phi_{\nu_i\nu} H[-1], \qquad \Psi_\nu H \gtrdot \Psi_{\nu_i\nu} H$$

with brick labels $\Phi_\nu S_i$ and $\Psi_\nu S_i$ respectively. Further, every covering relation in $\mathrm{tilt}(H)$ *which involves an element of* $\mathrm{tilt}^\pm(H)$ *is of the above form.*

Proof. That the relations written are covering with the given brick label is immediate.

Now if we have $K \lessdot \Psi_\nu H$ in $\mathrm{tilt}(H)$, then by theorem 2.2.2 the corresponding brick label b is a simple object of $\Psi_\nu H$ which lies in $H \cap \Psi_\nu H = V_\nu$ and K is the tilt of $\Psi_\nu H$ in the torsion class $\langle b \rangle$. In particular, we have $b = \Psi_\nu S_i$ for some $i \in \underline{\Delta} \setminus \mathfrak{J}$, so that theorem 6.1.1 ((3c)⇒(1)) gives us $\nu N > \nu_i\nu N$ and further $K = \Psi_{\nu_i\nu} H$.

An analogous reasoning shows covering relations of the form $K \gtrdot \Psi_\nu H$, $K \lessdot \Phi_\nu H[-1]$, and $K \gtrdot \Phi_\nu H[-1]$ are also of the given form. □

From proposition 3.4.4, we see that identifying positive paths from the respective unique maximal elements gives natural isomorphisms of posets

$$(\mathrm{tilt}^+ H, \leqslant) \simeq (\mathbf{MM}^N(R), \leqslant) \simeq (\mathrm{Cham}(\underline{\Delta}, \mathfrak{J}), \leqslant) \simeq (\mathrm{tilt}^- H, \leqslant)^{\mathrm{op}}.$$

Further, can be seen (e.g. by an argument similar to that in the proof of theorem 5.2.3) that any interval in $\mathrm{tilt}(H)$ whose end points lie in $\mathrm{tilt}^+(H)$ ($\mathrm{tilt}^-(H)$) must entirely lie in $\mathrm{tilt}^+(H)$ (resp. $\mathrm{tilt}^-(H)$) – that is to say if K is a heart satisfying $\Psi_\nu H \geqslant K \geqslant \Psi_{\nu'} H$ for some $\mathfrak{J}$-paths ν, ν', then there is a $\mathfrak{J}$-path ν'' such that $K = \Psi_{\nu''} H$.

§ 6.2 Dynamics of the nef monoid

The Hasse quiver, in principle, should say all one wants to hear about the poset alg-tilt(H). Lattice theoretic analyses of affine weak orders however remain elusive in practice. Fortunately, our understanding of the global structure of tilt(H) is greatly aided by the presence of the natural actions of various Picard groups – these play a role identical to that of coweight lattices in the study of Bruhat orders.

Note that tensoring by any line bundle $\mathscr{L} \in \operatorname{Pic} X$ naturally gives an autoequivalence of $\mathbf{D}^0 X$. We focus on the induced action on the poset of t-structures, where $\mathscr{L} \in \operatorname{Pic} X$ thus twists a heart K to

$$\mathscr{L} \otimes \mathsf{K} = \left\{ \mathscr{L} \otimes^{\mathbf{L}}_{X} k \mid k \in \mathsf{K} \right\}.$$

In what follows we study the orbit of $\mathsf{H} = \mathcal{per}(\frac{X}{Z})$ in relation to the poset tilt(H), setting up notation and stating key results. The proofs will follow (and rely on) a discussion of how these actions are observed on $\mathbf{K}$-theory and in the heart fan.

Certain twists of H can be described explicitly – namely, if we write $\mathscr{L}_i = \det(\mathcal{N}_i)$ $(i \in \Delta \setminus \mathfrak{J})$ for standard basis elements of $\operatorname{Pic}(X)$ with indexing as in lemma 4.1.1, then we have the following.

Proposition 6.2.1. *For any $i \in \Delta \setminus \mathfrak{J}$ and $n > 0$, the following statements hold.*

(1) The heart $(\mathscr{L}_i^{\vee})^{\otimes n} \otimes \mathsf{H}$ is intermediate with respect to H, and is the tilt of H in the smallest torsion class containing $\mathcal{O}_{C_i}(-n-1)[1]$.

(2) The heart $\mathscr{L}_i^{\otimes n} \otimes \mathsf{H}[-1]$ is intermediate with respect to H, and is the tilt of H in the smallest torsion-free class containing $\mathcal{O}_{C_i}(n-2)$.

If we denote by $\tau\colon X \to X_i$ the crepant partial contraction of $\bigcup_{j \in \Delta \setminus \{i\}} C_j$ (i.e. of all exceptional curves *except* C_i), then it is clear from the above description that we have

$$\begin{array}{ccccccccc} \mathsf{H} & > & \mathscr{L}_i^{\vee} \otimes \mathsf{H} & > & (\mathscr{L}_i^{\vee})^{\otimes 2} \otimes \mathsf{H} & > & \cdots & > & \mathcal{per}\left(\frac{X}{X_i}\right), \\ \mathsf{H}[-1] & < & \mathscr{L}_i \otimes \mathsf{H}[-1] & < & \mathscr{L}_i^{\otimes 2} \otimes \mathsf{H}[-1] & < & \cdots & < & \overline{\mathcal{per}}\left(\frac{X}{X_i}\right). \end{array}$$

Proposition 6.2.2. *For any $i \in \Delta \setminus \mathfrak{J}$, the torsion class $\mathcal{per}\left(\frac{X}{X_i}\right)[1] \cap \mathsf{H}$ in H is generated by the collection of objects $\{\mathcal{O}_{C_i}(n)[1] \mid n \leqslant -2\}$ while the torsion-free class $\overline{\mathcal{per}}\left(\frac{X}{X_i}\right) \cap \mathsf{H}$ is generated by $\{\mathcal{O}_{C_i}(n) \mid n \geqslant -1\}$. In particular, in the poset* tilt(H) *we have*

$$\begin{aligned} \mathcal{per}\left(\tfrac{X}{X_i}\right) &= \inf \left\{ (\mathscr{L}_i^{\vee})^{\otimes n} \otimes \mathsf{H} \mid n \geqslant 0 \right\}, \\ \overline{\mathcal{per}}\left(\tfrac{X}{X_i}\right) &= \sup \left\{ \mathscr{L}_i^{\otimes n} \otimes \mathsf{H}[-1] \mid n \geqslant 0 \right\}. \end{aligned}$$

In fact we realise many more geometric and semi-geometric hearts as limits of twists by line bundles on birational models. To explain this, note that if $W = \nu X$ is a birational model then the natural action of $\operatorname{Pic}(W)$ on $\mathbf{D}^0 W$ can be translated across the flop functor Ψ_ν to an action on $\mathbf{D}^0 X$. For brevity we continue to denote this action $\operatorname{Pic}(W) \circlearrowright \mathbf{D}^0 X$ by $\otimes$, i.e. for $\mathscr{L} \in \operatorname{Pic}(W)$ and $h \in \mathbf{D}^0 X$ we write

$$\mathscr{L} \otimes h = \Psi_\nu \left(\mathscr{L} \otimes^{\mathbf{L}}_{W} \Psi_\nu^{-1} h \right).$$

The induced action $\operatorname{Pic}(W) \circlearrowright$ t-str$(\mathbf{D}^0 X)$ on t-structures is denoted likewise.

Now $\operatorname{Pic}(W)$ sits as the lattice of integral points in the vector space $\operatorname{Pic}_{\mathbb{R}}(W)$, which we identify (as in proposition 5.1.10) with the hyperplane $\{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \mid \theta(\delta_{\mathfrak{J}}) = 0\}$. Cones in this hyperplane cut submonoids of $\operatorname{Pic}(W)$, in particular we can consider the various monoid actions $(\operatorname{Pic}(W) \cap \sigma) \circlearrowright \mathbf{D}^0 X$ determined by the choice of a birational model $W \in \operatorname{Bir}\left(\frac{X}{Z}\right)$ and a cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$.

The birational model W is σ-positive (definition 5.3.8) precisely when $\operatorname{Pic}(W) \cap \sigma$ lies in the nef cone. For such σ and W we can consider the $(\operatorname{Pic} W \cap \sigma)$-orbit of the reference heart $\mathsf{H} = \mathcal{per}(\frac{X}{Y})$. The following result shows that the orbit is independent of the choice of σ-positive birational model, lies in tilt(H), and limits to the maximal (semi-)geometric heart determined by σ.

Theorem 6.2.3. *Given a sequence of flops ν from X with corresponding birational model $W = \nu X$, the following statements hold in $\mathbf{D}^0 X$.*

(1) For any $\mathcal{L} \in \mathrm{Pic}(W)$, we have $H \geqslant \mathcal{L}^\vee \otimes H$ in the poset of t-structures on $\mathbf{D}^0 X$ if and only if $\mathcal{L}$ is nef, if and only if $\mathcal{L} \otimes H[-1] \geqslant H[-1]$. When these conditions hold, we have

$$H \ \geqslant\ \mathcal{L}^\vee \otimes H \ >\ \Psi_\nu\, \mathcal{coh}(W) \ >\ \Psi_\nu\, \overline{\mathcal{coh}}\, W \ >\ \mathcal{L} \otimes H[-1] \ \geqslant\ H[-1].$$

(2) Let $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ be a a cone such that W is σ-positive. Then

$$H^{tt}(\sigma) = \inf\{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \mathrm{Pic}(W) \cap \sigma\}, \qquad H_{tt}(\sigma) = \sup\{\mathcal{L} \otimes H[-1] \mid \mathcal{L} \in \mathrm{Pic}(W) \cap \sigma\}.$$

(3) Suppose $W' = \nu' X$ is another birational model, and the line bundles $\mathcal{L} \in \mathrm{Pic}(W)$ and $\mathcal{L}' \in \mathrm{Pic}(W')$ coincide as elements of $\Theta(\underline{\Delta} \setminus \mathfrak{J})$. If both $\mathcal{L}$ and $\mathcal{L}'$ are nef, then

$$\mathcal{L}'^\vee \otimes H = \mathcal{L}^\vee \otimes H \quad \textit{and} \quad \mathcal{L}' \otimes H[-1] = \mathcal{L} \otimes H[-1].$$

(4) Let $K \in \mathrm{tilt}(H)$ be of the form $\mathcal{L}'^\vee \otimes H$ or $\mathcal{L}' \otimes H[-1]$ for some line bundle $\mathcal{L}'$ on some birational model W'. If we have an inequality $K \leqslant \mathcal{coh}(W)$ or $K \geqslant \overline{\mathcal{coh}}(W)$, then there is an $\mathcal{L} \in \mathrm{Pic}(W)$ such that $K = \mathcal{L}^\vee \otimes H$ or $K = \mathcal{L} \otimes H[-1]$.

Given any non-zero cone $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$, theorem 5.3.9 gives a birational model $W = \nu X$ and a partial contraction $W \to Y$ such that $H^{tt}(\sigma) = \Psi_\nu \mathcal{per}(\frac{W}{Y})$. The monoid $\mathrm{Pic}\, W \cap \sigma$ can be naturally identified (as in proposition 5.1.10) with the nef cone of Y, and the limit of the twists $\mathcal{L}^\vee \otimes H$ ranging over $\mathcal{L} \in \mathrm{Pic}\, W \cap \sigma$ is precisely the heart $\mathcal{per}(\frac{W}{Y})$ which is geometric on the curves 'detected by' $\mathrm{Pic}\, W \cap \sigma$.

In particular the limit of all twists $\{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \mathrm{Pic}(W) \text{ nef}\}$ is the geometric heart $\Psi_\nu\, \mathcal{coh}\, W$, a 'fixed point' of the action of $\mathrm{Pic}(W)$.

Global order on Picard orbits. Suppose $W = \nu X$ is a birational model of X with exceptional curves C_i $(i \in \Delta \backslash \nu\mathfrak{J})$. The intersection pairing with these 1-cycles gives an isomorphism $\deg\colon \mathrm{Pic}(W) \to \mathbb{Z}^{(\Delta\backslash\nu\mathfrak{J})}$.

We endow $\mathbb{Z}^{(\Delta\backslash\nu\mathfrak{J})}$ with the product order (i.e. for tuples $(a_i), (b_i) \in \mathbb{Z}^{(\Delta\backslash\nu\mathfrak{J})}$ we have $(a_i) \leqslant (b_i)$ if and only if $a_i \leqslant b_i$ for each $i \in \Delta \setminus \nu\mathfrak{J}$) and write $\underline{0}, \underline{1} \in \mathbb{Z}^{(\Delta\backslash\nu\mathfrak{J})}$ for the tuples whose entries are all 0 or all 1 respectively. Thus for example a line bundle $\mathcal{L} \in \mathrm{Pic}(W)$ is nef if and only if $\underline{0} \leqslant \deg(\mathcal{L})$, and two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\, W$ satisfy $\deg(\mathcal{L}_1) \leqslant \deg(\mathcal{L}_2)$ if and only if $\mathcal{L}_2 \otimes \mathcal{L}_1{}^\vee$ is nef.

The partial order on various twists and shifts of $H = \mathcal{per}(\frac{X}{Z})$ can then be described as follows.

Corollary 6.2.4. *Given line bundles $\mathcal{L}_1, \mathcal{L}_2$ on some birational model $W \in \mathrm{Bir}(\frac{X}{Z})$ and integers $n_1, n_2 \in \mathbb{Z}$, we have $\mathcal{L}_1 \otimes H[n_1] \geqslant \mathcal{L}_2 \otimes H[n_2]$ in the poset of t-structures on $\mathbf{D}^0 X$ if and only if either $n_1 > n_2$, or $n_1 = n_2$ and $\deg(\mathcal{L}_1) \geqslant \deg(\mathcal{L}_2)$. Thus we have an isomorphism of posets*

$$\{\mathcal{L} \otimes H[n] \mid \mathcal{L} \in \mathrm{Pic}(W),\ n \in \mathbb{Z}\} \quad \simeq \quad \mathbb{Z} \times \mathbb{Z}^{(\Delta\backslash\mathfrak{J})}$$

where $\mathbb{Z} \times \mathbb{Z}^{(\Delta\backslash\mathfrak{J})}$ is ordered lexicographically on its two factors.

Proof. If $n_1 = n_2$, then it is clear that $\mathcal{L}_1 \otimes H[n_1] \geqslant \mathcal{L}_2 \otimes H[n_1]$ if and only if $H \geqslant \mathcal{L}_1{}^\vee \otimes \mathcal{L}_2 \otimes H$, which by theorem 6.2.3 (1) occurs if and only if $\deg(\mathcal{L}_1) \geqslant \deg(\mathcal{L}_2)$.

Thus to conclude it suffices to show $\mathcal{L}_1 \otimes H > \mathcal{L}_2 \otimes H[-1]$ for all $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\, W$. Now we may write $\mathcal{L}_1 = \mathcal{L}_-{}^\vee \otimes \mathcal{L}_+$ where the line bundles $\mathcal{L}_+, \mathcal{L}_- \in \mathrm{Pic}\, W$ are nef. Theorem 6.2.3 (1) shows there are inequalities $\mathcal{L}_+ \otimes H[-1] \geqslant H[-1]$ and $\mathcal{L}_-{}^\vee \otimes H > \Psi_\nu\, \mathcal{coh}\, W$. Applying the functor $\mathcal{L}_-{}^\vee \otimes (-)[1]$ to the first inequality thus yields $\mathcal{L}_1 \otimes H > \Psi_\nu\, \mathcal{coh}\, W$, and a similar argument gives $\Psi_\nu\, \overline{\mathcal{coh}}\, W > \mathcal{L}_2 \otimes H[-1]$ from which the conclusion follows. □

Restricting to the interval of intermediate hearts $\mathrm{tilt}(H) = [H[-1], H]$, we see that

$$\left\{K \in \mathrm{tilt}(H) \;\middle|\; K = \mathcal{L} \otimes H \text{ or } \mathcal{L} \otimes H[-1] \text{ for some } W \in \mathrm{Bir}(\tfrac{X}{Z}),\ \mathcal{L} \in \mathrm{Pic}(W)\right\}$$

decomposes into a union of sub-posets isomorphic to $(\mathbb{Z}_{\geqslant 0})^{|\Delta\backslash\mathfrak{J}|} \sqcup (\mathbb{Z}_{\leqslant 0})^{|\Delta\backslash\mathfrak{J}|}$ indexed over $\mathrm{Bir}(\frac{X}{Z})$. This highly regular sub-poset of $\mathrm{tilt}(H)$ provides anchors to track the remaining intermediate algebraic hearts.

Theorem 6.2.5. *For any algebraic heart* $K \in \mathrm{tilt}(H)$*, there is a birational model* W *of* X *and line bundles* $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(W)$ *with degrees*

$$\underline{0} \ \leqslant\ \deg(\mathcal{L}_1) \ <\ \deg(\mathcal{L}_2) \ \leqslant\ \deg(\mathcal{L}_1) + \underline{1},$$

such that one of the inequalities below holds:

$$\mathcal{L}_1{}^{\vee} \otimes H \ \geqslant\ K \ >\ \mathcal{L}_2{}^{\vee} \otimes H, \quad \textit{or} \quad \mathcal{L}_1 \otimes H[-1] \ \leqslant\ K \ <\ \mathcal{L}_2 \otimes H[-1].$$

Actions on the heart fan. In order to prove propositions 6.2.1 and 6.2.2 and theorems 6.2.3 and 6.2.5, we now examine the actions of Picard groups on t-structures, heart cones, and modifying modules. This builds upon the work of Hirano–Wemyss [HW23, §7] on the subject.

The action of $\mathrm{Pic}\,X$ on the Grothendieck group $\mathbf{K}X$ is straightforward to analyse after choosing an appropriate basis – choosing the classes $\delta_{\mathfrak{J}} = [\mathcal{O}_p]$ and $\alpha_i = [\mathcal{O}_{C_i}(-1)]$ $(i \in \Delta \setminus \mathfrak{J})$, the action $\mathrm{Pic}\,X \circlearrowright \mathbf{K}X$ is given by

$$\mathcal{L}_i \otimes \delta_{\mathfrak{J}} = \delta_{\mathfrak{J}}, \qquad \mathcal{L}_i \otimes \alpha_j = \begin{cases} \alpha_j + \delta_{\mathfrak{J}}, & j = i \\ \alpha_j, & \text{otherwise} \end{cases}. \tag{6.1}$$

This clearly preserves the root system, i.e. the transposed action on $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ preserves the intersection arrangement $\mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ and in particular takes chambers to chambers. This can be visualised by noting that the functionals $\{\alpha_i \mid i \in \Delta \setminus \mathfrak{J}\}$ give coordinates on the level set $\{\delta_{\mathfrak{J}} = 1\}$ and the action of $\mathrm{Pic}\,X$ on the level set restricts to translations along the lattice of integral points.

If $W = \nu X$ is a different birational model, then flop functor $\Psi_\nu : \mathbf{D}^0 W \to \mathbf{D}^0 X$ gives an isomorphism of $\mathbf{K}$-theory $\mathbf{K}W \to \mathbf{K}X$ which maps $\delta_{\nu\mathfrak{J}} \mapsto \delta_{\mathfrak{J}}$. Writing $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)]$ $(i \in \Delta \setminus \nu\mathfrak{J})$ for the classes of the simples of $\Psi_\nu \mathit{per}(\frac{W}{Z})$, we see that $\{\delta_{\mathfrak{J}}, \beta_i \mid i \in \Delta \setminus \mathfrak{J}\}$ is a basis for $\mathbf{K}X$ and the action $\mathrm{Pic}\,W \circlearrowright \mathbf{K}X$ in this basis is given by a formula analogous to (6.1). In particular the intersection arrangement is again preserved, and there is an induced action on the set of chambers.

For a bounded heart $K \subset \mathbf{D}^0 X$ and $\mathcal{L} \in \mathrm{Pic}\,W$, the chamber $\mathcal{L} \cdot \mathbf{C}(K)$ is the heart cone of $\mathcal{L}^{\vee} \otimes K$. The following preliminary lemma shows that this action plays well with the partial order on chambers.

Lemma 6.2.6. *Given a birational model* $W = \nu X$ *and nef line bundles* $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\,W$*, we have* $\mathcal{L}_1 C^+_{\mathfrak{J}} \geqslant \mathcal{L}_2 C^+_{\mathfrak{J}}$ *in the weak order if and only if* $\deg(\mathcal{L}_1) \leqslant \deg(\mathcal{L}_2)$.

Proof. If $\mathcal{L} \in \mathrm{Pic}(W)$ is nef and $i \in \Delta \setminus \nu\mathfrak{J}$, then the hyperplane $\{n\delta_{\mathfrak{J}} - \beta_i = 0\}$ separates $C^+_{\mathfrak{J}}$ and $\mathcal{L}\, C^+_{\mathfrak{J}}$ if and only if $0 < n \leqslant \deg(\mathcal{L}|_{C_i})$. It follows that if $\mathcal{L}_1, \mathcal{L}_2$ are nef such that $\mathcal{L}_1 C^+_{\mathfrak{J}} \geqslant \mathcal{L}_2 C^+_{\mathfrak{J}}$, then we necessarily have $\deg(\mathcal{L}_1) \leqslant \deg(\mathcal{L}_2)$.

Conversely consider any hyperplane $\{\alpha = 0\}$ defined by a restricted root α. Since α is simply the projection of a root in an affine Dynkin root system, we may write

$$\alpha = n\delta_{\mathfrak{J}} + \sum_{i \in \Delta \setminus \mathfrak{J}} a_i \beta_i,$$

for some $n \in \mathbb{Z}$ and some tuple of positive integers $\underline{a} = (a_i)$. Now for generic $\theta \in C^+_{\mathfrak{J}}$ normalised so that $\theta(\delta_{\mathfrak{J}}) = 1$, and for any $\mathcal{L} \in \mathrm{Pic}(W)$, we have

$$(\mathcal{L}\,\theta)(\alpha) = \theta(\alpha) + \langle \underline{a}, \deg \mathcal{L} \rangle$$

where $\langle -, - \rangle$ denotes the standard (Euclidean) inner product between two tuples in $\mathbb{Z}^{(\Delta \setminus \nu\mathfrak{J})}$. In particular if $\mathcal{L}$ is nef then $\langle \underline{a}, \deg \mathcal{L} \rangle \geqslant 0$, so the hyperplane $\{\alpha = 0\}$ separates $C^+_{\mathfrak{J}}$ and $\mathcal{L}\, C^+_{\mathfrak{J}}$ if and only if

$$\langle \underline{a}, \deg \mathcal{L} \rangle \geqslant \quad -\theta(\alpha) \quad \geqslant \quad 0.$$

It follows that if $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(W)$ are such that $\deg(\mathcal{L}_2) \geqslant \deg(\mathcal{L}_1) \geqslant \underline{0}$ then any hyperplane separating $C^+_{\mathfrak{J}}$ from $\mathcal{L}_1\, C^+_{\mathfrak{J}}$ also separates it from $\mathcal{L}_2\, C^+_{\mathfrak{J}}$ as required. $\square$

Actions on modifying modules. The following discussion pertains only to the $\dim Z = 3$ case of setup 1.0.2. Given a birational model $\pi\colon W \to Z$ and a line bundle $\mathcal{L} \in \operatorname{Pic}(W)$, the pushforward $\pi_*\mathcal{L}$ gives a reflexive R-module and this determines an injection $\operatorname{Pic}(W) \to \operatorname{Cl}(R)$ into the class group of R. It is clear that if W' is another birational model, then the inclusions of $\operatorname{Pic}(W)$ and $\operatorname{Pic}(W')$ into $\operatorname{Cl}(R)$ are compatible with the isomorphism $\operatorname{Pic}(W) \cong \operatorname{Pic}(W')$ induced by taking proper transforms, and we write $\operatorname{cl}(R) \subseteq \operatorname{Cl}(R)$ for the common image of all such inclusions. We remark that the equality $\operatorname{cl}(R) = \operatorname{Cl}(R)$ holds if and only if X is a minimal model of Z, this follows from [IW, proposition 9.1] and relies on the assumption that the singularity of Z is isolated.

Iyama and Wemyss [IW, §9.4] extensively study the action of $\operatorname{cl}(R)$ on the set $\operatorname{MM}^N(R)$, given by

$$L \cdot M = (L^* \otimes_R M)^{**}$$

for $L \in \operatorname{cl}(R)$, $M \in \operatorname{MM}^N(R)$, and $(-)^* = \operatorname{Hom}(-, R)$. In particular they show that the action is compatible with that on the intersection arrangement under the bijection $\mathbf{C}\colon \operatorname{MM}^N(R) \to \operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$. We translate the result into a form suitable for our purposes (see also [HW23, lemma 7.2]).

Proposition 6.2.7. *If $\pi : W \to Z$ is a birational model of the threefold X, then for every $\mathcal{L} \in \operatorname{Pic} W$ and $M \in \operatorname{MM}^N(R)$ we have $\mathcal{L} \cdot \mathbf{C}(M) = \mathbf{C}(\pi_*\mathcal{L} \cdot M)$.*

Proof. It suffices to consider $W = X$, and to prove the statement for the line bundles $\mathcal{L}_i = (\det \mathcal{N}_i)^\vee$ $(i \in \Delta \setminus \mathfrak{J})$ which generate $\operatorname{Pic}(X)$. Further, we may assume the basic modifying module N is *maximal modifying*, i.e.

$$\operatorname{add}(N) = \left\{M \in \operatorname{mod} R \text{ reflexive} \mid \operatorname{Ext}^1(N, M) = \operatorname{Ext}^1(M, N) = 0\right\}.$$

Indeed if not, then note that by [IW, theorem 9.1(1)] and [IW14, corollary 4.18] there is a modifying module N^c such that $N \oplus N^c$ is maximal modifying. This choice can clearly be made such that $N \oplus N^c$ is basic, and further [IW, theorem 9.5] shows that this $N \oplus N^c$ has at most $|\underline{\Delta}|$ indecomposable summands so we can index the summands of N^c as $N^c = \oplus_{i \in \mathfrak{I}} N_i$ for some $\mathfrak{I} \subset \mathfrak{J}$. Then we can work with the pair $(N \oplus N^c, \mathfrak{J} \setminus \mathfrak{I})$ instead of $(N, \mathfrak{J})$, using [IW, theorem 8.15] to translate the results back to N.

Consider as in the proof of theorem 4.2.4 a general elephant $\operatorname{Spec} \overline{R} \hookrightarrow \operatorname{Spec} R$ and the corresponding reduction functor $\mathbb{F}(-) = (-) \otimes_R \overline{R}$. Writing $\operatorname{modif}(R)$ for the set of modifying R-modules, Iyama–Wemyss [IW, definition 9.7] define a map $\operatorname{ind}\colon \operatorname{modif}(R) \to \Theta(\underline{\Delta} \setminus \mathfrak{J})$ as the composite

$$\begin{tikzcd} \operatorname{modif}(R) \arrow[d, "{[\operatorname{Hom}_R(N,-)]}"] & & \Theta(\underline{\Delta} \setminus \mathfrak{J}) \\ \mathbf{K}_{\text{split}}(\operatorname{proj} \Lambda) \arrow[rr, "\mathbb{F}(-)"] & & \mathbf{K}_{\text{split}}(\operatorname{proj} \mathbb{F}\Lambda) \arrow[u] \end{tikzcd} \tag{6.2}$$

where the final map sends class of the indecomposable projective $\mathbb{F}\Lambda_i = \mathbb{F}\operatorname{Hom}_R(N, N_i)$ to the vector $\theta_i \in \Theta(\underline{\Delta} \setminus \mathfrak{J})$ dual to $\alpha_i \in \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$. By construction, for a maximal modifying module M the cone $\mathbf{C}(M)$ is generated by the vectors $\operatorname{ind}(M_i)$ associated to indecomposable summands $M_i \subset M$. Now [IW, proposition 9.10 (1) and theorem 9.23 (2)] show that for any $M \in \operatorname{modif}(R)$ we have

$$\begin{aligned} \operatorname{ind}(\pi_*\mathcal{L}_i \cdot M) &= \operatorname{ind}(\det N_i \otimes_R M)^{**} \\ &= \operatorname{ind} M + \delta_{\mathfrak{J}}(\operatorname{ind} M) \cdot (\operatorname{ind}(\det N_i) - \operatorname{ind} R) \\ &= \operatorname{ind} M + \delta_{\mathfrak{J}}(\operatorname{ind} M) \cdot (\operatorname{ind} N_i - \operatorname{rk} N_i \cdot \operatorname{ind} N_0) \\ &= \operatorname{ind} M \circ (\mathcal{L}_i \otimes -) \end{aligned}$$

where the penultimate equality uses the identity $\operatorname{ind} N_i - \operatorname{ind}(\det N_i) = (\operatorname{rk} N_i - 1) \cdot \operatorname{ind} R$ which can be deduced, for example, from [IW, corollary 9.22]. Thus $\operatorname{ind}(\pi_*\mathcal{L}_i \cdot M) = \mathcal{L}_i \cdot \operatorname{ind} M$ for every $M \in \operatorname{modif}(R)$, and the result follows. □

In what follows we suppress the map π_* from notation, directly considering the action $\operatorname{Pic} W \circlearrowright \operatorname{MM}^N R$. Given $M \in \operatorname{MM}^N R$ and $\mathcal{L} \in \operatorname{Pic} W$, we write $\mathcal{L} \cdot \operatorname{End}_R(M) = \operatorname{End}_R(\mathcal{L}M)$. In particular $\mathcal{L}\Lambda$ is the endomorphism algebra of $\mathcal{L}N$. Note there is natural isomorphism of algebras $\varepsilon\colon \mathcal{L} \cdot \operatorname{End}_R(M) \to \operatorname{End}_R(M)$ and thus an equivalence $\varepsilon\colon \mathbf{D}^{\mathrm{fl}}(\mathcal{L} \operatorname{End}_R M) \to \mathbf{D}^{\mathrm{fl}}(\operatorname{End}_R M)$ which identifies standard hearts $(\varepsilon(H) = H)$.

Remark 6.2.8. To extend the above discussion to the $\dim Z = 2$ case of setup 1.0.2 we simply treat proposition 6.2.7 as definitive of the action $\operatorname{Pic} W \circlearrowright \mathrm{MM}^{N} R$ – that is, given a line bundle $\mathscr{L} \in \operatorname{Pic} W$ and a tagged modifying module (σ, M), we declare $\mathscr{L} \cdot (\sigma, M)$ to be the element $(\mathscr{L}\sigma, M) \in \mathrm{MM}^{N} R$.

Actions on t-structures. Given $\mathscr{L} \in \operatorname{Pic} W$, we examine the intermediacy of $\mathscr{L}^{\vee} \otimes H$ (with respect to H) by comparing it with the tautologically intermediate heart associated with $\mathscr{L}N \in \mathrm{MM}^{N} R$, i.e. the image of the Brenner–Butler map

$$\mathrm{fl}\,\mathit{mod}(\mathscr{L}\Lambda) \xrightarrow{\mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(\mathscr{L}\cdot N, N), -)} \mathbf{D}^{\mathrm{fl}}\Lambda.$$

Choosing an atomic sequence of mutations λ from N to $\mathscr{L} \cdot N$, the above map is simply Ψ_λ and the resulting intermediate heart is $\Psi_\lambda H$. Evidently this heart agrees with $\mathscr{L}^{\vee} \otimes H$ in $\mathbf{K}$-theory, and we now show that they are in fact equal when $\mathscr{L}$ is nef.

Proposition 6.2.9. *Let $\mathscr{L} \in \operatorname{Pic} W$ be a nef line bundle and λ an atomic path from N to $\mathscr{L}N$. Then the following diagram commutes on objects.*

$$\begin{tikzcd}
\mathbf{D}^0 X \arrow[rr, "\mathscr{L}^{\vee}\otimes(-)"] \arrow[d, "\mathrm{VdB}"] & & \mathbf{D}^0 X \arrow[d, "\mathrm{VdB}"] \\
\mathbf{D}^{\mathrm{fl}}\Lambda \arrow[r, "\varepsilon"] & \mathbf{D}^{\mathrm{fl}}\,\mathscr{L}\Lambda \arrow[r, "\Psi_\lambda"] & \mathbf{D}^{\mathrm{fl}}\Lambda
\end{tikzcd} \tag{6.3}$$

Proof. First consider the case when $\mathscr{L}$ is a nef bundle on X, i.e. when the sequence of flops ν is trivial. We induct on $\deg \mathscr{L}$, noting that the statement trivially holds for $\mathscr{L} = \mathscr{O}_X$.

So suppose the diagram (6.3) commutes for a nef $\mathscr{L} \in \operatorname{Pic} X$, and consider the bundle $\mathscr{L}_i \otimes \mathscr{L}$ for some $i \in \Delta \setminus \mathfrak{J}$. By lemmas 3.4.3 and 6.2.6, we can choose positive paths λ, λ_i whose composite $\lambda\lambda_i$ remains atomic such that

$$\mathscr{L}_i\, C^+_{\mathfrak{J}} = \lambda_i C^+_{\mathfrak{J}}, \qquad (\mathscr{L}_i \otimes \mathscr{L})\, C^+_{\mathfrak{J}} = \lambda\lambda_i C^+_{\mathfrak{J}}.$$

Further since λ is a reduced path from $\mathscr{L}_i\, C^+_{\mathfrak{J}}$ to $(\mathscr{L}_i \otimes \mathscr{L})\, C^+_{\mathfrak{J}}$, translating it along $\mathscr{L}_i^{\vee}$ gives a reduced (hence atomic) path λ' such that

$$\mathscr{L}\, C^+_{\mathfrak{J}} = \lambda' C^+_{\mathfrak{J}}.$$

By proposition 6.2.7, we have $\mathscr{L}_i N = \lambda N$, $(\mathscr{L}_i \otimes \mathscr{L})N = \lambda\lambda_i N$, and $\mathscr{L}N = \lambda' N$, and each path appearing in these expressions is atomic. Then the corresponding diagram (6.3) for $\mathscr{L}_i \otimes \mathscr{L}$ is precisely the outer circuit of the diagram below, and it suffices to prove that each component circuit (i)–(v) commutes.

$$\begin{tikzcd}
\mathbf{D}^0 X \arrow[rr, "\mathscr{L}^{\vee}\otimes(-)"] \arrow[d, "\mathrm{VdB}"] \arrow[drr, phantom, "\text{(i)}"] & & \mathbf{D}^0 X \arrow[rr, "\mathscr{L}_i^{\vee}\otimes(-)"] \arrow[d, "\mathrm{VdB}"] \arrow[drr, phantom, "\text{(ii)}"] & & \mathbf{D}^0 X \arrow[d, "\mathrm{VdB}"] \\
\mathbf{D}^{\mathrm{fl}}\Lambda \arrow[r, "\varepsilon"] \arrow[drr, bend right, "\varepsilon"'] & \mathbf{D}^{\mathrm{fl}}\,\mathscr{L}\Lambda \arrow[r, "\Psi_{\lambda'}"] \arrow[dr, "\varepsilon"] \arrow[d, phantom, "\text{(iii)}"] & \mathbf{D}^{\mathrm{fl}}\Lambda \arrow[r, "\varepsilon"] \arrow[d, phantom, "\text{(iv)}"] & \mathbf{D}^{\mathrm{fl}}\,\mathscr{L}_i\Lambda \arrow[r, "\Psi_{\lambda_i}"] \arrow[d, phantom, "\text{(v)}"] & \mathbf{D}^{\mathrm{fl}}\Lambda \\
 & & \mathbf{D}^{\mathrm{fl}}(\mathscr{L}_i \otimes \mathscr{L})\Lambda \arrow[ur, "\Psi_\lambda"] \arrow[urr, bend right, "\Psi_{\lambda\lambda_i}"'] & &
\end{tikzcd}$$

The pentagon (i) commutes by induction hypothesis, and (ii) is precisely the diagram shown to commute in [HW23, theorem 7.4]. The triangle (iii) commutes since there is a natural isomorphism of functors

$$(\pi_* \mathscr{L}_i^{\vee} \otimes_R -)^{**} \circ (\pi_* \mathscr{L}^{\vee} \otimes_R -)^{**} \quad \cong \quad (\pi_*(\mathscr{L}_i \otimes \mathscr{L})^{\vee} \otimes_R -)^{**},$$

so that the isomorphism of algebras $\Lambda \xrightarrow{\varepsilon} (\mathscr{L}_i \otimes \mathscr{L})\Lambda$ is the composite $\Lambda \xrightarrow{\varepsilon} \mathscr{L}\Lambda \xrightarrow{\varepsilon} (\mathscr{L}_i \otimes \mathscr{L})\Lambda$. The commutativity of (iv) is the content of [HW23, lemma 7.3], while the triangle (v) commutes by lemma 4.2.8 since $\lambda\lambda_i$ is an atomic path. Thus the whole diagram commutes as required.

Now suppose we are in the general case, i.e. $\mathscr{L}$ is a nef bundle on the flop $W = \nu X$ and λ is an atomic path from N to $\mathscr{L}N$. Write $M = \nu N$ for the modifying R-module generator associated to W, and choose an atomic path λ' from M to $\mathscr{L}M$. Further we may assume the sequence of flops ν is atomic, so that translating ν by $\mathscr{L}$ as above gives an atomic path ν' from $\mathscr{L}N$ to $\mathscr{L}M$.

We can thus construct a diagram

$$\begin{tikzcd}
\mathbf{D}^0 X \arrow[r, "\mathrm{flop}^{-1}"] \arrow[d, "\mathrm{VdB}"] \arrow[dr, phantom, "\text{(i)}"] & \mathbf{D}^0 W \arrow[rr, "\mathscr{L}^\vee \otimes (-)"] \arrow[d, "\mathrm{VdB}"] & \text{(ii)} & \mathbf{D}^0 W \arrow[r, "\mathrm{flop}"] \arrow[d, "\mathrm{VdB}"] \arrow[dr, phantom, "\text{(iii)}"] & \mathbf{D}^0 X \arrow[d, "\mathrm{VdB}"] \\
\mathbf{D}^{\mathrm{fl}} \Lambda \arrow[r, "\Psi_\nu^{-1}"] \arrow[drr, bend right, "\varepsilon"'] & \mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \arrow[r, "\varepsilon"] \arrow[d, phantom, "\text{(iv)}"] & \mathbf{D}^{\mathrm{fl}}\, \mathscr{L}\, {}_\nu\Lambda_\nu \arrow[r, "\Psi_{\lambda'}"] \arrow[d, "\Psi_{\nu'}"] & \mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \arrow[r, "\Psi_\nu"] \arrow[d, phantom, "\text{(v)}"] & \mathbf{D}^{\mathrm{fl}} \Lambda \\
 & & \mathbf{D}^{\mathrm{fl}}\, \mathscr{L}\Lambda \arrow[urr, bend right, "\Psi_\lambda"'] & &
\end{tikzcd}$$

where the squares (i) and (iii) commute by theorem 4.3.4, while the circuit (ii) commutes from the above discussion since $\mathscr{L}$ is a nef bundle on W (the trivial flop of W). The commutativity of (iv) is again the content of [HW23, lemma 7.3]. To show (v) commutes we use lemma 4.2.8, observing that the composite paths $\lambda'\nu$ and $\nu'\lambda$ are both atomic – indeed any hyperplane crossed by the atomic path ν must pass through the cones

$$C^+_{\mathfrak{J}} \cap \nu C^+_{\mathfrak{J}} \quad \supseteq \quad \bigcap_{i \in \Delta \setminus \nu\mathfrak{J}} \{\beta_i = 0\}$$

and hence is of the form $\left\{\sum_{i \in \Delta\setminus\nu\mathfrak{J}} b_i\beta_i = 0\right\}$ for some tuple of non-negative integers b_i. But then any generic vector $\theta \in \nu C^+_{\mathfrak{J}}$ and its translate $\mathscr{L}\theta \in \lambda'\nu C^+_{\mathfrak{J}}$ both evidently lie on the same side of such a hyperplane, which thereby cannot be crossed by the atomic path λ'. Similar reasoning shows the hyperplanes crossed by λ and ν' are distinct.

Thus the whole diagram commutes, and reading the outer circuit yields the required diagram (6.3). □

Intermediacy of twists. We now prove parts (1), (3), (2), and (4) of theorem 6.2.3 in that order, noting that each successive proof relies on the previous ones.

Proof of theorem 6.2.3 (1). If $\mathscr{L} \in \mathrm{Pic}(W)$ is nef, then proposition 6.2.9 shows that the heart $\mathscr{L}^\vee \otimes H$ coincides with $\Psi_\lambda H$ for some atomic path λ. In particular, it is intermediate with respect to H.

On the other hand given any $\mathscr{L} \in \mathrm{Pic}(W)$ such that $H \geqslant \mathscr{L}^\vee \otimes H$, we may express $\mathscr{L}$ as a difference of nef bundles $\mathscr{L} = \mathscr{L}_-{}^\vee \otimes \mathscr{L}_+$ to obtain the inequality $\mathscr{L}_-{}^\vee \otimes H \geqslant \mathscr{L}_+{}^\vee \otimes H$ in $\mathrm{tilt}(H)$. Noting $\mathscr{L}_-{}^\vee \otimes H = \Psi_\nu H$ and $\mathscr{L}_+{}^\vee \otimes H = \Psi_{\nu'} H$ for some paths ν, ν', by corollary 6.1.2 we thus have an inequality of chambers $\mathscr{L}_-\, C^+_{\mathfrak{J}} \geqslant \mathscr{L}_+\, C^+_{\mathfrak{J}}$ in the weak order. Then lemma 6.2.6 shows $\deg(\mathscr{L}_-) \leqslant \deg(\mathscr{L}_+)$, so that $\mathscr{L}$ is nef.

One can likewise show that the nef-ness of $\mathscr{L}$ is also equivalent to the inequality $\mathscr{L} \otimes H[-1] \geqslant H[-1]$, by first inverting the diagram (6.3) to show $\mathscr{L} \otimes H[-1] = \Phi_\lambda H[-1]$ for some $\mathfrak{J}$-path λ and then proceeding as above.

Lastly we show $\mathscr{L}^\vee \otimes H > \Psi_\nu(\mathit{coh}\, W)$ when $\mathscr{L} \in \mathrm{Pic}\, W$ is nef. Consider the $\mathbf{K}$-theory classes $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)]$ for $i \in \Delta \setminus \nu\mathfrak{J}$, and the vector $\beta_0^* \in \Theta(\underline{\Delta} \setminus \mathfrak{J})$ determined as

$$\beta_0^*(\beta_i) = \begin{cases} 1, & i = 0 \\ 0, & i \in \Delta \setminus \nu\mathfrak{J}. \end{cases}$$

One checks that $\beta_0^*(\delta_{\mathfrak{J}}) = 1$ and $\beta_0^*(\alpha_i) = 0$ for all $i \in \Delta \setminus \mathfrak{J}$, so we have $\beta_0^* \in C^+_{\mathfrak{J}}$ and thus the vector $\mathscr{L} \cdot \beta_0^*$ lies in $\mathscr{L} C^+_{\mathfrak{J}} = \mathbf{C}(\mathscr{L}^\vee \otimes H)$. Further since $\mathscr{L}$ is nef, we see that $\mathscr{L} \cdot \beta_0^* = \beta_0^* + \theta^0$ for some $\theta^0 \in \nu C^0_{\mathfrak{J}}$.

It follows that we have

$$\begin{aligned}\mathscr{L}^{\vee}\otimes \mathsf{H}[1]\cap \mathsf{H} &\subseteq \mathsf{H}^{\mathrm{tr}}(\mathscr{L}\cdot\beta_0^*)\\ &= \{h\in \mathsf{H} \mid \mathscr{L}\cdot\beta_0^*[f]\leqslant 0 \text{ for all factors } h\twoheadrightarrow f\}\\ &= \{h\in \mathsf{H} \mid \theta^0[f]\leqslant -\beta_0^*[f] \text{ for all factors } h\twoheadrightarrow f\}\\ &\subseteq \{h\in \mathsf{H} \mid \theta^0[f]< 0 \text{ for all non-zero factors } h\twoheadrightarrow f\neq 0\}\\ &= \mathsf{H}_{\mathrm{tr}}(\theta_0)\\ &\subseteq \Psi_\nu\, \mathit{coh}\, W[1]\cap \mathsf{H}\end{aligned}$$

and thus $\mathscr{L}^{\vee}\otimes \mathsf{H}\geqslant \Psi_\nu\,\mathit{coh}\,W$, with the inequality being necessarily strict since $\Psi_\nu\,\mathit{coh}\,W$ is not algebraic. The proof of the corresponding statement for $\mathscr{L}\otimes\mathsf{H}[-1]$ is analogous. □

Proof of theorem 6.2.3 (3). Given birational models $W=\nu X$ and $W'\in\nu' X$, we have seen that the action of any $\mathscr{L}\in \mathrm{Pic}(W)$ on $\mathbf{K}X$ coincides with that of its proper transform $\mathscr{L}'\in\mathrm{Pic}(W')$. Thus in particular $\mathscr{L}^{\vee}C_{\mathfrak{Z}}^{+}=\mathscr{L}'^{\vee}C_{\mathfrak{Z}}^{+}$, i.e. $\mathbf{C}(\mathscr{L}\otimes\mathsf{H})=\mathbf{C}(\mathscr{L}'\otimes\mathsf{H})$.

Now if $\mathscr{L}$ and $\mathscr{L}'$ are both nef, then $\mathscr{L}\otimes\mathsf{H}$ and $\mathscr{L}'\otimes\mathsf{H}$ are both intermediate with respect to H by theorem 6.2.3 (1). But the assignment $\mathbf{C}\colon \mathrm{tilt}(\mathsf{H})\to \mathrm{HFan}(\mathsf{H})$ restricts to a bijection between algebraic intermediate hearts and full dimensional heart cones, so $\mathscr{L}^{\vee}\otimes\mathsf{H}=\mathscr{L}'^{\vee}\otimes\mathsf{H}$. Likewise, the hearts $\mathscr{L}\otimes\mathsf{H}[-1]$ and $\mathscr{L}'\otimes\mathsf{H}[-1]$ are equal. □

Proof of theorem 6.2.3 (2). Suppose σ and W are as given, and consider a line bundle $\mathscr{L}\in\mathrm{Pic}(W)\cap\sigma$. If υX is another σ-positive birational model, then the proper transform of $\mathscr{L}$ is nef on υX and thus theorem 6.2.3 (1) and (3) give us $\mathscr{L}^{\vee}\otimes\mathsf{H}>\Psi_\upsilon\,\mathit{coh}(\upsilon X)$. Considering all such inequalities together, we obtain

$$\inf\{\mathscr{L}^{\vee}\otimes\mathsf{H}\mid \mathscr{L}\in\mathrm{Pic}(W)\cap\sigma\}\ \geqslant\ \sup\left\{\Psi_\upsilon\,\mathit{coh}(\upsilon X)\mid \sigma\subset\upsilon C_{\mathfrak{Z}}^{0}\right\}\ \overset{(5.3.9)}{=}\ \mathsf{H}^{\mathrm{tt}}(\sigma).$$

Conversely suppose $h\in\mathsf{H}$ lies in $\mathscr{L}^{\vee}\otimes\mathsf{H}$ for every $\mathscr{L}\in\mathrm{Pic}(W)\cap\sigma$, i.e. H lies in the torsion-free class associated to the infimum on the left-hand side above. Pick a line bundle $\mathscr{L}_0$ that lies *generically* in $\mathrm{Pic}(W)\cap\sigma$, i.e. $\mathscr{L}_0$ does not lie in $\mathrm{Pic}(W)\cap\sigma'$ for any proper face $\sigma'\subset\sigma$. Thus $\mathscr{L}_0$ determines a generic vector $\theta_0\in\sigma$. Considering the vector $\theta=\mathscr{L}_0^{\otimes n}\cdot\alpha_0^*$ where $n>0$ is some integer and α_0^* is the vector given by (5.9), one calculates

$$\theta=\alpha_0^*+n\cdot\theta_0.$$

Moreover θ clearly lies in $\mathscr{L}_0^{\otimes n}C_{\mathfrak{Z}}^{+}=\mathbf{C}((\mathscr{L}_0{}^{\vee})^{\otimes n}\otimes\mathsf{H})$, and thus we have $h\in\mathsf{H}_{\mathrm{tf}}(\theta)$. In other words, $\alpha_0^*[s]+n\cdot\theta_0[s]>0$ for every non-zero sub-object $s\hookrightarrow h$. But $n>0$ was arbitrary, so we must in fact have $\theta_0[s]\geqslant 0$ for every sub-object $s\hookrightarrow h$ – this is precisely the condition $h\in\mathsf{H}^{\mathrm{tf}}(\theta_0)$, and thus giving the opposite inequality

$$\inf\{\mathscr{L}^{\vee}\otimes\mathsf{H}\mid\mathscr{L}\in\mathrm{Pic}\,W\cap\sigma\}\ \leqslant\ \mathsf{H}^{\mathrm{tt}}(\theta_0)\ =\ \mathsf{H}^{\mathrm{tt}}(\sigma).$$

The statement for $\mathsf{H}_{\mathrm{tt}}(\sigma)$ can be proved analogously. □

Proof of theorem 6.2.3 (4). Suppose K is as given, in particular the heart cone $\mathbf{C}(\mathsf{K})$ is of the form $\mathscr{L}C_{\mathfrak{Z}}^{+}$ or $\mathscr{L}C_{\mathfrak{Z}}^{-}$ for a unique $\mathscr{L}\in\mathrm{Pic}(W)$ – without loss of generality the former. If this $\mathscr{L}$ were nef, then by theorem 6.2.3 (1) we would have that K and $\mathscr{L}^{\vee}\otimes\mathsf{H}$ are elements of $\mathrm{tilt}(\mathsf{H})$ with identical full-dimensional heart cones, whence they must be equal as required.

Noting $\mathbf{C}(\mathsf{K})$ lies in $\{\delta_{\mathfrak{Z}}\geqslant 0\}$, we see that the inequality $\mathsf{K}\leqslant\overline{\mathit{coh}}(W)$ cannot occur. Therefore we must show that $\mathscr{L}$ is necessarily nef if $\mathsf{K}\geqslant\mathit{coh}(W)$. By theorem 6.2.3 (2), this inequality can be rewritten as the containment of torsion classes

$$\mathsf{K}[1]\cap\mathsf{H}\ \subseteq\ \sup\{\mathscr{L}_0{}^{\vee}\otimes\mathsf{H}[1]\cap\mathsf{H}\mid\mathscr{L}_0\in\mathrm{Pic}(W)\text{ nef}\}.\tag{6.4}$$

Now for any two nef bundles $\mathscr{L}_1,\mathscr{L}_2\in\mathrm{Pic}(W)$, from theorem 6.2.3 (1) we see that both $\mathscr{L}_1{}^{\vee}\otimes\mathsf{H}$ and $\mathscr{L}_2{}^{\vee}\otimes\mathsf{H}$ are greater than $(\mathscr{L}_1\otimes\mathscr{L}_2)^{\vee}\otimes\mathsf{H}$ in the poset of t-structures. In other words, we have the containmen of torsion classes

$$(\mathscr{L}_1{}^{\vee}\otimes\mathsf{H}[1]\cap\mathsf{H})\ \cup\ (\mathscr{L}_2{}^{\vee}\otimes\mathsf{H}[1]\cap\mathsf{H})\quad\subseteq\quad(\mathscr{L}_1\otimes\mathscr{L}_2)^{\vee}\otimes\mathsf{H}[1]\cap\mathsf{H}.\tag{6.5}$$

It follows that the supremum in (6.4) is in fact a union. The torsion class $\mathsf{K}[1] \cap \mathsf{H}$ is finitely generated (for example by the finite set of brick labels in the interval $[\mathsf{K}, \mathsf{H}] \subset \mathrm{tilt}(\mathsf{H})$), so a sufficiently ample $\mathcal{L}_0 \in \mathrm{Pic}(W)$ determines a torsion class $\mathcal{L}_0 \otimes \mathsf{H}[1] \cap \mathsf{H}$ in this union that contains all the generators of $\mathsf{K}[1] \cap \mathsf{H}$, and hence contains $\mathsf{K}[1] \cap \mathsf{H}$.

Thus we have $\mathsf{H} \geqslant \mathsf{K} \geqslant \mathcal{L}_0{}^{\vee} \otimes \mathsf{H}$, and consequently (by corollary 6.1.2) the inequality of heart cones

$$C_{\mathfrak{J}}^{+} \geqslant \mathcal{L}\, C_{\mathfrak{J}}^{+} \geqslant \mathcal{L}_0\, C_{\mathfrak{J}}^{+}$$

for some for some nef $\mathcal{L}_0 \in \mathrm{Pic}(W)$. The nef-ness of $\mathcal{L}$ is then evident, e.g. by considering the hyperplanes $\{[\Psi_v \mathcal{O}_{C_i}(n)] = 0\}$ defined by curves $C_i \subset W$ and $n \in \mathbb{Z}$. □

The proofs of propositions 6.2.1 and 6.2.2 and theorem 6.2.5 now follow.

Proof of proposition 6.2.1. Given the line bundle $\mathcal{L} = \mathcal{L}_i^{\otimes n}$ ($i \in \Delta \setminus \mathfrak{J}$, $n > 0$), theorem 6.2.3 (1) shows that $\mathcal{L}^{\vee} \otimes \mathsf{H}$ is intermediate with respect to H. Further since it is an Artinian tilt, proposition 2.3.3 shows that the corresponding torsion class $\mathcal{L}^{\vee} \otimes \mathsf{H}[1] \cap \mathsf{H}$ is generated by the semibrick

$$\begin{aligned} \mathsf{S} &= \{\mathsf{b} \in \mathsf{H} \mid \mathsf{b}[-1] \text{ is a simple object of } \mathcal{L}^{\vee} \otimes \mathsf{H}\} \\ &= \{\mathsf{b} \in \mathsf{H} \mid \mathsf{b} \text{ is the brick label of a covering relation } \mathsf{K} \gtrdot \mathcal{L}^{\vee} \otimes \mathsf{H} \text{ in } \mathrm{tilt}(\mathsf{H})\}. \end{aligned}$$

In particular, the size of S is equal to the number of hearts in $\mathrm{tilt}(\mathsf{H})$ covered by $\mathcal{L}^{\vee} \otimes \mathsf{H}$. Equivalently (corollary 6.1.2), this is the number of covering relations of the form $\sigma \lessdot \mathcal{L}\, C_{\mathfrak{J}}^{+}$ in $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$.

Such covering relations in the weak order correspond to hyperplanes that support (i.e. contain a codimension one face of) $\mathcal{L}\, C_{\mathfrak{J}}^{+}$ and separate it from $C_{\mathfrak{J}}^{+}$. But the hyperplanes supporting $\mathcal{L}\, C_{\mathfrak{J}}^{+}$ are defined by the roots

$$\mathcal{L}^{\vee} \otimes \alpha_i = \alpha_i - n\delta_{\mathfrak{J}}, \quad \mathcal{L}^{\vee} \otimes \alpha_j = \alpha_j \quad (j \in \Delta \setminus \mathfrak{J},\ j \neq i), \quad \text{and} \quad \mathcal{L}^{\vee} \otimes \alpha_0 = \alpha_0 + n\delta_i \cdot \delta_{\mathfrak{J}}.$$

Of these only $\alpha_i - n\delta_{\mathfrak{J}}$ separates $C_{\mathfrak{J}}^{+}$ from $\mathcal{L}\, C_{\mathfrak{J}}^{+}$. Thus the semibrick S has precisely one element, and examining the simples of $\mathcal{L}^{\vee} \otimes \mathsf{H}$ shows that this element must be

$$\mathcal{L}^{\vee} \otimes \mathcal{O}_{C_i}(-1)[1] = \mathcal{O}_{C_i}(-n-1)[1].$$

The statement for $\mathcal{L} \otimes \mathsf{H}[-1]$ can be proved analogously. □

Proof of proposition 6.2.2. Immediate from proposition 6.2.1 and theorem 6.2.3 (2). □

Proof of theorem 6.2.5. We prove the statement for $\mathsf{K} = \Psi_v \mathsf{H}$ for some positive $\mathfrak{J}$-path v, the case for $\Phi_v \mathsf{H}[-1]$ being analogous. Now given a positive $\mathfrak{J}$-path v, there is a unique spherical $\mathfrak{J}$-path μ such that

$$\begin{aligned} vC_{\mathfrak{J}}^{+} &\subseteq \mu C_{\mathfrak{J}}^{0} + \mathbb{R}_{\geqslant 0} \cdot \alpha_0^* \\ &= \left\{\theta_0 + t\alpha_0^* \mid \theta_0 \in vC_{\mathfrak{J}}^{0},\ t \geqslant 0\right\}, \end{aligned}$$

where α_0^* is as in (5.9). Indeed, the half space $\{\delta_{\mathfrak{J}} \geqslant 0\}$ (which contains $\mathrm{Arr}^{+}(\underline{\Delta}, \mathfrak{J})$) can be written as the union of all such regions and they intersect pairwise in subsets of positive codimension. In particular for each class $\beta_i = [\Psi_v \mathcal{O}_{C_i}(-1)]$ ($i \in \Delta \setminus v\mathfrak{J}$), we have

$$vC_{\mathfrak{J}}^{+} \subseteq \{\theta \mid \theta(\beta_i) \geqslant 0\}.$$

A similar argument involving a decomposition of $\{\delta_{\mathfrak{J}} \geqslant 0\}$ shows that for each $i \in \Delta \setminus v\mathfrak{J}$ there is a unique integer d_i such that

$$vC_{\mathfrak{J}}^{+} \subseteq \{\theta \mid (d_i + 1) \cdot \theta(\delta_{\mathfrak{J}}) \geqslant \theta(\beta_i) \geqslant d_i \cdot \theta(\delta_{\mathfrak{J}})\}, \tag{6.6}$$

and clearly we must have $d_i \geqslant 0$ in this case. Then the vector $(d_i) \in \mathbb{Z}^{|\Delta \setminus v\mathfrak{J}|}$ determines line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(W)$ with degrees (d_i), $(d_i + 1)$ respectively – evidently we have

$$\underline{0} \leqslant \deg(\mathcal{L}_1) < \deg(\mathcal{L}_2) = \deg(\mathcal{L}_1) + \underline{1}.$$

Moreover, by construction we have the inequalities of chambers

$$\mathscr{L}_1\, C^+_{\mathfrak{J}} \;\geqslant\; \nu C^+_{\mathfrak{J}} \;\geqslant\; \mathscr{L}_2\, C^+_{\mathfrak{J}}. \tag{6.7}$$

To see why, suppose the hyperplane $\{\alpha = 0\}$ separates $C^+_{\mathfrak{J}}$ and $\mathscr{L}_1 C^+_{\mathfrak{J}}$, where (up to replacing α by its negative) we may write

$$\alpha = n\delta_{\mathfrak{J}} + \sum_{i \in \Delta\setminus\mu\mathfrak{J}} a_i\beta_i$$

for some $n \in \mathbb{Z}$ and a tuple of non-negative integers (a_i). For generic $\theta \in C^+_{\mathfrak{J}}$, the only way in which $\theta(\alpha)$ and $(\mathscr{L}_1\theta)(\alpha) = \theta(\alpha) + \sum_i a_i d_i\, \theta(\delta_{\mathfrak{J}})$ can have opposite signs is if $\theta(\alpha) \leqslant 0$ and $(\mathscr{L}_1\theta)(\alpha) \geqslant 0$. In particular choosing the vector $\theta = \alpha_0^*$ as in (5.9) which vanishes on all β_i ($i \in \Delta \setminus \mu\mathfrak{J}$), we obtain the inequality

$$n + \sum_{i \in \Delta\setminus\mu\mathfrak{J}} a_i d_i \geqslant 0.$$

It follows that for generic $\theta \in \nu C^+_{\mathfrak{J}}$ normalised so that $\theta(\delta_{\mathfrak{J}}) = 1$, we have

$$\theta(\alpha) \;=\; n + \sum_{i \in \Delta\setminus\mu\mathfrak{J}} a_i\theta(\beta_i) \;\geqslant\; n + \sum_{i \in \Delta\setminus\mu\mathfrak{J}} a_i d_i \;\geqslant\; 0.$$

In other words, the hyperplane cut by α must also separate $C^+_{\mathfrak{J}}$ and $\nu C^+_{\mathfrak{J}}$, showing $\mathscr{L}_1\, C^+_{\mathfrak{J}} \geqslant \nu C^+_{\mathfrak{J}}$ as required. The inequality $\nu C^+_{\mathfrak{J}} > \mathscr{L}_2\, C^+_{\mathfrak{J}}$ is analogous.

The desired inequality on K then follows by translating the inequality of heart cones (6.7) into inequalities of intermediate algebraic hearts via corollary 6.1.2. □

§ 6.3 Every torsion theory on the perverse heart is numerical

The above analysis of $\operatorname{tilt}(H)$ paves the way for this result, showing that the heart fan of H detects *every* intermediate t-structure and thus an arbitrary heart in $\operatorname{tilt}(H)$ must be one of the algebraic, geometric, or semi-geometric t-structures described in chapter 5.

Theorem 6.3.1. *Let K be an arbitrary tilt of H. Then the heart cone $\mathbf{C}(K)$ is non-zero.*

We prove theorem 6.3.1 over the course of this section. Thus, fix $K \in \operatorname{tilt}(H)$ as above. If K is algebraic, then $\mathbf{C}(K)$ is full-dimensional (in particular, non-zero) by theorem 2.4.4. So we may restrict to the case when K is non-algebraic. By theorem 5.2.3, $\mathbf{C}(K)$ is thus a (possibly zero) cone in $\operatorname{Arr}(\Delta, \mathfrak{J})$. The following lemma is then the key tool we exploit.

Lemma 6.3.2. *Given $K \in \operatorname{tilt}(H)$, suppose there is a non-zero cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ and a σ-positive birational model $W \in \operatorname{Bir}\left(\frac{X}{Z}\right)$ such that*

$$\mathscr{L}^\vee \otimes H \;\geqslant\; K \;\geqslant\; \mathscr{L} \otimes H[-1]$$

for every $\mathscr{L} \in \operatorname{Pic}(W) \cap \sigma$. Then the heart cone $\mathbf{C}(K)$ contains σ, and is in particular non-zero.

Proof. The conditions, combined with theorem 6.2.3 (2) show $H^{tt}(\sigma) \geqslant K \geqslant H_{tt}(\sigma)$ so the result follows from lemma 2.4.6. □

It is relatively straightforward to obtain the bound on one side for lemma 6.3.2.

Lemma 6.3.3. *If the heart $K \in \operatorname{tilt}(H)$ is not algebraic, then there is a non-zero cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ and a σ-positive birational model $W \in \operatorname{Bir}\left(\frac{X}{Z}\right)$ such that $\mathscr{L}^\vee \otimes H \geqslant K$ for every $\mathscr{L} \in \operatorname{Pic}(W) \cap \sigma$.*

Likewise, there is a (possibly different) non-zero cone $\sigma' \in \operatorname{Arr}(\Delta, \mathfrak{J})$ and a σ'-positive birational model $W' \in \operatorname{Bir}\left(\frac{X}{Z}\right)$ such that $K \geqslant \mathscr{L}' \otimes H[-1]$ for every $\mathscr{L}' \in \operatorname{Pic}(W') \cap \sigma'$.

Proof. We prove the first statement, the second being analogous. Since K is non-algebraic, corollary 6.1.2 shows that K is not covered by an algebraic heart. In particular if $K_n \in \operatorname{tilt}(H)$ is algebraic and satisfies $K_n > K$ then there is an algebraic heart K_{n+1} such that $K_n \gtrdot K_{n+1} > K$. Starting with the tautological relation $H = K_0 > K$, this produces an infinite chain $K_0 > K_1 > K_2 > \cdots > K$ approaching K using algebraic tilts of H, each of the form $\Psi_v H$ for some $\mathfrak{J}$-path v.

By theorem 6.2.5, for each K_n there is a birational model W_n and line bundles $\mathcal{L}_n, \mathcal{L}'_n$ in $\mathrm{Pic}(W_n)$ such that

$$\mathcal{L}_n{}^{\vee} \otimes H \geqslant K_n \geqslant \mathcal{L}'_n{}^{\vee} \otimes H, \quad \text{and} \quad \underline{0} \leqslant \deg(\mathcal{L}_n) < \deg(\mathcal{L}'_n) \leqslant \deg(\mathcal{L}_n) + \underline{1}.$$

But there are only finitely many birational models of X, so we can pass to a subsequence if necessary to assume all birational models W_n are equal. Thus there is a $W = vX$ and a sequence of nef line bundles $\mathcal{L}_1, \mathcal{L}_2, \ldots \in \mathrm{Pic}(W)$ such that $\mathcal{L}_n{}^{\vee} \otimes H > K$ for each n.

Claim that there is some integral exceptional curve $C_i \subset W$ such that $\deg(\mathcal{L}_n|_{C_i})$ attains arbitrarily large magnitude. Given the claim, we can write $\mathcal{L} \in \mathrm{Pic}(W)$ for the line bundle which has degree 1 on C_i and is trivial elsewhere, and observe that for every $N > 0$ there is an n such that $(\mathcal{L}^{\vee})^{\otimes N} \otimes H > \mathcal{L}_n{}^{\vee} \otimes H$. Thus $(\mathcal{L}^{\vee})^{\otimes N} \otimes H > K$ for all N. Choosing $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ to be the ray generated by $\mathcal{L}$ then yields the result.

We prove the claim by contradiction. Suppose the tuples $\deg(\mathcal{L}_n)$ are all bounded, so we can choose a line bundle $\mathcal{L}_\infty \in \mathrm{Pic}(W)$ such that $\deg \mathcal{L}_\infty \geqslant \deg \mathcal{L}_n + \underline{1} = \deg \mathcal{L}'_n$ for each n. It follows that $\mathcal{L}'_n{}^{\vee} \otimes H \geqslant \mathcal{L}_\infty{}^{\vee} \otimes H$ and hence $K_n \in [\mathcal{L}_\infty{}^{\vee} \otimes H, H]$ for each n. Noting $\mathcal{L}_\infty{}^{\vee} \otimes H = \Psi_v H$ for some $\mathfrak{J}$-path v, we see that the interval $[\Psi_v H, H] \subset \mathrm{tilt}(H)$ contains infinitely many elements. But we have seen (corollary 6.1.2) that there is a poset isomorphism

$$\mathrm{tilt}(H) \supset [\Psi_v H, H] \quad \simeq \quad [vC^+_{\mathfrak{J}}, C^+_{\mathfrak{J}}] \subset \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$$

and the latter interval is finite by lemma 3.4.1 – a contradiction. □

As an immediate consequence, we see that any heart bounded by a geometric one is numerical.

Lemma 6.3.4. *If $K \in \mathrm{tilt}(H)$ is such that $K \geqslant \Psi_v \operatorname{\mathcal{coh}} W$ or $\Psi_v \overline{\operatorname{\mathcal{coh}}} W \geqslant K$ for some birational model $W = vX$, then the heart cone $\mathbf{C}K$ is non-zero.*

Proof. Again we may assume K is non-algebraic and lies in $[\Psi_v \operatorname{\mathcal{coh}} W, H]$, so that lemma 6.3.3 gives a non-zero cone $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ and a σ-positive birational model W' such that $\mathcal{L}'^{\vee} \otimes H \geqslant K \geqslant \Psi_v \operatorname{\mathcal{coh}} W$ for every $\mathcal{L}' \in \mathrm{Pic}(W') \cap \sigma$.

But then theorem 6.2.3 (4) shows that for each $\mathcal{L}' \in \mathrm{Pic}(W') \cap \sigma$, we may choose a nef bundle $\mathcal{L} \in \mathrm{Pic}(W)$ (given by the proper transform of $\mathcal{L}'$) such that $\mathcal{L}'^{\vee} \otimes H = \mathcal{L}^{\vee} \otimes H$. In particular, W is σ-positive and we have $\mathcal{L}^{\vee} \otimes H \geqslant K$ for every $\mathcal{L} \in \mathrm{Pic}(W) \cap \sigma$.

On the other hand we also have the inequality $K \geqslant \Psi_v \operatorname{\mathcal{coh}} W \geqslant \mathcal{L} \otimes H[-1]$ for every $\mathcal{L} \in \mathrm{Pic}(W) \cap \sigma$, so lemma 6.3.2 yields the conclusion. □

We deduce analogous bounds on hearts $K \in \mathrm{tilt}(H)$ which contain skyscraper sheaves $\mathcal{O}_p$ (for closed points $p \in X$) or shifts thereof, noting that a priori each $\mathcal{O}_p$ is only a two-term complex in $K[0, 1]$.

Lemma 6.3.5. *Suppose $C_i \subset X$ is an integral exceptional curve with a closed point $p \in C_i$, and $\mathcal{L}_i \in \mathrm{Pic}(X)$ is the line bundle which has degree 1 on C_i and is trivial elsewhere. Writing $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ for the cone generated by $\mathcal{L}_i$, the following statements hold for any $K \in \mathrm{tilt}(H)$.*

(1) If $\mathcal{O}_p \in K$, then $K \geqslant \mathcal{L} \otimes H[-1]$ for every $\mathcal{L} \in \mathrm{Pic}(X) \cap \sigma$.

(2) If $\mathcal{O}_p \in K[1]$, then $\mathcal{L}^{\vee} \otimes H \geqslant K$ for every $\mathcal{L} \in \mathrm{Pic}(X) \cap \sigma$.

In particular if there are closed points $p, q \in C_i$ such that $\mathcal{O}_p \in K$ and $\mathcal{O}_q \in K[1]$, then $\mathbf{C}(K)$ is non-zero.

Proof. If $p \in C_i$ is a closed point such that $\mathcal{O}_p \in K$, then $\mathcal{O}_p$ lies in the torsion-free class $F = K \cap H$. The exact triangle

$$\mathcal{O}_{C_i}(-1) \to \mathcal{O}_p \to \mathcal{O}_{C_i}(-2)[1] \to \mathcal{O}_{C_i}(-1)[1]$$

shows $\mathcal{O}_{C_i}(-1)$ is a sub-object (in H) of $\mathcal{O}_p$, thus we also have $\mathcal{O}_{C_i}(-1) \in F$. Further, inductively considering extensions

$$\mathcal{O}_{C_i}(n) \to \mathcal{O}_{C_i}(n+1) \to \mathcal{O}_p \to \mathcal{O}_{C_i}(n)[1]$$

shows $\mathcal{O}_{C_i}(n) \in F$ for every $n \geqslant -1$, and thus F contains each torsion class $\mathcal{L}_i^{\otimes n} \otimes H[-1] \cap H$ by proposition 6.2.1. Statement (1) follows, and the proof of (2) is analogous. $\square$

Thus given an arbitrary non-algebraic t-structure K, we show $\mathbf{C}(K)$ is non-zero by comparing K against some appropriately chosen geometry. The following lemma informs this choice.

Lemma 6.3.6. *For any $K \in \operatorname{tilt}(H)$, there is a $W = \nu X \in \operatorname{Bir}(\frac{X}{Z})$ such that K lies in $\Psi_\nu \mathcal{P}er(\frac{W}{Z})[-1,0]$ and contains the objects $\Psi_\nu \mathcal{O}_{C_i}(-1)$ for each exceptional curve $C_i \subset W$.*

Proof. Consider the sub-poset of $\operatorname{tilt}(H)$ given by

$$[K, H] \ \cap \ \{\Psi_\nu \mathcal{P}er(\tfrac{\nu X}{Z}) \mid \nu \text{ a spherical } \mathfrak{J}\text{-path}\}.$$

This poset is non-empty (it contains H) and finite, so has a minimal element determined by some spherical $\mathfrak{J}$-path ν. This determines our birational model $W = \nu X$.

Continuing to write $\Psi_\nu H$ for the heart $\Psi_\nu \mathcal{P}er(\frac{\nu X}{Z}) = \Psi_\nu \operatorname{fl}\mathcal{mod}({}_\nu\Lambda_\nu)$, we evidently have the chain of inequalities $H \geqslant \Psi_\nu H \geqslant K \geqslant H[-1] \geqslant \Psi_\nu H[-1]$, so that K lies in $\Psi_\nu H[-1,0]$.

Suppose $i \in \Delta \setminus \nu\mathfrak{J}$ is such that $\Psi_\nu S_i = \Psi_\nu \mathcal{O}_{C_i}(-1)$ lies outside K, and hence outside the torsion-free class $K \cap \Psi_\nu H$ in $\Psi_\nu H$. Being a simple of $\Psi_\nu H$, this object must then lie in the torsion class $K[1] \cap \Psi_\nu H$. Noting $\langle \Psi_\nu S_i \rangle$ is precisely the torsion class $\Psi_\nu \Psi_i H[1] \cap \Psi_\nu H$ (theorem 6.1.1), we thus have $\Psi_\nu H \geqslant \Psi_\nu \Psi_i H \geqslant K$.

It follows that $\Psi_\nu \Psi_i H$ lies in the interval $[K, H]$, and in particular is an algebraic heart in $\operatorname{tilt}(H)$. Since its heart cone coincides with that of $\Psi_{\nu_i \nu} \mathcal{P}er(\frac{\nu_i \nu X}{Z})$ we conclude that the two hearts must be equal. But this contradicts the minimality of $\Psi_\nu H$, so we conclude that K must contain every simple $\Psi_\nu S_i$ for $i \in \Delta \setminus \nu\mathfrak{J}$. $\square$

Definition 6.3.7. Given a birational model $W = \nu X$, we say a heart $K \subset \mathbf{D}^0 X$ *lives on* W if it is intermediate with respect to $\Psi_\nu \mathcal{P}er(\frac{W}{Z})$ and contains the object $\Psi_\nu \mathcal{O}_{C_i}(-1)$ for every integral exceptional curve $C_i \subset W$, i.e. if there are inclusions

$$\langle \Psi_\nu \mathcal{O}_{C_i}(-1) \mid i \in \Delta \setminus \nu\mathfrak{J} \rangle \ \subseteq \ K \ \subseteq \ \Psi_\nu \mathcal{P}er(\tfrac{W}{Z})[-1,0].$$

Then lemma 6.3.6 explains that every non-algebraic heart in $\operatorname{tilt}(H)$ lives on *some* birational model, so up to the application of a flop functor we may assume K lives on X. This additional hypothesis gives us tremendous control on the torsion pair associated to K, in particular we show that every skyscraper on X is either torsion or torsion-free.

Lemma 6.3.8. *If $K \in \operatorname{tilt}(H)$ lives on X and $p \in X(\mathbb{C})$ is a closed point in the exceptional locus, then the sheaf $\mathcal{O}_p$ lies in a single cohomological degree with respect to K (i.e. we have $\mathcal{O}_p \in K$ or $\mathcal{O}_p \in K[1]$.)*

Proof. The intermediate heart K induces the torsion pair $H = (K[1] \cap H) * (K \cap H)$, and thus a filtration

$$t \to \mathcal{O}_p \to f \to t[1]$$

with $t \in K[1] \cap H$, $f \in K \cap H$. In particular, considering $\mathbf{K}$-theory classes

$$[t] = \sum_{i \in \Delta \setminus \mathfrak{J}} t_i \alpha_i, \qquad [\mathcal{O}_p] = \sum_{i \in \Delta \setminus \mathfrak{J}} \delta_i \alpha_i, \qquad [f] = \sum_{i \in \Delta \setminus \mathfrak{J}} f_i \alpha_i,$$

we see that for each i the t_i, δ_i, f_i are non-negative integers satisfying $t_i + f_i = \delta_i$, and the coefficients δ_i correspond to ranks of the summands of $\mathcal{V}(\frac{X}{Z})$ as in lemma 5.1.5. In particular we have $\delta_0 = \operatorname{rk}(\mathcal{O}_X) = 1$, so we must have $(t_0, f_0) = (0, 1)$ or $(t_0, f_0) = (1, 0)$.

If $f_0 = 0$, then f is filtered by the simples $S_i = \mathcal{O}_{C_i}(-1)$ for $i \in \Delta \setminus \mathfrak{J}$, and is in particular a pure sheaf. But this implies $\operatorname{Hom}(\mathcal{O}_p, f) = 0$, thus the map $t \to \mathcal{O}_p$ is an isomorphism and $\mathcal{O}_p$ lies in $K[1]$.

On the other hand if $t_0 = 0$, then t lies in the extension-closure of the sheaves $\mathcal{O}_{C_i}(-1)$, and hence in K (since K lives on X). It follows that $t \in K[1] \cap K$ must be the zero object, so that $\mathcal{O}_p = f \in K$. ☐

We are equipped to show that an arbitrary t-structure in $\mathrm{tilt}(H)$ is numerical.

Proof of theorem 6.3.1. We may assume K is non-algebraic and, applying a flop functor if necessary, lives on X. Thus by lemma 6.3.8, for every closed point p in the exceptional locus of X, either $\mathcal{O}_p$ or $\mathcal{O}_p[-1]$ lies in K.

If there is some curve C_i with two points $p, q \in C_i$ such that $\mathcal{O}_p \in K$ and $\mathcal{O}_q[-1] \in K$, then we obtain the conclusion using lemma 6.3.5.

Otherwise, since the exceptional locus of X is connected, either K or $K[1]$ contains the subcategory $\{\mathcal{O}_p \mid p \in X(\mathbb{C})\}$. If $\mathcal{O}_p \in K$ for every closed point p, then (by lemma 6.3.5) we have $K \geqslant \mathcal{L} \otimes H[-1]$ for every nef bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and thus $K \geqslant \overline{\mathit{coh}}\, X$. Further, the torsion-free part $K \cap H$ contains $H_{ss}(C^0_{\mathfrak{J}}) = \langle \mathcal{O}_p \mid p \in C \rangle$ so in fact $K \geqslant \mathit{coh}\, X$. By lemma 6.3.4, we conclude $\mathbf{C}(K) \neq 0$.

The case when $\mathcal{O}_p[-1] \in K$ for every p is handled likewise. ☐

§ 6.4 Classification of bricks

We show that every brick in $H = \mathit{per}(\frac{X}{Z})$ must arise from an algebraic or geometric simple.

Theorem 6.4.1. *Let* $b \in H$ *be a brick. Then either*

(1) there is a spherical $\mathfrak{J}$*-path* v *and a closed point* $p \in vX(\mathbb{C})$ *such that* $b = \Psi_v \mathcal{O}_p$ *, or*

(2) there is a $\mathfrak{J}$*-path* v *and an index* $i \in \underline{\Delta} \setminus v\mathfrak{J}$ *such that* $b \in \{\Psi_v S_i[1], \Phi_v S_i[-1]\}$.

In particular, the **K***-theory class* $[b] \in \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ *is a primitive restricted root.*

We prove the theorem over the course of this subsection. Thus fix a brick $b \in H$, and consider the torsion pair $H = T * F$ where T is the smallest torsion class containing b and accordingly $F = b^{\perp}$.

By lemma 6.3.6, the corresponding tilted heart $K = F * T[-1]$ lives on some birational model $W = \upsilon X$. We show that b then lies in a single cohomological degree with respect to $\Psi_\upsilon\, \mathit{coh}\, W$, i.e. $\Psi_\upsilon^{-1} b$ or $\Psi_\upsilon^{-1} b[-1]$ is a sheaf on W. To see this, first note that for every closed point $p \in W(\mathbb{C})$ we either have $\Psi_\upsilon \mathcal{O}_p \in T$ or $\Psi_\upsilon \mathcal{O}_p \in F$ (lemma 6.3.8). Then for each $p \in W(\mathbb{C})$, lemma 2.3.2 gives us the trichotomy

$$\mathrm{Hom}(\Psi_\upsilon \mathcal{O}_p, b) = 0 \qquad \text{or} \qquad b = \Psi_\upsilon \mathcal{O}_p \qquad \text{or} \qquad \mathrm{Hom}(b, \Psi_\upsilon \mathcal{O}_p) = 0$$

where the first two cases occur when $\Psi_\upsilon \mathcal{O}_p \in T$, while the last is equivalent to $\Psi_\upsilon \mathcal{O}_p \in F$.

The claimed result, proposition 6.4.3 below, will then follow from the above trichotomy and the following lemma on supports of sheaves.

Lemma 6.4.2. *Let* $\mathcal{F}$ *be a coherent sheaf on a variety, with* $\dim \mathrm{Supp}(\mathcal{F}) \leqslant 1$. *If* p *is a closed point such that* $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_p) = 0$ *or* $\mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{F}) = 0$, *then* p *lies outside* $\mathrm{Supp}(\mathcal{F})$.

Proof. The conclusion is immediate from the first hypothesis.

On the other hand if $\mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{F}) = 0$, then in particular we have $\mathcal{H}om(\mathcal{O}_p, \mathcal{F}) = 0$ where we note that the homomorphism sheaf has zero dimensional support and is therefore determined by its module of global sections. Likewise we can deduce (e.g. from the Grothendieck spectral sequence associated to $\Gamma \circ \mathbf{R}\mathcal{H}om(\mathcal{O}_p, -)$) that $\Gamma(\mathcal{E}xt^1(\mathcal{O}_p, \mathcal{F})) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{F}) = 0$ and hence $\mathcal{E}xt^1(\mathcal{O}_p, \mathcal{F}) = 0$.

Taking stalks at p, we thus have $\mathrm{Hom}(\mathcal{O}_p, \mathcal{F}_p) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{F}_p) = 0$ where $\mathcal{F}_p$ is a finitely generated module over the local ring at p, and $\mathcal{O}_p$ is the unique simple module. It follows that $\mathcal{F}_p$ is either zero (i.e. $p \notin \mathrm{Supp}(\mathcal{F})$) or $\mathrm{depth}(\mathcal{F}_p) \geqslant 2$ [Mat08, theorem 16.6] – but the latter situation is absurd since the depth of a module is bounded above by the dimension of its support [Stacks, tag 00LK]. ☐

Proposition 6.4.3. *Let* $b \in H$ *be a brick,* $H = T * F$ *the torsion pair whose torsion class* T *is generated by* b*, and* $W = \upsilon X$ *a birational model such that* $K = F * T[-1]$ *lives on* W*. Then one of the following (mutually exclusive) possibilities is true.*

(1) *Either* $b = \Psi_\upsilon \mathcal{O}_q$ *for some closed point* $q \in W(\mathbb{C})$,

(2) *or* $b = \Psi_\upsilon \mathcal{F}[1]$ *for a pure sheaf* $\mathcal{F} \in \mathit{coh}(W)$ *with one-dimensional support,*

(3) *or* $b = \Psi_\upsilon \mathcal{G}$ *for a pure sheaf* $\mathcal{G} \in \mathit{coh}(W)$ *with one-dimensional support.*

Proof. If $b = \Psi_\upsilon \mathcal{O}_p$ for some p then there is nothing to prove, so we assume that doesn't happen. Thus from the above discussion, for every p we have $\mathrm{Hom}(\Psi_\upsilon \mathcal{O}_p, b) = 0$ or $\mathrm{Hom}(b, \Psi_\upsilon \mathcal{O}_p) = 0$.

Considering the torsion pair $H = (\Psi_\upsilon\, \mathit{coh}\, W[1] \cap H) * (\mathit{coh}\, W \cap H)$ shows b lies in an exact triangle

$$\Psi_\upsilon \mathcal{F}[1] \to b \to \Psi_\upsilon \mathcal{G} \to \Psi_\upsilon \mathcal{F}[2] \tag{6.8}$$

where $\mathcal{F} \in \mathit{coh}(W) \cap \Psi_\upsilon^{-1} H[-1]$ and $\mathcal{G} \in \mathit{coh}(W) \cap \Psi_\upsilon^{-1} H$ are the cohomology objects of $\Psi_\upsilon^{-1} b$ seen as a complex of sheaves on W.

For any $p \in W(\mathbb{C})$, note that $\mathcal{O}_p$ lies in the heart $\Psi_\upsilon^{-1} H$ and thus $\mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0$. On the other hand, long exact sequences show that $\mathrm{Hom}(b, \Psi_\upsilon \mathcal{O}_p) = \mathrm{Hom}(\mathcal{G}, \mathcal{O}_p)$, and $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{F}) \subset \mathrm{Hom}(\Psi_\upsilon \mathcal{O}_p, b)$.

It follows that if $\mathrm{Hom}(b, \Psi_\upsilon \mathcal{O}_p) = 0$ then $p \notin \mathrm{Supp}(\mathcal{G})$, while if $\mathrm{Hom}(\Psi_\upsilon \mathcal{O}_p, b) = 0$ then $p \notin \mathrm{Supp}(\mathcal{F})$. Thus $\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G}) = \varnothing$, so that the sequence (6.8) splits. But b is a brick (in particular indecomposable), so one of $\mathcal{F}$ or $\mathcal{G}$ must vanish.

In the case $b = \Psi_\upsilon \mathcal{F}[1]$, the fact that $\mathcal{F}$ is pure of one-dimensional support follows from the observation that $\mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0$ for all p.

In the case $b = \Psi_\upsilon \mathcal{G}$, if $\mathcal{G}$ has zero-dimensional support then the condition $\mathrm{End}(\mathcal{G}) = \mathbb{C}$ ensures $\mathcal{G} = \mathcal{O}_q$ for some $q \in W(\mathbb{C})$ so that we are reduced to the first case. On the other hand if $\mathcal{G}$ has one-dimensional support, then for every point $p \in \mathrm{Supp}(\mathcal{G})$ we have $\mathrm{Hom}(b, \Psi_\upsilon \mathcal{O}_p) = \mathrm{Hom}(\mathcal{G}, \mathcal{O}_p) \neq 0$, so that necessarily $\mathrm{Hom}(\mathcal{O}_p, \mathcal{G}) = \mathrm{Hom}(\Psi_\upsilon \mathcal{O}_p, b) = 0$. It follows that $\mathcal{G}$ is pure in this case. □

The proof of theorem 6.4.1 is then reduced to an analysis of each of the cases arising in the proposition above. If $b = \Psi_\upsilon \mathcal{O}_q$ for a closed point then there is nothing to prove. The other two cases are addressed by the lemmas below.

Lemma 6.4.4. *If* $b = \Psi_\upsilon \mathcal{F}[1]$ *for some sheaf* $\mathcal{F} \in \mathit{coh}(W) \cap \Psi_\upsilon^{-1} H[-1]$*, then the heart* K *is algebraic of the form* $\Psi_\nu H$ *for some* $\mathfrak{J}$*-path* ν*. Consequently,* $b = \Psi_\nu S_i[1]$ *for some* $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$.

Proof. By (the dual statement to) proposition 2.3.3 the torsion-free class $\Psi_\upsilon\, \mathit{coh}(W) \cap H$ is generated by the semibrick $\{\Psi_\upsilon \mathcal{O}_p \mid p \in W(\mathbb{C})\}$. The constraints on b guarantee that every $\Psi_\upsilon \mathcal{O}_p$ lies in F, and thus $\Psi_\upsilon\, \mathit{coh}(W) \cap H \subset F$. Equivalently, we have the containment of torsion classes

$$T \subseteq \Psi_\upsilon\, \mathit{coh}(W)[1] \cap H = \sup\{\mathcal{L}^\vee \otimes H[1] \cap H \mid \mathcal{L} \in \mathrm{Pic}(W) \text{ nef}\}.$$

The supremum on the right hand side is an infinite union (since any two elements in the set are dominated by a third as in (6.5)). It follows that b, and hence also T, is contained in the torsion class $\mathcal{L}^\vee \otimes H[1] \cap H$ for some $\mathcal{L} \in \mathrm{Pic}(W)$ nef. In other words, K is contained in the interval $[\mathcal{L}^\vee \otimes H, H] \subset \mathrm{tilt}^+(H)$. The result follows. □

Lemma 6.4.5. *If* $b = \Psi_\upsilon \mathcal{G}$ *for some purely one-dimensional sheaf* $\mathcal{G} \in \mathit{coh}(W) \cap \Psi_\upsilon^{-1} H$*, then* b *is a simple object in an algebraic heart* $\Phi_\nu H[-1]$ *determined by some* $\mathfrak{J}$*-path* ν*. Consequently,* $b = \Phi_\nu S_i[-1]$ *for some* $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$.

Proof. Consider the torsion pair $H = T' * F'$ where F' is the smallest torsion-free class in H that contains b. By (the dual statement to) proposition 2.3.3, b is a simple in the tilted heart $K' = F' * T'[-1]$.

Since $\mathcal{G}$ is pure of one dimensional support, we have $\mathrm{Hom}(\Psi_\nu \mathcal{O}_p, b) = \mathrm{Hom}(\mathcal{O}_p, \mathcal{G}) = 0$ for all $p \in W(\mathbb{C})$, and hence each $\Psi_\nu \mathcal{O}_p$ lies in T'. It follows that so does the torsion class $\Psi_\nu\, \overline{\mathit{coh}}(W)[1] \cap H$, which is generated by the semibrick $\{\Psi_\upsilon \mathcal{O}_p \mid p \in W(\mathbb{C})\}$. We can then conclude in a similar fashion as the proof of lemma 6.4.4. □

On irreducible contractions

(Chapter 7)

In their study [HW24] of flopping contractions $\pi\colon X \to Z$, Hara and Wemyss found that the category $\mathcal{C} = \ker \mathbf{R}\pi_*$ is especially cooperative when it comes to classifications of spherical objects, bricks, and more generally, objects that 'look like they belong in the heart of a t-structure' – so much so, it was decided that whatever precise definition the term ***Iyama dream category*** might assume in times to come, $\mathcal{C}$ must fit the mould [HW25].

This chapter advocates a similar case for the category $\mathbf{D}^0 X$ when the exceptional fibre C is irreducible, by exhibiting a classification of objects with no self-extensions in negative degrees. A multitude of corollaries follow, including the classifications of (semi)bricks and tilting complexes, and the connectedness of the stability manifold. We have highlighted the use and importance of the irreducibility hypothesis in all arguments; remarks at the end of the introduction discuss why the categories associated to contractions of more than one curve might not be so dreamy after all.

§ 7.1 A laundromat of functors

Fix a flopping contraction $\pi\colon X \to Z = \operatorname{Spec}(R, \mathfrak{m})$ as before, where Z is a complete local Gorenstein $\mathbb{C}$-scheme of dimension 3 (or 2) with an at worst terminal (resp. canonical) singularity. For the rest of this chapter we impose the additional hypothesis that the exceptional fibre $\underline{C} = \pi^{-1}[\mathfrak{m}]$ is irreducible, i.e. X has Picard rank 1.

The notation and combinatorics can be simplified with this assumption. If $\mathfrak{J} \subset \Delta \subset \underline{\Delta}$ is the Dynkin data associated to π as in § 4.1, then any representative slice of the $\mathfrak{J}$-cone arrangement is piecewise linearly equivalent to the inclusion $\mathbb{Z} \subset \mathbb{R}$. More precisely, writing $C^n = \{\, (\lambda\mu, \lambda) \mid \lambda \geqslant 0, \quad n \geqslant \mu \geqslant n-1\}$ for the cone in $\mathbb{R}^2$ over an interval $[n-1, n] \times \{1\}$, the assignments $C^+_{\mathfrak{J}} \mapsto C^0$, $\nu_0 C^+_{\mathfrak{J}} \mapsto C^1$ extend naturally to an isomorphism of fans $\operatorname{Arr}^+(\underline{\Delta}, \mathfrak{J}) \simeq \bigcup_{n\in\mathbb{Z}} \operatorname{faces}(C^n)$.

Write $N^0 = N$ for the R-module $\pi_*\mathcal{N}$ furnished via Van den Bergh's construction (§ 4.1), and more generally, $N^n \in \operatorname{MM}^N(R)$ for the modifying R-module associated to the chamber C^n via Iyama–Wemyss's bijection (theorem 4.2.4). Writing $\Lambda_n = \operatorname{End}_R N^n$, the simple mutation functors of the form Ψ_i, $\Psi_{\iota(i)}$ ($i \in \underline{\Delta} \setminus \mathfrak{J}$) can be denoted with this simplified indexing as

$$\mathbf{D}^b \Lambda_n \underset{\Psi_n := \mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(N^{n+1}, N^n), -)}{\overset{\Psi_n := \mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(N^n, N^{n+1}), -)}{\leftrightarrows}} \mathbf{D}^b \Lambda_{n+1}.$$

On the geometric side, the set $\operatorname{Bir}(\frac{X}{Z})$ contains precisely two birational models – X and its flop, which we denote W – corresponding to the modifying generators N^0 and N^{-1} respectively.

As in § 6.2, the class group $\pi_* \operatorname{Pic} X = \operatorname{cl}(R)$ acts on $\operatorname{MM}^N(R)$. Choosing a primitive ample divisor $\mathcal{O}_X(1) \in \operatorname{Pic} X$, this action is determined as the index-shift

$$(\pi_*\mathcal{O}_X(-1)^* \otimes_R N^n)^{**} = N^{n+\wp}$$

for some positive integer $\wp \geqslant 1$ that Donovan–Wemyss [DW25, §2.1] call the ***helix period*** associated to π. Accordingly, we have natural equivalences $\varepsilon_k\colon \mathbf{D}^{\mathrm{fl}} \Lambda_n \xrightarrow{\sim} \mathbf{D}^{\mathrm{fl}} \Lambda_{n+k\wp}$ for $n, k \in \mathbb{Z}$ induced by isomorphisms of algebras, and natural isomorphisms of functors $\Psi_{n+k\wp} \simeq \varepsilon_{-k} \circ \Psi_n \circ \varepsilon_k$ [HW23, lemma 7.3].

Remark 7.1.1. The helix period $\wp$ is a subtle invariant of the flopping contraction $\pi\colon X \to Z$ with no known immediate algebraic or algebro-geometric interpretation other than the one above. It is however completely determined by the length invariant ℓ via the table below.

Length (ℓ)	1	2	3	4	5	6
Period ($\wp$)	1	2	4	6	10	12

The latter invariant has a multitude of descriptions – combinatorially ℓ is the label δ_i on the unique vertex $i \in \Delta \setminus \mathfrak{J}$ determined by the Dynkin data for π (see § 3.2), and geometrically it is the length (i.e. generic rank) of $\mathcal{O}_{\underline{C}} = \pi^*\mathcal{O}_{[m]}$ as an $\mathcal{O}_C$-module [see DW25, § 2.1].

Omitting the ε_ks from the notation, we can make the identifications $\mathbf{D}^{\mathrm{fl}}\Lambda_{n+k\wp} = \mathbf{D}^{\mathrm{fl}}\Lambda_n$, $\Psi_{n+k\wp} = \Psi_n$ to treat indices modulo $\wp$, and therefore reckon with the finitely many categories and equivalences

$$\mathbf{D}^b X \xrightarrow{\mathrm{VdB}} \mathbf{D}^b\Lambda_0, \qquad \mathbf{D}^b W \xrightarrow{\mathrm{VdB}} \mathbf{D}^b\Lambda_{\wp-1}, \qquad \mathbf{D}^b\Lambda_n \overset{\Psi_n}{\underset{\Psi_n}{\leftrightarrows}} \mathbf{D}^b\Lambda_{n+1} \quad (n \in \mathbb{Z}/\wp\mathbb{Z}). \tag{7.1}$$

Viewing the categories as basepoints and equivalences as homotopy classes of paths, the system (7.1) is a representation of the fundamental groupoid of a $(\wp+2)$-punctured sphere (or a $\wp$-punctured annulus) as in the figure below.

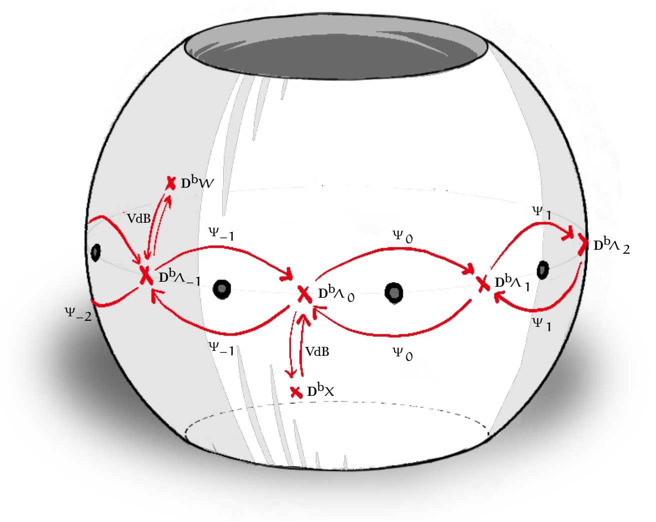


Figure 7.1. Labelling basepoints (marked ✗) with categories, and homotopy classes of paths with functors as shown gives a representation of the fundamental groupoid of a $\wp$-punctured annulus with $\wp+1$ basepoints.

Loops at a fixed basepoint then give autoequivalences of the label, for instance the anti-clockwise monodromy of $\mathbf{D}^b X$ around the north pole is given by the functor $\mathrm{VdB}^{-1} \circ \Psi_{-1} \circ \cdots \circ \Psi_0 \circ \mathrm{VdB}$, computed to coincide with the tensor–action of a generator of $\operatorname{Pic} X$. Likewise monodromy around the south pole recovers the action of $\operatorname{Pic} W$, and monodromies around the equatorial punctures correspond to twist functors described in [DW25, §6.3].

Ivan Smith once pointed out that this circular arrangement bears the semblance of a laundromat; we shall see that it indeed functions as an automaton that untangles complexes into their cleanest forms.

§ 7.2 A sorting of objects

The goal of this section is to establish the following classification.

Theorem 7.2.1. *If a complex* $x \in \mathbf{D}^b X$ *is supported on* C *and satisfies* $\operatorname{Hom}^{<0}(x, x) = 0$*, then up to a shift and iterated applications of the functors appearing in* (7.1) *and their inverses,* x *is either*

(c1) a Λ_i*-module for some* $i \in \mathbb{Z}/\wp\mathbb{Z}$*, or*

(c2) a direct sum of shifts of coherent sheaves with zero-dimensional support on X *(or on* W*), or*

(c3) a coherent sheaf on X *(or on* W*), or a two-term complex thereof determined as a Yoneda-ext* $x \in \operatorname{Ext}^2(\mathcal{G}, \mathcal{F})$ *between coherent sheaves* $\mathcal{F}, \mathcal{G}$ *such that* $\operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0$ *and* $\dim(\operatorname{Supp} \mathcal{G}) = 0$.

To this end, we fix a complex $x \in \mathbf{D}^b X$ that is supported on C and satisfies $\operatorname{Hom}^i(x, x) := \operatorname{Hom}(x, x[i]) = 0$ whenever $i < 0$.

Passing across Van den Bergh's equivalence, we work with respect to the standard heart $\mathsf{H} = \mathrm{fl}\mathit{mod}\,\Lambda_0$ in $\mathbf{D}^{\mathrm{fl}}\Lambda_0$ and view x as an extension of the objects $\{H^i(x)[-i] \mid i \in \mathbb{Z}\}$, each the shift of a finite length Λ_0-module. In particular, if $a \leqslant b$ are integers such that $H^i(x) = 0$ whenever $i \notin [-b, -a]$, then x lies in the full subcategory $\mathsf{H}[a, b]$.

Notation 7.2.2. We write $x \in \mathsf{H}\llbracket a, b]$ to mean $x \in \mathsf{H}[a, b]$ and $x \notin \mathsf{H}[a+1, b]$ (equivalently, $x \in \mathsf{H}[a, b]$ and $H^{-a}(x) \neq 0$). Other variations such as $x \in \mathsf{H}[a, b\rrbracket$ and $x \in \mathsf{H}\llbracket a, b\rrbracket$ are defined likewise. The notation naturally extends when measuring cohomologies with respect to other bounded hearts $\mathsf{K} \subset \mathbf{D}^{\mathrm{fl}}\Lambda_0$.

Now we may, up to replacing x with a shift thereof, assume that x lies in $\mathsf{H}\llbracket 0, n\rrbracket$ for some integer $n \geqslant 0$ (called the *spread* of x.) If the equality $n = 0$ held true then $x \in \mathsf{H}$ would be a finite length Λ_0–module as described in theorem 7.2.1 (c1), so it suffices to only consider the case $n > 0$ and attempt to reduce the spread by updating H.

Lemma 7.2.3. *Suppose* $\mathsf{H} = \mathsf{T} * \mathsf{F}$ *is a torsion class, corresponding to the (positively) tilted heart* $\mathsf{K} = \mathsf{F}[1] * \mathsf{T}$.

(i) The torsion class T *contains the top cohomology* $H^0(x)$ *if and only if* x *lies in* $\mathsf{K}\llbracket 0, n]$.

(ii) The torsion-free class F *contains the bottom cohomology* $H^{-n}(x)$ *if and only if* x *lies in* $\mathsf{K}[-1, n-1\rrbracket$.

Proof. By construction, K lies in $\mathsf{H}[0, 1]$ and hence H (in particular, each cohomology $H^{-i}(x)$) lies in $\mathsf{K}[-1, 0]$. Now x is an extension of the objects $H^{-i}(x)[i] \in \mathsf{K}[i-1, i]$ for $i \in [0, n]$, of which all except $H^0(x)$ clearly lie in $\mathsf{K}[0, n]$. Thus if $H^0(x)$ lies in $\mathsf{T} \subseteq \mathsf{K}$, then $H^0(x)$ (and hence also x itself) lies in $\mathsf{K}[0, n]$. It is also clear that in this case x cannot lie in $\mathsf{K}[1, n] \subset \mathsf{H}[1, n+1]$, since $H^0(x) \neq 0$ by assumption. Hence x lies in $\mathsf{K}\llbracket 0, n]$.

Conversely if x lies in $\mathsf{K}\llbracket 0, n]$, then $\mathrm{Hom}(x, k) = 0$ for all $k \in \mathsf{K}[< 0]$ from whence it follows (e.g. by truncating to the coaisle of H) that $\mathrm{Hom}(H^0 x, k) = 0$ for all $k \in \mathsf{K}[< 0]$. Thus $H^0(x)$ lies in $\mathsf{K}[\geqslant 0]$, and hence in $\mathsf{T} = \mathsf{K}[\geqslant 0] \cap \mathsf{H}$. This proves (i); the proof of (ii) is similar. $\square$

Since H is an algebraic Abelian category, there is a minimal torsion class containing $H^0(x)$ with corresponding tilted heart $\overline{\mathsf{K}}$, and a minimal torsion-free class containing $H^{-n}(x)$ with corresponding tilted heart $\underline{\mathsf{K}}$. Lemma 7.2.3 then characterises these hearts in terms of the partial order as

$$\big[\mathsf{H}, \overline{\mathsf{K}}\,\big] = \big\{\mathsf{K} \in \big[\mathsf{H}, \mathsf{H}[1]\big] \;\big|\; x \in \mathsf{K}\llbracket 0, n]\big\}\,, \qquad \big[\underline{\mathsf{K}}, \mathsf{H}[1]\big] = \big\{\mathsf{K} \in \big[\mathsf{H}, \mathsf{H}[1]\big] \;\big|\; x \in \mathsf{K}[-1, n-1\rrbracket\big\}\,.$$

Lemma 7.2.4. *We have the inequality* $\underline{\mathsf{K}} \leqslant \overline{\mathsf{K}}$. *In particular, the poset of t-structures on* $\mathbf{D}^0 X$ *contains a* non-empty *interval* $\big[\underline{\mathsf{K}}, \overline{\mathsf{K}}\,\big]$ *containing hearts* K *such that* x *lies in* $\mathsf{K}\llbracket 0, n-1\rrbracket$.

Proof. The statement claims the containment of torsion classes $\overline{\mathsf{K}} \cap \mathsf{H} \subseteq \underline{\mathsf{K}} \cap \mathsf{H}$, hence it suffices to show that $H^0(x)$ is contained in $\underline{\mathsf{K}} \cap \mathsf{H} = {}^{\perp}(H^{-n}x)$. In other words, we must show $\mathrm{Hom}(H^0 x, H^{-n} x) = 0$. But this readily follows from the hypothesis $\mathrm{Hom}(x, x[-n]) = 0$, by truncating first to the aisle and then to the coaisle and noting that the truncation functors are adjoint to inclusions (alternatively, the same conclusion can be deduced by using the spectral sequence $E_2^{p,q} = \bigoplus_i \mathrm{Hom}^p(H^i x, H^{i+q} x) \Longrightarrow \mathrm{Hom}^{p+q}(x, x)$, which shows $\mathrm{Hom}(x, x[-n]) = \mathrm{Hom}(H^0 x, H^{-n} x)$.) $\square$

Thus the spread of x can *always* be improved by tilting once, in particular if x has spread $n = 1$ then it is an object in some tilted heart $\mathsf{K} \in \big[\mathsf{H}, \mathsf{H}[1]\big]$. We understand these hearts well following chapter 6, the key results are recalled below.

The algebraic elements in $\big[\mathsf{H}, \mathsf{H}[1]\big]$ all arise as images of standard hearts $\mathsf{H} = \mathrm{fl}\mathit{mod}\,\Lambda_i$ under the mutation functors (7.1), and form a subposet with Hasse quiver

$$\begin{tikzcd}[column sep=small]
& \Psi_0(\mathrm{fl}\mathit{mod}\,\Lambda_1)[1] \arrow[r] & \Psi_0\Psi_1(\mathrm{fl}\mathit{mod}\,\Lambda_2)[1] \arrow[r] & \cdots \arrow[r] & \Psi_{-1}^{-1}\Psi_{-2}^{-1}(\mathrm{fl}\mathit{mod}\,\Lambda_{-2}) \arrow[r] & \Psi_{-1}^{-1}(\mathrm{fl}\mathit{mod}\,\Lambda_{-1}) \arrow[dr] & \\
\mathsf{H}[1] \arrow[ur] \arrow[dr] & & & & & & \mathsf{H}\,. \\
& \Psi_{-1}(\mathrm{fl}\mathit{mod}\,\Lambda_{-1})[1] \arrow[r] & \Psi_{-1}\Psi_{-2}(\mathrm{fl}\mathit{mod}\,\Lambda_{-2})[1] \arrow[r] & \cdots \arrow[r] & \Psi_0^{-1}\Psi_1^{-1}(\mathrm{fl}\mathit{mod}\,\Lambda_2) \arrow[r] & \Psi_0^{-1}(\mathrm{fl}\mathit{mod}\,\Lambda_1) \arrow[ur] &
\end{tikzcd} \tag{7.2}$$

The two decreasing chains starting at $H[1]$ above have infima $\mathrm{VdB}(\mathit{coh}\, X)[1]$ and $\Psi_0 \circ \mathrm{VdB}(\mathit{coh}\, W)[1]$ respectively, likewise the two increasing chains starting at H have suprema $\mathrm{VdB}(\overline{\mathit{coh}}\, X)[1]$ and $\Psi_0 \circ \mathrm{VdB}(\overline{\mathit{coh}}\, W)[1]$. Passing across equivalences, the intervals $\left[\mathit{coh}\, X[1], \overline{\mathit{coh}}\, X[1]\right]$ and $\left[\mathit{coh}\, W[1], \overline{\mathit{coh}}\, W[1]\right]$ are both isomorphic to a boolean lattice on the set of closed points of $\mathbb{P}^1$. Since this accounts for all covering relations in the poset, the list obtained is exhaustive and H has no other tilts.

Theorem 7.2.5. *Let $K \in \left[H, H[1]\right]$ be the heart of a bounded t-structure in $\mathbf{D}^{\mathrm{fl}}\Lambda_0$. Then K admits one of the following descriptions.*

(h1) The heart K is the image of some standard heart $\mathrm{fl}\,\mathit{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}}\Lambda_i$ ($i = \mathbb{Z}/\wp\mathbb{Z}$) under a functor $\mathbf{D}^{\mathrm{fl}}\Lambda_i \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ given by one of the composites $[1]\circ(\Psi_{m\wp-1}\cdots\Psi_{i+1}\Psi_i)$, $[1]\circ(\Psi_{-m\wp}\cdots\Psi_{i-2}\Psi_{i-1})$, $(\Psi_{m\wp+i-1}\cdots\Psi_1\Psi_0)^{-1}$, or $(\Psi_{i-m\wp}\cdots\Psi_{-2}\Psi_{-1})^{-1}$ (for some $m \in \mathbb{N}$).

(h2) Identifying $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \simeq \mathbf{D}^0 X$ across the equivalence VdB, the heart K is a tilt of the standard heart $\mathit{coh}\, X$ in a (possibly trivial) torsion class containing no sheaves with one–dimensional support. In other words, there is a subset $U \subseteq X$ such that $K = \langle\{\mathcal{O}_p \mid p \in U\} \cup \{\mathcal{F}[1] \mid \mathcal{F} \in \mathit{coh}\, X,\ \mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0$ for all $p \in U\}\rangle$.

(h3) Identifying $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \simeq \mathbf{D}^0 W$ across the equivalence $\Psi_0 \circ \mathrm{VdB}$, the heart $K \subset \mathbf{D}^0 W$ is a tilt of the standard heart $\mathit{coh}\, W$ in a (possibly trivial) torsion class containing no sheaves with one–dimensional support. In other words, there is a subset $U \subseteq W$ such that $K = \langle\{\mathcal{O}_p \mid p \in U\} \cup \{\mathcal{F}[1] \mid \mathcal{F} \in \mathit{coh}\, W,\ \mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0$ for all $p \in U\}\rangle$.

The following consequence is immediate.

Corollary 7.2.6. *If the complex $x \in \mathbf{D}^{\mathrm{fl}}\Lambda_0$ lies in $H[0, 1]$, then x adheres to one of the descriptions in theorem 7.2.1.*

Proof. Lemma 7.2.4 guarantees that x lies in some heart K obtained as a tilt of $H = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$. If K admits the description (h1) in theorem 7.2.5, then $K = \Psi\,\mathrm{fl}\,\mathit{mod}\,\Lambda_i$ for some $i \in \mathbb{Z}/\wp\mathbb{Z}$ and Ψ a composite of shifts and mutation functors, hence $\Psi^{-1}(x) \in \mathrm{fl}\,\mathit{mod}\,\Lambda_i$ is a Λ_i–module as in theorem 7.2.1 (c1).

Otherwise the heart K admits one of the descriptions (h2) or (h3), say (without loss of generality) the former. Thus K is a tilt of $\mathit{coh}\, X$ in a torsion class $\langle \mathcal{O}_p \mid p \in U\rangle \subset \mathit{coh}\, X$, determined by some subset $U \subset X$. It follows that the object $x \in K$ can be written as an extension $\mathcal{F}[1] \to x \to \mathcal{G} \to \mathcal{F}[2]$, where $\mathcal{F}$ and $\mathcal{G} \in \mathit{coh}\, X$ are torsion–free (resp. torsion) with respect to the torsion pair relating K and $\mathit{coh}\, X$. Thus $\mathcal{G} \in \langle \mathcal{O}_p \mid p \in U\rangle$ has zero–dimensional support, $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$, and x is determined by the morphism $\mathcal{G} \to \mathcal{F}[2]$ seen as an element of $\mathrm{Ext}^2(\mathcal{G}, \mathcal{F})$. ☐

Induction, dream scenario. Given $x \in H[\![0, n]\!]$ for $n \geqslant 2$, we may hope that the interval $\left[\underline{K}, \overline{K}\right]$ contains a heart of the form ΨH for some *auto*equivalence $\Psi\colon \mathbf{D}^{\mathrm{fl}}\Lambda_0 \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ of desired form, so that replacing x by $\Psi^{-1}(x) \in H[\![0, n-1]\!]$ allows us to rinse and repeat until corollary 7.2.6 kicks in.

And we are correct in hoping so — the sub-poset of $\left[H, H[1]\right]$ containing only algebraic hearts is given by the Hasse quiver (7.2), so unless the window $\left[\underline{K}, \overline{K}\right]$ is exceptionally narrow it is bound to contain an image of $H = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$. In fact if we allow ourselves an additional tilt, then we can arrange for this to be true as long as $\left[\underline{K}, \overline{K}\right]$ contains *at least one* algebraic heart.

Lemma 7.2.7. *If either $\overline{K}$ or $\underline{K}$ is algebraic, then there is an autoequivalence $\Psi\colon \mathbf{D}^{\mathrm{fl}}\Lambda_0 \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ composed of shifts, mutation functors, and their inverses such that x lies in $\Psi(H)[0, n-1]$.*

Before moving on to the proof we sketch an illustrative albeit contrived example. Writing $S_0, S_1 \in \mathrm{fl}\,\mathit{mod}\,\Lambda_1$ for the two simple modules indexed as in [DW25, §3.1], the non–zero cohomologies of the complex $x = \Psi_0 S_0[3] \oplus \Psi_0 S_1[1] \in \mathbf{D}^{\mathrm{fl}}\Lambda_0$ with respect to $H = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$ are given by $H^0(x) = \Psi_0 S_1[1]$ and $H^{-3}(x) = \Psi_0 S_0$. In particular $H^0(x)$ is simple in H and $H^{-3}(x)$ generates the torsion–free class $H^0(x)^{\perp}$, so the only heart in $\left[H, H[1]\right]$ which improves the spread of $x \in H[\![0, 3]\!]$ is $\underline{K} = \overline{K} = \Psi_0(\mathrm{fl}\,\mathit{mod}\,\Lambda_1)[1]$. But looking at $x' = \Psi_0^{-1}(x)[-1] \in H'[\![0, 2]\!]$, where $H' = \mathrm{fl}\,\mathit{mod}\,\Lambda_1$, it is easy to see that further tilting H' in any torsion class containing S_1 will not worsen the spread of x'. Thus x' has spread $\leqslant 2$ with respect to each of the hearts $\Psi_0(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)[1]$, $\Psi_0\Psi_{-1}(\mathrm{fl}\,\mathit{mod}\,\Lambda_{-1})$, $\Psi_0\Psi_{-1}\Psi_{-2}(\mathrm{fl}\,\mathit{mod}\,\Lambda_{-2})$, ...

It follows that each of the objects $(\Psi_0\Psi_{-1}\cdots\Psi_{\wp-1})^{-k} \circ \Psi_0^{-1}\Psi_0^{-1} x[-2]$ $(k \geqslant 0)$ lies in $H[0, 2]$.

Proof of lemma 7.2.7. The hypothesis guarantees that there is a heart $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda_i) \in [H, H[1]]$, where Ψ is composed of shifts and (inverse) mutation functors, such that x lies in $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda_i)[\![0, n-1]\!]$. Equivalently, $x' = \Psi^{-1}(x)$ lies in $H'[\![0, n-1]\!]$ where $H' = \mathrm{fl}\,\mathit{mod}\,\Lambda_i$.

Now the same argument as in lemma 7.2.3 shows that the poset $[H', H'[1]]$ contains intervals

$$[H', \overline{K}'] = \{K \in [H', H'[1]] \mid x' \in K[\![0, n-1]\!]\}, \qquad [\underline{K}', H'[1]] = \{K \in [H', H'[1]] \mid x' \in K[-1, n-2]\!]\}$$

defined using the top and bottom cohomologies of x' with respect to H'. The vanishing of $\mathrm{Hom}^{<0}(x', x')$ implies the inequality $\underline{K}' \leqslant \overline{K}'$ as in lemma 7.2.4.

This implies that the union $[H', \overline{K}'] \cup [\underline{K}', H'[1]]$ contains an increasing chain from H', with each K in the chain satisfying $x' \in K[m, m+n-1]$ for some $m \in \{0, -1\}$. Since an algebraic heart cannot cover or be covered by a non-algebraic heart (corollary 6.1.2) and the Hasse quiver of algebraic hearts in $[H', H'[1]]$ is identical to (7.2) with indices shifted up by i, we see that the chain must contain some K of the form $\Phi(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)$ where Φ is composed of mutation functors. Then $\Phi^{-1}(x')[-m] = \Phi^{-1}\Psi^{-1}(x)[-m]$ lies in $(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)[0, n-1]$ as required. □

Proverbial spanners in the inductive works We now address complexes for which the induction outlined above *fails* to continue, i.e. complexes $x \in H[\![0, n]\!]$ $(n \geqslant 2)$ for which both $\underline{K}$ and $\overline{K}$ are non-algebraic – therefore fitting the descriptions (h2) or (h3) in theorem 7.2.5.

Note that hearts described in (h2) all contain the simple object $\omega_{\underline{C}}[1]$ of H, while those described in (h3) contain the other simple. Tilts of $\mathit{coh}\,X$ are thus incomparable with tilts of $\Psi_0\,\mathit{coh}\,W$, so replacing (x, Λ_0, X) with $(\Psi_{-1}^{-1} x, \Lambda_{-1}, W)$ if necessary, we may assume the hearts $\underline{K}, \overline{K}$ are both tilts of $\mathit{coh}\,X$ in some torsion classes containing only sheaves with zero–dimensional support.

We show in this case that x fits the description (c2) in theorem 7.2.1, i.e. decomposes into shifts of sheaves contained in the extension–closure $\langle \mathcal{O}_p \mid p \in C \rangle$. It is classical [see e.g. Stacks, tag 00KY] that such sheaves (henceforth dubbed *zero–dimensional sheaves*) can be characterised by the property that their support is zero–dimensional, equivalently the support (being closed in an irreducible curve) has non-empty complement in C.

We have also seen that the support of a coherent sheaf can detected by looking at morphisms from skyscrapers (and extensions thereof) as follows.

Lemma 7.2.8. *Suppose non-zero sheaves $u, y \in \mathit{coh}\,X$ (supported within C) are such that $u \in \langle \mathcal{O}_p \mid p \in X \rangle$, and either $\mathrm{Hom}(y, u) = 0$ or $\mathrm{Hom}(u, y) = \mathrm{Ext}^1(u, y) = 0$ holds. Then $\dim(\mathrm{Supp}\, y) = 0$.*

Proof. Immediate from lemma 6.4.2. □

The result above can be extended to objects of the heart $H = \mathit{per}(\frac{X}{Z})$.

Lemma 7.2.9. *An object $y \in H$ lies in the extension closure $\langle \mathcal{O}_p \mid p \in X \rangle$ if any of the following conditions hold.*

(i) $\mathrm{Hom}(u, y) = \mathrm{Ext}^1(u, y) = 0$ *for some non-zero* $u \in \langle \mathcal{O}_p \mid p \in C \rangle$.

(ii) $\mathrm{Hom}(y, u) = \mathrm{Ext}^1(y, u) = 0$ *for some non-zero* $u \in \langle \mathcal{O}_p \mid p \in C \rangle$.

(iii) $\mathrm{Hom}(u, y) = \mathrm{Hom}(y, v) = 0$ *for some (not necessarily distinct) non-zero* $u, v \in \langle \mathcal{O}_p \mid p \in C \rangle$.

Proof. Recall that by definition of $\mathit{per}(\frac{X}{Z})$, the object $y \in H$ sits in an exact triangle $y'[1] \to y \to y'' \to y'[2]$ where $y', y'' \in \mathit{coh}\,X$ satisfy $\pi_*(y') = \mathbf{R}^1\pi_*(y'') = 0$ and $\mathrm{Hom}(\mathcal{O}_C(-1), y') = 0$. In particular y' has no subsheaves with zero–dimensional support, i.e. $\mathrm{Hom}(u, y') = 0$ for all $u \in \langle \mathcal{O}_p \mid p \in C \rangle$.

For a non-zero $u \in \langle \mathcal{O}_p \mid p \in C \rangle$, a straightforward long exact sequence argument shows there is an inclusion $\mathrm{Ext}^1(u, y') \subseteq \mathrm{Hom}(u, y)$, thus if either (i) or (iii) holds then the hypothesis $\mathrm{Hom}(u, y) = 0$ implies $\mathrm{Ext}^1(u, y') = 0$ which cannot happen unless $\dim(\mathrm{Supp}\, y') < 1$ by lemma 7.2.8. This necessarily implies $y' = 0$, and hence $y = y''$ is a sheaf. The same lemma then lets us deduce the condition on $\mathrm{Supp}(y)$ from either pair of hypotheses.

On the other hand if (ii) holds, then the vanishing of $\mathrm{Hom}(y'', u) = \mathrm{Hom}(y, u)$ shows $\mathrm{Supp}(y'')$ is disjoint from $\mathrm{Supp}(u)$. The long exact sequence $\cdots \to \mathrm{Ext}^1(y, u) \to \mathrm{Hom}(y', u) \to \mathrm{Ext}^2(y'', u) \to \cdots$ then shows $\mathrm{Hom}(y', u)$ vanishes too, which can only happen if $y' = 0$ and hence $y = y''$ is of the claimed form. □

We are now sufficiently equipped to witness the decomposition of x – the constraints on $\overline{K}$ show the cohomology object $H^0(x) \in H$ surjects onto a skyscraper sheaf, which we then exploit against lemma 7.2.9.

Lemma 7.2.10. *Suppose* $x \in H[\![0, n]\!]$ *(*$n \geqslant 2$*) is such that the hearts* $\underline{K}, \overline{K}$ *are both tilts of* $\mathit{coh}\, X$ *in (possibly trivial) torsion classes containing sheaves with zero-dimensional support. Then each cohomology object* $H^i(x) \in H$ *lies in* $\langle \mathcal{O}_p \mid p \in X\rangle$*, and cohomologies in distinct degrees have disjoint supports.*

Proof that $H^0(x)$ *and* $H^{-n}(x)$ *have zero-dimensional supports.* The conditions on $\overline{K}$ guarantee the containments of torsion classes

$$\mathit{coh}\, X[1] \cap H \quad \subsetneq \quad \overline{K} \cap H \quad \subseteq \quad \langle \mathit{coh}\, X[1] \cap H, \{\mathcal{O}_p \mid p \in C\}\rangle,$$

and the first containment must be proper – indeed, by theorem 6.2.3 the torsion class $\mathit{coh}\, X[1] \cap H$ is the supremum (i.e. union) of a strictly increasing chain $0 \subset T_1 \subset T_2 \subset \cdots$ in $\mathrm{tors}(H)$, so if $\overline{K} \cap H$ is equal to this union then $H^0(x) \in \overline{K} \cap H$ is contained in some T_i. But this is absurd, since $\overline{K} \cap H$ is by definition the minimal torsion class containing $H^0(x)$.

Thus $H^0(x)$ lies in $\langle \mathit{coh}\, X[1] \cap H, \{\mathcal{O}_p \mid p \in C\}\rangle \setminus (\mathit{coh}\, X[1] \cap H)$, sitting in a triangle $y[1] \to H^0(x) \to z \to y[2]$ where $y, z \in \mathit{coh}\, X$ are sheaves such that $y \in H[-1]$, and $z \in \langle \mathcal{O}_p \mid p \in C\rangle$ is non-zero.

Now the analogous condition $\underline{K}[-1] \cap H \subseteq \mathit{coh}\, X \cap H$ implies that $H^{-n}(x)$ is a sheaf, so $\mathrm{Hom}^{<0}(y, H^{-n}x)$ vanishes and the above exact triangle gives the inclusions $\mathrm{Hom}(z, H^{-n}x[i]) \subseteq \mathrm{Hom}(H^0x, H^{-n}x[i])$ for $i = 0, 1$ via a long exact sequence.

But the conditions $\mathrm{Hom}(x, x[-n]) = \mathrm{Hom}(x, x[1-n]) = 0$, after truncating to the aisle and the coaisle of H as in the proof of lemma 7.2.4, imply

$$\mathrm{Hom}(H^0x, H^{-n}x) = \mathrm{Hom}^1(H^0x, H^{-n}x) = 0. \tag{7.3}$$

Thus we conclude $\mathrm{Hom}(z, H^{-n}x) = \mathrm{Hom}^1(z, H^{-n}x) = 0$, so that $H^{-n}(x)$ is a sheaf with zero–dimensional support outside $\mathrm{Supp}(z)$ by lemma 7.2.9. It immediately follows (from (7.3) and lemma 7.2.9) that $H^0(x)$ is also a sheaf with zero–dimensional support, in particular $H^0(x) = z$ and $y = 0$. □

Proof that the remaining cohomologies have zero–dimensional supports. One can continue to use truncation functors and analyse long exact sequences to compare the cohomologies $H^i(x)$ against $H^0(x)$ and $H^{-n}(x)$, we find it more convenient to instead examine the spectral sequence [see e.g. GM96, IV.2, Exercise 2]

$$E_2^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^p(H^i x, H^{i+q}x) \Longrightarrow \mathrm{Hom}^{p+q}(x, x)$$

and inductively show enough rows vanish on the second page. Suppose we have shown that the terms $E_2^{p,q}$ all vanish for $q \leqslant j$, from the above discussion this is true for $j = -n$. Since the terms also vanish if $p < 0$, we see that the differentials in and out of $E_2^{0,1+j}$ and $E_2^{1,1+j}$ are trivial so that these terms persist in future pages of the sequence, arising as subquotients of $\mathrm{Hom}^{1+j}(x, x)$ and $\mathrm{Hom}^{2+j}(x, x)$ respectively.

In particular if $j < -2$, then $E_2^{0,1+j} = E_2^{1,1+j} = 0$ and we have

$$\begin{aligned} \mathrm{Hom}\ (H^0x, H^{1+j}x) &= \mathrm{Hom}\ (H^{-n-j-1}x, H^{-n}x) = 0, \\ \mathrm{Hom}^1(H^0x, H^{1+j}x) &= \mathrm{Hom}^1(H^{-n-j-1}x, H^{-n}x) = 0. \end{aligned}$$

Since $H^0(x)$ and $H^{-n}(x)$ both lie in $\langle \mathcal{O}_p \mid p \in X\rangle$, lemma 7.2.9 and the above equalities show that the same holds for $H^{1+j}(x)$ and $H^{-n-j-1}(x)$. In fact an identical reasoning for lower rows shows $H^i(x) \in \langle \mathcal{O}_p \mid p \in X\rangle$ for each of the values $i \in \{-n, -n+1, \ldots, j+1\} \cup \{-n-j-1, -n-j, \ldots, 0\}$. For these values of i, the vanishing of $\mathrm{Hom}(H^i x, H^{i+j+1}x) \subset E_2^{0,1+j}$ then shows that $H^i(x)$ and $H^{i+j+1}(x)$ have disjoint supports, so that $E_2^{p,j+1} = 0$ for remaining values of p too.

Thus for each $q \leqslant -2$, we have $E_2^{0,q} = E_2^{1,q} = 0$, and hence $H^q(x)$ and $H^{-n-q}(x)$ have zero-dimensional supports as above.

If $n \geqslant 3$ this accounts for all cohomologies of x. If $n = 2$ the above argument fails to address the cohomology in degree -1, but note that in this case the vanishing of $E_2^{0,-1} = \mathrm{Hom}(H^0x, H^{-1}x) \oplus \mathrm{Hom}(H^{-1}x, H^{-2}x)$ alone suffices to apply lemma 7.2.9 and deduce that $\dim(\mathrm{Supp}(H^{-1}x)) = 0$. □

Proof that cohomologies have mutually disjoint supports. Since x is an extension of shifts of skyscrapers, it admits a decomposition $x = \bigoplus_{p \in C} x|_p$ indexed over closed points of C, where the summand $x|_p$ is a complex supported within $\{p\}$. In particular the top non-zero cohomology of $x|_p$ admits a non-trivial morphism (which factors via $\mathcal{O}_p$) to the bottom non-zero cohomology of $x|_p$. But $x|_p$ also satisfies $\mathrm{Hom}^{<0}(x|_p, x|_p) = 0$, whence $x|_p$ must be a sheaf concentrated in a single cohomological degree. Since $H^i(x) = \bigoplus_{p \in C} H^i(x|_p)$, the result follows. □

The above result, combined with lemma 7.2.7 and corollary 7.2.6, finishes our analysis of all complexes supported on C that have only non-negative self extensions, describing a complete proof of theorem 7.2.1.

§ 7.3 A washload of corollaries

We sketch how finite semibricks, hence also bricks, simple-minded collections, algebraic t-structures, and tilting complexes in $\mathbf{D}^0X$, and hence also stability conditions and autoequivalences, are all readily classified using theorem 7.2.1. Throughout, the phrase ***up to mutation*** means up to shifts and iterated application of functors appearing in (7.1) and their inverses.

Theorem 7.3.1. *Every finite semibrick $\{x_0, \ldots, x_m\} \subset \mathbf{D}^0X$ must, up to mutation, either be a collection of simple Λ_i-modules for some $i \in \mathbb{Z}/\wp\mathbb{Z}$, or be of the form $\{\mathcal{O}_{p_0}[n_0], \ldots, \mathcal{O}_{p_m}[n_m]\}$ for distinct closed points $p_0, \ldots, p_m$ in X (or in W) and integers $n_0, \ldots, n_m \in \mathbb{Z}$.*

Proof. Consider the object $x = x_0 \oplus \cdots \oplus x_m$, which clearly satisfies $\mathrm{Hom}^{<0}(x, x) = 0$ and is therefore classified by theorem 7.2.1. If it fits the description (c2), then each x_i must be a shift of some sheaf $\mathcal{O}_p$ ($p \in X$) since those are the only brick sheaves with zero–dimensional support (to see this, note all objects in $\langle \mathcal{O}_p \rangle$ have a non-zero endomorphism which factors via $\mathcal{O}_p$).

On the other hand if x fits the description (c1), i.e. the semibrick $\{x_0, \ldots, x_m\}$ up to mutation lies in some $H' = \mathrm{fl}\,\mathit{mod}\,\Lambda_j$, then consider the smallest torsion class $T \subset H'$ that contains x. By (the proof of) theorem 6.4.1, each x_i is a simple object in the heart K obtained by tilting H' in this torsion class. Examining the Hasse quiver of $[H', H'[1]]$, we see that K must also be a tilt of $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)[1]$ or $\Psi^{-1}(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)$ for some mutation functor Ψ. Thus K is, up to mutation, a tilt of $\mathrm{fl}\,\mathit{mod}\,\Lambda_0$ and required descriptions of the simple objects $x_0, \ldots, x_m \in K$ can be read off from theorem 7.2.5.

To address the last case, suppose x is a two-term complex of coherent sheaves as in (c3), without loss of generality on X. Each summand x_i must then also admit the same description, sitting in an exact triangle

$$\mathcal{F}_i[1] \to x_i \to \mathcal{G}_i \to \mathcal{F}_i[2]$$

with $\mathcal{G}_i \in \langle \mathcal{O}_p \mid p \in X \rangle$, $\mathcal{F}_i \in \mathit{coh}\,X$, and $\mathrm{Hom}(\mathcal{G}_i, \mathcal{F}_i) = 0$.

Claim for each i, either $\mathcal{F}_i$ or $\mathcal{G}_i$ vanishes. Indeed if $\mathcal{F}_i \neq 0$, it cannot be a summand in $\mathcal{G}_i[-2]$ so the first map in the triangle above is a non–zero element in $\mathrm{Hom}(\mathcal{F}_i[1], x_i)$. This element is annihilated by the natural map $\mathrm{Hom}(\mathcal{F}_i[1], x_i) \to \mathrm{Hom}(\mathcal{G}_i[-1], x_i)$ since any two consecutive maps in an exact triangle compose to zero [Stacks, tag 0146]. Thus in the long exact sequence

$$\cdots \to \underbrace{\mathrm{Hom}(\mathcal{F}_i[2], x_i)}_{0} \longrightarrow \mathrm{Hom}(\mathcal{G}_i, x_i) \longrightarrow \underbrace{\mathrm{Hom}(x_i, x_i)}_{\mathbb{C}} \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{F}_i[1], x_i) \longrightarrow \mathrm{Hom}(\mathcal{G}_i[-1], x_i) \to \cdots$$

the final map has non-trivial kernel so that α is injective. It follows that $\mathrm{Hom}(\mathcal{G}_i, x_i) = 0$, and considering the long exact sequence associated to $\mathrm{Hom}(\mathcal{G}_i, -)$ shows $\mathrm{Hom}(\mathcal{G}_i, \mathcal{F}_i[1])$ vanishes too. If $\mathcal{G}_i$ were non–zero then lemma 7.2.9 would show $\mathrm{Supp}(\mathcal{F}_i)$ is zero–dimensional hence disjoint from $\mathrm{Supp}(\mathcal{G}_i)$, but this is absurd since we would then have a non-trivial splitting $x_i = \mathcal{F}_i[1] \oplus \mathcal{G}_i$ of a brick. Thus there is a $k \geqslant 0$ and a reordering of the summands of x so that $x_i = \mathcal{G}_i$ for $i < k$, and $x_i = \mathcal{F}_i[1]$ for $i \geqslant k$. If $k \geqslant 1$ then lemma 7.2.8 shows each $\mathcal{F}_i$ ($i \geqslant k$) has zero–dimensional support since $\mathrm{Ext}^{-1}(x_0, x_i) = \mathrm{Hom}(\mathcal{G}_0, \mathcal{F}_i)$ and $\mathrm{Hom}(x_0, x_i) = \mathrm{Ext}^1(\mathcal{G}_0, \mathcal{F}_i)$ both vanish, whence x also admits the description (c2) and can be addressed as in the first paragraph.

On the other hand if $k = 0$ (so that $x \in \mathit{coh}\, X[1]$), then we can consider (as in lemma 7.2.4) the maximal tilt $\overline{K}$ of $H = \mathrm{fl}\mathit{mod}\, \Lambda_0$ satisfying $x \in \overline{K}$. We have $\overline{K} \geqslant \mathit{coh}\, X[1]$ (i.e. $\overline{K} \cap H \subseteq \mathit{coh}\, X[1] \cap H$), and equality cannot hold since the torsion class $\mathit{coh}\, X[1] \cap H$ is the union of an increasing chain in $\mathrm{tors}(H)$ while the torsion class $\overline{K} \cap H$ is the intersection of all torsion classes containing $H^0(x)$ (see proof of lemma 7.2.10). It follows from theorem 7.2.5 that $\overline{K}$ is an algebraic heart, so that $x \in \overline{K}$ also admits the description (c1) and is addressed by the second paragraph. □

Two special cases of the above theorem are of note.

Corollary 7.3.2. *Every brick $x \in \mathbf{D}^0 X$ is, up to mutation, either a skyscraper sheaf $\mathcal{O}_p$ at a closed point in X or in W, or a simple Λ_i-module for some $i \in \mathbb{Z}/\wp\mathbb{Z}$. In the latter case, up to shifts, mutation functors, and the Grothendieck duality functor, x is one of the sheaves $\mathcal{O}_C, \mathcal{O}_{2C}, \ldots, \mathcal{O}_{\ell C}$, or the unique non-split extension $\mathcal{Z}$ of $\mathcal{O}_{2C}$ by $\mathcal{O}_{3C}$ which exists when $\ell \geqslant 5$.*

We remind the reader that the Grothendieck duality functor on $\mathit{Coh}\, X$ is, by definition, $\mathbb{D} = \mathbf{R}\mathcal{H}\mathit{om}(-, \varpi^! \mathbb{C})$ where $\varpi \colon X \to \mathbb{C}$ is the structure morphism defining X as a complex scheme. It can equivalently be given as $\mathbf{R}\mathcal{H}\mathit{om}(-, \pi^! E)$, where $E \in R\text{-}\mathit{Mod}$ is the injective hull of the residue field $\mathbb{C} \simeq R/\mathfrak{m}$.

Proof. Considering the finite semibrick $\{x\}$ shows x must be of the required form. The explicit form of simple Λ_i-modules can then be read off from Donovan–Wemyss's computation [DW25, §1.2]. □

Corollary 7.3.3. *Any algebraic t-structure in $\mathbf{D}^0 X$ has heart, up to mutation, given by the standard heart $\mathrm{fl}\mathit{mod}\, \Lambda_i \subset \mathbf{D}^{\mathrm{fl}} \Lambda_i$ for some $i \in \mathbb{Z}/\wp\mathbb{Z}$.*

Proof. The simples of any algebraic heart K form a finite semibrick, and the only heart containing this semibrick must contain (and hence be equal to) K itself. The result follows by noting collections of skyscrapers cannot be simples of an algebraic heart. □

Tilting complexes The above classification of algebraic t-structures in $\mathbf{D}^0 X$ can be lifted to a classification of tilting complexes in the bigger category $\mathbf{D}^b X$, following [HW23, §5.4].

Recall that a complex $t \in \mathbf{D}^b X$ is *tilting* if it satisfies $\mathrm{Hom}^i(t, t) = 0$ for all $i \neq 0$ and the smallest thick subcategory of $\mathbf{D}^b X$ containing t also contains all perfect complexes. We emphasise that such t cannot be simply plugged into theorem 7.2.1, since t would necessarily be supported on the generic point of X (and hence outside the exceptional fiber).

Given an algebra Λ and a derived equivalence $\Psi : \mathbf{D}^b \Lambda \to \mathbf{D}^b X$, the image $\Psi(\Lambda)$ of the regular Λ-module is a tilting object in $\mathbf{D}^b X$. The converse is Rickard's theorem [Ric89, theorem 6.4], that any tilting object $t \in \mathbf{D}^b X$ arises in the above fashion from the R-algebra $\Lambda = \mathrm{End}(t)$ and the R-linear equivalence determined as $\Psi^{-1} = \mathbf{R}\mathrm{Hom}(t, -)$. If the complex t (equivalently the algebra Λ) is *basic*, i.e. if distinct indecomposable summands of t are non-isomorphic, then Morita theory[1] [see e.g. Lam99, proposition 18.37] picks out t as the unique basic projective generator in the heart $\Psi(\mathit{mod}\, \Lambda) \subset \mathbf{D}^b X$.

In this fashion, each equivalence $\mathbf{D}^b \Lambda_i \to \mathbf{D}^b X$ composed of the functors appearing in (7.1) and their inverses gives a unique basic tilting object in $\mathbf{D}^b X$. These tilting objects sit in a connected tetravalent exchange graph so that each one can be obtained from $\mathrm{VdB}^{-1}(\Lambda_0) = \mathcal{N}$ by iterated left or right mutation in indecomposable summands (i.e. by *simple mutation*). Other tilting complexes include non–trivial shifts of those already considered, these can be obtained by simple mutation from $\mathcal{N}[n]$.

We show this list is exhaustive, in particular $\{\Lambda_i \mid i \in \mathbb{Z}/\wp\mathbb{Z}\}$ is a complete derived-equivalence class of basic algebras.

[1] This requires Λ to be semi-perfect, which is true since it is a module-finite algebra over the complete local domain R.

Corollary 7.3.4. *Any basic tilting complex* $t \in \mathbf{D}^b X$ *can be obtained by iteratively mutating the tilting complex* $(\mathcal{N} \oplus \mathcal{O}_X)[n]$ *in indecomposable summands, for some* $n \in \mathbb{Z}$.

Proof. Consider the module-finite R-algebra $\Lambda = \mathrm{End}(t)$ and the induced equivalence $\Psi\colon \mathbf{D}^b\Lambda \to \mathbf{D}^b X$ with inverse $\mathbf{R}\mathrm{Hom}(t,-)$. By R-linearity, the functor Ψ restricts to an equivalence $\Psi\colon \mathbf{D}^{\mathrm{fl}}\Lambda \to \mathbf{D}^0 X$ and hence the image of $\mathrm{fl}\,\mathit{mod}\,\Lambda$ is an algebraic heart in $\mathbf{D}^0 X$. By theorem 7.3.1, we thus see that $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda)$ agrees with some heart $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda_i)[n]$ for $i \in \mathbb{Z}/\wp\mathbb{Z}$, $n \in \mathbb{Z}$, and Ψ a composite of functors appearing in (7.1) and their inverses.

But by [HW23, lemma 5.7], the subcategory $\mathrm{fl}\,\mathit{mod}\,\Lambda \subset \mathbf{D}^b\Lambda$ determines the natural t-structure via

$$\mathit{mod}\,\Lambda\,[\geqslant 0] = \left\{ x \in \mathbf{D}^b\Lambda \mid \mathrm{Hom}^j(x,y) = 0 \text{ for all } j < 0 \text{ and all } y \in \mathrm{fl}\,\mathit{mod}\,\Lambda \right\}.$$

Likewise the subcategory $\mathrm{fl}\,\mathit{mod}\,\Lambda_i \subset \mathbf{D}^b\Lambda_i$ determines the natural t-structure with heart $\mathit{mod}\,\Lambda_i$, thus we see that the equality $\Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda) = \Psi(\mathrm{fl}\,\mathit{mod}\,\Lambda_i)[n]$ extends to an equality $\Psi(\mathit{mod}\,\Lambda) = \Psi(\mathit{mod}\,\Lambda_i)[n]$. In particular $t = \Psi(\Lambda)$ must equal the basic projective generator $\Psi(\Lambda_i)[n] \in \Psi(\mathit{mod}\,\Lambda_i)[n]$, so that t is obtained by iterated simple mutation from $(\mathcal{N} \oplus \mathcal{O}_X)[n]$ as required. □

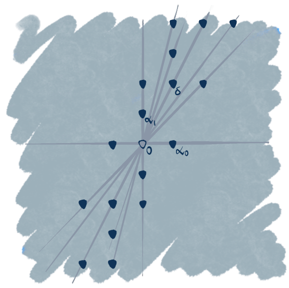
Remark 7.3.5. The shift n appearing above is unique, and can be read off as the cohomological degree of the generic stalk $\Bbbk \otimes_R t$ where $\Bbbk = \mathbb{C}(Z)$ is the function field of Z. Indeed considering $t \in \mathbf{D}^b\Lambda_0$ (across Van den Bergh's equivalence), the localisation $\Bbbk \otimes t$ is a tilting module over $\Bbbk \otimes \Lambda_0$ [Nam23, lemma 6.4]. But $\Bbbk \otimes \Lambda_0$ is a matrix ring over the field $\Bbbk$ (see lemma 6.2 *ibid*) so every tilting $(\Bbbk \otimes \Lambda_0)$–module is the shift of a $\Bbbk$–vector space. It follows that t is obtained by simple mutation from $\Lambda_0[n]$ if and only if $\Bbbk \otimes t[-n] = \Bbbk \otimes \Lambda_0$ is a $\Bbbk$–vector space.

Stability conditions. We conclude by addressing the connectivity of the stability manifold of $\mathbf{D}^0 X$, referring the reader to [Bri06, §3] for definitions and generalities. The stability manifold $\mathrm{Stab}(\mathbf{D}^0 X)$ has been extensively analysed [Bri09; Tod08; HW23] with a key result being the existence of a connected component $\mathrm{Stab}^\circ(\mathbf{D}^0 X)$ that contains all stability conditions (Z, P) for which some heart $P(\varphi, \varphi+1]$ agrees with some $\mathrm{fl}\,\mathit{mod}\,\Lambda_j$ up to mutation. We show that every locally finite stability condition on $\mathbf{D}^0 X$ is of this form.

Corollary 7.3.6. *For any locally finite Bridgeland stability condition* (Z, P) *on* $\mathbf{D}^0 X$, *there is a phase* $\varphi \in \mathbb{R}$ *such that the heart* $P(\varphi, \varphi+1]$ *is algebraic, and therefore equal to some standard heart* $\mathrm{fl}\,\mathit{mod}\,\Lambda_i$ *up to mutation. In particular the stability manifold* $\mathrm{Stab}(\mathbf{D}^0 X)$ *is connected.*

Proof. Given the stability condition $\sigma = (Z, P)$, consider the set $\{\varphi \in \mathbb{R} \mid P(\varphi) \neq \emptyset\}$ of phases occupied by σ–stable objects. Now a σ–stable object $x \in \mathbf{D}^0 X$ of phase φ is necessarily simple in the Abelian category $P(\varphi)$ and therefore is a brick, equal to a simple Λ_i–module or a skyscraper sheaf up to mutation (theorem 7.3.1).

But theorem 6.4.1 then shows that the $\mathbf{K}$–theory class $[x]$ is a restricted affine root in the rank two lattice $\mathbf{K}X \simeq \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$. The corresponding affine hyperplane arrangements in $\mathbb{R}^2$ is locally finite away from the origin; it follows that the possible phases of the complex number $Z[x]$ must be restricted to a countable set of values with a discrete set of accumulation points. For example, the figure alongside is the affine D_4 restricted root system associated to a length 2 flop – clearly all phases accumulate towards that of $\pm\delta$.



In particular σ has a 'non-trivial phase gap', i.e. there is a $\varphi \in \mathbb{R}$ and $\epsilon > 0$ such that $P(\varphi, \varphi+\epsilon) = \emptyset$. If the image of $Z : \mathbf{K}X \to \mathbb{C}$ is a discrete subgroup of $\mathbb{C}$, then we may use [Bri08, lemma 4.4] to conclude that $P(\varphi, \varphi+1] = P(\varphi+\epsilon, \varphi+1]$ is an algebraic Abelian category. On the other hand if $\mathrm{img}(Z)$ is a non-discrete rank 2 subgroup of $\mathbb{C}$, then this image must lie in the $\mathbb{R}$-span of some non-zero complex number $e^{i\pi\varphi}$ so that the phase of each stable object is an integer away from φ. Then local finiteness allows us to conclude that $P(\varphi, \varphi+1] = P(\varphi+1)$ is algebraic. □

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