Amenability and singular functions on non-Hausdorff groupoids

by

Julian J. Gonzales

A thesis submitted in fulfilment of the requirements
for the degree of

Doctor of Philosophy

at the

School of Mathematics & Statistics
College of Science & Engineering
University of Glasgow

VIA VERITAS VITA

June 2026

Abstract

This thesis investigates structural properties of non-Hausdorff groupoid C^*-algebras. The two main points of study are groupoid amenability, and the nature of functions belonging to the singular ideal. Conclusive results are obtained relating amenability of a non-Hausdorff étale groupoid to nuclearity of its C^*-algebras, generalising well-known results from the Hausdorff setting. In relation to the singular ideal, algebraic techniques are established for the building of explicit *elementary* singular functions within it. For a large class of amenable groupoids, these elementary functions are shown to span a dense subspace of the singular ideal. In the context of Steinberg algebras, an analogous family of elementary singular functions are shown to linearly span the algebraic singular ideal of any ample groupoid.

A key tool throughout is the Hausdorff cover groupoid. It is used to obtain the aforementioned results regarding amenability, and to study ideals in non-Hausdorff groupoid C^*-algebras. In most instances, the Hausdorff cover is used to reduce a given problem to the Hausdorff setting. Under certain conditions on the Hausdorff cover, an explicit formula is provided for computing the singular ideal.

Contents

Acknowledgments

Firstly, I thank Xin Li for his guidance and encouragement, and for always making time for me throughout the PhD.

Secondly I thank Jeremy Hume for his collaboration, and for many valuable conversations relating to topics in this thesis.

I extend thanks to past and present members of the analysis group at Glasgow for creating such a supportive environment whilst I conducted my PhD.

I am grateful to all my friends in Glasgow, particularly the Breakfast Club, for the hikes, climbing, lunch games, breakfasts, trips, and parkruns that have made these last four years memorable. I'm especially grateful to Philipp and Isacco, and for all the memories at the Old Schoolhouse.

Lastly, I thank my family for their constant support, and for giving me confidence in all of my endeavours.

Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

Groupoids have been a prominent feature in the theory of C^*-algebras since the pioneering work of Renault [58]. There are many natural ways to construct a C^*-algebra given certain dynamical, combinatorial, or algebraic data, and groupoids provide a means to unify many of these constructions. The class of C^*-algebras that admit a groupoid model is large ([46]), and many structural properties of a C^*-algebra are seen by a groupoid that models it ([3, 41]).

Since the early development of the theory, it has been observed that many interesting and important examples of groupoids are non-Hausdorff. Some of the earliest such examples are due to Connes in his survey on operator algebras associated with foliations ([19]). He observed that the holonomy groupoid of a foliation may fail to be Hausdorff, and addresses some of the modifications that should be made when defining the C^*-algebras of non-Hausdorff groupoids. As well as these examples from geometry, many examples of non-Hausdorff groupoids arise naturally in dynamics as groupoids of germs associated with inverse semigroup actions. A class of groupoids arising in this manner, that have gained notable recent attention, are those associated with self-similar group actions (see [30, 52, 53, 67]).

Despite an abundance of examples, progress has been slow in the development of a robust theory for non-Hausdorff groupoids and their C^*-algebras. While there are many results relating properties of Hausdorff groupoids with structural properties of their C^*-algebras, analogues for non-Hausdorff groupoids have proved difficult to attain. This is in part due to the additional technical difficulties that arise when working with non-Hausdorff groupoids, but also because their C^*-algebras often exhibit strange and unexpected behaviour (see [26]). Many instances of this strange behaviour are due to the existence of an ideal of "singular" functions, known as the *singular ideal* J (see [18, 27, 43]). There has been significant recent progress towards understanding this ideal (see [38, 48]), but many questions about its structure remain unanswered.

This thesis addresses some of the difficulties associated with studying non-Hausdorff groupoids. The two main topics of interest are the notion of amenability, and the singular ideal J of a non-Hausdorff groupoid. This thesis aims to clarify the notion of amenability for non-Hausdorff groupoids, as well as to prove results relating amenability with properties of groupoid C^*-algebras. Conclusive results are obtained in this direction. In particular, it is shown that an étale groupoid is amenable if and only if its reduced C^*-algebra is nuclear. The second main object of study is the singular ideal J, with the aim of finding dense subspaces within it, and thus describing its elements. Notably, this thesis addresses the problem of when the intersection $J \cap \mathscr{C}_c(G)$ with Connes' algebra $\mathscr{C}_c(G)$ of "compactly supported" functions is dense in J. Results are obtained that characterise when this density holds for amenable and second-countable étale groupoids. In addition to this, the first concrete techniques are established for the building of elements of the singular ideal, and it is shown that singular elements constructed via these techniques have dense linear span in $J \cap \mathscr{C}_c(G)$ for any étale groupoid.

A key tool throughout is the Hausdorff cover groupoid. The Hausdorff cover is used to reduce problems concerning non-Hausdorff groupoids to the Hausdorff setting. Time is devoted to describing the structure of the Hausdorff cover, before it is used to obtain results on amenability and the singular ideal.

The thesis opens with a discussion of amenability in the context of non-Hausdorff groupoids. To motivate this, there is a brief discussion of amenability in the Hausdorff setting, including some observations from the book [3] – the main reference for the topic of amenability for Hausdorff groupoids. As in the case for groups, the notion of groupoid amenability admits many equivalent characterisations. This allows the authors in [3] to study applications of amenability with a great degree of flexibility. Later on in the book, several results are proved regarding the operator algebras associated with Hausdorff amenable groupoids. In particular, [3] proves the following results about the full and reduced C^*-algebras of a second-countable locally compact Hausdorff groupoid G: if G is amenable, then the canonical surjection $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism, and $C_r^*(G)$ is nuclear.

For non-Hausdorff groupoids, the situation is not so clear-cut. In particular, some of the useful descriptions of amenability for Hausdorff groupoids can fail (see Subsection 2.2). Care must therefore be taken when proving results for amenable non-Hausdorff groupoids, but nevertheless, amenability is still a powerful property of these groupoids and much can be said about their operator algebras. Indeed, it was known to experts that the results mentioned above also hold for C^*-algebras of non-Hausdorff groupoids, with virtually the same proof. Chapter 2 gives a proof of these "folklore" results, now adapted to the non-Hausdorff setting (see Theorem 2.4.1).

Proving results about non-Hausdorff groupoids directly can be very technical as many of the tools available for Hausdorff groupoids don't apply in this setting. A solution to this is presented by the Hausdorff cover. Any non-Hausdorff groupoid G can be used to build a Hausdorff groupoid $\tilde{G}$, known as the *Hausdorff cover* ([68]). This groupoid provides a means to avoid many of the technical difficulties involved in proving results about non-Hausdorff groupoids by simply reducing the given problem to the Hausdorff setting. Chapter 3 of this thesis discusses two applications of the Hausdorff cover. The first is to amenability (Theorem A), and the second is to ideals in groupoid C^*-algebras (Theorem B). The Hausdorff cover is defined for any locally compact groupoid, but we will only discuss it in the context of étale groupoids.

The "folklore" result mentioned earlier states that any amenable groupoid has nuclear C^*-algebra. For Hausdorff étale groupoids, the converse holds as well (see [3, Corollary 6.2.14(ii)], [11, Theorem 5.6.18]). With the aid of the Hausdorff cover, this converse can be shown also for non-Hausdorff étale groupoids.

Theorem A (See Corollary 3.2.10)**.** *Let G be a (non-Hausdorff) étale groupoid. Consider the following statements.*

(I) G is amenable.

(II) $\tilde{G}$ is amenable.

(III) $C^*_r(G)$ is nuclear.

(IV) $C^*_r(\tilde{G})$ is nuclear.

(V) The canonical surjection $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism.

(VI) The canonical surjection $C^*(\tilde{G}) \twoheadrightarrow C^*_r(\tilde{G})$ is an isomorphism.

The following implications hold: (I) $\Leftrightarrow$ (II) $\Leftrightarrow$ (III) $\Leftrightarrow$ (IV) $\Rightarrow$ (VI) $\Rightarrow$ (V).

Nuclearity of C^*-algebras passes to quotients, and therefore the reduced C^*-algebras $C^*_r(G)$ and $C^*_r(\tilde{G})$ can be replaced by the full C^*-algebras $C^*(G)$ and $C^*(\tilde{G})$ in statements (III) and (IV) of the previous theorem. It follows from [70] that the implication (VI) $\Rightarrow$ (II) does not hold in general. In Example 4.8.5, it is shown that the implication (V) $\Rightarrow$ (VI) can also fail.

An important property of amenability for Hausdorff étale groupoids is that ideals in the C^*-algebra are determined by their restrictions to isotropy groups. This observation originally dates back to a conjecture of Effros and Hahn from [23], which can be phrased as follows: if G is the transformation groupoid associated with the action of a countable amenable group on a second-countable locally compact Hausdorff space, then every primitive ideal in $C^*(G)$ is induced from an isotropy group. This conjecture has since been verified, and actually holds in a much more general setting than originally stated. The most general form appears in [39], where the transformation groupoid G is replaced by an arbitrary Hausdorff, amenable, and second-countable étale groupoid. A consequence of this deep result appears in [17, Theorem 2.10]. In order to state this theorem, let us introduce some notation. Given an étale groupoid G and a unit $x \in G^{(0)}$, it is shown in [17] that the (restriction) map $\eta_x \colon C^*(G) \to C^*(G_x^x)$ sends closed ideals to closed ideals. Given an ideal $I \trianglelefteq C^*(G)$, define $I_x := \eta_x(I) \trianglelefteq C^*(G_x^x)$, and call I_x the *isotropy fibre* of I at x. Then, given ideals $I, K \trianglelefteq C^*(G)$, [17, Theorem 2.10] states that $I = K$ if and only if $I_x = K_x$ for all $x \in G^{(0)}$.

This brings us to the second application of the Hausdorff cover $\tilde{G}$. Any ideal I in a non-Hausdorff groupoid C^*-algebra $C^*(G)$ naturally generates an ideal $\tilde{I}$ in $C^*(\tilde{G})$. Moreover, the assignment $I \mapsto \tilde{I}$ is one-to-one (Lemma 3.3.1), and the isotropy fibres of $\tilde{I}$ can be computed using those of I (Proposition 3.3.2). It follows that the Hausdorff cover yields a generalisation of [17, Theorem 2.10] to the non-Hausdorff setting.

Theorem B (See Theorem 3.3.3)**.** *Let G be an amenable and second-countable (non-Hausdorff) étale groupoid. Let $I, K \trianglelefteq C^*(G)$ be ideals. Then $I = K$ if and only if $I_x = K_x$ for all $x \in G^{(0)}$.*

The theorems above demonstrate that, with the right definitions, many results known for Hausdorff groupoids and their C^*-algebras extend to the non-Hausdorff setting. Of course, not everything can be expected to generalise so smoothly. Chapter 4 of this thesis investigates the key antagonist within the study of non-Hausdorff groupoid C^*-algebras: the singular ideal J.

Let us describe two standard properties of Hausdorff groupoid C^*-algebras that fail for non-Hausdorff groupoids (see [26] for more). A simple and convenient property of a Hausdorff groupoid G is that the reduced norm $\|f\|_{C_r^*(G)}$ of any $f \in C_c(G)$ can be calculated using a dense subset of the unit space $G^{(0)}$. Secondly, given an effective étale groupoid G, the subalgebra $C_0(G^{(0)}) \subseteq C_r^*(G)$ has the *ideal intersection property.* That is, any non-zero ideal in the reduced C^*-algebra $C_r^*(G)$ has non-zero intersection with $C_0(G^{(0)})$. Both of these facts fail for non-Hausdorff groupoids, and in both cases the culprit is the *singular ideal* $J \trianglelefteq C_r^*(G)$. This ideal is defined as the set of elements in $C_r^*(G)$ whose strict support has empty interior (as a function on G). Elements of J are far from continuous, and therefore dense subspaces are of no use in describing a typical element of $C_r^*(G)$. Moreover, the singular ideal J *always* has trivial intersection with $C_0(G^{(0)})$. For Hausdorff groupoids the entire singular ideal is always trivial. It is therefore unsurprising that simplicity and the ideal intersection property for non-Hausdorff groupoid C^*-algebras cannot be described by "naive" analogues of results that hold in the Hausdorff setting.

In the form described above, the singular ideal J first appeared in the papers [18, 27, 43]. Much of the research conducted on the singular ideal J centres around conditions that ensure its vanishing (see [18, 30, 55, 67]). An important problem is to look for characterisations for the simplicity of $C_r^*(G)$, and for this to hold it is evidently necessary that $J = \{0\}$. It has been known since [58] that minimal and effective Hausdorff étale groupoids have simple reduced C^*-algebra. However, due to the existence of J, these conditions are not sufficient to ensure simplicity in the non-Hausdorff setting. Observations in [18] revealed that they are sufficient if one assumes in addition that $J = \{0\}$, and in [43] it was shown that these conditions are always sufficient to ensure simplicity of the quotient $C_r^*(G)/J$. By the results of [41], there now exists an adequate characterisation for the simplicity of the quotient $C_r^*(G)/J$ for any étale groupoid (covered by countably many open bisections). This provided yet more motivation for studying the vanishing of J. A recent breakthrough in [38] uncovers a connection between the singular ideal and quasi-regular representations of isotropy groups. Indeed, the isotropy fibres of the singular ideal are always described by kernels of quasi-regular representations (see [38, Theorem 5.5]). This yields representation-theoretic characterisations for the vanishing of J and $J \cap \mathscr{C}_c(G)$ (see also [9, Theorem 4.2]). Therefore, by combining the results of [38] and [41], there now exists a characterisation for the simplicity of $C_r^*(G)$ for any étale groupoid that is covered by countably many open bisections. Note the following: it is still an open problem whether $J = \{0\}$ if and only if $J \cap \mathscr{C}_c(G) = \{0\}$.

Looking beyond simplicity of $C_r^*(G)$ and the vanishing of J, one is led to ask about the structure of J. Very little work has been done in this direction as the early papers constructed elements of J on an ad hoc basis. Regarding structure of J, it seems reasonable to begin with the following two broad questions. Firstly, how might one describe a typical element of $J \cap \mathscr{C}_c(G)$? Here, $\mathscr{C}_c(G)$ denotes Connes' algebra of "compactly supported" functions on a non-Hausdorff groupoid. In principle, elements of $\mathscr{C}_c(G)$ are easier to understand than general elements of $C_r^*(G)$, and therefore it is logical to study $J \cap \mathscr{C}_c(G)$ before studying all of J. The second, more concrete question, is the following: for which groupoids is $J \cap \mathscr{C}_c(G)$ dense in J? The first examples of groupoids for which $J \cap \mathscr{C}_c(G)$ is not dense in J were provided recently in [48]. Two example classes are given in [48]: the first contains ample groupoids with equal source and range map, and the second contains minimal ample groupoids. All examples are non-amenable. Indeed, it is still an open question whether $J \cap \mathscr{C}_c(G)$ is dense in J for all amenable étale groupoids.

These two questions regarding the singular ideal J are addressed in Chapter 4. The techniques used are inspired by the work of [38], and exploit the fact that the isotropy fibres of J can be described by kernels of quasi-regular representations (see Section 4.2).

The first question asks about the typical elements of $J \cap \mathscr{C}_c(G)$. Section 4.3 addresses this by providing explicit instructions for building functions in $J \cap \mathscr{C}_c(G)$. Functions constructed in this fashion will be called *elementary* (see Definition 4.3.3), and it is shown that they have dense linear span in all of $J \cap \mathscr{C}_c(G)$. In order to describe these elementary singular functions, some notation must be established. Given a unit $x \in G^{(0)}$, the set $\mathcal{X}(x)$ denotes an explicit family of subgroups in the isotropy group G_x^x (see Definition 4.2.3). For each $X \in \mathcal{X}(x)$, let $\lambda_{G_x^x/X} \colon \mathbb{C}[G_x^x] \to \mathcal{L}(\ell^2(G_x^x/X))$ denote the associated quasi-regular representation.

- Take an (extremely dangerous) unit $x \in G^{(0)}$ and an element $b \in \mathbb{C}[G_x^x]$ belonging to the kernels $\ker(\lambda_{G_x^x/X})$ for all $X \in \mathcal{X}(x)$. Let $\{g_1, \ldots, g_n\}$ denote the support of b.
- Take $h \in G_x$, and let $V_1, \ldots, V_n$ be open bisections such that $hg_i \in V_i$ for each i.
- For a sufficiently small open neighbourhood $W \subseteq \bigcap_{i=1}^n s(V_i)$ of x the function

  $$f^\psi := \sum_{i=1}^n b(g_i)(\psi \circ s|_{V_i}) \tag{$\star$}$$

  belongs to $J \cap \mathscr{C}_c(G)$ for any $\psi \in C_c(W)$. A function f^ψ of this form is called an *elementary* singular function.

Theorem C (See Theorem 4.3.5)**.** *Let G be an étale groupoid that is covered by countably many open bisections. The elementary singular functions have dense linear span in $J \cap \mathscr{C}_c(G)$ with respect to any C^*-norm.*

See Subsection 4.3.1 for a detailed description of elementary singular functions. This subsection also makes precise the (technical) condition that W must satisfy to ensure that the function f^ψ from ($\star$) belongs to J.

The second question asks when $J \cap \mathscr{C}_c(G)$ is dense in J. Section 4.5 makes progress towards answering this question by providing a characterisation for the density of $J \cap \mathscr{C}_c(G)$ in J for amenable and second-countable étale groupoids (see Theorem 4.5.1). This characterisation follows easily from Theorem B, and only relies on the representation theory of isotropy groups. In particular, $J \cap \mathscr{C}_c(G)$ is dense in J whenever all isotropy groups satisfy the *density property* (see Definition 4.5.3). Section 4.6 contains an in depth discussion of the density property, and it is shown to be satisfied by all finite-by-nilpotent groups.

Theorem D (See Corollary 4.6.14)**.** *Let G be an amenable and second-countable étale groupoid. For each extremely dangerous unit $x \in G^{(0)}$ suppose that one of the following holds.*

(I) The isotropy group G_x^x is finite-by-nilpotent.

(II) All subgroups $X \in \mathcal{X}(x)$ are finite.

(III) All subgroups $X \in \mathcal{X}(x)$ are co-finite in G_x^x.

Then $J \cap \mathscr{C}_c(G)$ is dense in J.

As mentioned earlier, it is an open question whether $J \cap \mathscr{C}_c(G)$ is dense in J for any amenable and second-countable étale groupoid. This question has a positive answer if and only if every amenable discrete group satisfies the density property (see Remark 4.5.6).

For a particularly nice class of groupoids, we can do better than Theorems C and D, and can provide an explicit formula for the singular ideal J. The amenability and second-countability assumptions of Theorem D can also be relaxed.

Theorem E (See Theorem 4.7.2)**.** *Let G be an étale groupoid that is covered by countably many open bisections.*

Assume that G has finite-by-nilpotent isotropy groups, that $\tilde{G}$ is inner-exact, and that $\tilde{G}_{\mathrm{sing}}$ is a discrete groupoid with a single orbit. Then $J \cap \mathscr{C}_c(G)$ is dense in J and

$$J \cong J_{G_x^x, \mathcal{X}(x)} \otimes \mathcal{K}_n$$

for any extremely dangerous unit $x \in G^{(0)}$.

The groupoid $\tilde{G}_{\mathrm{sing}}$ denotes the singular part of the Hausdorff cover (see Definition 3.1.8). The assumption that $\tilde{G}$ be inner-exact is only necessary to ensure that J embeds into $C_r^*(\tilde{G}_{\mathrm{sing}})$ (see Subsection 3.1.2). The assumption that $\tilde{G}_{\mathrm{sing}}$ is discrete is fairly strong, but is satisfied in many natural examples (see Section 4.8). Next, $J_{G_x^x, \mathcal{X}(x)}$ represents an explicit ideal in the isotropy group C^*-algebra $C_r^*(G_x^x)$ (see Definition 4.2.1 and also [38, Definition 5.1]). Lastly, $\mathcal{K}_n$ denotes the C^*-algebra of compact operators on a Hilbert space with Hilbert dimension n, where n is the cardinality of $\tilde{G}_{\text{sing}}^{(0)}$. The requirement that $\tilde{G}_{\text{sing}}$ has a single orbit is only included here to simplify notation. A more general version of this theorem is stated in Theorem 4.7.2.

The results above mainly consider C^*-algebras of groupoids, but for ample groupoids there are other algebras to consider. A class of algebras that have gained significant attention over the past decade are *Steinberg algebras* ([65]). These objects are purely algebraic in nature. Given an ample groupoid G, a Steinberg algebra RG is defined for any ring R and, for non-Hausdorff ample groupoids, there is a notion of singular ideal $J_R \trianglelefteq RG$ analogous to that in the C^*-setting. Section 4.4 investigates the typical element of these singular ideals J_R.

The story is similar to the C^*-setting: the singular ideals J_R have historically been an obstruction to understanding simplicity of Steinberg algebras. For an ample groupoid G and a field $\mathbb{K}$, it is shown in [66] that the quotient $\mathbb{K}G/J_{\mathbb{K}}$ is simple if and only if G is minimal and topologically free. Then, for $\mathbb{K} = \mathbb{C}$, the results of [38] yield a characterisation for the vanishing of $J_{\mathbb{C}}$. By combining these results, one obtains the following for any ample groupoid G that is covered by countably many open bisections: the Steinberg algebra $\mathbb{C}G$ is simple if and only if G is minimal, topologically free, and $\bigcap_{X \in \mathcal{X}(x)} \ker(\lambda_{G_x^x/X}) = \{0\}$ for every (extremely dangerous) unit $x \in G^{(0)}$. Here, $\lambda_{G_x^x/X} \colon \mathbb{C}[G_x^x] \to \mathcal{L}(\ell^2(G_x^x/X))$ denotes the quasi-regular representation associated with the group $X \subseteq G_x^x$. As noted in [38], an analogue can be stated for Steinberg algebras over any field (or indeed any ring, see [9, Theorem 4.2]).

Let us explain how the techniques of this thesis apply in the context of ample groupoids. For notational convenience, we primarily focus on the Steinberg algebra $\mathbb{C}G$, known as the *complex Steinberg algebra*. In analogy with Theorem C, Section 4.4 describes instructions for building elementary singular functions in $J_{\mathbb{C}}$, and these functions provide a *genuine* spanning set for $J_{\mathbb{C}}$. The instructions are as follows.

- Take an (extremely dangerous) unit $x \in G^{(0)}$ and an element $b \in \mathbb{C}[G_x^x]$ belonging to the kernels $\ker(\lambda_{G_x^x/X})$ for all $X \in \mathcal{X}(x)$. Let $\{g_1, \ldots, g_n\}$ denote the support of b.

- Take $h \in G_x$, and let $V_1, \ldots, V_n$ be open bisections such that $hg_i \in V_i$ for each i.

- For any sufficiently small compact open neighbourhood $W \subseteq \bigcap_{i=1}^n s(V_i)$ of x the function
$$f^W := \sum_{i=1}^n b(g_i) \mathbb{1}_{U_i} \tag{$\star\star$}$$

belongs to $J_{\mathbb{C}}$, where $U_i := V_i \cap s^{-1}(W)$. A function f^W of this form is called an (algebraic) *elementary* singular function.

Theorem F (See Theorem 4.4.2). *Let G be an ample groupoid that is covered by countably many open bisections. The (algebraic) elementary singular functions linearly span $J_{\mathbb{C}}$.*

See Subsection 4.4.1 for a detailed description of algebraic elementary singular functions in $J_{\mathbb{C}}$. This subsection also makes precise the (technical) condition that W must satisfy to ensure that the function f^W from ($\star\star$) belongs to J.

By definition, the complex Steinberg algebra $J_{\mathbb{C}}$ is a subalgebra of $J \cap \mathscr{C}_c(G)$. The constructions of elementary singular functions in $J \cap \mathscr{C}_c(G)$ and $J_{\mathbb{C}}$ are noticeably similar. The density result of Theorem C yields the following.

Theorem G (See Theorem 4.4.4). *Let G be any ample groupoid. The algebraic singular ideal $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to any C^*-norm.*

This discussion of the singular ideal is closed by addressing a particularly important and well-studied class of non-Hausdorff ample groupoids – those associated with self-similar group actions. The C^*-algebra associated with a self-similar group action was introduced by Nekrashevych in [53]. This C^*-algebra is (canonically) isomorphic to $C^*(G)$, where G is the groupoid of germs associated with the action of the self-similar group (see [53, Theorem 5.1]). These groupoids are ample, minimal, and effective, and therefore the quotient $C^*_r(G)/J$ is simple. In recent years, characterisations have been obtained for the vanishing of $J_{\mathbb{C}}$ and J for groupoids associated with *contracting* self-similar groups (see also [72]). In [67] an algorithm is provided capable of deciding when $J_{\mathbb{C}} = \{0\}$ for such groupoids. In [30] a simplification of this algorithm is given, and it is shown that $J = \{0\}$ if and only if $J_{\mathbb{C}} = 0$. This last observation can be improved by combining Theorems D(II) and G.

Corollary H (See Corollary 4.6.15). *Let G be the groupoid associated with a contracting self-similar group. Then $J_{\mathbb{C}}$ is dense in J.*

The structure of this thesis is as follows. Chapter 1 introduces the basic definitions and notation associated with groupoids and their algebras. This includes some of the (modern) terminology relevant exclusively to non-Hausdorff groupoids. The later parts of Chapter 1 include background on measured groupoids, continuous systems of measures on locally compact groupoids, and isotropy fibres of ideals for étale groupoids.

Chapter 2 is focused on groupoid amenability. This chapter works in the setting of general locally compact groupoids. It opens with a discussion of amenability in the context of Hausdorff groupoids, stating well-known characterisations of amenability from [3]. These results are then contrasted with the non-Hausdorff setting, where most "continuous" characterisations of amenability fail. A simple example is provided to demonstrate this failure. Chapter 2 closes with the proof of the "folklore" result that for any (non-Hausdorff) amenable and second-countable locally compact groupoid G, the canonical surjection $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism, and $C_r^*(G)$ is nuclear.

Chapters 3 and 4 only consider étale groupoids. The main object of study in Chapter 3 is the Hausdorff cover $\tilde{G}$ of a non-Hausdorff groupoid G. The chapter opens with a thorough description of the cover $\tilde{G}$ and its properties, as well as some of the basic structure it inherits from G. The later subsections of this chapter give two applications whereby the Hausdorff cover is used to prove Theorems A and B.

Chapter 4 is focused solely on the singular ideal J. It includes all of the results from Theorem C to Corollary H stated above.

The thesis contains a recurring group of examples. First introduced in Section 1.6, the examples will reappear at the end of each of the following chapters. The first set of examples (from Subsection 1.6.1) are very simple, yet still demonstrate some of the different topological phenomena possible in non-Hausdorff groupoids. Each of these examples is chosen to have a specific set of properties, and each is the "simplest possible" example of a groupoid with these properties. The last example from Subsection 1.6.1 is more complicated (see Example 1.6.5). It is designed so that its full and reduced C^*-algebras are canonically isomorphic, but the full and reduced C^*-algebras of the Hausdorff cover are not. The other examples introduced in Section 1.6 come from dynamics. They include the groupoids of germs associated with actions of Bieri-Strebel groups, as well as groupoids associated with self-similar group actions.

This thesis is broadly based on results that appear in the papers [9] and [31]. The first, [9], is an investigation into the Hausdorff cover, and contains the contents of Sections 3.1 and 3.2, as well as Lemma 4.1.3. The second paper, [31], studies the singular ideal. Its main results appear in Section 3.3 and Sections 4.3 to 4.6. The techniques of these two papers are further developed in Section 4.7 to give a formula for computing the singular ideal in certain cases.

Chapter 1

Preliminaries

This preliminary chapter introduces the necessary background material for the thesis. Section 1.1 covers the basic definitions surrounding locally compact groupoids, and Section 1.3 introduces the C^*-algebras associated with groupoids. Section 1.2 introduces continuous systems of measures – these will be important for the discussion of amenability in Chapter 2. Section 1.4 covers the background on measured groupoids necessary for Section 2.4. Section 1.5 recalls the definitions of isotropy fibres of ideals – these will be key during Chapter 4. Lastly, Section 1.6 gives some examples of non-Hausdorff groupoids.

§ 1.1 | Étale and locally compact groupoids

This section begins by establishing basic definitions and notation. Later, Subsection 1.1.2 will introduce terminology and properties relevant to non-Hausdorff groupoids.

§ 1.1.1 | Definitions

Definition 1.1.1. A *groupoid* is a set G equipped with a map $(g, h) \mapsto gh$ defined on a set $G^{(2)} \subseteq G \times G$, and a map $g \mapsto g^{-1}$ defined on the entirety of G, satisfying the following conditions for all $g, h, k \in G$.

(i) If $(g, h), (h, k) \in G^{(2)}$, then $(gh, k), (g, hk) \in G^{(2)}$ and $(gh)k = g(hk)$.

(ii) $(g, g^{-1}), (g^{-1}, g) \in G^{(2)}$.

(iii) If $(g, h) \in G^{(2)}$ then $g^{-1}(gh) = h$ and $(gh)h^{-1} = g$.

The map $(g, h) \mapsto gh$ is known as the *product map*, and the map $g \mapsto g^{-1}$ as the *inverse map*. A pair $(g, h) \in G \times G$ is called *composable* if it belongs to $G^{(2)}$.

Let G be a groupoid.

- The *unit space* is the set $G^{(0)} := \{gg^{-1} : g \in G\}$. Its elements are called *units.*
- The *range* map $r : G \to G^{(0)}$ is defined by $r(g) := gg^{-1}$, and *source* map $s : G \to G^{(0)}$ by $s(g) := g^{-1}g$.

Definition 1.1.2. Let G be a groupoid and let $x \in G^{(0)}$ be a unit.

- The *range fibre* and *source fibre* at x are defined by $G^x := r^{-1}(\{x\}) \subseteq G$ and $G_x := s^{-1}(\{x\}) \subseteq G$ respectively.
- The *isotropy group* at x is $G_x^x := G^x \cap G_x$.
- The *orbit* of x is defined to be $G(x) := \{y \in G^{(0)} : \exists g \in G \; s(g) = x, r(g) = y\}$.

A subset $A \subseteq G^{(0)}$ is said to be *invariant* if it is a union of orbits. Given an invariant subset $A \subseteq G^{(0)}$, let A/G denote the quotient of A by orbits, and define $G|_A := s^{-1}(A)$. A subgroupoid $H \subseteq G$ is *invariant* if $H = G|_A$ for some invariant subset $A \subseteq G^{(0)}$.

Definition 1.1.3. A *locally compact groupoid* is a groupoid G equipped with a topology satisfying the following conditions.

- The inverse $g \mapsto g^{-1}$ and product $(g, h) \mapsto gh$ maps are continuous.
- Every element of G has a compact Hausdorff neighbourhood.
- The unit space $G^{(0)}$ is Hausdorff.

Here, $G \times G$ is equipped with the product topology, and $G^{(2)} \subseteq G \times G$ and $G^{(0)} \subseteq G$ are equipped with their respective subspace topologies.

Note. Continuity of the range map r ensures that $G^{(0)}$ is also locally compact as a topological space. Range (and hence source) fibres are automatically Hausdorff. This is because the diagonal $\Delta G^x = \{(g, h) \in G^x \times G^x : \; (g, h^{-1}) \in G^{(2)}, gh^{-1} = x\}$ is closed in $G^x \times G^x$ for any $x \in G^{(0)}$.

Definition 1.1.4. A locally compact groupoid G is called *étale* if the range map $r : G \to G^{(0)}$ is a local homeomorphism. An étale groupoid is *ample* if $G^{(0)}$ is totally disconnected. A *bisection* is a subset $B \subseteq G$ on which both range and source maps are injective.

Note. If G is a locally compact groupoid with open range and source maps, then G is étale if and only if it has a basis of open bisections, and G is ample if and only if it has a basis of compact open bisections.

The following two properties of étale groupoids will appear occasionally but will not be central to the thesis.

- An étale groupoid G is *minimal* if every orbit is dense in $G^{(0)}$.
- An étale groupoid G is *effective* if $G^{(0)} = \operatorname{int}\left(\bigcup_{x\in G^{(0)}} G_x^x\right)$.

Definition 1.1.5. Let G be a locally compact groupoid.

For a function $f\colon G \to \mathbb{C}$, the *strict support* of f is defined to be

$$\operatorname{supp}^\circ(f) \coloneqq \{g \in G\colon f(g) \neq 0\}.$$

Given an open Hausdorff set $U \subseteq G$, we abuse notation and write $C_c(U)$ for the set of functions $f\colon G \to \mathbb{C}$ such that $\operatorname{supp}^\circ(f) \subseteq U$ and $f|_U \in C_c(U)$. Then, define

$$\mathscr{C}_c(G) \coloneqq \operatorname{span}\{f\colon G \to \mathbb{C} \;:\; f \in C_c(U) \text{ for some open Hausdorff set } \; U \subseteq G\}.$$

Define $\mathscr{C}_{c,r}(G)$ to be the set of functions $f\colon G \to \mathbb{C}$ such that the pointwise product $(\chi \circ r).f$ belongs to $\mathscr{C}_c(G)$ for all $\chi \in C_c(G^{(0)})$.

It is simple to check that $\mathscr{C}_c(G) \subseteq \mathscr{C}_{c,r}(G)$. Note that functions in $\mathscr{C}_c(G)$ need not be continuous on G.

Notation. Write $\mathscr{C}_c(G)^+$ for the set of functions in $\mathscr{C}_c(G)$ that are non-negative on all of G. Similar notation will be used for other functions spaces.

When G is Hausdorff, $C_c(G) = \mathscr{C}_c(G)$, and we define $C_{c,r}(G) \coloneqq \mathscr{C}_{c,r}(G)$. This notation is consistent with [3, page 19]. In this case, any element of $C_{c,r}(G)$ is a continuous function on G.

When G is an étale groupoid, the function space $\mathscr{C}_c(G)$ has a cleaner description. For any $f \in \mathscr{C}_c(G)$, there exist open bisections $U_1, \ldots, U_n$ and functions $f_i \in C_c(U_i)$ for each i such that $f = \sum_{i=1}^n f_i$ (see [42, Lemma 1.3], for instance).

Convention. Locally compact groupoids are not assumed to be Hausdorff or second-countable unless this is explicitly stated.

§ 1.1.2 | Basic properties of non-Hausdorff groupoids

This section records some terminology and facts that are relevant exclusively to non-Hausdorff groupoids. We begin with a well-known lemma that characterises when a locally compact groupoid is Hausdorff.

Lemma 1.1.6. *A locally compact groupoid G is Hausdorff if and only if $G^{(0)}$ is closed.*

Proof. Suppose that G is Hausdorff, and take a net of units $(x_\alpha) \subset G^{(0)}$ converging to some $g \in G$. By continuity of the source map, the net also converges to $s(g)$. However, since G is Hausdorff, limits are unique, so that $g = s(g) \in G^{(0)}$.

Conversely, assume that $G^{(0)}$ is closed. Then the diagonal set $\Delta G := \{(g, g) \in G^2 \colon g \in G\}$ is closed. This is because $X := \{(g, h) \in G^2 \colon s(g) = s(h)\}$ is closed in G^2, and $\Delta \subseteq X$ is the preimage of $G^{(0)}$ under the continuous map $(g, h) \mapsto gh^{-1}$. It follows that G is Hausdorff. □

This lemma motivates the following definition.

Definition 1.1.7 ([43, Definition 7.14])**.** Let G be a locally compact groupoid. A unit $x \in G^{(0)}$ is said to be *Hausdorff* if $\overline{G^{(0)}} \cap G^x_x = \{x\}$, and is *dangerous* otherwise. Let $C \subseteq G^{(0)}$ denote the set of Hausdorff units, and $D \subseteq G^{(0)}$ the set of dangerous units.

Definition 1.1.8 ([55])**.** Let G be an étale groupoid. A unit $x \in G^{(0)}$ is said to be *extremely dangerous* if there exists an open set $U \subseteq G^{(0)}$, isotropy group elements $g_1, \ldots, g_n \in G^x_x \setminus \{x\}$ and open bisections $U_1, \ldots, U_n$ with $g_i \in U_i$ for each i such that

$$U \setminus \bigcup_{i=1}^{n} U_i$$

contains x, but has empty interior. Let $D_0 \subseteq D$ denote the set of extremely dangerous units.

The term "dangerous point" first appeared in [43], and describes the units in $G^{(0)}$ where "Hausdorffness" fails. The term "extremely dangerous" was coined in [55] in an attempt to capture further pathological behaviour relating to the singular ideal (see Chapter 4). For étale groupoids that are covered by countably many open bisections, we will see an alternative description of extremely dangerous points using the Hausdorff cover (see Lemma 3.1.9). This alternative characterisation readily yields a third (first noted in [38, Corollary 5.6]) that will be useful in the analysis of the singular ideal in Chapter 4 (see Definition 4.2.3).

Remark 1.1.9. For groupoids that are not étale, the definition of an extremely dangerous unit should be modified: each open bisection U_i should be replaced by a compact Hausdorff neighbourhood of g_i. With this modification, the alternative characterisation in Lemma 3.1.9 still holds (at least for groupoids that are covered by countably many open Hausdorff sets). We do not discuss this further.

We record a series of lemmas about dangerous points. These lemmas will be used implicitly throughout, and will play a particularly important role in Subsection 3.1.2.

Lemma 1.1.10. *Let G be a locally compact groupoid with open range and source maps, and let $C \subseteq G^{(0)}$ denote the set of Hausdorff units. Let $g \in G$. The following statements are equivalent.*

(i) $s(g) \in C$.

(ii) Any net converging to g has a unique limit.

In particular, the set of Hausdorff units C, and set of dangerous units D, are invariant subsets of $G^{(0)}$.

The set D_0 of extremely dangerous units is also an invariant subset of $G^{(0)}$. This fact can be seen by a direct computation, but we refer the reader to Proposition 3.1.9 instead.

Proof. Assume that $s(g) \in C$, and let (g_α) be a net in G converging to g. Suppose that (g_α) also converges to a point $h \in G$. Consider the net of units (x_α) defined as $x_\alpha := s(g_\alpha)$. By continuity of multiplication and the inverse map, the net (x_α) converges to both $g^{-1}g = s(g)$ and $h^{-1}g$. The unit $s(g)$ is Hausdorff, and therefore $h^{-1}g = s(g)$, so that $h = g$.

Conversely, assume that $s(g) \in D$. Write $x := s(g)$, and let $h \in \overline{G^{(0)}} \cap G_x^x \setminus \{x\}$. Take any open neighbourhoods U_g and U_h of g and h respectively. Then $s(U_g)$ is an open subset of $G^{(0)}$ containing x, and $U_h \cap G^{(0)}$ is an open subset of $G^{(0)}$ with x in its closure. Therefore, the intersection $s(U_g) \cap U_h$ is an open subset of $G^{(0)}$ with x in its closure. In particular, this intersection is non-empty. The open neighbourhoods U_g and U_h were arbitrary, so we can construct a net (g_α) in G converging to g, and such that $s(g_\alpha)$ converges to h. Now $g_\alpha = g_\alpha s(g_\alpha)$ for all α, so continuity of multiplication implies that this net also converges to gh. Therefore, the net (g_α) has at least two distinct limit points.

To see that C is an invariant subset of $G^{(0)}$, let $g \in G$ be such that $s(g) \in C$. By implication "(i) $\implies$ (ii)", any net converging to g has a unique limit point. Therefore, by continuity of the inverse map, any net converging to g^{-1} has a unique limit point. Implication "(ii) $\implies$ (i)" then ensures that the unit $s(g^{-1}) = r(g)$ belongs to C. $\square$

Functions in $\mathscr{C}_c(G)$ need not be continuous on G, however, the following lemma says that such functions are continuous at Hausdorff points. For étale groupoids, this lemma is proven in [43, Lemma 7.15].

Lemma 1.1.11. *Let G be a locally compact groupoid with open range and source maps, and let $f \in \mathscr{C}_c(G)$. Then f is continuous at any $g \in G|_C$.*

Proof. By linearity of continuity, it suffices to prove the lemma in the case that f belongs to $C_c(U)$ for some open Hausdorff set $U \subseteq G$. By definition, f is continuous on U, so it suffices to prove that f is continuous at each $g \in G|_C \setminus U$. Let $K \subseteq U$ be a compact set containing $\operatorname{supp}^\circ(f)$, and choose some net (g_α) in G that converges to g. By Lemma 1.1.10, g is the unique limit point of any subnet of (g_α), so it follows that the net (g_α) eventually belongs to $G \setminus K$. Therefore, $f(g_\alpha) \to 0 = f(g)$, as desired. □

The following lemma says that the set $D \subseteq G^{(0)}$ of dangerous units is small, provided that the groupoid G is not too large. For étale groupoids, this lemma is proven in [43, Lemma 7.15].

Lemma 1.1.12. *Let G be a locally compact groupoid that is covered by countably many open Hausdorff sets. Then D is a meagre subset of $G^{(0)}$, hence has empty interior.*

Recall that a subset of a topological space is meagre if it is a countable union of nowhere dense sets, and a set is nowhere dense if its closure has empty interior. The Baire-category theorem is the key ingredient of the following proof.

Proof. Let $\{U_n\}_{n\geq 1}$ be a countable family of open Hausdorff sets that covers G, and let $x \in D$ be a dangerous unit. Choose $g \in \overline{G^{(0)}} \cap G_x^x \setminus \{x\}$, and let n be such that $g \in U_n$. Since U_n is Hausdorff, it does not contain x. Therefore, the intersection $V_n := G^{(0)} \cap U_n$ is an open subset of $G^{(0)}$ that contains x in its boundary. That is, $x \in \partial V_n$, where $\partial V_n := \overline{V_n}^{G^{(0)}} \setminus V_n$. Note that ∂V_n is a closed subset of $G^{(0)}$ with empty interior. The dangerous unit x was arbitrary, and therefore $D \subseteq \bigcup_{n\geq 1} \partial V_n$. This implies that D is meagre. The fact that D has empty interior then follows from the Baire-category theorem, since $G^{(0)}$ is locally compact and Hausdorff. □

§ 1.2 | Continuous systems of measures

Locally compact groupoids can be equipped with extra measure-theoretic structure. In particular, it is desirable for a locally compact groupoid to admit a Haar system of measures. The existence of such a Haar system will be important for us to build groupoid C^*-algebras in the following section.

§ 1.2.1 | Definitions and Haar systems

Definition 1.2.1. Let G be a locally compact groupoid.

- A *continuous system of measures* on G is a collection $\{m^x\}_{x \in G^{(0)}}$ of positive (possibly zero) measures satisfying the following.
  (i) Each m^x is a Radon measure on G^x.
  (ii) For any $f \in \mathscr{C}_c(G)$, the map $x \mapsto m^x(f)$ is continuous on $G^{(0)}$, where

$$m^x(f) \coloneqq \int_{G^x} f(g)\ \mathrm{d}m^x(g). \tag{1.2.1}$$

  If in addition each m^x is a probability measure on G^x, then $\{m^x\}_{x \in G^{(0)}}$ is called a *continuous system of probability measures* on G.

- A *Haar system* on G is a continuous system of non-zero measures $\{\lambda^x\}_{x \in G^{(0)}}$ satisfying the invariance property

$$\lambda^{s(g)}(A) = \lambda^{r(g)}(g \cdot A)$$

  for any $g \in G$ and Borel $A \subseteq G^{s(g)}$. Here $g \cdot A \coloneqq \{gh \colon h \in A\}$.

By a Radon measure, we mean a Borel measure that is finite on compacts, inner regular on open sets, and outer regular on all Borel sets. Radon measures are not assumed to be σ-finite.

Remark 1.2.2. If $\{\lambda^x\}_{x \in G^{(0)}}$ is a Haar system, then $\lambda^x(U \cap G^x) > 0$ for any open set $U \subseteq G$ that intersects G^x. It follows that the existence of a Haar system ensures the range and source maps r and s are open maps.

Notation. Given a continuous system of measures $\{m^x\}_{x \in G^{(0)}}$ and a function f on G (not necessarily in $\mathscr{C}_c(G)$), we write $m^x(f) \coloneqq \int_{G^x} f(g)\ \mathrm{d}m^x(g)$ whenever this integral is defined. If $m^x(f)$ is defined for all $x \in G^{(0)}$, we let $m(f)$ denote the function $x \mapsto m^x(f)$. Note that $m(f)$ is a continuous function on $G^{(0)}$ for any $f \in \mathscr{C}_{c,r}(G)$.

If f is a non-negative Borel function on G, we let $m(f)$ denote the function $x \mapsto m^x(f)$ that takes values in the extended non-negative real numbers.

Notation. An étale groupoid G will always be equipped with the Haar system of counting measures. Precisely, we have $\lambda^x(A) = |A|$ for each $A \subseteq G^x$. It is elementary to check that this is indeed a Haar system on any étale groupoid.

Definition 1.2.3. Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. The function space $\mathscr{C}_c(G)$ can be equipped with the following involution and convolution product.

$$f_1 * f_2(g) \coloneqq \int_{G^{r(g)}} f_1(h) f_2(h^{-1}g)\ \mathrm{d}\lambda^{r(g)}(h) \qquad f^*(g) \coloneqq \overline{f(g^{-1})}.$$

It is clear that the involution is well-defined on $\mathscr{C}_c(G)$. For a proof that the convolution product is well-defined, see [42], for example.

Remark 1.2.4. It follows that (finite) products of open sets are open in G. We take this opportunity to point out that finite products of compact sets are compact in G. This is clear as $G^{(2)}$ is closed in $G \times G$, and hence $(K \times L) \cap G^{(2)}$ is compact whenever $K, L \subseteq G$ are compact.

The following is based on [3, Lemma 1.1.2], and says that any Haar system is "proper" in the sense of [3]. Note that if G is étale then one can take $\varphi := \mathbb{1}_{G^{(0)}}$, and paracompactness of the unit space is not needed.

Lemma 1.2.5. *Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$ and paracompact unit space. Then there exists $\varphi \in \mathscr{C}_{c,r}(G)^+$ with $\lambda(\varphi) \equiv 1$.*

Proof. For each unit $x \in G^{(0)}$ choose an open neighbourhood $V_x \subseteq G$, and let $f_x \in C_c(V_x)^+$ be such that $f_x(x) > 0$. Write $U_x := G^{(0)} \cap V_x$. The unit space $G^{(0)}$ is locally compact, Hausdorff, and paracompact, therefore there exists a partition of unity $\{\rho_x\}_{x \in G^{(0)}} \subseteq C_c(G^{(0)})$ for $G^{(0)}$ subordinate to the cover open cover $\{U_x\}_{x \in G^{(0)}}$. Define $\psi, \varphi \colon G \to \mathbb{C}$ by

$$\psi(g) := \sum_x \rho_x \circ r(g).f_x(g) \qquad \varphi(g) := \frac{\psi(g)}{\int_{G^{r(g)}} \psi \, \mathrm{d}\lambda^{r(g)}}$$

respectively. By local finiteness of the partition of unity, we have $\psi \in \mathscr{C}_{c,r}(G)^+$. Since $\operatorname{supp}^\circ(\psi)$ is open and intersects every range fibre, Remark 1.2.2 ensures that the map $x \mapsto \int_{G^x} \psi \, \mathrm{d}\lambda^x$ is strictly positive and continuous on $G^{(0)}$. It follows that $\varphi \in \mathscr{C}_{c,r}(G)^+$. It is clear that $\lambda(\varphi) \equiv 1$. $\square$

The following lemma shows that continuous systems of measures are also "Borel", provided that G is not too large. The lemma can fail if G is not covered by countably many open Hausdorff sets, even for étale groupoids.

Lemma 1.2.6. *Let G be a locally compact groupoid and $\{m^x\}_{x \in G^{(0)}}$ a continuous system of measures on G.*

Assume that G is covered by countably many open Hausdorff sets. For any non-negative Borel function f on G, the function $m(f)$ is Borel on $G^{(0)}$.

Proof. Let us first prove the lemma in the case that $f = \mathbb{1}_U$ for some open Hausdorff set $U \subseteq G$. Given any compact set $K \subseteq U$, there exists a function $f_K \in C_c(U)$ with $0 \leq f_K \leq 1$ and such that $f_K \equiv 1$ on K. Then $m(f_K)$ is a continuous function on $G^{(0)}$ satisfying $0 \leq m(f_K) \leq m(\mathbb{1}_U)$ pointwise. Since the measures m^x are Radon measures (hence inner regular on open sets), it is clear that $m(\mathbb{1}_U)$ is equal to the pointwise supremum $\sup_K m(f_K)$, where the supremum is taken over all compact subsets of U. In particular, $m(\mathbb{1}_U)$ is a pointwise supremum of continuous functions and is therefore lower-semicontinuous, hence Borel.

The remainder of the proof follows standard techniques. Since G is covered by countably many open Hausdorff sets, the Borel σ-algebra on G is generated by the open Hausdorff sets. The set of open Hausdorff sets is closed under finite intersections, therefore, a simple application of the π-λ lemma shows that $m(\mathbb{1}_B)$ is a Borel function on $G^{(0)}$ for any Borel $B \subseteq G$. By linearity, $m(\varphi)$ is Borel on G whenever φ is *simple*, in the sense that $\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{B_i}$ for some $\alpha_i \geq 0$ and $B_i \subseteq G$ Borel. Finally, any non-negative Borel function f on G is a pointwise limit of an increasing sequence of simple functions. Therefore, $m(f)$ is a pointwise limit of an increasing sequence of Borel functions, and is thus Borel. □

§ 1.2.2 | Continuous systems of measures on Hausdorff groupoids

In Section 2.1, we will need the following two results concerning continuous systems of measures on Hausdorff groupoids. Neither result holds for non-Hausdorff groupoids. The first is based on [3, Lemma 2.2.3].

Lemma 1.2.7. *Let G be a Hausdorff locally compact groupoid, and let $\{m^x\}_{x\in G^{(0)}}$ be a continuous system of measures satisfying $m^x(G^x) > \tau$ for all x in some compact set $L \subseteq G^{(0)}$. There exists a compact set $M \subseteq G$ such that*

$$m^x(M \cap G^x) > \tau$$

for all $x \in L$.

Proof. For each $x \in L$, let $M_x \subseteq G^x$ be compact such that

$$m^x(M_x) > \tau.$$

Take $\chi_x \in C_c(G)$ such that $0 \leq \chi_x \leq 1$, and $\chi_x \equiv 1$ on M_x. Then there exists an open set $U_x \subseteq G^{(0)}$ such that

$$\int_{G^y} \chi_x \, \mathrm{d}m^y > \tau$$

for all $y \in U_x$. Let $x_1, \ldots, x_n \in L$ be such that $U_{x_1}, \ldots, U_{x_n}$ cover L, and let $\rho_1, \ldots, \rho_n \in C_c(G^{(0)})$ be a partition of unity for L subordinate to this cover. Define

$$\chi(g) := \sum_{i=1}^n (\rho_i \circ r).\chi_{x_i}(g).$$

Then $M := \operatorname{supp}(\chi)$ has the desired properties. □

The next result implies that the "convolution" of a continuous function with a continuous system of measures is a continuous function. It is based on [71, Proposition 3.3]. Following the notation of [3], write $G \rtimes G = \{(k,g) \colon r(g) = r(k)\}$.

Proposition 1.2.8. *Let $\{m^x\}_{x \in G^{(0)}}$ be a continuous system of measures on a Hausdorff locally compact groupoid G.*

Let $f \in C_c(G \times G)$, and define the function $\rho(f) \colon G \to \mathbb{C}$ by

$$\rho(f)(g) \coloneqq \int_{G^{r(g)}} f(k,g) \, \mathrm{d}m^{r(g)}(k)$$

for $g \in G$. Then $\rho(f)$ is continuous on G. In fact, it suffices for f to be a continuous function on $G \rtimes G$ of the form $f(k,g) = \psi(k^{-1}g)$ for some $\psi \in C_c(G)$.

Proof. Let $\mathscr{A}$ denote the vector space of functions $f \in C_c(G \times G)$ for which the function $\overline{\rho}(f) \colon G \times G^{(0)} \to \mathbb{C}$ given by

$$\overline{\rho}(f)(g,x) \coloneqq \int_{G^x} f(k,g) \, \mathrm{d}m^x(k) \tag{1.2.2}$$

is continuous on $G \times G^{(0)}$. We claim that $\mathscr{A} = C_c(G \times G)$. Suppose $f \in C_c(G \times G)$ decomposes as $f(k,g) = \varphi_1(k)\varphi_2(g)$ for all $g,k \in G$, where $\varphi_1, \varphi_2 \in C_c(G)$. Then

$$\overline{\rho}(f)(g,x) = \varphi_2(g) \int_{G^x} \varphi_1(k) \, \mathrm{d}m^x(k)$$

and hence $\overline{\rho}(f) \in C_c(G \times G^{(0)})$ by continuity of the system of measures $\{m^x\}_{x \in G^{(0)}}$. So $f \in \mathscr{A}$ in this case. Let $\mathscr{B} \subseteq \mathscr{A}$ denote the vector space generated by functions that decompose in this way. Observe that $\mathscr{B}$ is a *-algebra that separates points and vanishes nowhere, so by the Stone-Weierstrass theorem $\mathscr{B}$ is dense in $C_0(G \times G)$.

Take arbitrary $f \in C_c(G \times G)$, and let (f_n) be a sequence in $\mathscr{B}$ converging uniformly to f. Let $\chi_1 \in C_c(G)$ be such that $0 \leq \chi_1 \leq 1$ and $\chi \equiv 1$ on a compact set $K \subseteq G$ satisfying $\operatorname{supp}(f) \subseteq K \times G$. Define the functions $\tilde{f}_n \in C_c(G \times G)$ by $\tilde{f}_n(k,g) \coloneqq f_n(k,g)\chi_1(k)$ for $k,g \in G$. Then $(\tilde{f}_n)$ is a sequence in $\mathscr{B}$ converging uniformly to f and satisfying $\operatorname{supp}(\tilde{f}_n) \subseteq L \times G$ for all n, where $L \coloneqq \operatorname{supp}(\chi_1)$. So

$$\begin{aligned}\left|\overline{\rho}(f)(g,x) - \overline{\rho}\left(\tilde{f}_n\right)(g,x)\right| &\leq \int_{G^x \cap L} \left|f(k,g) - \tilde{f}_n(k,g)\right| \, \mathrm{d}m^x(k) \\ &\leq m^x(G^x \cap L) \left\|f - \tilde{f}_n\right\|_\infty\end{aligned}$$

for any $(g,x) \in G \times G^{(0)}$. The values $m^x(G^x \cap L)$ are uniformly bounded over $x \in G^{(0)}$, and therefore $\overline{\rho}\left(\tilde{f}_n\right)$ converges to $\overline{\rho}(f)$ uniformly, showing that $f \in \mathscr{A}$. This proves the claim, and hence the first part of the proposition.

For the second part of the proposition, take $\psi \in C_c(G)$, and define $f(k,g) \coloneqq \psi(k^{-1}g)$ for all $(k,g) \in G \rtimes G$. We must show that $\rho(f)$ is continuous on G. To prove this, it suffices to fix a precompact open set $U \subseteq G$ and prove that $\rho(f)$ is continuous on U (since G is covered by its precompact open sets). Let $\chi_2 \in C_c(G)$ be such that $\chi_2 \equiv 1$ on U and define $\tilde{f}(k,g) := \chi_2(g)\psi(k^{-1}g)$ for $(k,g) \in G \rtimes G$. By Remark 1.2.4 the function $\tilde{f}$ is continuous and compactly supported on the closed set $G \rtimes G$, so by the Tietze extension theorem there exists an extension $\tilde{f} \in C_c(G \times G)$. By the first part of the proposition the function $\overline{\rho}(\tilde{f})$ is continuous on $G \times G^{(0)}$. It follows that $\rho(f)$ is continuous on U because $\rho(f)(g) = \overline{\rho}(\tilde{f})\big(g, r(g)\big)$ for all $g \in U$. □

Remark 1.2.9. The action of a Hausdorff locally compact groupoid G on itself by left multiplication gives rise to another Hausdorff locally compact groupoid: the transformation groupoid $G \rtimes G$. Following the notation of [3], write $G \rtimes G = \{(k,g) \colon r(g) = r(k)\}$. The unit space of $G \rtimes G$ can be identified with G, and the source and range maps as $\overline{r}, \overline{s} \colon G \rtimes G \to G$ are given by $\overline{r}(k,g) = k$ and $\overline{s}(k,g) = g^{-1}k$. Given a continuous system of measures $\{m^x\}_{x \in G^{(0)}}$ on G and an element $k \in G$, define the measure $\overline{m}^k$ on the range fibre $(G \rtimes G)^k$ by $\overline{m}^k(A) := m^{r(k)}\Big(\{g \in G^{r(k)} \colon (g,k) \in A\}\Big)$. Continuity of the function $\overline{\rho}$ from (1.2.2) shows that $\{\overline{m}^k\}_{k \in G}$ is a continuous system of measures over the range fibres of $G \rtimes G$.

§ 1.3 | C^*-algebras and Steinberg algebras

This section recalls the definitions of the full and reduced C^*-algebras associated with a locally compact groupoid. For ample groupoids, we will recall the definitions of the associated Steinberg algebras.

§ 1.3.1 | Groupoid C^*-algebras

In order to define the full C^*-algebra of a locally compact groupoid G, the following definition will be needed.

Definition 1.3.1. Let G be a locally compact groupoid. Given $f \in \mathscr{C}_c(G)$, a net $(f_\alpha) \subset \mathscr{C}_c(G)$ is said to converge to f in the *inductive limit topology* if $f_\alpha \to f$ uniformly, and there exists a compact set $K \subseteq G$ such that $\operatorname{supp}^\circ(f_\alpha) \subseteq K$ eventually.

The following norm will be useful later on.

Definition 1.3.2 ([58])**.** Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. The norm $\|\cdot\|_I$ is defined as

$$\|f\|_I := \max\left\{ \sup_{x \in G^{(0)}} \int_{G^x} |f(g)| \, \mathrm{d}\lambda^x(g), \quad \sup_{x \in G^{(0)}} \int_{G^x} \left|f(g^{-1})\right| \, \mathrm{d}\lambda^x(g) \right\}.$$

for Borel functions $f \colon G \to \mathbb{C}$.

It is clear that convergence in the inductive limit topology implies convergence with respect to the norm $\|\cdot\|_I$ (see [57, Proposition 2.2.2] for example). Note also that $\|f^*\|_I = \|f\|_I$ and $\|f_1 * f_2\|_I \leq \|f_1\|_I \|f_2\|_I$ for any $f, f_1, f_2 \in \mathscr{C}_c(G)$.

A version of the norm $\|\cdot\|_I$ first appeared in [35] for measured groupoids. The definitions differ in that here we do not include the modular function Δ (see Definition 1.4.2). In Section 1.4 we will see a version of the norm $\|\cdot\|_I$ for measured groupoids. There the norm will depend on a measure (class) μ on $G^{(0)}$, and thus we will denote it $\|\cdot\|_{I,\mu}$.

We now define the C^*-algebras associated with a locally compact groupoid. In the following, let λ_x denote the measure on the source fibre G_x given by $\lambda_x(A) \coloneqq \lambda^x(A^{-1})$ for Borel $A \subseteq G_x$.

Definition 1.3.3. Let G be a locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$.

- For each $x \in G^{(0)}$, let $\pi_x \colon \mathscr{C}_c(G) \to \mathcal{L}\Big(L^2(G_x, \lambda_x)\Big)$ be the *-representation given by convolution as follows:

$$\Big(\pi_x(f)\xi\Big)(g) \coloneqq \int_{G_x} f(gh^{-1})\xi(h)\ \mathrm{d}\lambda_x(h)$$

  for $f \in \mathscr{C}_c(G)$, $\xi \in L^2(G_x, \lambda_x)$, and $g \in G_x$. We call π_x the *left-regular representation* corresponding to x.

- The *reduced C^*-algebra* of G, denoted $C^*_r(G)$, is the completion of $\mathscr{C}_c(G)$ with respect to the C^*-norm

$$\|f\|_{C^*_r(G)} \coloneqq \sup_{x\in G^{(0)}} \|\pi_x(f)\|.$$

It is simple to check that $\|f\|_{C^*_r(G)} \leq \|f\|_I$ for all $f \in \mathscr{C}_c(G)$.

Definition 1.3.4. Let G be a locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$.

The *(full) C^*-algebra* of G, denoted $C^*(G)$, is the completion of $\mathscr{C}_c(G)$ with respect to the C^*-norm

$$\|f\|_{C^*(G)} \coloneqq \sup_{\pi} \|\pi(f)\|$$

for $f \in \mathscr{C}_c(G)$, where the supremum is taken over all *-representations $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ that are continuous with respect to the inductive limit topology.

Remark 1.3.5. In the previous definition, $\mathcal{L}(H)$ was equipped with the norm topology. This choice of topology was not important. For a second-countable locally compact groupoid G, it follows from [51, Theorem 7.8] that if a (non-degenerate) *-representation $\mathscr{C}_c(G) \to \mathcal{L}(H)$ is continuous with respect to the inductive limit and weak operator topologies, then it is automatically bounded with respect to the norm $\|\cdot\|_I$.

§ 1.3.2 | C^*-algebras of étale groupoids

If G is an étale groupoid, then any *-representation is automatically bounded with respect to the norm $\|\cdot\|_I$ (see [4, Corollary 5.8]), and is therefore continuous with respect to the inductive limit topology (see [57, Proposition 2.2.2]). Alternatively, it can be seen directly that any *-representation is continuous with respect to the inductive limit topology using the argument sketched below.

Let $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ be a *-representation. The unit space $G^{(0)}$ is an open Hausdorff subset, and therefore π restricts to a *-homomorphism on $C_c(G^{(0)})$. Given any precompact open set subset $U \subseteq G^{(0)}$, π restricts further to a *-homomorphism on the C^*-algebra $C_0(U)$, and is thus bounded with respect to the supremum norm. Next, given an open bisection $V \subseteq G$ and a function $f \in C_c(V)$, the convolution product $f^* * f$ is supported on $G^{(0)}$, so that $\|\pi(f)\|^2 \leq \|f^* * f\|_\infty = \|f\|_\infty^2$. Now, take a net $(f_\alpha) \subset \mathscr{C}_c(G)$ converging to $0 \in \mathscr{C}_c(G)$ with respect to the inductive limit topology. Let $K \subseteq G$ be a compact set such that $\operatorname{supp}^\circ(f_\alpha) \subseteq K$ eventually, and let $U_1, \ldots, U_n$ be open bisections whose union covers K. Whenever $\operatorname{supp}^\circ(f_\alpha) \subseteq K$, the proof of [9, Theorem 3.13(i)] ensures that there exist $f_\alpha^i \in C_c(U_i)$ for $i = 1, \ldots, n$ such that $f_\alpha = f_\alpha^1 + \cdots + f_\alpha^n$ and $\|f_\alpha^i\|_\infty \leq \|f_\alpha\|_\infty$ for all i. Since $\|f_\alpha\|_\infty \to 0$ it follows that $\pi(f_\alpha) \to 0$, and hence π is continuous with respect to the inductive limit topology.

We now discuss the reduced C^*-algebra of an étale groupoid G. There is a canonical way to view elements of $C_r^*(G)$ as bounded Borel functions on G (see [58, Proposition II.4.2]). Given $a \in C_r^*(G)$, we write

$$a(g) \coloneqq \langle \delta_g, \pi_{s(g)}(a)\delta_{s(g)} \rangle.$$

Here, the inner product is linear in the second variable, and δ_g and $\delta_{s(g)}$ denote elements of the orthonormal basis in $\ell^2(G_{s(g)})$. It is clear from this formula that a is bounded and that its supremum norm $\|a\|_\infty$ is dominated by $\|a\|_{C_r^*(G)}$. This formula will be used implicitly throughout the remainder of the thesis.

§ 1.3.3 | Steinberg algebras

In this subsection, we consider a class of algebras associated with ample groupoids. These algebras were first introduced in [65], and are now known as *Steinberg algebras*.

Definition 1.3.6 ([65])**.** Let G be an ample groupoid and R a ring. The *Steinberg* *algebra* RG over R is defined as

$$RG \coloneqq \operatorname{span}_R \{r_U \colon U \subseteq G \text{ a compact open bisection}, r \in R\}$$

where $r_U \colon G \to R$ is the function that takes constant value r on U, and is equal to 0 elsewhere.

The Steinberg algebra RG is equipped with the same convolution and involution operations as $\mathscr{C}_c(G)$. Concretely, we define

$$f_1 * f_2(g) \coloneqq \sum_{h \in G^{r(g)}} f_1(h) f_2(h^{-1}g) \qquad\qquad f^*(g) \coloneqq \overline{f(g^{-1})}$$

for $f, f_1, f_2 \in RG$ and $g \in G$.

When $R = \mathbb{K}$ is a field, then $\mathbb{K}G$ is an algebra over $\mathbb{K}$ in the usual sense of the word. More generally, if R is commutative and unital, then RG is an associative R-algebra. For an arbitrary ring R (not necessarily commutative or unital), it is perhaps not appropriate to refer to RG as an algebra. Nevertheless, it is itself a ring with the multiplication given by $*$, and is also clearly a left R-module.

§ 1.4 | Measured groupoids

In Section 2.4 we will need to make use of results that concern measured groupoids. The main result we will need is known as the Renault disintegration theorem (see Theorem 1.4.11). This theorem says that all *-representations of $\mathscr{C}_c(G)$ (continuous with respect to the inductive limit topology) can be understood as representations of measured groupoids. In this preliminary section we introduce all necessary background on measured groupoids and their operator algebras (see [60], [3, § 3 & 6] for more details).

§ 1.4.1 | Definitions and notation

Definition 1.4.1. A *Borel groupoid* is a groupoid G equipped with a Borel structure such that $G^{(2)}$ is a Borel subset of $G \times G$, $G^{(0)}$ is a Borel subset of G, and the multiplication $(g, h) \mapsto gh$ and inverse $g \mapsto g^{-1}$ maps are Borel.

A *Borel system of measures* on a Borel groupoid G is a collection $\{m^x\}_{x \in G^{(0)}}$ of positive (possibly zero) measures where each m^x is a σ-finite Borel measure on G^x, and such that the map $x \mapsto m^x(f)$ is Borel on $G^{(0)}$ for any non-negative Borel function f on G.

A *Borel Haar system* on a Borel groupoid G is a Borel system of non-zero measures

$\{\lambda^x\}_{x\in G^{(0)}}$ satisfying the invariance property

$$\lambda^{s(g)}(A) = \lambda^{r(g)}(g \cdot A) \tag{1.4.1}$$

for any $g \in G$ and Borel $A \subseteq G^{s(g)}$.

Let $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$ be a Borel Haar system on a Borel groupoid G. Given a σ-finite measure μ on $G^{(0)}$, let $\mu \circ \lambda$ be the Borel measure on G defined as

$$\mu \circ \lambda(A) := \int_{G^{(0)}} \int_{G^x} \mathbb{1}_A(g)\ \mathrm{d}\lambda^x(g)\ \mathrm{d}\mu(x) \tag{1.4.2}$$

for Borel subsets $A \subseteq G$. We say that μ is *quasi-invariant* with respect to λ if the measure $\mu\circ\lambda$ is equivalent to the measure $(\mu\circ\lambda)^{-1}$, where $(\mu\circ\lambda)^{-1}(A) := \mu\circ\lambda(A^{-1})$.

Definition 1.4.2. A *measured groupoid* is a triple (G, λ, μ) where G is a Borel groupoid with an analytic Borel structure, $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$ is a Borel Haar system on G, and μ is a σ-finite measure on $G^{(0)}$ that is quasi-invariant with respect to λ.

Given a measured groupoid (G, λ, μ) we let $\Delta\colon G \to \mathbb{R}_{>0}$ denote a Radon-Nikodym derivative of $\mu \circ \lambda$ with respect to $(\mu \circ \lambda)^{-1}$, and call Δ a *modular function*.

Any modular function Δ is a homomorphism almost everywhere (with respect to the canonical choice of measure on $G^{(2)}$).

Notation. Let (G, λ, μ) be a measured groupoid. For each $x \in G^{(0)}$, we let λ_x denote the measure on the source fibre G_x defined by $\lambda_x(A) := \lambda^x(A^{-1})$ for Borel $A \subseteq G_x$. For any integrable function $\varphi \in L^1(G, (\mu \circ \lambda)^{-1})$ note that

$$\int_G \varphi\ \mathrm{d}(\mu\circ\lambda)^{-1} = \int_{G^{(0)}} \int_{G_x} \varphi(g)\ \mathrm{d}\lambda_x(g)\mathrm{d}\mu(x).$$

Remark 1.4.3. In [47] (see also [34, Definition 2.3]) Mackey gave a definition of "measure groupoid" that is similar to the one above, but only depends on the measure class $[\mu\circ\lambda]$ of the measure from (1.4.2). Given a measured groupoid in our sense, it is simple to check that $(G, [\mu \circ \lambda])$ is a "measure groupoid" in the sense of Mackey. In [34] it is shown that the converse is essentially true. That is, up to null sets, a "measure groupoid" in the sense of Mackey admits the structure of a measured groupoid in our sense. Therefore, the two definitions essentially agree.

For locally compact groupoids, we saw in Section 1.3 that the function space $\mathscr{C}_c(G)$ can be equipped with the structure of a *-algebra. There is also a natural *-algebra of functions on measured groupoids. The following norm is the analogue of $\|\cdot\|_I$ (Definition 1.3.2) for measured groupoids. We follow [60] and [3].

Definition 1.4.4. Given a Borel function $\psi\colon G \to \mathbb{C}$ define

$$\|\psi\|_{I,\mu} := \max\left\{\operatorname*{ess\,sup}_{x\in G^{(0)}} \int_{G^x} |\psi(g)| \, \mathrm{d}\lambda^x(g), \quad \operatorname*{ess\,sup}_{x\in G^{(0)}} \int_{G^x} \left|\psi(g^{-1})\right| \, \mathrm{d}\lambda^x(g)\right\}. \tag{1.4.3}$$

We let $I(G,\lambda,\mu)$ denote the set of Borel functions $\psi\colon G \to \mathbb{C}$ with $\|\psi\|_{I,\mu} < \infty$.

Note that the norm $\|\cdot\|_{I,\mu}$ does not include the modular function Δ, and therefore differs from the norm $\|\cdot\|_I$ in [35]. The space $I(G,\lambda,\mu)$ is equipped with the same involution and convolution product as $\mathscr{C}_c(G)$:

$$\psi_1 * \psi_2(g) := \int_{G^{r(g)}} \psi_1(h)\psi_2(h^{-1}g) \, \mathrm{d}\lambda^{r(g)}(h) \qquad \psi^*(g) := \overline{\psi(g^{-1})}.$$

With this structure, $I(G,\lambda,\mu)$ becomes a Banach *-algebra (see [60, § 2.1]).

Remark 1.4.5. Let us establish the relationship between locally compact groupoids and measured groupoids. Fix a *second-countable* locally compact groupoid G with Haar system $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$. Although G does not need to be Hausdorff, it is still covered by countably many open Hausdorff sets, and therefore its Borel structure is standard (hence analytic). By Lemma 1.2.6, the Haar system λ is also a Borel Haar system. Therefore, given any quasi-invariant σ-finite Borel measure μ on $G^{(0)}$, the triple (G,λ,μ) is a measured groupoid. If μ is finite on compact subsets of $G^{(0)}$, then any function $f \in \mathscr{C}_c(G)$ belongs to $I(G,\lambda,\mu)$ (μ is actually a Radon measure on $G^{(0)}$ in this case, see [29, Theorem 7.8]).

§ 1.4.2 | Unitary representations of measured groupoids

In order to define unitary representations of measured groupoids, we will need some background on Hilbert bundles (see [22] for more details). We follow the notation and conventions of [51, § 7].

Definition 1.4.6. Let X be a locally compact Hausdorff space, and let $\mathscr{H} = \{H(x)\}_{x\in X}$ be a collection of Hilbert spaces. Write $X * \mathscr{H} = \{(x,h)\colon h \in H(x)\}$, and define $\pi\colon X * \mathscr{H} \to X$ by $\pi(x,h) := x$.

We say that $X * \mathscr{H}$ is a *(Borel) Hilbert bundle* over X if it has an analytic Borel structure for which π is Borel, and satisfies the other regularity conditions spelled out in [51, Definition 7.2]. A *Borel section* is a Borel map $\varphi\colon X \to X * \mathscr{H}$ such that $\pi \circ \varphi = \mathrm{id}_X$. By a slight abuse of notation, we assume that $\varphi(x) \in H(x)$ for all $x \in X$.

Given a Borel measure μ on X, the Hilbert space $L^2(X * \mathscr{H}, \mu)$ is defined to be the quotient of the space $\{\varphi \text{ a Borel section}, x \mapsto \|\varphi(x)\|^2 \text{ integrable}\}$ where sections that agree μ-almost everywhere are identified. We equip $L^2(X * \mathscr{H}, \mu)$ with the obvious inner product:

$$\langle \varphi_1, \varphi_2 \rangle := \int_X \langle \varphi_1(x), \varphi_2(x) \rangle_{H(x)} \, \mathrm{d}\mu(x)$$

for $\varphi_1, \varphi_2 \in L^2(X * \mathscr{H}, \mu)$. Here, $\langle \cdot, \cdot \rangle_{H(x)}$ denotes the inner product on $H(x)$.

The set of *decomposable* operators on $L^2(X * \mathscr{H}, \mu)$, denoted $\mathcal{D}$, is the set of bounded linear operators T for which there exists a collection of operators $T_x \in \mathcal{L}(H(x))$ for $x \in X$ satisfying

$$(T\varphi)(x) = T_x(\varphi(x)) \tag{1.4.4}$$

for all $\varphi \in L^2(X * \mathscr{H}, \mu)$ and μ-almost all $x \in X$.

The set of *diagonalisable* operators on $L^2(G^{(0)} * \mathscr{H}, \mu)$, denoted $\mathcal{T}$, is the set of (decomposable) linear operators T for which there exists a function $f \in L^\infty(X, \mu)$ satisfying

$$(T\varphi)(x) = f(x)\varphi(x) \tag{1.4.5}$$

for all $\varphi \in L^2(X * \mathscr{H}, \mu)$ and μ-almost all $x \in X$.

Definition 1.4.7. Let (G, λ, μ) be a measured groupoid. A *unitary representation* of the measured groupoid is a pair $(G^{(0)} * \mathscr{H}, L)$ consisting of a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ over $G^{(0)}$ and an assignment $g \mapsto L_g$ satisfying the following.

- For each $g \in G$, L_g is a unitary from $H(s(g))$ to $H(r(g))$.
- For any Borel sections $\varphi_1, \varphi_2 \colon G^{(0)} \to G^{(0)} * \mathscr{H}$ the map

  $$g \mapsto \Big\langle L_g\Big(\varphi_1(s(g))\Big), \varphi_2\Big(r(g)\Big)\Big\rangle_{H(r(g))}$$

  is Borel on G.

Given a unitary representation $(G^{(0)} * \mathscr{H}, L)$, its associated *integrated form* is the *-representation $\mathbb{L}$ of $I(G, \lambda, \mu)$ on $L^2(G^{(0)} * \mathscr{H}, \mu)$ given by

$$\Big\langle \mathbb{L}(\psi)\varphi_1, \varphi_2 \Big\rangle = \int_G \psi(g) \Big\langle L_g\Big(\varphi_1(s(g))\Big), \varphi_2\Big(r(g)\Big)\Big\rangle_{H(r(g))} \Delta(g)^{-\frac{1}{2}} \, \mathrm{d}(\mu \circ \lambda)(g) \tag{1.4.6}$$

for $\psi \in I(G, \lambda, \mu)$ and $\varphi_1, \varphi_2 \in L^2(G^{(0)} * \mathscr{H}, \mu)$.

For any unitary representation, its integrated form $\mathbb{L}$ is bounded with respect to the norm $\|\cdot\|_{I,\mu}$. Note also that for μ-almost all $x \in G^{(0)}$ we have

$$\Big\langle \Big(\mathbb{L}(\psi)\varphi_1\Big)(x), \eta \Big\rangle = \int_{G^x} \psi(g) \Big\langle L_g\Big(\varphi_1(s(g))\Big), \eta \Big\rangle_{H(r(g))} \Delta(g)^{-\frac{1}{2}} \, \mathrm{d}\lambda^x(g) \tag{1.4.7}$$

for any $\eta \in H(x)$.

As was the case for locally compact groupoids, any measured groupoid has a left-regular representation. There is also a full C^*-algebra whose norm is given by taking the supremum over all unitary representations.

Definition 1.4.8. Let (G, λ, μ) be a measured groupoid.

- The *left regular representation* of (G, λ, μ) is the unitary representation $(G^{(0)} * \mathscr{H}, L)$ on the Hilbert bundle $\mathscr{H} = \{L^2(G^x, \lambda^x)\}_{x \in G^{(0)}}$ given by
$$(L_g \xi)(h) := \xi(g^{-1}h)$$
for $\xi \in L^2(G^{s(g)}, \lambda^{s(g)})$ and $h \in G^{r(g)}$. We let Reg_μ denote the associated integrated form on $L^2(G^{(0)} * \mathscr{H}, \mu)$.

- The *(full) C^*-algebra*, denoted $C^*(G, \lambda, \mu)$, is the completion of $I(G, \lambda, \mu)$ with respect to the C^*-norm
$$\|\psi\| = \sup_{(G^{(0)} * \mathscr{H}, L)} \|\mathbb{L}(\psi)\|$$
for $\psi \in I(G, \lambda, \mu)$, where the supremum is taken over all unitary representations of (G, λ, μ).

§ 1.4.3 | Unitary representations and locally compact groupoids

In the previous subsection, we saw how unitary representations of a measured groupoid (G, λ, μ) give *-representations of the *-algebra $I(G, \lambda, \mu)$ via the so-called integrated form (see Definition 1.4.7). As discussed in Remark 1.4.5, any second-countable locally compact groupoid G with Haar system $\lambda = \{\lambda^x\}_{x \in G^{(0)}}$ and quasi-invariant Radon measure μ on $G^{(0)}$ gives a measured groupoid (G, λ, μ), where G is equipped with its associated Borel structure. Any function $f \in \mathscr{C}_c(G)$ belongs to $I(G, \lambda, \mu)$, and therefore the integrated form $\mathbb{L}$ of any unitary representation of (G, λ, μ) restricts to a *-representation of $\mathscr{C}_c(G)$.

In this subsection, we see how the theory of unitary representations for measured groupoids can be used to study *-representations of locally compact groupoids. The main result is the Renault disintegration theorem (Theorem 1.4.11), which says that any *-representation of $\mathscr{C}_c(G)$ that continuous with respect to the inductive limit topology is unitarily equivalent to the integrated form of some unitary representation.

First, we prove a simple lemma saying that $\mathscr{C}_c(G)$ and $I(G, \lambda, \mu)$ generate the same von Neumann algebra.

Lemma 1.4.9. *Let G be a second-countable locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x \in G^{(0)}}$, and let μ be a Radon measure on $G^{(0)}$ that is quasi-invariant with respect to λ.*

Let $(G^{(0)} * \mathscr{H}, L)$ be a unitary representation of the measured groupoid (G, λ, μ) with integrated form $\mathbb{L}\colon I(G, \lambda, \mu) \to \mathcal{L}\Big(L^2(G^{(0)} * \mathscr{H}, \mu)\Big)$. Then $\mathbb{L}(\mathscr{C}_c(G))$ is dense in $\mathbb{L}(I(G, \lambda, \mu))$ with respect to the weak operator topology (WOT).

The proof is simple, but it is included anyway.

Proof. Let $\psi \in I(G,\lambda,\mu)$. First suppose that ψ is bounded, and also that $\operatorname{supp}^\circ(\psi)$ is contained in some open Hausdorff set $U \subseteq G$ with $\mu \circ \lambda(U) < \infty$. The space U is second-countable, locally compact, and Hausdorff, and therefore there exists a sequence $(f_n) \subset C_c(U)$ that is uniformly bounded in supremum norm and converges to ψ almost everywhere with respect to $\mu \circ \lambda$. A simple application of the dominated convergence theorem shows that $\mathbb{L}(f_n)$ converges to $\mathbb{L}(\psi)$ with respect to the WOT.

To deal with the general case, note that $\mu \circ \lambda$ is finite on compacts, and therefore G is covered by a countable union of open Hausdorff sets $U \subseteq G$ with $\mu \circ \lambda(U) < \infty$. So, to deal with an arbitrary element $\psi \in I(G,\lambda,\mu)$, choose a countable disjoint covering $G = \bigsqcup_{n=1}^{\infty} B_n$ by Borel sets B_n each contained in an open Hausdorff set U as above, and such that $\psi|_{B_n}$ is bounded. Writing $\psi_n := \psi|_{B_n}$ for each n, it is simple to check that $\mathbb{L}(\sum_{i=1}^{n} \psi_i)$ converges to $\mathbb{L}(\psi)$ with respect to the weak operator topology. $\square$

Let us comment on the connection between the left-regular representations of measured groupoids and locally compact groupoids (see Definitions 1.3.3 and 1.4.8). Given a measured groupoid (G,λ,μ) let $\mathscr{H} = \{L^2(G^x,\lambda^x)\}_{x\in G^{(0)}}$. The Hilbert space $L^2(G^{(0)} * \mathscr{H},\mu)$ is canonically isomorphic to $L^2(G,\mu\circ\lambda)$. It follows that the integrated form Reg_μ on $I(G,\lambda,\mu)$ is unitarily equivalent to the representation π_1 on $L^2(G,\mu\circ\lambda)$ defined as follows: for $\psi \in I(G,\lambda,\mu)$ and $\varphi \in L^2(G,\mu\circ\lambda)$, define

$$\Big(\pi_1(\psi)\varphi\Big)(g) := \int_{G^{r(g)}} \psi(h)\varphi(h^{-1}g)\ \Delta(h)^{-\frac{1}{2}}\ \mathrm{d}\lambda^{r(g)}(h).$$

Next, given $\varphi \in L^2(G,\mu\circ\lambda)$ define $\tilde{\varphi} \in L^2(G,(\mu\circ\lambda)^{-1})$ by $\tilde{\varphi}(g) := \varphi(g)\Delta(g)^{\frac{1}{2}}$. The mapping $\varphi \mapsto \tilde{\varphi}$ is an isomorphism. Therefore, the representation π_1 is unitarily equivalent to the representation π_2 on the Hilbert space $L^2(G,(\mu\circ\lambda)^{-1})$ defined as follows: for $\psi \in I(G,\lambda,\mu)$ and $\varphi \in L^2(G,(\mu\circ\lambda)^{-1})$, define

$$\Big(\pi_2(\psi)\varphi\Big)(g) := \int_{G_{s(g)}} \psi(gh^{-1})\varphi(h)\ \mathrm{d}\lambda_{s(g)}(h).$$

That is, π_2 implements an action of $I(G,\lambda,\mu)$ on $L^2(G,(\mu\circ\lambda)^{-1})$ by convolution. The following is now obvious.

Lemma 1.4.10. *Let G be a second-countable locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$, and let μ be a Radon measure on $G^{(0)}$ that is quasi-invariant with respect to λ. Then*

$$\left\| Reg_\mu(f) \right\| \leq \|f\|_{C_r^*(G)}\,.$$

for any function $f \in \mathscr{C}_c(G)$.

We now state the Renault disintegration theorem. The original result is due to Renault

[59, Proposition 4.2], however, the non-Hausdorff generalisation we state here is proven in [51]. This result will be key in proving Theorem 2.4.1.

Theorem 1.4.11 ([51, Theorem 7.8])**.** *Let G be a second-countable locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x\in G^{(0)}}$, and let $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ be any non-degenerate *-representation that is continuous with respect to the inductive limit topology.*

Then π is bounded with respect to the norm $\|\cdot\|_I$, and there exists a quasi-invariant Radon measure μ on $G^{(0)}$ and a unitary representation $(G^{(0)} * \mathscr{H}, L)$ of the measured groupoid (G, λ, μ) whose integrated form $\mathbb{L}$ is unitarily equivalent to π on $\mathscr{C}_c(G)$.

In [51] this result is stated and proved in a much more general setting. The version stated above will be sufficient for us.

§ 1.5 | Isotropy fibres of ideals

This short section discusses restriction maps and isotropy fibres of ideals on groupoid C^*-algebras (see [16] and [17] for further details). The contents of this section are only stated for étale groupoids, and will be particularly important in Chapter 4.

Let G be an étale groupoid. Given a unit $x \in G^{(0)}$, consider the restriction map $\eta_x\colon \mathscr{C}_c(G) \to \mathbb{C}[G_x^x]$, where $\mathbb{C}[G_x^x]$ denotes the group algebra of the isotropy group G_x^x (thought of as finitely supported functions $G_x^x \to \mathbb{C}$). The map η_x extends to completely positive contractions $\eta_x\colon C^*(G) \to C^*(G_x^x)$ and $\eta_{x,r}\colon C_r^*(G) \to C_r^*(G_x^x)$ (see [16, Lemma 1.2] for example). Both maps send closed ideals to closed ideals (see [17, Lemma 2.1]).

Definition 1.5.1 ([17, Definition 2.2])**.** Let $I \trianglelefteq C^*(G)$ be a closed ideal. Given $x \in G^{(0)}$, we write $I_x \coloneqq \eta_x(I)$ and call I_x the *isotropy fibre* of I at x.

Let $I \trianglelefteq C_r^*(G)$ be a closed ideal. Given a unit $x \in G^{(0)}$ for which G_x^x is amenable, we write $I_x \coloneqq \eta_{x,r}(I)$ and call I_x the *isotropy fibre* of I at x.

Remark 1.5.2. The amenability requirement in the second part of Definition 1.5.1 may seem a bit odd, but we include it to avoid a clash of terminology with [17] and [38]. If the isotropy group G_x^x is not amenable, then $C_r^*(G_x^x)$ may not be the “correct” codomain for the map $\eta_{x,r}$. One should instead replace $C_r^*(G_x^x)$ with an exotic C^*-algebra (see [16]).

Let us introduce some more notation. For the remainder of this section, we only consider full C^*-algebras. Given $g \in G_x^x$, denote by $\delta_g \in \mathbb{C}[G_x^x]$ the function that takes value 1 at g and zero elsewhere. Each $g \in G$ induces an isomorphism of group algebras $\mathrm{Ad}_g \colon \mathbb{C}\Big[G_{s(g)}^{s(g)}\Big] \to \mathbb{C}\Big[G_{r(g)}^{r(g)}\Big]$ given by $\delta_h \mapsto \delta_{ghg^{-1}}$, and this map extends to an isomorphism of C^*-algebras $\Psi_g \colon C^*\Big(G_{s(g)}^{s(g)}\Big) \to C^*\Big(G_{r(g)}^{r(g)}\Big)$. Let $(K_x)_{x\in G^{(0)}}$ be a family of ideals satisfying $K_x \trianglelefteq C^*(G_x^x)$ for each $x \in G^{(0)}$. We say that the family is *invariant* if $\Psi_g(K_{s(g)}) = K_{r(g)}$ for all $g \in G$. Given an ideal $I \trianglelefteq C^*(G)$, its isotropy fibres $(I_x)_{x\in G^{(0)}}$ form an invariant family of ideals. Going in the other direction, if $\mathcal{K} = (K_x)_{x\in G^{(0)}}$ is an invariant family of ideals, define

$$I(\mathcal{K}) \coloneqq \{a \in C^*(G) \colon \eta_x(a^*a) \in K_x \text{ for all } x \in G^{(0)}\}.$$

By [17, Corollary 2.9], $I(\mathcal{K})$ is an ideal in $C^*(G)$. The following lemma isolates an observation from [17], and will be useful later in Section 3.3. It states that $I(\mathcal{K}) \subseteq C^*(G)$ is the largest C^*-subalgebra such that $\eta_x(I(\mathcal{K})) \subseteq K_x$ for all $x \in G^{(0)}$.

Lemma 1.5.3. *Let G be an étale groupoid. Let $\mathcal{K} = (K_x)_{x\in G^{(0)}}$ be an invariant family of ideals, and let $A \subseteq C^*(G)$ be a C^*-subalgebra. Then $A \subseteq I(\mathcal{K})$ if and only if $\eta_x(A) \subseteq K_x$ for all $x \in G^{(0)}$.*

Proof. Let $x \in G^{(0)}$ and assume that A satisfies $A \subseteq I(\mathcal{K})$. By definition of $I(\mathcal{K})$, we have $\eta_x(a) \in K_x$ for all positive elements $a \in A^+$. Since A is spanned by its positive elements, it follows that $\eta_x(a) \in K_x$ for all $a \in A$. Conversely, assume that $\eta_x(A) \subseteq K_x$ for all $x \in G^{(0)}$. For any $a \in A$ we have $\eta_x(a^*a) \in K_x$ for all $x \in G^{(0)}$, and therefore $a \in I(\mathcal{K})$ as desired. □

§ 1.6 | Examples

This section introduces several examples of non-Hausdorff groupoids. These examples will be used to describe the various concepts appearing in the later chapters, as well as demonstrate the applicability of later results.

We introduce three classes of examples. The first contains groupoids that have equal source and range maps – that is, groupoids that have the structure of a bundle of groups fibred over the unit space. The second class contains the groupoids of germs associated with canonical actions of Thompson's groups F and T, as well as actions of Bieri-Strebel groups. The third contains groupoids associated with (contracting) self-similar group actions.

The first four bundle of group examples are very simple. They are included because despite their simplicity, all exhibit different, interesting phenomena with regards to the Hausdorff cover and singular ideals of Chapters 3 and 4.

§ 1.6.1 | Bundle of groups examples

The constructions of the first five examples are similar. Start with a locally compact Hausdorff space X and a non-trivial discrete group Γ. Equip $\Gamma \times X$ with the product topology and the following groupoid structure: inverses are defined by $(g,x)^{-1} = (g^{-1},x)$, and the pairs (g,x) and (h,y) are composable if and only if $x = y$, in which case the product is $(g,x)(h,x) = (gh,x)$. Note that the source and range maps are equal, and correspond to the projection $(g,x) \mapsto (e,x)$. In fact, $\Gamma \times X$ is a trivial bundle of groups. Given such a product $\Gamma \times X$, equip it with an equivalence relation $\sim$ satisfying the following conditions.

- The equivalence classes of $\sim$ are contained within the source/range fibres.
- Given any $x \in X$, there exists a normal subgroup $N_x \subseteq \Gamma$ such that the equivalence classes of $\sim$ in the fibre $\{(g,x) \colon g \in \Gamma\}$ correspond to the cosets of the normal subgroup N_x.
- The union $\bigcup_{x \in X}\{(g,x) \colon (g,x) \sim (e,x)\}$ is open in $\Gamma \times X$.

The quotient $G \coloneqq \Gamma \times X/\sim$ is an étale groupoid when equipped with the quotient topology and the induced groupoid structure (it is étale by the third point above). Elements of G are equivalence classes, denoted $[g,x] \coloneqq \{(h,x) \colon (h,x) \sim (g,x)\}$. We will identify the unit space $G^{(0)}$ with X via the homeomorphism $[e,x] \mapsto x$. In each example, the unit space $G^{(0)}$ is not closed, so that G is non-Hausdorff by Lemma 1.1.6.

The first example appeared in [42, Example 1.2]. It is possibly the simplest example of a non-Hausdorff groupoid.

Example 1.6.1. Let X be a locally compact Hausdorff space with a non-isolated point $x_0 \in X$. For example, one could take $X = \mathbb{N} \cup \{\infty\}$, the one-point compactification of the natural numbers, and $x_0 = \infty$. Let Γ be any non-trivial discrete group. Define the equivalence relation $\sim$ on $\Gamma \times X$ by $(g,x) \sim (h,x)$ if and only if $x \neq x_0$. Let $G \coloneqq \Gamma \times X/\sim$ so that G is an étale groupoid. The isotropy group $G_{x_0}^{x_0}$ is isomorphic to Γ, and all other isotropy groups of G are trivial. Abusing notation, we can write $G = X \setminus \{x_0\} \cup \Gamma$. The unit space contains a single dangerous point, namely x_0. In fact, $x_0 \in X$ is an extremely dangerous point.

Example 1.6.2. Let $X = [0,2]$ and let Γ be any non-trivial discrete group. Define the equivalence relation $\sim$ on $\Gamma \times [0,2]$ by $(g,x) \sim (h,x)$ if and only if $x \in [0,1)$. Let $G \coloneqq (\Gamma \times [0,2])/\sim$. The isotropy groups G_x^x are isomorphic to Γ when $x \in [1,2]$ and are trivial otherwise. The unit $1 \in X$ is the only dangerous point. It is not extremely dangerous.

The next example contains an extremely dangerous point, yet its singular ideal vanishes (see Example 4.8.3). It is arguably the simplest such example.

Example 1.6.3. Take three copies of the unit interval $I = [0,1]$ equipped with the usual topology, and label them A, B, C so that we have a disjoint union $I \times \{A\} \sqcup I \times \{B\} \sqcup I \times \{C\}$. Identify the three points $(0,A)$, $(0,B)$ and $(0,C)$ and let 0 denote this common point. Let X be the resulting space and write $X = \{0\} \cup \left\{t_g \colon t \in (0,1], g \in \{A,B,C\}\right\}$. Next, let $\Gamma = V_4$ be the Klein four-group, written $V_4 = \{e,A,B,C\}$. Equip $V_4 \times X$ with the following equivalence relation: for any distinct $g,h,k \in \{A,B,C\}$ and $t \in (0,1]$ declare that $(e,t_g) \sim (g,t_g) \nsim (h,t_g) \sim (k,t_g)$. Let $G \coloneqq (V_4 \times X)/\sim$. The isotropy group G_0^0 is isomorphic to V_4 and all other isotropy groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The unit $0 \in X$ is the only dangerous point, and is extremely dangerous.

Example 1.6.4. Let $X = [0,1]$ and let $\Gamma = V_4$ be the Klein four-group, written $V_4 = \{e,A,B,C\}$. Choose a sequence $x_n \in [0,1)$ converging to 1. Define the equivalence relation $\sim$ on $V_4 \times X$ as follows: $(e,x) \sim (A,x) \sim (B,x) \sim (C,x)$ if $x \in X \setminus \{1, x_1, x_2, \dots\}$, and $(e,x) \sim (A,x) \nsim (B,x) \sim (C,x)$ if $x \in \{x_1, x_2, \dots\}$. Let $G \coloneqq (V_4 \times X)/\sim$. The isotropy group G_1^1 is isomorphic to V_4, and the isotropy groups G_x^x are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $x \in \{x_1, x_2, \dots\}$. All other isotropy groups are trivial. The units $\{1, x_1, x_2, \dots\} \subseteq G^{(0)}$ are the only dangerous points, and are all extremely dangerous (in Example 3.4.4 we will see that the unit $1 \in X$ is somehow even more dangerous than the rest).

In each of the previous three examples, one could replace intervals with Cantor sets (or indeed other totally disconnected spaces with sufficiently many accumulation points). The resulting groupoids would then be ample.

The next example is more complicated. It combines elements of the groupoid from Example 1.6.3 with a construction from [37, § 2]. In Example 4.8.5 it is shown this groupoid has the property that the canonical surjection $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism but $C^*(\tilde{G}) \twoheadrightarrow C_r^*(\tilde{G})$ is not. Before describing this groupoid, let us first recall the "HLS groupoid" construction from [70].

Let $\mathbb{F}_2$ denote the free group on two generators. For each $n \geq 1$ define

$$K_n \coloneqq \bigcap \{\ker(\varphi) \colon \varphi \colon \mathbb{F}_2 \to \Gamma \text{ a group homomorphism, } |\Gamma| \leq n\} \tag{1.6.1}$$

as in [70, Lemma 2.8], and write K_∞ for the trivial subgroup of $\mathbb{F}_2$. Let $X = \mathbb{N} \cup \{\infty\}$ denote the one-point compactification of the natural numbers. Define the relation $\sim$ on $\mathbb{F}_2 \times X$ as follows: $(s,n) \sim (t,n)$ if and only if $sK_n = tK_n$. Define

$$G_{\mathrm{HLS}} \coloneqq (\mathbb{F}_2 \times X)/\sim\, . \tag{1.6.2}$$

Then G_{HLS} is isomorphic to a "HLS groupoid" as defined in [70, Definition 2.2]. The isotropy group $(G_{\mathrm{HLS}})_\infty^\infty$ is isomorphic to $\mathbb{F}_2$, and the isotropy groups $(G_{\mathrm{HLS}})_n^n$ are isomorphic to the finite quotients $\mathbb{F}_2/K_n$. By [70, Lemmas 2.7 and 2.8] the canonical surjection $C^*(G_{\mathrm{HLS}}) \twoheadrightarrow C^*_r(G_{\mathrm{HLS}})$ is an isomorphism.

The next example combines this HLS groupoid with Example 1.6.3. Before giving details, let us first describe the idea behind its construction. Take three copies of the groupoid G_{HLS} from above and "glue" them together at the isotropy group G^∞_∞. Take the product of this new groupoid with the Klein four-group. The resulting (Hausdorff) étale groupoid is equipped with an equivalence relation analogous to that in Example 1.6.3. The quotient is the (non-Hausdorff) étale groupoid we need.

Example 1.6.5. Take three copies of the compact space $Y := \mathbb{N} \cup \{\infty\}$, and label them A, B, C so that we have a disjoint union $Y \times \{A\} \sqcup Y \times \{B\} \sqcup Y \times \{C\}$. Identify the three points (∞, A), (∞, B) and (∞, C) and let ∞ denote this common point. Let X be the resulting space and write $X = \{\infty\} \cup \left\{n_g \colon n \in \mathbb{N}, g \in \{A, B, C\}\right\}$. Next, let $\Gamma = \mathbb{F}_2 \times V_4$ where $V_4 = \{e, A, B, C\}$ is the Klein four-group. Let $K_n \subseteq \mathbb{F}_2$ be as in (1.6.1) for each $n \in \mathbb{N}$. Equip $\Gamma \times X$ with the following equivalence relation: for any $s, t \in \mathbb{F}_2$, $n \in \mathbb{N}$, and $g \in V_4$ declare that $\big((s, g), n_g\big) \sim \big((t, g), n_g\big)$ if and only if $sK_n = tK_n$, and for any distinct $g, h, k \in \{A, B, C\}$, $s \in \mathbb{F}_2$, and $n \in \mathbb{N}$ declare that $\big((s, e), n_g\big) \sim \big((s, g), n_g\big) \nsim \big((s, h), n_g\big) \sim \big((s, k), n_g\big)$. Let $G := (\Gamma \times X)/\sim$. The isotropy group G^∞_∞ is isomorphic to Γ. Given $n \in \mathbb{N}$ and $g \in \{A, B, C\}$ the isotropy group $G^{n_g}_{n_g}$ is isomorphic to the finite group $(\mathbb{F}_2/K_n) \times \mathbb{Z}/2\mathbb{Z}$. The unit $\infty \in X$ is the only dangerous point, and is extremely dangerous.

In order to construct examples of non-Hausdorff groupoids that are not étale, one can simply take the product of a non-Hausdorff étale groupoid with a locally compact group. The product of two locally compact groupoids is always a locally compact groupoid when equipped with the product topology. Here is a "twisted" example that is not just the product of an étale groupoid with a group. Thanks is given to B. Kwaśniewski for introducing the author to the following example inspired by [21, § 3].

Example 1.6.6. Let $X_1 = [0, 1]$, $x_1 = 1 \in X_1$ and $X_2 = [1, 2]$, $x_2 = 1 \in X_2$. Write $\Gamma_1 = \{e_1, a_1\} \cong \mathbb{Z}/2\mathbb{Z}$ and $\Gamma_2 = \{e_2, a_2\} \cong \mathbb{Z}/2\mathbb{Z}$. Construct two copies of the groupoid from Example 1.6.1: $G_1 := \Gamma_1 \times X_1/\sim_1$ and $G_2 := \Gamma_2 \times X_2/\sim_2$. The groupoid G_1 looks like the interval $[0, 1]$ where $1 \in [0, 1]$ has been replaced by the group Γ_1. Write $G_1 = [0, 1) \cup \Gamma_1$. Similarly, write $G_2 = \Gamma_2 \cup (1, 2]$. Take the disjoint union $G_1 \sqcup G_2$ of these two groupoids, and let $H := (G_1 \sqcup G_2) \times S^1$ be the product with the unit circle S^1. Finally, define an equivalence relation $\sim$ on H as follows: $(e_1, t) \sim (e_2, t^{-1})$ and $(a_1, t) \sim (a_2, t^{-1})$ for all $t \in S^1$. Then the quotient $G := H/\sim$ is a non-Hausdorff locally compact groupoid that is not étale. The groupoid G is a bundle of groups fibred over the unit space $G^{(0)} \cong [0, 2]$. The isotropy group G^1_1 is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S^1$, and all other isotropy groups are isomorphic to S^1.

§ 1.6.2 | Groupoid of germs

The next class of examples comes from dynamics. Take a locally compact Hausdorff space X and let $\mathcal{P}$ be a family of partial homeomorphisms $f\colon U \to V$ between open subsets $U, V \subseteq X$ of X satisfying the following conditions.

- $X = \bigcup_{f\in\mathcal{P}} \operatorname{dom}(f)$.
- Whenever $f \in \mathcal{P}$, the inverse f^{-1} also belongs to $\mathcal{P}$.
- Whenever $f_1, f_2 \in \mathcal{P}$, the composition $f_1 \circ f_2|_{f_2^{-1}(\operatorname{dom}(f_1))}$ also belongs to $\mathcal{P}$.

We call $\mathcal{P}$ a *pseudogroup* on X (this definition of pseudogroup is not standard, compare with [50, Example 5.23] and [52, Definition 4.1.1] for example). Note that $\mathcal{P}$ is an inverse semigroup under composition. The space $\mathcal{P} \ltimes X := \{(f, x) \in \mathcal{P} \times X\colon x \in \operatorname{dom}(f)\}$ can be equipped with composition and inverses: inverses are defined by $(f, x)^{-1} := \big(f^{-1}, f(x)\big)$, and the pairs (f_1, x_1) and (f_2, x_2) are composable if and only if $x_1 = f_2(x_2)$, in which case the product is $\big(f_1, f_2(x_2)\big)(f_2, x_2) := (f_1 \circ f_2, x_2)$. Note that $P \ltimes X$ might not be a groupoid with these operations (Definition 1.1.1 (iii) may fail). The pseudogroup $\mathcal{P}$ is equipped with the discrete topology, and the space $\mathcal{P} \ltimes X$ is equipped with the product topology. Next, define an equivalence relation $\sim$ on $\mathcal{P} \ltimes X$ as follows : $(f_1, x_1) \sim (f_2, x_2)$ if and only if $x_1 = x_2$ and there exists an open set $U \subseteq \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$ containing x_1 such that $f_1|_U = f_2|_U$. The *groupoid of germs* of the action is defined to be the quotient

$$\operatorname{Germ}(\mathcal{P} \curvearrowright X) := \mathcal{P} \ltimes X/\sim.$$

It is an effective étale groupoid with the induced groupoid structure and the quotient topology, and is typically non-Hausdorff. Indeed, it is Hausdorff if and only if the open set $\operatorname{int}(\{x \in X\colon f(x) = x\})$ is closed for all $f \in \mathcal{P}$ (see [56, Lemma 4.3] for instance).

Instead of working with a concrete pseudogroup $\mathcal{P}$, one can instead consider the action of an inverse semigroup. Given an inverse semigroup S, an action of S on X is given by a surjective semigroup homomorphism $S \to \mathcal{P}$, where $\mathcal{P}$ is some pseudogroup on X. Define the set $S \ltimes X := \{(f, x)\colon x \in \operatorname{dom}(f)\}$, and equip it with the same topology, operations, and equivalence relation $\sim$ as $\mathcal{P} \ltimes X$. As above, the *groupoid of germs* $\operatorname{Germ}(S \curvearrowright X)$ is then defined to be the quotient $S \ltimes X/\sim$. Note that there exists an obvious isomorphism $\operatorname{Germ}(S \curvearrowright X) \cong \operatorname{Germ}(\mathcal{P} \curvearrowright X)$.

We will only consider examples for which the homomorphism $S \to \mathcal{P}$ is injective, and in this case it is clear that our definition agrees with the one in [57, p.140] (see also [25, Definition 4.6]).

Notation. Elements of $\mathrm{Germ}(S \curvearrowright X)$ are equivalence classes, denoted $[f, x] := \{(\tilde{f}, x) : (\tilde{f}, x) \sim (f, x)\}$. We will call $[f, x]$ the *germ* of f at x. The unit space will be identified with X via the homeomorphism $[f^{-1} \circ f, x] \mapsto x$ (here f is any element of S with $x \in \mathrm{dom}(f)$). For each $f \in S$, we let $[f, \mathrm{dom}(f)]$ denote the open bisection $\{[f, x] : x \in \mathrm{dom}(f)\}$. Note that the open bisections $[f, \mathrm{dom}(f)]$ cover the groupoid $\mathrm{Germ}(S \curvearrowright X)$, and therefore its topology be understood entirely through these bisections.

§ 1.6.3 | Examples from Thompson's groups

In this subsection we look at groupoids of germs inspired by the canonical actions of Thompson's groups F and T on the unit interval and circle respectively. These groupoids have previously been considered in [40] and [63]. For an introduction to Thompson's groups, see [14]. Let us begin by recalling their definitions.

Let $\mathbb{Z}[\frac{1}{2}]$ denote the set of dyadic rationals. Thompson's group F is the group of orientation-preserving, piecewise linear homeomorphisms of the unit interval $I = [0, 1]$ that have non-differentiable points in $\mathbb{Z}[\frac{1}{2}] \cap I$ and slopes a power of 2. Next, let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the circle, so that each element of $\mathbb{T}$ can be described uniquely by some $t \in [0, 1)$. Thompson's group T is the group of orientation-preserving, piecewise linear homeomorphisms of $\mathbb{T}$ that map $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ to itself, have non-differentiable points in $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$, and have slopes a power of 2.

The fact that $\mathbb{Z}[\frac{1}{2}]$ is dense in $\mathbb{R}$ allows us to make the following observations. Given any $x, y \in (0, 1)$ there exists $f \in F$ with $f(x) = y$ if and only if y belongs to the additive coset $x + \mathbb{Z}[\frac{1}{2}]$. Similarly, given any $t, s \in \mathbb{T}$, there exists $f \in T$ with $f(s) = t$ if and only if t belongs to the additive coset $s + \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

Example 1.6.7 (Action of Thompson's group F on I)**.** By definition, Thompson's group F is a group of homeomorphisms of I, hence acts canonically on I. Let $G := \mathrm{Germ}(F \curvearrowright I)$ be the associated groupoid of germs. Given any unit $x \in (0, 1)$, its orbit in the unit space is equal to $(x + \mathbb{Z}[\frac{1}{2}]) \cap (0, 1)$ by the observation above. Therefore, excluding the units $0, 1 \in I$, all orbits are dense in I. Any germ $[f, x]$ is uniquely determined by its source, range, and the slopes of f either side of x. That is, given $f, \tilde{f} \in F$ and $x \in I$, the germs $[f, x]$ and $[\tilde{f}, x]$ are equal if and only if $\tilde{f}(x) = f(x)$ and the slopes of f and $\tilde{f}$ are equal either side of x. The set of dangerous points D is equal to $\mathbb{Z}[\frac{1}{2}] \cap (0, 1)$ and, moreover, all dangerous points are extremely dangerous (i.e. $D_0 = D$). For dangerous points $x \in D$, the isotropy groups G_x^x are isomorphic to $\mathbb{Z}^2$. All other isotropy groups are isomorphic to $\mathbb{Z}$.

Example 1.6.8 (Action of Thompson's group T on $\mathbb{T}$)**.** Let $G := \mathrm{Germ}(T \curvearrowright \mathbb{T})$ be the groupoid of germs associated with the canonical action of T on $\mathbb{T}$. Given any unit

$t \in \mathbb{T}$, its orbit in the unit space is equal to $t + \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ by the observation above. Hence, the groupoid G is minimal. As in the previous example, any germ $[f, t]$ is uniquely determined by its source, range, and the slopes of f either side of t. The set of dangerous points D is equal to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ and, moreover, all dangerous points are extremely dangerous (i.e. $D_0 = D$). For dangerous points $t \in D$, the isotropy groups G_t^t are isomorphic to $\mathbb{Z}^2$. All other isotropy groups are isomorphic to $\mathbb{Z}$.

We now consider some generalisations of the previous examples, again coming from groups of orientation-preserving piecewise linear homeomorphisms of I and $\mathbb{T}$. Specifically, we look at Bieri-Strebel groups – see [8] for a thorough discussion of these groups. These groups split into two families: one generalises Thompson's group F, and the other T. These are not to be confused with the groups known as *Stein groups* ([64]), that generalise Thompson's group V.

Throughout the following, P is a subgroup of the multiplicative group $\mathbb{R}_{>0}$, and A is a $\mathbb{Z}[P]$-submodule of the additive group $\mathbb{R}$. Additionally, we assume that P is non-trivial and that $P \subseteq A$.

Definition 1.6.9 ([8])**.** The *Bieri-Strebel group* $\Gamma(I; A, P)$ is the set of orientation-preserving homeomorphisms f of the unit interval $I = [0, 1]$ satisfying the following conditions.

- f is piecewise-linear, and has only finitely many non-differentiable points.
- The non-differentiable points of f belong to the set $A \cap (0, 1)$.
- The slopes of f all belong to the multiplicative group P.
- f maps $A \cap (0, 1)$ to itself.

The *Bieri-Strebel group* $T(\mathbb{T}; A, P)$ is the set of orientation-preserving homeomorphisms f of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ satisfying the following conditions.

- f is piecewise-linear, and has only finitely many non-differentiable points.
- The non-differentiable points of f belong to the set $A/\mathbb{Z}$.
- The slopes of f all belong to the multiplicative group P.
- f maps $A/\mathbb{Z}$ to itself.

Bieri-Strebel groups are groups under composition. Note that Thompson's group F is equal to the Bieri-Strebel group $\Gamma\Big(I;\ \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle\Big)$, and Thompson's group T is equal to the Bieri-Strebel group $T\Big(\mathbb{T};\ \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle\Big)$, where $\langle 2 \rangle = \{2^n : n \in \mathbb{Z}\}$.

The assumption that P and A are non-trivial ensures that A is dense in the real line $\mathbb{R}$. Similarly, if $IP \cdot A$ denotes the submodule of A generated by elements of the form $(p-1)a$ for $p \in P$, $a \in A$, then $IP \cdot A$ is also dense in $\mathbb{R}$. It follows from [8, Theorem 1] that given any $x, y \in (0,1)$, there exists $f \in \Gamma(I; A, P)$ with $f(x) = y$ if and only if y belongs to the additive coset $x + IP \cdot A$. Similarly, given any $t, s \in \mathbb{T}$ there exists $f \in T(\mathbb{T}; A, P)$ with $f(s) = t$ if and only if t belongs to the additive coset $s + (IA \cdot P)/\mathbb{Z}$ (this uses that $\mathbb{Z} \subseteq A$).

Example 1.6.10. Let $G := \mathrm{Germ}\Big(\Gamma(I; A, P) \curvearrowright I\Big)$ be the groupoid of germs associated with the canonical action of the Bieri-Strebel group $\Gamma(I; A, P)$ on the unit interval. Given any unit $x \in (0,1)$, its orbit in the unit space is equal to $(x + IP \cdot A) \cap (0,1)$ by the observation above. Therefore, excluding the units $0, 1 \in I$, all orbits are dense in I. As in Example 1.6.7, any germ $[f, x]$ is uniquely determined by its source, range, and the slopes of f either side of x. The set of dangerous points D is equal to $A \cap (0,1)$ and, moreover, all dangerous points are extremely dangerous (i.e. $D_0 = D$). For dangerous points $x \in D$, the isotropy groups G_x^x are isomorphic to P^2. All other isotropy groups are isomorphic to P.

Example 1.6.11. Let $G := \mathrm{Germ}\Big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\Big)$ be the groupoid of germs associated with the canonical action of $T(\mathbb{T}; A, P)$ on $\mathbb{T}$. Given any unit $t \in \mathbb{T}$, its orbit in the unit space is equal to $t + (IA \cdot P)/\mathbb{Z}$ by the observation above. Hence, the groupoid G is minimal. Any germ $[f, t]$ is uniquely determined by its source, range, and the slopes of f either side of t. The set of dangerous points D is equal to $A/\mathbb{Z}$ and, moreover, all dangerous points are extremely dangerous. For dangerous points $t \in D$, the isotropy groups G_t^t are isomorphic to P^2. All other isotropy groups are isomorphic to P.

Since $P \subseteq A$, the group $\Gamma(I; A, P)$ embeds canonically into $T(\mathbb{T}; A, P)$. It follows that $H := \mathrm{Germ}\Big(\Gamma(I; A, P) \curvearrowright I\Big)$ is an open subgroupoid of $G := \mathrm{Germ}\Big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\Big)$ via the inclusion $[f, x] \mapsto [f, q(x)]$, where $q \colon I \to \mathbb{T}$ is the quotient map. Moreover, the two groupoids only differ by a "small" set: the only elements of G not belonging to H are the germs of the form $[f, 0] \in G$, where $f \in T(\mathbb{T}; A, P)$ has $f(0) \neq 0$. Despite this, we continue to discuss both example classes in the later chapters.

§ 1.6.4 | Groupoids from self-similar groups

A well-studied class of groupoids are those associated with self-similar groups (see [52–54]). These groupoids are typically non-Hausdorff. Characterisations for when these groupoids are Hausdorff appear in [12, Corollary 6.7] and [44, Corollary 6.14]. Let us begin by recalling the definition of a self-similar group.

Fix a finite alphabet A. Let T denote the tree of finite words in the alphabet A, and let $\mathrm{Aut}(T)$ denote the set of tree automorphisms of T. A subgroup $\Gamma \subseteq \mathrm{Aut}(T)$ is called a *self-similar group* on A if for every $g \in \Gamma$ and $a \in A$ there exists an element $g|_a$ of Γ such that $g(av) = g(a)g|_a(v)$ for all $v \in T$. In this case, note that for any $u \in T$ there exists $g|_u \in \Gamma$ such that $g(uv) = g(u)g|_u(v)$ for all $v \in T$.

There are various (equivalent) ways to describe the groupoid associated with a self-similar group Γ on a finite alphabet A. We follow [53, § 5.2] in describing it as the groupoid of germs of an inverse semigroup action (see Subsection 1.6.2). Write $X := \partial T$ for the boundary of the tree T, and equip it with the usual topology (i.e. the product topology on $A^{\mathbb{N}}$). The group Γ acts by homeomorphisms on X as follows: for $g \in \Gamma$ and $\omega = a_1 a_2 a_3 \cdots \in X$ define

$$g(\omega) := g(a_1)\ g|_{a_1}(a_2)\ g|_{a_1 a_2}(a_3) \ldots$$

Given any finite word $u \in T$, there is an associated (partial) homeomorphism $u\colon X \xrightarrow{\cong} uX$ given by $\omega \mapsto u\omega$. Let u^* denote the inverse $uX \xrightarrow{\cong} X; u\omega \mapsto \omega$. Define S_Γ to be the inverse semigroup of partial homeomorphisms of X generated by Γ and $\{u, u^*\colon u \in T\}$. We can write $S_\Gamma = \{ugv^*\colon g \in \Gamma, u, v \in T\}$. We then define the groupoid G associated with the self-similar group Γ to be the groupoid of germs $G := \mathrm{Germ}(S_\Gamma \curvearrowright X)$. Explicitly we have

$$G = \{[ugv^*, \omega]\colon g \in \Gamma, u, v \in T, \omega \in vX\} \tag{1.6.3}$$

where $[ugv^*, \omega]$ denotes the germ of ugv^* at ω. In this context, germs have the following explicit description. Take two finite words $v_1, v_2 \in T$ with v_2 at least as long as v_1, and let $\omega_1 \in v_1 X, \omega_2 \in v_2 X$. The germs $[u_1 g_1 v_1^*, \omega_1]$ and $[u_2 g_2 v_2^*, \omega_2]$ are equal if and only if $\omega_1 = \omega_2$ and there exist $y, z \in T$ such that $\omega_1 \in v_2 z X$, $v_2 = v_1 y$, $u_2 g_2(z) = u_1 g_1(yz)$ and $g_2|_z = g_1|_{yz}$. In this case, the inverse semigroup elements $u_1 g_1 v_1^*$ and $u_2 g_2 v_2^*$ agree on the open set $v_2 z X$. Overall, G is an ample (minimal effective) groupoid with Cantor unit space.

From now on, we will only be concerned with contracting self-similar groups. A self-similar group Γ on A is said to be *contracting* if there exists a finite set $N \subseteq \Gamma$ such that for all $g \in \Gamma$, there exists $l \in \mathbb{N}$ such that $g|_u \in N$ whenever $u \in T$ is a word of length at least l. In this case, the *nucleus* N is defined to be the smallest such set. Note that $n|_u \in N$ for any $n \in N$ and $u \in T$. Note also the following property of the groupoid G associated with Γ: for every $g \in \Gamma$ and $\omega \in X$, there exist $u, v \in T$ and $n \in N$ such that $[g, \omega] = [unv^*, \omega]$.

The following lemma describes the closure $\overline{G^{(0)}}$ of the unit space in terms of the open bisections $\{[n, X]\colon n \in N\}$, where N is the nucleus. Since the nucleus is finite, it follows that $\overline{G^{(0)}}$ is not too large. Given $ugv^* \in S_\Gamma$, write $u[g, X]v^* = [ugv^*, vX]$ for the set of germs $\{[ugv^*, v\omega]\colon \omega \in X\}$. Note that $u[g, X]v^*$ is an open bisection. Then, given $u \in T$, $g \in \Gamma$, and $\omega \in X$ write $u[g,\omega]u^* := [ugu^*, u\omega]$, and note that the map $[g,X] \to u[g,X]u^*;\ [g,\omega] \mapsto u[g,\omega]u^*$ is a homeomorphism.

Lemma 1.6.12. *Let Γ be a contracting self-similar group with nucleus N. Let G denote the associated groupoid of germs.*

(i) The closure of the unit space $\overline{G^{(0)}}$ satisfies

$$\overline{G^{(0)}} = \bigcup_{u \in T,\, n \in N} u(\overline{[n,X] \cap G^{(0)}}^{[n,X]})u^*. \tag{1.6.4}$$

(ii) We have $\left|G_\omega^\omega \cap \overline{G^{(0)}}\right| \leq |N|$ for all $\omega \in X$.

Proof. We first prove part (i). Let us first prove the inclusion from right to left in (1.6.4). Take any $u \in T$, $n \in N$. The homeomorphism $[n,X] \to u[n,X]u^*; [n,\dot{\omega}] \mapsto u[n,\dot{\omega}]u^*$ maps the open set $[n,X] \cap G^{(0)}$ to $u[n,X]u^* \cap G^{(0)}$. Therefore, $u[n,\omega]u^*$ belongs to $G^{(0)}$ if and only if $[n,\omega]$ belongs to $\overline{[n,X] \cap G^{(0)}}^{[n,X]}$. The inclusion from right to left follows. To prove the opposite inclusion, let $[ugv^*, v\omega] \in \overline{G^{(0)}}$. By definition of "contracting", we can assume that $g = n$ belongs to the nucleus N. Next, if we had $u \neq v$, then the open bisection $u[n,X]v^*$ would be disjoint from the unit space. Therefore $u = v$. Now, by the comment above, the germ $[unu^*, u\omega] = u[n,\omega]u^*$ belongs to $G^{(0)}$ if and only if $[n,\omega]$ belongs to $\overline{[n,X] \cap G^{(0)}}^{[n,X]}$. But, the germ $[unu^*, u\omega]$ belongs to $G^{(0)}$ by assumption, and therefore, $[unu^*, u\omega] \in u(\overline{[n,X] \cap G^{(0)}}^{[n,X]})u^*$, as desired. This completes the proof of part (i).

To prove part (ii), fix $\omega \in X$, and use part (i) to deduce that an arbitrary element of $G_\omega^\omega \cap \overline{G^{(0)}}$ is of the form $u[n, u^*(\omega)]u^*$ for some $[n, u^*(\omega)] \in \overline{G^{(0)}}$. Let ℓ denote the length of the word u. It is clear that $u = \omega_\ell$, where ω_ℓ denotes the word consisting of just the first ℓ letters of ω, and hence $u[n, u^*(\omega)]u^* = [\omega_\ell n \omega_\ell^*, \omega]$. This shows that $G_\omega^\omega \cap \overline{G^{(0)}}$ is contained in the set $G_\omega^\omega \cap \{[\omega_\ell n \omega_\ell^*, \omega] \colon n \in N, \ell \geq 0\}$. Next, for any $n \in N$ and $\ell \geq 1$, with $[\omega_\ell n \omega_\ell^*, \omega] \in G_\omega^\omega$, note that there exists $n' \in N$ such that $[\omega_\ell n \omega_\ell^*, \omega] = [\omega_{\ell+1} n' \omega_{\ell+1}^*, \omega]$. Finiteness of N then ensures that there exists $L \geq 0$ such that $G_\omega^\omega \cap \{[\omega_\ell n \omega_\ell^*, \omega] \colon n \in N, \ell \geq 0\} \subseteq \{[\omega_L n \omega_L^*, \omega] \colon n \in N\}$. $\square$

When describing examples of self-similar groups, it is helpful to use Moore diagrams. Given $g, g' \in \Gamma$ and $a, a' \in A$, the following diagram says that $g(a) = a'$ and $g|_a = g'$.

We now discuss two concrete examples of well-known self-similar groups. The first is the Grigorchuk group ([32, 33]) and the second is the Grigorchuk-Erschler group ([24, 33]).

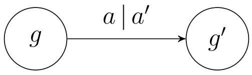

Figure 1.1: Arrow in a Moore diagram

Both are self-similar groups acting on the alphabet $A = \{0, 1\}$, and they are generated by their respective nuclei. The groupoid associated with the Grigorchuk group is discussed in [18, § 5.6] and [30, Example 6.10], and the groupoid associated with the Grigorchuk-Erschler group is discussed in [67, Example 7.7] and [30, Example 6.11].

Example 1.6.13. The Grigorchuk group was the first known group of intermediate growth. It can be realised as a contracting self-similar group on the alphabet $\{0, 1\}$. Its structure is described by the following Moore diagram, where $N = \{e, a, b, c, d\}$ denotes the nucleus and e is the identity element (see [52, § 1.6] for further details).

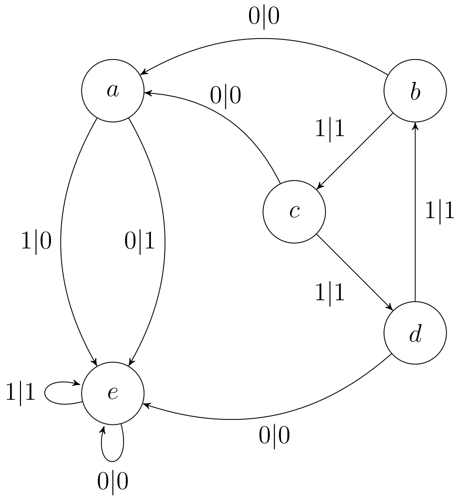

Figure 1.2: Moore diagram for the Grigorchuk group

The elements a, b, c, d have order two, and the set $\{e, b, c, d\}$ forms a Klein four-group. Let G denote the associated groupoid of germs. Let $1^\infty \in X = G^{(0)}$ denote the infinite word containing only 1. The set of dangerous points $D \subseteq G^{(0)}$ is equal to $\{u1^\infty : u \in T\}$ and forms a single orbit in $G^{(0)}$. All dangerous units are extremely dangerous. The isotropy group of any dangerous unit contains a copy of the Klein four-group $\{e, b, c, d\}$. We will see later (Example 4.8.6) that the topological structure close to these Klein four-groups resembles the structure of the groupoid from Example 1.6.3.

Example 1.6.14. The next example is known as the Grigorchuk-Erschler group. It is an example of a group with intermediate growth that is not torsion. As was the case with the Grigorchuk group, it can be realised as a contracting self-similar group on the alphabet $\{0, 1\}$. Its structure is described by the Moore diagram below, where $N = \{e, h, \alpha, \beta, \gamma\}$ is the nucleus and e is the identity element.

The elements h, α, β, γ have order two, and the set $\{e, \alpha, \beta, \gamma\}$ forms a Klein four-group. Let G denote the associated groupoid of germs. The set of dangerous points $D \subseteq G^{(0)}$

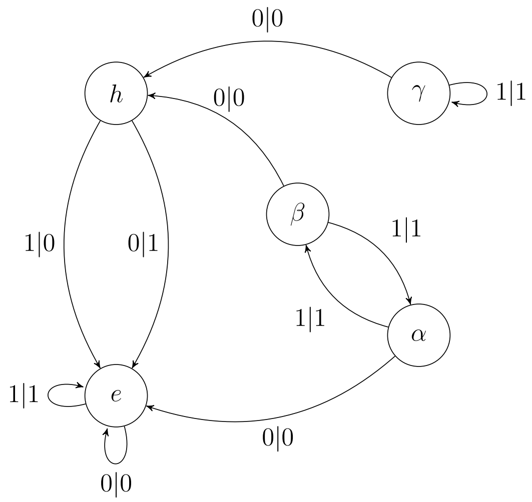

Figure 1.3: Moore diagram for the Grigorchuk-Erschler group

is equal to $\{u1^\infty \colon u \in T\}$ and forms a single orbit. All dangerous units are extremely dangerous.

The following lemma describes the dangerous isotropy groups of G. It will be useful later on. Let $1^m \in T$ denote the finite word of length m containing only 1, and write $1 := 1^1$. Let $1^\infty \in X$ denote the infinite word containing only 1. Let φ denote the automorphism of $V_4 \cong \{e, \alpha, \beta, \gamma\}$ given by $\alpha \mapsto \beta$, $\beta \mapsto \alpha$, $\gamma \mapsto \gamma$, and write $V_4 \rtimes_\varphi \mathbb{Z}$ for the associated semi-direct product.

Lemma 1.6.15. *Let G be the groupoid associated with the Grigorchuk-Erschler group.*

The isotropy group $G_{1^\infty}^{1^\infty}$ is isomorphic to $V_4 \rtimes_\varphi \mathbb{Z}$ via the map

$$\begin{aligned} [\alpha, 1^\infty] &\mapsto \alpha & \qquad [\beta, 1^\infty] &\mapsto \beta \\ [\gamma, 1^\infty] &\mapsto \gamma & \qquad [1, 1^\infty] &\mapsto 1. \end{aligned}$$

Proof. It is clear that the germs $[e, 1^\infty], [\alpha, 1^\infty], [\beta, 1^\infty], [\gamma, 1^\infty]$ are all distinct and form a Klein four-subgroup of the isotropy group $G_{1^\infty}^{1^\infty}$. Take arbitrary $[ugv^*, 1^\infty] \in G_{1^\infty}^{1^\infty}$. It is clear that $u = 1^\ell$ and $v = 1^k$ for some integers $\ell, k \geq 0$. We also have $g(1^\infty) = 1^\infty$ and therefore, by definition of nucleus, there exists some even integer m satisfying $g|_{1^m} \in \{e, \alpha, \beta, \gamma\}$. The actions of g and $g|_{1^m}$ agree on the open set $1_m X$, so that $[g, 1^\infty] = [g|_{1^m}, 1^\infty]$, and hence $[1^\ell g(1^k)^*, 1^\infty] = [1^\ell g|_{1^m}(1^k)^*, 1^\infty]$. It follows that the isotropy group $G_{1^\infty}^{1^\infty}$ is generated by the germs $[\alpha, 1^\infty], [\beta, 1^\infty], [\gamma, 1^\infty]$ and $[1, 1^\infty]$. The following relations are simple to check.

$$\begin{aligned} [1\alpha 1^*, 1^\infty] &= [\beta, 1^\infty] \\ [1\beta 1^*, 1^\infty] &= [\alpha, 1^\infty] \\ [1\gamma 1^*, 1^\infty] &= [\gamma, 1^\infty] \end{aligned}$$

These are exactly the relations in the group $V_4 \rtimes_\varphi \mathbb{Z}$. $\square$

If G denotes the groupoid associated with the Grigorchuk group, then a similar argument shows that the isotropy group $G_{1^\infty}^{1^\infty}$ is isomorphic to a semidirect product $V_4 \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on V_4 by permuting its non-trivial elements.

Chapter 2

Groupoid amenability

In this chapter we discuss the concept of amenability for locally compact groupoids possessing a Haar system. The definition we choose is the following.

Definition 2.0.1. Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. We say that G is *(topologically) amenable* if there exists a net (f_i) of functions in $\mathscr{C}_c(G)^+$ satisfying the following.

- $\lambda^x(f_i) \leq 1$ for all $x \in G^{(0)}$ and all i.
- $\lambda^x(f_i) \to 1$ uniformly on compact subsets of $G^{(0)}$ (with respect to x).
- $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \,\mathrm{d}\lambda^{r(g)}(h) \to 0$ uniformly on compact subsets of G (with respect to g).

This definition agrees with [61, Definition 2.7]. One goal of this chapter is to compare this definition with other candidates. It is necessary to do this separately in the Hausdorff and non-Hausdorff settings. Section 2.1 considers Hausdorff groupoids, and follows [3] in giving a characterisation of amenability using continuous systems of probability measures. In contrast, it is shown in Section 2.2 that continuous systems of probability measures cannot be used to describe amenability for non-Hausdorff groupoids. Section 2.3 then uses results of [61] to list some Borel characterisations of amenability.

A second goal of this chapter, as well as Section 3.2 of the next, is to show that the definition above has the important qualities one would like in a definition of amenablity. Section 2.4 discusses how amenability can be applied in the study of operator algebras. Explicitly, we use Renault's disintegration theorem to prove that any (non-Hausdorff) amenable second-countable locally compact groupoid has nuclear C^*-algebra. In the following chapter, we will see that a (non-Hausdorff) étale groupoid is amenable if and only if its reduced (or full) C^*-algebra is nuclear. Both are generalisations of well-known results for Hausdorff groupoids to the non-Hausdorff setting. In the next chapter, we will also see that an étale groupoid is amenable if and only if its Hausdorff cover is amenable, thus showing that amenability of a non-Hausdorff groupoid is witnessed (canonically) by that of a Hausdorff groupoid. Overall, there is now strong evidence to suggest that Definition 2.0.1 is the correct definition of amenability for non-Hausdorff groupoids.

Remark 2.0.2. The first bullet point of Definition 2.0.1 is redundant. We keep it here to be consistent with previous definitions. If G is σ-compact, the net can be replaced by a sequence.

Proof of Remark 2.0.2. Assume that there is a net in $\mathscr{C}_c(G)^+$ satisfying the second and third conditions. Fix $\varepsilon > 0$ and compact sets $K \subseteq G$ and $L \subseteq G^{(0)}$, and let $\check{L}$ be a compact neighbourhood of $s(K) \cup r(K) \cup L$. By assumption there exists $f \in \mathscr{C}_c(G)^+$ such that $|1 - \lambda^x(f)| \leq \varepsilon$ for $x \in \check{L}$ and $\int_{G^{r(g)}} |f(g^{-1}h) - f(h)| \, \mathrm{d}\lambda^{r(g)}(h) \leq \varepsilon$ for $g \in K$. Let $\chi \in C_c(G^{(0)})^+$ be such that $0 \leq \chi_n \leq (1+\varepsilon)^{-1}$, $\chi_n \equiv (1+\varepsilon)^{-1}$ on $s(K) \cup r(K) \cup L$ and $\operatorname{supp}(\chi_n) \subseteq \check{L}$. Define $\tilde{f} := (\chi \circ r).f \in \mathscr{C}_c(G)^+$. Then $\lambda^x(\tilde{f}) \leq 1$ for all $x \in G^{(0)}$, $\lambda^x(\tilde{f}) \geq 1 - \frac{2\varepsilon}{1+\varepsilon}$ for all $x \in L$, and $\int_{G^{r(g)}} \left|\tilde{f}(g^{-1}h) - \tilde{f}(h)\right| \mathrm{d}\lambda^{r(g)}(h) \leq \varepsilon$ for all $g \in K$. Since ε, K and L are arbitrary, there exists a net in $\mathscr{C}_c(G)^+$ satisfying all three conditions of Definition 2.0.1.

Now assume that G is σ-compact, and let $K_1, K_2, \ldots$ be a sequence of compact sets in G satisfying $K_1 \subseteq \operatorname{int}(K_2) \subseteq K_2 \subseteq \operatorname{int}(K_3) \subseteq \ldots$ and such that $G = \bigcup_{n=1}^{\infty} K_n$. If G is amenable, then there exists a sequence (f_n) in $\mathscr{C}_c(G)^+$ satisfying $\lambda^x(f_n) \leq 1$ for all $x \in G^{(0)}$, $|1 - \lambda^x(f_n)| \leq \frac{1}{n}$ for $x \in r(K_n)$ and $\int_{G^{r(g)}} |f_n(g^{-1}h) - f_n(h)| \, \mathrm{d}\lambda^{r(g)}(h) \leq \frac{1}{n}$ for $g \in K_n$. Then (f_n) satisfies all three conditions of Definition 2.0.1. $\square$

§ 2.1 | Amenability for Hausdorff groupoids

For a deep and thorough investigation into the nature of amenability for Hausdorff groupoids, see [3], where it is shown that the "correct" way to define amenability for Hausdorff groupoids is in terms of "amenable maps". This framework allows the authors to work in a very general setting.

Our definition of amenability from Definition 2.0.1 is not the one given in [3], but it is shown to be equivalent in [3, Proposition 2.2.13]. This proposition shows that amenability for Hausdorff locally compact groupoids admits characterisations both in terms of continuous functions and in terms of continuous systems of probability measures.

This proposition is stated and proved below. The proof follows [3]. It is included for completeness, and because G is not assumed to be second-countable.

Proposition 2.1.1 ([3, Proposition 2.2.13])**.** *Let G be a* Hausdorff *locally compact groupoid with Haar system $\{\lambda^x\}_{x\in G^{(0)}}$. Consider the following statements.*

1. *G is amenable.*
2. *There exists a net (f_i) of functions in $C_{c,r}(G)^+$ such that*
   - *(i) $\lambda^x(f_i) = 1$ for all $x \in G^{(0)}$ and all i;*
   - *(ii) $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \, \mathrm{d}\lambda^{r(g)}(h) \to 0$ uniformly on compact subsets of G (with respect to g).*
3. *There exists a net (f_i) of non-negative continuous functions on G such that*
   - *(i) $\lambda^x(f_i) \leq 1$ for all $x \in G^{(0)}$ and all i;*
   - *(ii) $\lambda^x(f_i) \to 1$ uniformly on compact subsets of $G^{(0)}$ (with respect to x);*
   - *(iii) $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \, \mathrm{d}\lambda^{r(g)}(h) \to 0$ uniformly on compact subsets of G (with respect to g).*
4. *There exists a net (m_i) of continuous systems of measures such that*
   - *(i) $\|m_i^x\|_1 \leq 1$ for all $x \in G^{(0)}$ and all i;*
   - *(ii) $\|m_i^x\|_1 \to 1$ uniformly on compact subsets of $G^{(0)}$ (with respect to x);*
   - *(iii) $\left\|g \cdot m_i^{s(g)} - m_i^{r(g)}\right\|_1 \to 0$ uniformly on compact subsets of G (with respect to g).*
5. *There exists a net (m_i) of continuous systems of measures such that*
   - *(i) $\|m_i^x\|_1 = 1$ for all $x \in G^{(0)}$ and all i;*
   - *(ii) $\left\|g \cdot m_i^{s(g)} - m_i^{r(g)}\right\|_1 \to 0$ uniformly on compact subsets of G (with respect to g).*

We have (1) ⇔ (3) ⇔ (4). Moreover, if $G^{(0)}$ is paracompact then all statements are equivalent.

Proof. (1) ⇒ (2). Fix $\varepsilon \in (0,1)$ and a compact set $K \subseteq G$, and let $L \subseteq G^{(0)}$ be a compact neighbourhood of $r(K) \cup s(K)$. Let $\psi \in C_c(G)^+$ satisfy the following three conditions.

(I) $\lambda^x(\psi) \leq 1$ for all $x \in G^{(0)}$.

(II) $\lambda^x(\psi) > 1 - \varepsilon$ for all $x \in L$.

(III) $\int_{G^{r(g)}} |\psi(g^{-1}h) - \psi(h)| \, \mathrm{d}\lambda^{r(g)}(h) < \varepsilon$ for all $g \in K$.

Using continuity of the map $x \mapsto \lambda^x(\psi)$ on $G^{(0)}$, we can assume that $\lambda(\psi) \equiv 1$ on L. Applying Lemma 1.2.5, let $\varphi \in C_{c,r}(G)^+$ be such that $\lambda^x(\varphi) = 1$ for all $x \in G^{(0)}$. Let $\chi \in C_c(G^{(0)})$ be such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $r(K) \cup s(K)$, and $\operatorname{supp}(\chi) \subseteq \operatorname{int}(L)$. Define

$$f(g) \coloneqq (\chi \circ r).\psi(g) + \Big(1 - \chi \circ r\Big).\varphi(g)$$

for $g \in G$. Then $f \in C_{c,r}(G)^+$ with $\lambda(f) \equiv 1$. Moreover, $f(k) = \psi(k)$ whenever $r(k) \in r(K) \cup s(K)$ so that

$$\int_{G^{r(g)}} \Big|f(g^{-1}h) - f(h)\Big| \, \mathrm{d}\lambda^{r(g)}(h) < \varepsilon$$

for any $g \in K$.

(2) $\Rightarrow$ (5) and (1) $\Rightarrow$ (4) follow from letting $m_i^x \coloneqq f_i \, \mathrm{d}\lambda^x$ for each i and $x \in G^{(0)}$.

(5) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (3). Fix $\varepsilon > 0$ and compact sets $K \subseteq G$ and $L \subseteq G^{(0)}$. Let $\{m^x\}_{x \in G^{(0)}}$ be a continuous system of measures on G satisfying the following conditions

- $\|m^x\|_1 \leq 1$ for all $x \in G^{(0)}$.
- $\|m^x\|_1 > 1 - \varepsilon$ for all $x \in L$.
- $\Big\|g \cdot m^{s(g)} - m^{r(g)}\Big\|_1 < \varepsilon$ for all $g \in K$.

Applying Lemma 1.2.7, let $M \subseteq G$ be compact and such that $m^x(G^x \cap M) > 1 - \varepsilon$ for all $x \in L \cup r(K)$. Choose a function $\chi \in C_c(G)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $K^{-1}M \cup M$ (this is possible by Remark 1.2.4, as products of compact sets are compact). Let $\varphi \in C_c(G)^+$ be such that $\lambda^x(\varphi) \leq 1$ for all $x \in G^{(0)}$ and $\lambda^x(\varphi) = 1$ for all $x \in s(M)$. Define

$$f(g) \coloneqq \int_{G^{r(g)}} \chi(k)\varphi(k^{-1}g) \, \mathrm{d}m^{r(g)}(k) \tag{2.1.1}$$

for all $g \in G$. Since the function $(k, g) \mapsto \chi(k)\varphi(k^{-1}g)$ is compactly supported, Proposition 1.2.8 implies that f is continuous and non-negative on G. Compact support of the integrand also implies that the Fubini-Tonelli theorems can be applied freely. We show that f satisfies the following three conditions.

(I) $\lambda^x(f) \leq 1$ for all $x \in G^{(0)}$.

(II) $\lambda^x(f) > 1 - \varepsilon$ for all $x \in L$.

(III) $\int_{G^{r(g)}} |f(g^{-1}h) - f(h)| \, \mathrm{d}\lambda^{r(g)}(h) < 4\varepsilon$ for all $g \in K$.

Condition (I) is clear as $\lambda^x(f) \leq \|m^x\|_1 \leq 1$ for all $x \in G^{(0)}$. Then, for $x \in L$ we have

$$\begin{aligned}
\lambda^x(f) &= \int_{G^x} \int_{G^x} \chi(k)\varphi(k^{-1}g)\ \mathrm{d}m^x(k)\mathrm{d}\lambda^x(g) \\
&\geq \int_{G^x \cap M} \left[\int_{G^x} \varphi(k^{-1}g)\ \mathrm{d}\lambda^x(g)\right] \chi(k)\ \mathrm{d}m^x(k) \\
&= \int_{G^x \cap M} \chi(k)\ \mathrm{d}m^x(k) \\
&\geq m^x(G^x \cap M) \\
&> 1 - \varepsilon
\end{aligned}$$

so that f satisfies (II). Next, observe that

$$\begin{aligned}
&\int_{G^{r(g)}} \left|f(g^{-1}h) - f(h)\right| \mathrm{d}\lambda^{r(g)}(h) \\
&= \int_{G^{r(g)}} \left|\int_{G^{r(g)}} \chi(g^{-1}k)\varphi(k^{-1}h)\ \mathrm{d}g\cdot m^{s(g)}(k) - \int_{G^{r(g)}} \chi(k)\varphi(k^{-1}h)\ \mathrm{d}m^{r(g)}(k)\right| \mathrm{d}\lambda^{r(g)}(h) \\
&\leq \int_{G^{r(g)}} \left[\int_{M\cap G^{r(g)}} \left|\varphi(k^{-1}h)\right| \mathrm{d}\left|g\cdot m^{s(g)} - m^{r(g)}\right|(k)\right] \mathrm{d}\lambda^{r(g)}(h) \\
&\quad + \int_{G^{r(g)}} \left[\int_{G^{r(g)}\setminus M} \left|\chi(g^{-1}k)\varphi(k^{-1}h)\right| \mathrm{d}g\cdot m^{s(g)}(k)\right] \mathrm{d}\lambda^{r(g)}(h) \\
&\quad + \int_{G^{r(g)}} \left[\int_{G^{r(g)}\setminus M} \left|\chi(k)\varphi(k^{-1}h)\right| \mathrm{d}m^{r(g)}(k)\right] \mathrm{d}\lambda^{r(g)}(h) \\
&= \int_{M\cap G^{r(g)}} \left[\int_{G^{r(g)}} \varphi(k^{-1}h)\ \mathrm{d}\lambda^{r(g)}(h)\right] \mathrm{d}\left|g\cdot m^{s(g)} - m^{r(g)}\right|(k) \\
&\quad + \int_{G^{r(g)}\setminus M} \chi(g^{-1}k)\ \left[\int_{G^{r(g)}} \varphi(k^{-1}h)\ \mathrm{d}\lambda^{r(g)}(h)\right] \mathrm{d}g\cdot m^{s(g)}(k) \\
&\quad + \int_{G^{r(g)}\setminus M} \chi(k)\ \left[\int_{G^{r(g)}} \varphi(k^{-1}h)\ \mathrm{d}\lambda^{r(g)}(h)\right] \mathrm{d}m^{r(g)}(k) \\
&\leq \int_{G^{r(g)}} 1\ \mathrm{d}\left|g\cdot m^{s(g)} - m^{r(g)}\right|(k) + \int_{G^{r(g)}\setminus M} 1\ \mathrm{d}g\cdot m^{s(g)}(k) + \int_{G^{r(g)}\setminus M} 1\ \mathrm{d}m^{r(g)}(k) \\
&= \left\|g\cdot m^{s(g)} - m^{r(g)}\right\|_1 + g\cdot m^{s(g)}\Big(G^{r(g)}\setminus M\Big) + m^{r(g)}\Big(G^{r(g)}\setminus M\Big)
\end{aligned}$$

for any $g \in K$. By assumption, the first and third terms have value less than ε. It then follows that the second term has value less than 2ε, and therefore f satisfies (III).

(3) $\Rightarrow$ (1). Fix $\varepsilon > 0$ and compact sets $K \subseteq G$ and $L \subseteq G^{(0)}$. Let f be a nonnegative continuous function on G satisfying the following.

- $\lambda^x(f) \leq 1$ for all $x \in G^{(0)}$.
- $\lambda^x(f) > 1 - \varepsilon$ for all $x \in L$.
- $\int_{G^{r(g)}} |f(g^{-1}h) - f(h)|\,\mathrm{d}\lambda^{r(g)}(h) < \varepsilon$ for all $g \in K$.

Apply Lemma 1.2.7 to the continuous system of measures $f\ \mathrm{d}\lambda^x$, and let $M \subseteq G$ be compact and such that $\int_{G^x \cap M} f\ \mathrm{d}\lambda^x > 1-\varepsilon$ for all $x \in L \cup s(K) \cup r(K)$. Let $\chi \in C_c(G)$ be such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on M, and define $\psi := \chi.f \in C_c(G)$. We show that ψ satisfies the following three conditions.

(I) $\lambda^x(\psi) \leq 1$ for all $x \in G^{(0)}$.

(II) $\lambda^x(\psi) > 1 - \varepsilon$ for all $x \in L$.

(III) $\int_{G^{r(g)}} |\psi(g^{-1}h) - \psi(h)| \, \mathrm{d}\lambda^{r(g)}(h) < 3\varepsilon$ for all $g \in K$.

Condition (I) is clear as $\chi \leq 1$. For $x \in L$ we have

$$\lambda^x(\psi) \geq \int_{G^x} \mathbb{1}_M(g) f(g) \, \mathrm{d}\lambda^x(g) > 1 - \varepsilon$$

so that ψ satisfies (II). Finally, observe that for $g \in K$

$$\begin{aligned}
&\int_{G^{r(g)}} \left|\psi(g^{-1}h) - \psi(h)\right| \mathrm{d}\lambda^{r(g)}(h) \\
&\leq \int_{G^{r(g)}} \left|\psi(g^{-1}h) - f(g^{-1}h)\right| \mathrm{d}\lambda^{r(g)}(h) + \int_{G^{r(g)}} \left|f(g^{-1}h) - f(h)\right| \mathrm{d}\lambda^{r(g)}(h) \\
&+ \int_{G^{r(g)}} |f(h) - \psi(h)| \, \mathrm{d}\lambda^{r(g)}(h) \\
&\leq \int_{G^{r(g)}} \Big(1 - \chi(g^{-1}h)\Big) f(g^{-1}h) \, \mathrm{d}\lambda^{r(g)}(h) + \varepsilon + \int_{G^{r(g)}} \Big(1 - \chi(h)\Big) f(h) \, \mathrm{d}\lambda^{r(g)}(h) \\
&\leq \int_{G^{s(g)}} \Big(1 - \mathbb{1}_M(\dot{h})\Big) f(\dot{h}) \, \mathrm{d}\lambda^{s(g)}(\dot{h}) + \varepsilon + \int_{G^{r(g)}} \Big(1 - \mathbb{1}_M(h)\Big) f(h) \, \mathrm{d}\lambda^{r(g)}(h) \\
&< 3\varepsilon.
\end{aligned}$$

Hence ψ satisfies (III) as required. □

Remark 2.1.2. Condition (i) is redundant in statements (3) and (4). This follows from Remark 2.0.2 and by asking that f satisfies $1 - \varepsilon < \lambda^x(f) < 1 + \varepsilon$ for all $x \in L \cup s(K) \cup r(K)$ in implication (3)⇒(1) of the preceding proof. If G is σ-compact, the net can be replaced by a sequence in each of the statements (1) – (5). Similar comments are made in [3, Proposition 2.2.13].

As noted at the start of [3, § 2], this proposition proves that amenability does not depend on the choice of Haar system. Also, the proof of implication (2)⇒(5) shows that the continuous systems of probability measures $\{m_i^x\}_{x \in G^{(0)}}$ can be chosen such that m_i^x is absolutely continuous with respect to λ^x for all $x \in G^{(0)}$ and all i.

We finish this section with the well-known "L^2" characterisation of amenability for Hausdorff groupoids (see [3, Proposition 2.2.13(iv)] for example). This will be useful in Chapter 3. In the following, the norm $\|\cdot\|_{L^2(G^x)}$ denotes the Hilbert space norm on the fibre G^x with respect to the measure λ^x.

Proposition 2.1.3. *Let G be a* Hausdorff *locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. The following statements are equivalent.*

1. *G is amenable.*

2. *There exists a net (f_i) of functions in $C_c(G)^+$ such that*

(i) $\|f_i\|_{L^2(G^x)} \leq 1$ *for all* $x \in G^{(0)}$ *and all* i*;*

(ii) $(f_i * f_i^*)(g) \to 1$ *uniformly on compact subsets of* G *(with respect to* g*).*

Remark 2.1.4. Unlike in Proposition 2.1.1, condition (i) of statement (2) is not redundant. Indeed, it is necessary that $\lambda^x(f_i^2)$ is uniformly bounded over all i.

Proof. Given a net $(f_i) \subset C_c(G)^+$, write $\zeta_i := f_i^2 \in C_c(G)^+$. We will show that the net (f_i) satisfies the conditions of statement (2) if and only if the net (ζ_i) satisfies the conditions of Definition 2.0.1.

Let $f \in C_c(G)^+$ be such that $\|f\|_{L^2(G^x)} \leq 1$ for all $x \in G^{(0)}$. Given $g \in G$, calculate

$$\begin{aligned}\int_{G^{r(g)}} |f(h) - f(g^{-1}h)|^2 \,\mathrm{d}\lambda^{r(g)}(h) &= \int_{G^{r(g)}} \big(f(h) - f(g^{-1}h)\big)\big(f(h) - f(g^{-1}h)\big) \,\mathrm{d}\lambda^{r(g)}(h) \\ &= \int_{G^{r(g)}} \big(f(h)^2 + f(g^{-1}h)^2 - 2f(h)f(g^{-1}h)\big) \,\mathrm{d}\lambda^{r(g)}(h) \\ &= \|f\|^2_{L^2(G^{r(g)})} + \|f\|^2_{L^2(G^{s(g)})} - 2(f * f^*)(g)\,.\end{aligned}$$

Therefore, given a net $(f_i) \subset C_c(G)^+$ satisfying $\|f_i\|_{L^2(G^x)} \leq 1$ for all $x \in G^{(0)}$ and all i, the net $(f_i * f_i^*)$ converges to 1 uniformly on compact subsets of G if and only if $\int_{G^{r(g)}} |f_i(h) - f_i(g^{-1}h)|^2 \,\mathrm{d}\lambda^{r(g)}(h)$ converges to 0 uniformly on compact subsets of G (with respect to g), and $\|f_i\|^2_{L^2(G^x)}$ converges to 1 uniformly on compact subsets of $G^{(0)}$ (with respect to x).

Define $\zeta := f^2 \in C_c(G)^+$. For any $g \in G$ we have

$$\begin{aligned}\int_{G^{r(g)}} |f(h) - f(g^{-1}h)|^2 \,\mathrm{d}\lambda^{r(g)}(h) &\leq \int_{G^{r(g)}} |\zeta(h) - \zeta(g^{-1}h)| \,\mathrm{d}\lambda^{r(g)}(h) \\ &\leq 2\bigg(\int_{G^{r(g)}} |f(h) - f(g^{-1}h)|^2\bigg)^{\frac{1}{2}} \,\mathrm{d}\lambda^{r(g)}(h)\end{aligned}$$

where the first inequality uses that $f \geq 0$, and the second uses Hölder's inequality and the assumption that $\|f\|_{L^2(G^x)} \leq 1$ for all $x \in G^{(0)}$. The result follows by combining these inequalities with the observations from the first part of the proof. □

§ 2.2 | Amenability for non-Hausdorff groupoids

The previous section gave several equivalent characterisations of amenability for Hausdorff groupoids. Notably, it gave a characterisation in terms of continuous systems of measures, as well as an "L^2" characterisation in terms of $C_c(G)$ functions. In each case, the proofs relied on the fact that $C_c(G)$ is closed under pointwise products.

For non-Hausdorff groupoids, we work with functions that are not continuous. As a result, we will see that statements (4) & (5) of Proposition 2.1.1 are not equivalent to amenability for non-Hausdorff groupoids. It is therefore not appropriate to use continuous systems of measures to define amenability of non-Hausdorff groupoids. Similarly, there is no analogue of Proposition 2.1.3 in terms of functions in $\mathscr{C}_c(G)$. Of all the results in the previous section, only the equivalence "(1) $\Leftrightarrow$ (2)" from Proposition 2.1.1 has an exact analogue in the non-Hausdorff setting.

Before giving the details of these results, let us digress from the topic of amenability and remind the reader of the definition of the function space $\mathscr{C}_c(G)$ (Definition 1.1.5). Given an open Hausdorff set $U \subseteq G$, we abuse notation and let $C_c(U)$ denote the set of functions $f\colon G \to \mathbb{C}$ such that $f|_U \in C_c(U)$ and $\operatorname{supp}^\circ(f) \subseteq U$. The function space $\mathscr{C}_c(G)$ is defined to be the linear span

$$\mathscr{C}_c(G) \coloneqq \operatorname{span}\{f\colon G \to \mathbb{C}\colon f \in C_c(U) \text{ for some open Hausdorff } U \subseteq G\}.$$

In Definition 2.0.1, amenability is defined via functions in $\mathscr{C}_c(G)^+$, that is, functions in $\mathscr{C}_c(G)$ that are non-negative on all of G. It is natural to ask whether functions $f \in \mathscr{C}_c(G)^+$ admit a "non-negative decomposition", in the sense that $f = \sum_{i=1}^n f_i$ for some non-negative functions $f_i \in C_c(U_i)$ supported on open Hausdorff sets $U_i \subseteq G$. In Subsection 2.2.2 we give positive answer to this question.

§ 2.2.1 | Equivalent definitions of amenability for non-Hausdorff groupoids

Let us begin this subsection by proving that the equivalence "(1) $\Leftrightarrow$ (2)" from Proposition 2.1.1 has an analogue in the non-Hausdorff setting. In the example that follows, we will see that no other characterisation of amenability from Section 2.1 has an analogue for non-Hausdorff groupoids.

Proposition 2.2.1. *Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x\in G^{(0)}}$ and paracompact unit space. Then the following are equivalent.*

1. *G is amenable.*
2. *There exists a net (f_i) of functions in $\mathscr{C}_{c,r}(G)^+$ such that*
   - *(i) $\lambda^x(f_i) = 1$ for all $x \in G^{(0)}$ and all i;*
   - *(ii) $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)|\, \mathrm{d}\lambda^{r(g)}(h) \to 0$ uniformly on compact subsets of G (with respect to g).*

Proof. The proof of "(1)$\Rightarrow$(2)" is identical to "(1)$\Rightarrow$(2)" of Proposition 2.1.1. For the converse, fix $\varepsilon > 0$ and compact sets $K \subseteq G$ and $L \subseteq G^{(0)}$. Take $\psi \in \mathscr{C}_{c,r}(G)$ satisfying $\lambda^x(\psi) = 1$ for all $x \in G^{(0)}$ and $\int_{G^{r(g)}} |\psi(g^{-1}h) - \psi(h)|\, \mathrm{d}\lambda^{r(g)}(h) < \varepsilon$ for $g \in K$. Let $\chi \in C_c(G^{(0)})$ be such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on $s(K) \cup r(K) \cup L$. Define $f \coloneqq (\chi \circ r).\psi \in \mathscr{C}_c(G)$. Then f satisfies the following three conditions.

- $\lambda^x(f) \leq 1$ for all $x \in G^{(0)}$.
- $f(g) = \psi(g)$ whenever $r(g) \in L$ so that $\lambda^x(f) = 1$ for all $x \in L$.
- $f(g) = \psi(g)$ whenever $r(g) \in r(K) \cup s(K)$ so that

$$\int_{G^{r(g)}} \left| f(g^{-1}h) - f(h) \right| \, \mathrm{d}\lambda^{r(g)}(h) < \varepsilon$$

for all $g \in K$. □

The following is an example of an amenable non-Hausdorff groupoid for which statements (4) and (5) of Proposition 2.1.1 fail. Also, statement (2) of Proposition 2.1.3 fails if one replaces $C_c(G)^+$ with $\mathscr{C}_c(G)^+$.

Example 2.2.2. Let $X = [0,1]$ be the closed unit interval and define $x_0 \coloneqq 1 \in X$. Define $\Gamma \coloneqq \{e, a\} = \mathbb{Z}/2\mathbb{Z}$, and let G be the resulting groupoid from Example 1.6.1. Write $G = [0,1) \cup \{e, a\}$, where $e \in G^{(0)}$ and a is the unique element of G not in $G^{(0)}$. Define $f\colon G \to \mathbb{C}$ to take the value 1 on $[0,1)$ and the value $\frac{1}{2}$ on e and a. It is clear that f belongs to $\mathscr{C}_c(G)^+$, and therefore G is amenable.

Let us now show that the groupoid G does not admit a continuous system of measures satisfying statements (4) and (5) of Proposition 2.1.1. Take an arbitrary continuous system of measures $\{m^x\}_{x \in G^{(0)}}$. It is simple to check that the function φ given by $\varphi(g) \coloneqq m^{r(g)}(\{g\})$ is continuous on G. Therefore

$$\|m^e\|_1 = \varphi(e) + \varphi(a) = 2 \lim_{\substack{x \to 1 \\ x \in (0,1)}} \varphi(x) = 2 \lim_{\substack{x \to 1 \\ x \in (0,1)}} \|m^x\| \, .$$

It follows that either $\|m^e\|_1 \geq \frac{4}{3}$ or $\|m^x\|_1 \leq \frac{2}{3}$ for some $x \in (0,1)$. This shows that it is impossible for a continuous system of measures to satisfy either condition (4)(ii) or condition (5)(i) in Proposition 2.1.1.

We now show that there does not exist a net of functions in $\mathscr{C}_c(G)^+$ satisfying condition (2) of Proposition 2.1.3. Take a function $f \in \mathscr{C}_c(G)^+$ with $\|f\|_{L^2(G^x)} \leq 1$ for all $x \in X = G^{(0)}$. The convolution $\psi \coloneqq f * f^*$ belongs to $\mathscr{C}_c(G)$, and therefore the function $x \mapsto \lambda^x(\psi)$ is continuous on $G^{(0)}$. But then

$$\psi(e) + \psi(a) = \lim_{\substack{x \to 1 \\ x \in (0,1)}} f(x)^2 \leq 1.$$

This shows that $\psi(e)$ and $\psi(a)$ cannot both be close to 1, and therefore condition (2)(ii) of Proposition 2.1.3 cannot be satisfied.

§ 2.2.2 | The function space $\mathscr{C}_c(G)^+$

Any element $f \in \mathscr{C}_c(G)$ admits a decomposition of the form $f = \sum_{i=1}^n f_i$ for which there exist open Hausdorff sets $U_1, \dots, U_n$ such that $f_i \in C_c(U_i)$ for each i. When $f \in \mathscr{C}_c(G)$ is non-negative, we show that the f_i can also be chosen non-negative. Therefore, Definition 2.0.1 simplifies somewhat. To prove this, we will need the following technical lemma.

Lemma 2.2.3. *Let $V \subseteq G$ be an open Hausdorff subset, and let $f \in C_c(V)$.*

(i) Let $U_1, U_2 \subseteq G$ be open, and assume that U_1 is Hausdorff. If $\operatorname{supp}^\circ(f) \subseteq U_1 \cup U_2$, then f is continuous on $U_1 \setminus U_2$.

(ii) Now suppose that $f \geq 0$, and that there exist open Hausdorff sets $V_1, \dots, V_n \subseteq G$ and a compact set $K \subseteq V_1 \cup \dots \cup V_n$ such that $\operatorname{supp}^\circ(f) \subseteq K$. Then there exist $f_i \in C_c(V_i)^+ \cap C_c(V)^+$ such that

$$f = \sum_{i=1}^n f_i.$$

The non-trivial part of this lemma comes from the fact that K is not required to be contained in V. If this were the case, the existence of the f_i would be obvious by a partition of unity argument.

Proof. (i): Define $\check{U} := U_1 \setminus U_2$ and $W := \operatorname{supp}^\circ(f) \cap \check{U}$. Take a net (g_α) in $\check{U}$ converging to $g \in \check{U}$. If $g \in W$, then continuity of f on the open set $\operatorname{supp}^\circ(f)$ ensures that $f(g_\alpha)$ converges to $f(g)$, and if $g \notin \overline{W}^{\check{U}}$ then $f(g) = 0$ and $f(g_\alpha) = 0$ eventually. All that remains is to prove that $f(g_\alpha) \to f(g)$ when g is in the boundary $\partial_{\check{U}} W = \overline{W}^{\check{U}} \setminus W$. In this case $f(g) = 0$. Suppose for a contradiction that $f(g_\alpha)$ doesn't converge to 0, and let $\varepsilon > 0$ be such that there exists a subnet (g_β) with $|f(g_\beta)| \geq \varepsilon$ for all β. Since $\operatorname{supp}^\circ(f)$ is contained in a compact subset of V and $g_\beta \in \operatorname{supp}^\circ(f)$ for all β, there exists a further subnet (g_γ) converging to some limit $h \in V$. Continuity of f on V ensures that $|f(h)| \geq \varepsilon$ so in particular $h \in \operatorname{supp}^\circ(f)$. Since each $g_\gamma \notin U_2$, we also have $h \notin U_2$, and therefore $h \in U_1$. However, U_1 is Hausdorff and $g \in U_1$, so uniqueness of limits implies that $g = h$. This is a contradiction, as $g \notin \operatorname{supp}^\circ(f)$.

(ii) The set K is compact, so there exist open subsets $U_i \subseteq V_i$ with compact closure $\overline{U_i}^{V_i}$ relative to V_i such that $\operatorname{supp}^\circ(f) \subseteq U_1 \cup \dots \cup U_n$. Let us construct, inductively on $m \in \{1, \dots, n-1\}$, functions $f_i \in C_c(V_i)^+ \cap C_c(V)^+$, for $1 \leq i \leq m$, such that $\operatorname{supp}^\circ(f - f_1 - \ldots - f_m) \subseteq U_{m+1} \cup \ldots \cup U_n$ and $f - f_1 - \ldots - f_m \in C_c(V)^+$.

Define $\check{V}_1 := V_1 \setminus (U_2 \cup \ldots \cup U_n)$. The function f is nonnegative and lower-semicontinuous on G, and by part (i) it is continuous on $\check{V}_1$. Take a compact subset $K_1 \subseteq V_1$ such that $\overline{U_1}^{V_1} \subseteq \operatorname{int}(K_1)$. Observe that K_1 is normal and Hausdorff, and f is continuous on the closed subset $\check{K}_1 \coloneqq K_1 \setminus (U_2 \cup \ldots \cup U_n)$. Therefore, by [69, Theorem 2], lower-semicontinuity of f allows us to find a continuous function $\rho\colon K_1 \to [0,\infty)$ extending f on $\check{K}_1$ (i.e., $\rho \equiv f$ on $\check{K}_1$) and satisfying $\rho \leq f$ on all of K_1. Let $\chi \in C_c(V_1)$ be such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $\overline{U_1}^{V_1}$, and $\chi \equiv 0$ on $V_1 \setminus K_1$. Define $f_1\colon V_1 \to [0,\infty)$ by $f_1 \equiv \rho \cdot \chi$ on K_1 and $f_1 \equiv 0$ on $V_1 \setminus K_1$. Then $f_1 \in C_c(V_1)^+$ satisfies $f_1 \equiv f$ on $\check{V}_1$ and $f_1 \leq f$ on all of V_1.

We claim that $\operatorname{supp}^\circ(f - f_1) \subseteq U_2 \cup \ldots \cup U_n$. Let $g \notin U_2 \cup \ldots \cup U_n$. If $g \in \check{V}_1$, then $f(g) = f_1(g)$ by construction, so that $(f - f_1)(g) = 0$. Now assume that $g \notin \check{V}_1$. Then $g \notin U_1 \cup \ldots \cup U_n$ and hence $f(g) = 0$. At the same time, $g \notin V_1$ and hence $f_1(g) = 0$. We conclude that $(f - f_1)(g) = 0$, as desired.

The inequalities $0 \leq f_1 \leq f$ imply that $\operatorname{supp}^\circ(f_1) \subseteq \operatorname{supp}^\circ(f)$, and therefore $f_1 \in C_c(V)$ by part (i). It follows that $f - f_1 \in C_c(V)^+$.

Proceeding inductively, we obtain $f_i \in C_c(V_i) \cap C_c(V)$ such that $f_n \coloneqq f - f_1 - \ldots - f_{n-1} \in C_c(V)^+$ satisfies $\operatorname{supp}^\circ(f_n) \subseteq U_n$. Part (i) now implies that f_n is continuous on V_n, and thus we conclude that $f_n \in C_c(V_n)$. □

We can now prove the main result of this subsection.

Proposition 2.2.4. *For any $f \in \mathscr{C}_c(G)^+$, there exist open Hausdorff sets $V_1, \ldots, V_m \subseteq G$ and functions $f_i \in C_c(V_i)^+$ such that*

$$f = \sum_{i=1}^{n} f_i.$$

In particular, f is lower-semicontinuous, and $\operatorname{supp}^\circ(f)$ *is open.*

Proof. Let us prove the following claim by induction on n: for any $0 \leq n \leq N$, open Hausdorff sets $V_1, \ldots, V_N \subseteq G$, and functions $f_i \in C_c(V_i)$ satisfying $\sum_{i=1}^N f_i \geq 0$, there exist $\eta_i \in C_c(V_i)$ such that $\eta_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^N \eta_i = \sum_{i=1}^N f_i$. The proposition then follows by taking $N = n$.

Taking real parts, we can assume that all functions are real-valued. The base case $n = 0$ holds trivially. Assume the claim holds for some $n \geq 0$ and all $N \geq n$. Let $N \geq n+1$, and take open Hausdorff sets $V_1, \ldots, V_N \subseteq G$, and functions $f_i \in C_c(V_i)$ such that $\sum_{i=1}^N f_i \geq 0$. By the inductive hypothesis there exist $\eta_i \in C_c(V_i)$ such that $\eta_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^N \eta_i = \sum_{i=1}^N f_i$. Define $\eta_i^+, \eta_i^- \in C_c(V_i)^+$ via $\eta_i^+ \coloneqq \max(\eta_i, 0)$ and $\eta_i^- \coloneqq \max(-\eta_i, 0)$ so that $\eta_i = \eta_i^+ - \eta_i^-$. Since $\sum_{i=1}^N \eta_i \geq 0$, we have $\operatorname{supp}^\circ(\eta_{n+1}^-) \subseteq \bigcup_{i \neq n+1} \operatorname{supp}^\circ(\eta_i^+)$. By Lemma 2.2.3(ii) we can write $\eta_{n+1}^- = \sum_{i \neq n+1} \rho_i$ where $\rho_i \in C_c(V_i)^+$ for each $i \in \{1, \ldots, N\} \setminus \{n+1\}$. Define $\eta_i' \coloneqq \eta_i -$

$\rho_i \in C_c(V_i)$ for $i \in \{1,\dots,N\} \setminus \{n+1\}$. For $i = 1,\dots,n$ we have $\operatorname{supp}^\circ(\eta_i'^-) \subseteq \operatorname{supp}^\circ(\rho_i) \subseteq \operatorname{supp}^\circ(\eta_{n+1}^-)$ as $\eta_i \geq 0$, and therefore $\operatorname{supp}^\circ(\eta_i'^-) \cap \operatorname{supp}^\circ(\eta_{n+1}^+) = \emptyset$. It follows that $\sum_{i=1}^n \eta_i' + \sum_{i=n+2}^N \eta_i'^+ \geq 0$. Applying the inductive hypothesis again, there exist $\eta_i'' \in C_c(V_i)$ for $i \in \{1,\dots,N\} \setminus \{n+1\}$ such that $\eta_i'' \geq 0$ for $i = 1,\dots,n$ and $\sum_{i \neq n+1} \eta_i'' = \sum_{i=1}^n \eta_i' + \sum_{i=n+2}^N \eta_i'^+$. Define $f_{n+1}' \coloneqq \eta_{n+1}^+ \in C_c(V_{n+1})^+$, $f_i' \coloneqq \eta_i'' \in C_c(V_i)^+$ for $i = 1,\dots,n$ and $f_i' \coloneqq \eta_i'' - \eta_i'^- \in C_c(V_i)$ for $i = n+2,\dots,N$. This proves the claim for $n+1$ because

$$\sum_{i=1}^N f_i' = \eta_{n+1}^+ + \sum_{i \neq n+1} \eta_i'' - \sum_{i=n+2}^N \eta_i'^- = \eta_{n+1}^+ + \sum_{i \neq n+1} \eta_i' = \sum_{i=1}^N \eta_i = \sum_{i=1}^N f_i.$$

□

§ 2.3 | Amenability via Borel functions

The definition of topological amenability uses the existence of certain continuous functions, and thus it would be reasonable to assume that amenability of a groupoid depends on its topology. Nevertheless, by [61] it turns out that topological amenability is a Borel property. In this section, we state the main result of [61] (Proposition 2.3.1) and then prove a generalisation that holds for locally compact groupoids that are not σ-compact (Corollary 2.3.7).

§ 2.3.1 | Topological amenability is a Borel property

This subsection states results from [61]. Part of Proposition 2.3.1 below states that amenability of non-Hausdorff groupoids *can* be characterised in terms of Borel systems of measures. This is in contrast to the previous subsection, where we saw that continuous systems of measures cannot be used.

Recall from Definition 1.4.1 that a Borel system of measures is a collection $\{m^x\}_{x \in G^{(0)}}$ of positive measures where each m^x is a σ-finite Borel measure on G^x, and such that the map $x \mapsto m^x(f)$ is Borel on $G^{(0)}$ for any non-negative Borel function f on G.

Proposition 2.3.1. *[61, Proposition 2.4 & Theorem 2.14] Let G be a σ-compact locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. The following are equivalent.*

1. *G is amenable.*
2. *There exists a sequence (f_n) of non-negative Borel functions on G such that*
   - *(i) $\lambda^x(f_n) \leq 1$ for all $x \in G^{(0)}$ and all n;*
   - *(ii) $\lambda^x(f_n) \to 1$ pointwise on $G^{(0)}$ (with respect to x);*
   - *(iii) $\int_{G^{r(g)}} |f_n(g^{-1}h) - f_n(h)| \, \mathrm{d}\lambda^{r(g)}(h) \to 0$ pointwise on G (with respect to g).*

3. *There exists a sequence (m_n) of Borel systems of measures such that*

   (i) $\|m_n^x\|_1 \leq 1$ *for all* $x \in G^{(0)}$ *and all* n;

   (ii) $\|m_n^x\|_1 \to 1$ *pointwise on* $G^{(0)}$ *(with respect to* x*);*

   (iii) $\left\| g \cdot m_n^{s(g)} - m_n^{r(g)} \right\|_1 \to 0$ *pointwise on* G *(with respect to* g*).*

4. *There exists a sequence (m_n) of Borel systems of measures such that*

   (i) $\|m_n^x\|_1 = 1$ *for all* $x \in G^{(0)}$ *and all* n;

   (ii) $\left\| g \cdot m_n^{s(g)} - m_n^{r(g)} \right\|_1 \to 0$ *pointwise on* G *(with respect to* g*).*

Note in this proposition that convergence is only required to be pointwise. The price we pay is that we must work with sequences rather than nets.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is the content of [61, Theorem 2.14], and the equivalence (2) $\Leftrightarrow$ (3) is the content of [61, Proposition 2.4]. To prove (3) $\Rightarrow$ (4), note that the unit space $G^{(0)}$ is σ-compact, so by Lemma 1.2.5 there exists a non-negative Borel function ξ on G satisfying $\lambda^x(\xi) = 1$ for all x. By Lemma 1.2.6 the measures $\tilde{m}^x := \xi \, \mathrm{d}\lambda^x$ give a Borel system of probability measures on G. Defining $\tilde{m}_n^x := \frac{m_n^x}{\|m_n^x\|}$ if $\|m_n^x\| \neq 0$ and $\tilde{m}_n^x := \tilde{m}^x$ otherwise, one obtains a sequence of Borel systems of measures satisfying conditions (i) and (ii) of (4) (see also the comment before [61, Remark 2.2]). $\square$

Remark 2.3.2. It follows from statements (3) or (4) of the proposition that amenability of a σ-compact locally compact groupoid does not depend on the choice of Haar system. As was the case in Proposition 2.1.1, condition (i) is redundant in statements (2) and (3) (see the comment after [61, Definition 2.1]).

Remark 2.3.3. The paper [61] gives two definitions of topological amenability, and it is claimed in [61, Proposition 2.8] that they are equivalent. However, the first ([61, Definition 2.6]) uses continuous systems of measures, and as noted in Subsection 2.2.1 this does not work for non-Hausdorff groupoids. Therefore, [61, Proposition 2.8] is false (see also [13, Remark 5.3]). Despite this, the main result of [61] still holds with the definition of topological amenability from [61, Definition 2.7]. To close the remark, note that this definition agrees with our one from Definition 2.0.1.

We now state an "L^2" characterisation of amenability using Borel functions. This is again in contrast to the previous section, where we saw that there is no "L^2" characterisation of amenability using functions in $\mathscr{C}_c(G)$. The proof is omitted as it is identical to that of Proposition 2.1.3.

Proposition 2.3.4. *Let G be a σ-compact locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^{(0)}}$. The following statements are equivalent.*

1. *G is amenable.*
2. *There exists a sequence (f_n) of non-negative Borel functions on G such that*
   (i) $\|f_n\|_{L^2(G^x)} \leq 1$ *for all $x \in G^{(0)}$ and all n;*
   (ii) $(f_n * f_n^*)(g) \to 1$ *pointwise on G (with respect to g).*

§ 2.3.2 | Generalisation to non σ-compact groupoids

In this subsection, we prove a generalisation of Proposition 2.3.1 that applies to locally compact groupoids that are not necessarily σ-compact (see Corollary 2.3.7). This result will be used later in Theorem 3.2.2. In passing, we will also prove two permanence properties of amenability for non-Hausdorff groupoids.

We will need two preliminary lemmas. In the following, we let $B_c(G)$ denote the set of bounded Borel functions on G that have strict support contained in a compact set.

Lemma 2.3.5. *Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x\in G^{(0)}}$. Suppose that there exists a net of non-negative functions $(f_i) \subset B_c(G)$ such that*

- $\lambda^x(f_i) \leq 1$ *for all $x \in G^{(0)}$ and all i;*
- $\lambda^x(f_i) \to 1$ *uniformly on compact subsets of $G^{(0)}$ (with respect to x);*
- $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \,\mathrm{d}\lambda^{r(g)}(h) \to 0$ *uniformly on compact subsets of G (with respect to g).*

Then every open σ-compact subgroupoid of G is amenable.

In contrast with Proposition 2.3.1, we now require uniform convergence rather than pointwise convergence. The proof is based on techniques from [3] and [71].

Proof. Let H be an open σ-compact subgroupoid of G. The Haar system $\{\lambda^x\}_{x\in G^{(0)}}$ restricts to a Haar system on H. Call this restricted Haar system $\{\lambda_H^x\}_{x\in G^{(0)}}$. The unit space $H^{(0)}$ is an open subset of $G^{(0)}$, and hence $G^{H^{(0)}} := \{g \in G : r(g) \in H^{(0)}\}$ is an open subset of G. Define an equivalence relation $\sim$ on $G^{H^{(0)}}$ as follows: $g_1 \sim g_2$ if and only if there exists $h \in H$ such that $g_1 = hg_2$. Define $\Omega := G^{H^{(0)}}/\sim$, and equip Ω with the quotient topology. In other words, Ω is the orbit space of $G^{H^{(0)}}$ under the action of H by left-multiplication. Let $q\colon G^{H^{(0)}} \to \Omega$ denote the quotient map. Then q is an open map. To see this, take an open set $U \subseteq G^{H^{(0)}}$ and observe that the product HU is open by Remark 1.2.4. Therefore, $q^{-1}(q(U)) = HU$ is open.

Define the map $s^\Omega\colon \Omega \to q(H^{(0)})$ by $s^\Omega(H\cdot g) := q\big(s(g)\big)$ for $g \in G^{H^{(0)}}$ (note that this is independent of the representative g and thus well-defined). The map $q|_{H^{(0)}}\colon H^{(0)} \to$

$q(H^{(0)})$ is a homeomorphism, and $s\colon G^{H^{(0)}} \to H^{(0)}$ is a continuous open map. It is then easily seen that s^Ω is also a continuous open map. We claim that s^Ω is locally injective, from which it follows that s^Ω is a local homeomorphism. Fix $g \in G^{H^{(0)}}$. The range $r(g) = gg^{-1}$ belongs to the open subset H. Therefore, continuity of the product and inverse maps ensures that there exists an open neighbourhood $U \subseteq G^{H^{(0)}}$ of g such that whenever $g_1, g_2 \in U$ have equal source, the product $g_1 g_2^{-1}$ belongs to H. In other words, $g_1, g_2 \in U$ have equal source if and only if $g_1 \sim g_2$. Consider the image set $q(U) \subseteq \Omega$. It is open in Ω because q is an open map, and by construction s^Ω is injective on it. This proves that claim, and therefore s^Ω is a local homeomorphism. In particular, since $H^{(0)}$ is Hausdorff, it follows that Ω is locally compact and locally Hausdorff.

Given $f \in B_c(G)$ define the function f^Ω on Ω by

$$f^\Omega(H \cdot g) \coloneqq \int_{H^{r(g)}} f(h^{-1}g)\ \mathrm{d}\lambda_H^{r(g)}(h)$$

for $g \in G^{H^{(0)}}$. This is independent of the representative g because $\{\lambda_H^x\}_{x \in G^{(0)}}$ is a Haar system on H.

We claim that for any $g \in G^{H^{(0)}}$ there exists an open Hausdorff neighbourhood U such that $f^\Omega \in C_c\big(q(U)\big)$ for all $f \in C_c(U)$. Fix $g \in G^{H^{(0)}}$ and let $V^\Omega \subseteq \Omega$ be an open Hausdorff neighbourhood of $H \cdot g$ on which s^Ω restricts to a homeomorphism. Then, let $U \subseteq q^{-1}(V^\Omega)$ be an open Hausdorff neighbourhood of g so that given $g_1, g_2 \in U$ we have $g_1 \sim g_2$ if and only if $s(g_1) = s(g_2)$. For $g' \in G^{H^{(0)}}$ note that $s(g') = q|_{H^{(0)}}^{-1} \circ s^\Omega(H \cdot g')$, so that $s(g')$ varies continuously on Ω. Then, observe that

$$f^\Omega(H \cdot g') = \int_{G^{s(g')}} f(k^{-1})\ \mathrm{d}\lambda^{s(g')}(k).$$

for any $f \in C_c(U)$ and $g' \in U$. Continuity of the Haar system $\{\lambda^x\}_{x \in G^{(0)}}$ ensures that f^Ω is continuous on $q(U)$. Also, writing K for the support of f in $C_c(U)$, it is clear that f^Ω is supported on the compact set $q(K)$. Therefore $f^\Omega \in C_c\big(q(U)\big)$, proving the claim.

We now proceed to prove that H is amenable. Let $K_1 \subseteq K_2 \subseteq \cdots \subseteq H$ be an increasing sequence of compact subsets whose union is H and satisfy $K_n \subseteq \operatorname{int}(K_{n+1})$ for all n. In order to prove that H is amenable, we prove that for any $n \geq 1$ and $\varepsilon > 0$ there exists a non-negative Borel function f on H satisfying the following three conditions.

(i) $\lambda_H^x(f) \leq 1$ for all $x \in H^{(0)}$.

(ii) $\lambda_H^x(f) > 1 - \varepsilon$ for all $x \in r(K_n)$.

(iii) $\int_{H^{r(g)}} |f(g^{-1}h) - f(h)|\ \mathrm{d}\lambda_H^{r(g)}(h) < \varepsilon$ for all $g \in K_n$.

By assumption, there exists a non-negative function $f' \in B_c(G)$ satisfying conditions (i), (ii), (iii) with the Haar system $\{\lambda_H^x\}_{x\in H^{(0)}}$ replaced by $\{\lambda^x\}_{x\in G^{(0)}}$. We can assume that $r(\mathrm{supp}^\circ(f')) \subseteq r(K_n)$, so let $L \subseteq G^{H^{(0)}}$ be a compact set containing $\mathrm{supp}^\circ(f')$. By the claim, we can cover L with open sets $U_1, \ldots, U_n \subseteq G^{H^{(0)}}$ such that $\varphi_j^\Omega \in C_c\big(q(U_j)\big)$ for any $\varphi_j \in C_c(U_j)$. Choose $\varphi_j \in C_c(U_j)^+$ such that $\varphi := \sum_{j=1}^n \varphi_j$ satisfies $L \subseteq \mathrm{supp}^\circ(\varphi)$. The compact set $q(L)$ is contained in the union $\bigcup_{j=1}^n q(\mathrm{supp}^\circ(\varphi_j))$. Moreover, it is readily verified that φ_j^Ω is strictly positive on $q(\mathrm{supp}^\circ(\varphi_j))$ for each j (see Remark 1.2.2), and therefore $\varphi^\Omega = \sum_{j=1}^n \varphi_j^\Omega$ is strictly positive on $q(L)$. Lower-semicontinuity of φ^Ω then ensures that there exists a number $\delta > 0$ and an open set $V^\Omega \subseteq \Omega$ containing $q(L)$ such that $\varphi^\Omega(H\cdot g) > \delta$ for all $H \cdot g \in V^\Omega$. Define

$$\tilde{\varphi}(g) := \frac{\mathbb{1}_{q^{-1}(V^\Omega)}(g)}{\varphi^\Omega(H\cdot g)}\varphi(g)$$

for $g \in G^{H^{(0)}}$. Then $\tilde{\varphi}$ is a non-negative bounded Borel function on $G^{H^{(0)}}$ whose strict support is contained in a compact set, and satisfies $\tilde{\varphi}^\Omega = \mathbb{1}_{V^\Omega}$. In particular, we have $\tilde{\varphi}^\Omega(H\cdot g) = 1$ for all $g \in L$. Define

$$f(g) := \int_{G^{r(g)}} \tilde{\varphi}(g^{-1}k) f'(k)\, \mathrm{d}\lambda^{r(g)}(k) \tag{2.3.1}$$

for $g \in H$. This is a convolution product of two functions in $B_c(G)$, and therefore f is a non-negative Borel function. The function $(k, g) \mapsto \tilde{\varphi}(g^{-1}k)f'(k)$ is bounded and has strict support contained in a compact subset of $G^{H^{(0)}} \times G^{H^{(0)}}$, therefore we can apply the Fubini-Tonelli theorems freely. We check that f satisfies all three conditions (i), (ii), (iii) above. The computations are similar to those in the proof of Proposition 2.1.1 (4) $\Rightarrow$ (3).

Condition (i) is clear as $\lambda_H^x(f) = \int_{G^x} \tilde{\varphi}^\Omega(H\cdot k)f'(k)\,\mathrm{d}\lambda^x(k) \le 1$ for all $x \in H^{(0)}$. Then, for $x \in r(K_n)$ we have

$$\begin{aligned}
\lambda_H^x(f) &= \int_{H^x}\int_{G^x} \tilde{\varphi}(g^{-1}k)f'(k)\,\mathrm{d}\lambda^x(k)\,\mathrm{d}\lambda_H^x(g) \\
&= \int_{G^x\cap L}\left[\int_{H^x} \tilde{\varphi}(g^{-1}k)\,\mathrm{d}\lambda_H^x(g)\right] f'(k)\,\mathrm{d}\lambda^x(k) \\
&= \int_{G^x} f'(k)\,\mathrm{d}\lambda^x(k) \\
&> 1-\varepsilon
\end{aligned}$$

so that f satisfies (ii). Lastly, observe that

$$\begin{aligned}
&\int_{H^{r(g)}} \left|f(g^{-1}h) - f(h)\right|\,\mathrm{d}\lambda_H^{r(g)}(h) \\
&\quad= \int_{H^{r(g)}}\left|\int_{G^{s(g)}} \tilde{\varphi}(h^{-1}gk)f'(k)\,\mathrm{d}\lambda^{s(g)}(k) - \int_{G^{r(g)}} \tilde{\varphi}(h^{-1}k)f'(k)\,\mathrm{d}\lambda^{r(g)}(k)\right|\mathrm{d}\lambda_H^{r(g)}(h) \\
&\quad\le \int_{H^{r(g)}}\int_{G^{r(g)}} \tilde{\varphi}(h^{-1}k)\left|f'(g^{-1}k) - f'(k)\right|\,\mathrm{d}\lambda^{r(g)}(k)\,\mathrm{d}\lambda_H^{r(g)}(h) \\
&\quad= \tilde{\varphi}^\Omega(H\cdot k)\int_{G^{r(g)}}\left|f'(g^{-1}k) - f'(k)\right|\,\mathrm{d}\lambda^{r(g)}(k) < \varepsilon
\end{aligned}$$

for all $g \in K_n$. The non-negative Borel function f satisfies all three conditions (i), (ii), (iii), and therefore the σ-compact groupoid H is amenable by Proposition 2.3.1. $\square$

Lemma 2.3.6. *Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x\in G^{(0)}}$. Suppose that every open σ-compact subgroupoid is amenable. Then G is also amenable.*

Proof. We construct an explicit net in $\mathscr{C}_c(G)^+$ satisfying the conditions of Definition 2.0.1. Let $\mathscr{S}$ be the set of triples (H, K, n) where $H \subseteq G$ is an open σ-compact subgroupoid, $K \subseteq H$ is compact, and $n \geq 1$. Equip $\mathscr{S}$ with the following order: we declare that $(H_1, K_1, n_1) \leq (H_2, K_2, n_2)$ if and only if $H_1 \subseteq H_2$, $K_1 \subseteq K_2$ and $n_1 \leq n_2$. With this ordering $\mathscr{S}$ becomes a directed set. By assumption, any open σ-compact subgroupoid is amenable. Therefore, given any triple $s = (H, K, n) \in \mathscr{S}$ there exists a function $f_s \in \mathscr{C}_c(H)^+$ satisfying $\lambda^x(f_s) \leq 1$ for $x \in H^{(0)}$, $|1 - \lambda^x(f_s)| \leq \frac{1}{n}$ for $x \in r(K)$ and $\int_{H^{r(g)}} |f_s(g^{-1}h) - f_s(h)| \,\mathrm{d}\lambda^{r(g)}(h) \leq \frac{1}{n}$ for $g \in K$. Since H is open in G, each f_s also belongs to $\mathscr{C}_c(G)^+$. Note that any compact subset of G is contained in an open σ-compact subgroupoid. It follows that the net $(f_s)_{s\in\mathscr{S}}$ satisfies the conditions of Definition 2.0.1. Therefore G is amenable, as required. $\square$

The following is now an immediate consequence of the two preceeding lemmas.

Corollary 2.3.7. *Let G be a locally compact groupoid with Haar system $\{\lambda^x\}_{x\in G^{(0)}}$. Then G is amenable if and only if there exists a net of non-negative functions $(f_i) \subset B_c(G)$ such that*

- $\lambda^x(f_i) \leq 1$ *for all $x \in G^{(0)}$ and all i;*
- $\lambda^x(f_i) \to 1$ *uniformly on compact subsets of $G^{(0)}$ (with respect to x);*
- $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \,\mathrm{d}\lambda^{r(g)}(h) \to 0$ *uniformly on compact subsets of G (with respect to g).*

As another consequence, we observe that amenability of any locally compact groupoid is inherited by its open subgroupoids. This fact was proved for second-countable non-Hausdorff groupoids in [48, Proposition 2.17].

Corollary 2.3.8. *Let G be a locally compact groupoid with Haar system. If G is amenable, then any open subgroupoid is also amenable.*

Proof. Let H be an open subgroupoid of the amenable groupoid G. Any open σ-compact subgroupoid of H is an open σ-compact subgroupoid of G, and is thus amenable by Lemma 2.3.5. Lemma 2.3.6 then implies that H is amenable. $\square$

§ 2.4 | Nuclearity via Renault's disintegration theorem

The goal of this section is to discuss the following theorem.

Theorem 2.4.1. *Let G be a second-countable (non-Hausdorff) locally compact groupoid with Haar system. If G is amenable, the following two statements hold.*

(i) The canonical surjection $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism.

(ii) The full C^*-algebra $C^*(G)$ is nuclear.

For Hausdorff groupoids these results were proved in [3, Proposition 6.1.8 & Corollary 6.2.14], and were known by experts to hold also for non-Hausdorff groupoids. Indeed, the proof from [3] can be adapted to the non-Hausdorff setting because, for the most part, it only uses the Borel structure of the underlying groupoid. However, it is a standing assumption throughout [3] that all groupoids are Hausdorff. We take this opportunity to give a proof in the non-Hausdorff setting, as no proof currently exists in the literature. We follow [3] closely.

For both parts of the theorem, the key tool is Renault's disintegration theorem (stated as Theorem 1.4.11). This result allows us to view *-representations of $\mathscr{C}_c(G)$ as integrated forms of unitary representations (see Subsection 1.4.2). By taking results from [3], [58] and [60], we will obtain analogues of Theorem 2.4.1 in the measured setting. We then apply the Renault disintegration theorem to lift these results to the setting of locally compact groupoids. Overall, topological amenability of the groupoid G is not actually needed in Theorem 2.4.1. As noted in [3], it suffices for G to be "measurewise amenable" (see Definition 2.4.3). It is easily verified that any topologically amenable groupoid is measurewise amenable (Lemma 2.4.4), and thus Theorem 2.4.1 can be deduced from Corollary 2.4.7 and Corollary 2.4.9 in the following sections.

Throughout this section, groupoids are assumed to be second-countable. This is necessary because the proof of Renault's disintegration theorem from [51] makes this assumption.

Let us briefly discuss the converse implications of Theorem 2.4.1. It has been known for some time that statement (i) is equivalent to amenability for any locally compact group. However, it does not imply amenability for groupoids. In fact, there exist non-amenable, ample, Hausdorff groupoids for which statement (i) holds (see [70]). Statement (ii) fails to imply amenability even for groups: all connected Lie groups have nuclear reduced C^*-algebra, and they are certainly not all amenable. On the other hand, it is known that statement (ii) is equivalent to amenability for Hausdorff étale groupoids. In Section 3.2 we will see that it is also equivalent to amenability for non-Hausdorff étale groupoids.

§ 2.4.1 | Amenability for measured groupoids

In order to prove Theorem 2.4.1, we will need to work with measured groupoids as defined in Section 1.4. Let us define what it means for a measured groupoid to be amenable.

Definition 2.4.2 ([3])**.** A measured groupoid (G, λ, μ) is *amenable* if there exists a net of non-negative Borel functions (f_i) on G satisfying the following conditions.

(i) $\lambda^x(f_i) \leq 1$ for all $x \in G^{(0)}$ and all i.

(ii) $\lambda^x(f_i) \to 1$ in the weak* topology on $L^\infty(G^{(0)}, \mu)$ (with respect to x).

(iii) $\int_{G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \, \mathrm{d}\lambda^{r(g)}(h) \to 0$ in the weak* topology on $L^\infty(G, \mu \circ \lambda)$ (with respect to g).

The definition above is not the one given in [3, Definition 3.2.8]. However, the two definitions are equivalent by [3, Proposition 3.2.14 & Remark 3.2.15]. In condition (i) it suffices to have $\lambda^x(f_i) \leq 1$ for μ-almost all $x \in G^{(0)}$ and all i.

Let us now discuss the connection between topological amenability and amenability for a measured groupoid. It is convenient to introduce another definition.

Definition 2.4.3 ([3, Definition 3.3.1])**.** Let G be a second-countable locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x \in G^{(0)}}$. We say that G is *measurewise amenable* if (G, λ, μ) is an amenable measured groupoid for any σ-finite Radon measure μ on $G^{(0)}$ that is quasi-invariant with respect to λ.

Given our choice of definition for amenability of a measured groupoid in Definition 2.4.2, the following is obvious.

Lemma 2.4.4 ([3, Proposition 3.3.2])**.** *Let G be a second-countable locally compact groupoid with Haar system.*

If G is topologically amenable, then it is measurewise amenable.

Remark 2.4.5. In [61, Corollary 2.16] it is noted that the converse to this lemma holds in certain cases – even for non-Hausdorff groupoids. Indeed, if G is a second-countable locally compact groupoid with countable orbits in $G^{(0)}$, then G is topologically amenable if and only if it is measurewise amenable. In particular, this holds for any second-countable étale groupoid.

§ 2.4.2 | Amenability implies the coincidence of full and reduced C^*-algebras

In this subsection, we show that for any measurewise amenable groupoid, the full and reduced C^*-algebras coincide ([3, Proposition 6.1.8]). By the Renault disintegration theorem (Theorem 1.4.11) it suffices to show that the integrated form of any unitary representation is continuous with respect to the reduced norm $\|\cdot\|_{C_r^*(G)}$ on $\mathscr{C}_c(G)$. If one views G as a measured groupoid, this is precisely the contents of [60, Proposition 3.4].

Proposition 2.4.6 ([60, Proposition 3.4])**.** *Let (G, λ, μ) be an amenable measured groupoid.*

Any unitary representation of (G, λ, μ) is weakly contained in the regular representation. That is, given any unitary representation $(G^{(0)} * \mathscr{H}, L)$ with integrated form $\mathbb{L}$, then

$$\|\mathbb{L}(\psi)\| \leq \left\|Reg_\mu(\psi)\right\|$$

for all $\psi \in I(G, \lambda, \mu)$.

We can now prove the desired result. The proof here is identical to that in [3, Proposition 6.1.8].

Corollary 2.4.7. *Let G be a second-countable (non-Hausdorff) locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x \in G^{(0)}}$.*

If G is measurewise amenable, then the canonical surjection $C^*(G) \to C_r^*(G)$ is an isomorphism.

Proof. Let $\pi \colon C^*(G) \to \mathscr{L}(H)$ be a faithful representation of the full C^*-algebra $C^*(G)$. By Theorem 1.4.11, there exists unitary representation of a measured groupoid (G, λ, μ) whose integrated form $\mathbb{L}$ is unitarily equivalent to π on $\mathscr{C}_c(G)$. By assumption, (G, λ, μ) is an amenable measured groupoid. Therefore, by Proposition 2.4.6, we have $\|\mathbb{L}(\psi)\| \leq \left\|\mathrm{Reg}_\mu(\psi)\right\|$ for any $\psi \in I(G, \lambda, \mu)$, and thus $\|\pi(f)\| \leq \left\|\mathrm{Reg}_\mu(f)\right\|$ for any $f \in \mathscr{C}_c(G)$. The result follows because $\left\|\mathrm{Reg}_\mu(f)\right\| \leq \|f\|_{C_r^*(G)}$ for any $f \in \mathscr{C}_c(G)$ (Lemma 1.4.10). □

§ 2.4.3 | Amenability implies nuclearity of the reduced C^*-algebra

We finish the section by showing that any measurewise amenable groupoid has nuclear C^*-algebra ([3, Corollary 6.2.14(i)]). The idea involves showing that any unitary representation of an amenable measured groupoid (G, λ, μ) has injective von Neumann algebra. In [3], this is shown by reducing the problem to one concerning unitary representations of the transformation groupoid $G \rtimes G$. The groupoid $G \rtimes G$ is Morita equivalent to the space $G^{(0)}$, and therefore all its representations have Type I von Neumann algebra.

In order to keep our proof as self-contained as possible, we use the more direct argument of [58, Proposition II.3.5] instead (see also [73, Theorem 2.1]). We follow its proof verbatim.

Proposition 2.4.8 ([3, Corollary 6.2.2], [58, Proposition II.3.5]). *Let (G, λ, μ) be an amenable measured groupoid, and let $(G^{(0)} * \mathscr{H}, L)$ be any unitary representation of (G, λ, μ) with integrated form $\mathbb{L}$.*

The von Neumann algebra $\mathcal{M}$ generated by the image of $\mathbb{L}$ is injective.

In the following proof, we let $\mathcal{D}$ denote the set of decomposable operators, and let $\mathcal{T}$ denote the set of diagonalisable operators on $L^2(G^{(0)} * \mathscr{H}, \mu)$ (see Definition 1.4.6).

Proof. It suffices to prove that the commutant $\mathcal{M}'$ is injective. The von Neumann algebra $\mathcal{D}$ is injective as it is equal to the commutant of the (commutative) algebra $\mathcal{T}$. Next, note that $\mathcal{T} \subseteq \mathcal{M}$ and hence $\mathcal{M}' \subseteq \mathcal{D}$. It is then simple to check that $\mathcal{M}'$ is equal to the set of operators $T \in \mathcal{D}$ satisfying $L_g \circ T_{s(g)} = T_{r(g)} \circ L_g$ for $\mu \circ \lambda$-almost all $g \in G$. In order to prove that $\mathcal{M}'$ is injective, we construct a conditional expectation $P \colon \mathcal{D} \to \mathcal{M}'$.

Let (f_i) be a net of non-negative Borel functions as in Definition 2.4.2. For each i, define the map $P_i \colon \mathcal{D} \to \mathcal{D}$ as follows. Given $T \in \mathcal{D}$ and $x \in G^{(0)}$, let $(P_i T)_x$ be the bounded linear operator on $H(x)$ satisfying

$$\langle (P_i T)_x \xi, \eta \rangle = \int_{G^x} f_i(g) \Big\langle L_g \circ T_{s(g)} \circ L_g^*(\xi), \eta \Big\rangle \, \mathrm{d}\lambda^x(g)$$

for $\xi, \eta \in H(x)$ (for each i, the operators $(P_i T)_x$ are defined for μ-almost all $x \in G^{(0)}$). Then $P_i \colon \mathcal{D} \to \mathcal{D}$ is a well-defined linear map. Moreover, $\|P_i\| \leq 1$ by condition (i) of Definition 2.4.2. Since $\mathcal{D}$ is a von Neumann algebra, its unit ball is compact in the weak operator topology (WOT) and, therefore, the space of linear maps $Q \colon \mathcal{D} \to \mathcal{D}$ with $\|Q\| \leq 1$ is compact in the point-WOT topology. We may thus select a subnet (P_j) that converges in the point-WOT topology to some linear map $P \colon \mathcal{D} \to \mathcal{D}$ with $\|P\| \leq 1$. We claim that P is a conditional expectation $\mathcal{D} \to \mathcal{M}'$.

Let us show that P restricts to the identity on $\mathcal{M}'$. Indeed, given $T \in \mathcal{M}'$, then

$$\begin{aligned} \langle (P_j T)_x \xi, \eta \rangle &= \int_{G^x} f_j(g) \Big\langle L_g \circ T_{s(g)} \circ L_g^*(\xi), \eta \Big\rangle \, \mathrm{d}\lambda^x(g) \\ &= \Big\langle T_x(\xi), \eta \Big\rangle \int_{G^x} f_j(g) \, \mathrm{d}\lambda^x(g) \end{aligned}$$

for μ-almost all $x \in G^{(0)}$ and $\xi, \eta \in H(x)$. After integrating over $G^{(0)}$ with respect to μ, it follows from condition (ii) of Definition 2.4.2 that $P_j T \to T$ in the WOT.

It remains to prove that P maps $\mathcal{D}$ into the commutant $\mathcal{M}'$. Take $T \in \mathcal{D}$. We will show that $\mathbb{L}(\psi)(PT) = (PT)\mathbb{L}(\psi)$ for all $\psi \in I(G, \lambda, \mu)$. Given $\psi \in I(G, \lambda, \mu)$ and $\varphi_1, \varphi_2 \in L^2(G^{(0)} * \mathscr{H}, \mu)$ calculate

$$\Big\langle \mathbb{L}(\psi)(P_jT)\varphi_1, \varphi_2 \Big\rangle = \int_G \psi(g) \Big\langle L_g\Big((P_jT)\varphi_1(s(g))\Big), \varphi_2(r(g))\Big\rangle \Delta(g)^{\frac{1}{2}} \, \mathrm{d}(\mu \circ \lambda)(g)$$
$$= \int_G \int_{G^{s(g)}} \psi(g) f_j(h) \Big\langle L_{gh} \circ T_{s(h)} \circ L_h^*\Big(\varphi_1(s(g))\Big), \varphi_2(r(g))\Big\rangle \, \mathrm{d}\lambda^{s(g)}(h) \; \Delta(g)^{\frac{1}{2}} \mathrm{d}(\mu \circ \lambda)(g)$$

and

$$\Big\langle (P_jT)\mathbb{L}(\psi)\varphi_1, \varphi_2 \Big\rangle = \int_{G^{(0)}} \int_{G^x} f_j(h) \Big\langle L_h \circ T_{s(h)} \circ L_h^*\Big(\mathbb{L}(\psi)\varphi_1(x)\Big), \varphi_2(x)\Big\rangle \, \mathrm{d}\lambda^x(h) \mathrm{d}\mu(x)$$
$$= \int_G \int_{G^{s(g)}} \psi(g) f_j(g\dot{h}) \Big\langle L_{g\dot{h}} \circ T_{s(h)} \circ L_{\dot{h}}^*\Big(\varphi_1(s(g))\Big), \varphi_2(r(g))\Big\rangle \, \mathrm{d}\lambda^{s(g)}(\dot{h}) \; \Delta(g)^{\frac{1}{2}} \mathrm{d}(\mu \circ \lambda)(g)$$

using (1.4.7) as well as invariance of the Borel Haar system λ (see (1.4.1)). Now, take the difference and compute

$$\begin{aligned} &\Big|\Big\langle \Big(\mathbb{L}(\psi)(P_jT) - (P_jT)\mathbb{L}(\psi)\Big)\varphi_1, \varphi_2 \Big\rangle\Big| \leq \qquad (2.4.1) \\ &\|T\| \int_G |\psi(g)| \, \|\varphi_1(s(g))\| \, \|\varphi_2(r(g))\| \left(\int_{G^{r(g)}} \Big| f_j(g^{-1}h) - f_j(h) \Big| \, \mathrm{d}\lambda^{r(g)}(h) \right) \Delta(g)^{\frac{1}{2}} \mathrm{d}(\mu \circ \lambda)(g). \end{aligned}$$

The Cauchy-Schwarz inequality yields the bound

$$\int_G |\psi(g)| \, \|\varphi_1(s(g))\| \, \|\varphi_2(r(g))\| \, \Delta(g)^{\frac{1}{2}} \, \mathrm{d}(\mu \circ \lambda)(g) \leq \|\psi\|_{I,\mu} \, \|\varphi_1\| \, \|\varphi_2\| \, .$$

Therefore, it follows from condition (iii) of Definition 2.4.2 that the value in (2.4.1) converges to 0. Since P_jT converges to PT in the WOT, we have $(PT)\mathbb{L}(\psi) = \mathbb{L}(\psi)(PT)$, and therefore $PT \in \mathcal{M}'$. □

The following corollary is now a simple consequence of results we have already seen. By combining it with Lemma 2.4.4, we complete the proof of Theorem 2.4.1.

Corollary 2.4.9. *Let G be a second-countable (non-Hausdorff) locally compact groupoid with Haar system $\lambda = \{\lambda^x\}_{x \in G^{(0)}}$.*

If G is measurewise amenable, then $C^*(G)$ is nuclear.

Proof. Let $\pi_\Phi \colon C^*(G) \to \mathcal{L}(H_\Phi)$ denote the universal representation of $C^*(G)$ so that the double commutant $\pi_\Phi\Big(C^*(G)\Big)''$ is equal to the enveloping von Neumann algebra of $C^*(G)$. By Theorem 1.4.11 there exists a unitary representation of a measured groupoid (G, λ, μ) whose integrated form $\mathbb{L}$ is unitarily equivalent to π_Φ on $\mathscr{C}_c(G)$. Since $\mathbb{L}(\mathscr{C}_c(G))$ is dense in $\mathbb{L}\Big(I(G, \lambda, \mu)\Big)$ with respect to the weak operator topology

(Lemma 1.4.9), the von Neumann algebra $\pi_\Phi\big(C^*(G)\big)''$ is isomorphic to the von Neumann algebra generated by the image of $I(G, \lambda, \mu)$ under $\mathbb{L}$. By Proposition 2.4.8, it follows that $\pi_\Phi\big(C^*(G)\big)''$ is injective, and therefore, $C^*(G)$ is nuclear by [15]. $\square$

§ 2.5 | Examples

In this short section we discuss the amenability of the groupoids appearing Section 1.6. However, this discussion will be brief for the following reasons – if one wants to check amenability of a non-Hausdorff groupoid in practice, one should instead check amenability of the Hausdorff cover $\tilde{G}$ (see Chapter 3). The Hausdorff cover $\tilde{G}$ is a Hausdorff groupoid one can associate with any locally compact groupoid G. In particular, it is amenable if and only if G is amenable (at least for étale groupoids, see Theorem 3.2.2). As seen earlier in the chapter (Example 2.2.2), amenability is not as easily described for non-Hausdorff groupoids as it is in the Hausdorff case and, therefore, it will always be preferable to check amenability of $\tilde{G}$ instead of G.

Let us discuss the groupoids from Examples 1.6.1 to 1.6.6. Each of these examples has equal range and source maps. A locally compact groupoid with equal range and source maps is amenable if and only if its isotropy groups are all amenable. Therefore the Examples 1.6.3, 1.6.4 and 1.6.6 are amenable, and the groupoids from Examples 1.6.1 and 1.6.2 are amenable if and only if the group Γ is amenable. The groupoid from Example 1.6.5 has an isotropy group isomorphic to $\mathbb{F}_2 \times V_4$, and is therefore non-amenable.

Next, let us discuss amenability for groupoids associated with contracting self-similar groups (see Subsection 1.6.4). Given any contracting self-similar group Γ acting on a finite alphabet, its associated groupoid $G := \mathrm{Germ}(S_\Gamma \curvearrowright X)$ is always amenable. This was shown for Hausdorff G in [53], and was then extended to non-Hausdorff groupoids in [49, Corollary 2.19] (the arguments from the Hausdorff setting can be applied with the aid of [61]).

Take a discrete group Γ acting on a locally compact space X, and let $\mathrm{Germ}(\Gamma \curvearrowright X)$ denote the associated groupoid of germs from Subsection 1.6.2. By applying Reiter's condition for amenability of a group, it is easily verified that $\mathrm{Germ}(\Gamma \curvearrowright X)$ is amenable whenever Γ is amenable.

Subsection 1.6.3 introduced the groupoids of germs associated with the actions of Thompson's groups and, more generally, Bieri-Strebel groups. It goes without saying that whether Thompson's groups F and T are amenable is not known, so the observation above is of no use. Nevertheless, it was shown in [63, Theorem 3.3] that the groupoid of germs $\mathrm{Germ}(T \curvearrowright \mathbb{T})$ is "Borel amenable". We now discuss this example, as well as the proof from [63], in more detail. In Subsection 3.4.3 of the next chapter, we will use the Hausdorff cover to prove amenability for groupoids of germs associated with some other Bieri-Strebel groups. This computation using the Hausdorff cover follows a similar approach to the one discussed here, but it becomes simpler once we start working with a Hausdorff groupoid (see Example 3.4.11).

Example 2.5.1 ([63])**.** We follow [63] in showing that the groupoid $G \coloneqq \mathrm{Germ}(T \curvearrowright \mathbb{T})$ satisfies statement (2) of Proposition 2.3.1. It then follows that G is amenable. Recall from Example 1.6.8 that the set of dangerous points $D \subseteq G^{(0)}$ equals $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$, and therefore the set of Hausdorff units is $C = \mathbb{T} \setminus (\mathbb{Z}[\frac{1}{2}]/\mathbb{Z})$. We work on the Borel subgroupoids $G|_D$ and $G|_C$ separately. First consider the groupoid $G|_D$. This is a transitive groupoid with isotropy group $\mathbb{Z}^2$. Since it is countable, its Borel structure is discrete. By amenability of $\mathbb{Z}^2$, it is clear that this discrete groupoid is amenable.

Next, consider the groupoid $G|_C$. We recall the canonical action of T on the Cantor space $\{0,1\}^{\mathbb{N}}$. Define the surjective continuous map $\psi\colon \{0,1\}^{\mathbb{N}} \to \mathbb{T}$ by $\psi(\alpha) \coloneqq \sum_{n=1}^{\infty} \alpha_n 2^{-n}$. Writing $Q \coloneqq \psi^{-1}\Big(\mathbb{T} \setminus (\mathbb{Z}[\frac{1}{2}]/\mathbb{Z})\Big)$, the map ψ restricts to a homeomorphism $\psi|_Q\colon Q \to \mathbb{T} \setminus (\mathbb{Z}[\frac{1}{2}]/\mathbb{Z})$. Since $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ is an orbit of the action of T on $\mathbb{T}$, there is an induced action of T on Q. This then extends uniquely to an action on all of $\{0,1\}^{\mathbb{N}}$. By [56, Proposition 4.10] (see also [10, Example 3.3]), the groupoid $\mathrm{Germ}(T \curvearrowright \{0,1\}^{\mathbb{N}})$ is isomorphic to the Cuntz groupoid O_2 from [58]. In particular, this groupoid is amenable by [58, Proposition III.2.5]. Define the map

$$\begin{aligned} \Psi \,:\, G|_C &\longrightarrow \mathrm{Germ}(T \curvearrowright \{0,1\}^{\mathbb{N}}) \\ [f,t] &\mapsto [f, \psi^{-1}(t)]. \end{aligned}$$

This map is a Borel isomorphism onto its image. Moreover, its image is an *invariant* Borel subgroupoid of the amenable groupoid $\mathrm{Germ}(T \curvearrowright \{0,1\}^{\mathbb{N}})$, and therefore there exists a sequence (f_n) of non-negative Borel functions on $G|_C$ satisfying the conditions of Proposition 2.3.1(2). Combining this with amenability of the discrete groupoid $G|_D$, we can construct a sequence $(\tilde{f}_n)$ of non-negative Borel functions defined on all of G, and satisfying the conditions of Proposition 2.3.1(2). Therefore, G is amenable.

The following is now immediate by Theorem 2.4.1.

Corollary 2.5.2. *Let $G \coloneqq \mathrm{Germ}(T \curvearrowright \mathbb{T})$ be the groupoid of germs associated with the canonical action of Thompson's group T on the circle $\mathbb{T}$.*

The canonical surjection $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism, and $C^*_r(G)$ is nuclear.

We finish this chapter by briefly commenting on permanence properties. For Hausdorff groupoids, amenability is known to satisfy a variety of permanence properties (see [3] and also [71]). One can ask whether these permanence results also hold in the non-Hausdorff setting, and in many cases this can be verified by a direct computation. Indeed, in Subsection 2.3.2 it was shown directly that amenability passes to open subgroupoids (see Corollary 2.3.8). However, it is usually more efficient to use the Hausdorff cover groupoid instead (see Chapter 3). The Hausdorff cover $\tilde{G}$ sees many of the structural properties of the original groupoid G, and, as mentioned earlier, it is amenable if and only if G is amenable (at least for étale groupoids, see Theorem 3.2.2).

Chapter 3

The Hausdorff cover

This chapter is focused on the Hausdorff cover of a non-Hausdorff groupoid. Originally defined in [68], the Hausdorff cover serves as a tool for reducing questions about non-Hausdorff groupoids to the Hausdorff setting. In general, many technical challenges can arise when proving a given result regarding non-Hausdorff groupoids. These technical challenges may not be present in the Hausdorff setting, and in many cases, it may be possible to avoid these challenges altogether with the aid of the Hausdorff cover. In this chapter, we demonstrate two instances of this.

In Section 3.2, we will use the Hausdorff cover to prove that a non-Hausdorff étale groupoid is amenable if and only if its reduced (or full) C^*-algebra is nuclear. The proof amounts to showing that a groupoid is amenable if and only if its Hausdorff cover is amenable, and also that its C^*-algebra is nuclear if and only if the cover has nuclear C^*-algebra. The result then follows by known results in the Hausdorff case. Then, Section 3.3 describes how the ideal structure in a non-Hausdorff groupoid C^*-algebra can be understood in terms of the cover, and proves a result regarding isotropy fibres of ideals using known results from the Hausdorff setting. An important feature of the Hausdorff cover is that it is relatively simple to compute. The final section of this chapter computes the Hausdorff cover for each of the examples introduced in Section 1.6.

In this chapter and the next, we will exclusively be concerned with étale groupoids. The definition of the Hausdorff cover extends to locally compact groupoids that are not étale, as is discussed in [68], but we do not consider such groupoids here.

Sections 3.1 and 3.2 of this chapter are based around results in [9]. Section 3.3 is based on [31, § 2].

§ 3.1 | Topology and structure

§ 3.1.1 | Definitions and preliminaries

Let G be a non-Hausdorff étale groupoid. The Hausdorff cover is constructed in the space $\mathfrak{C}(G)$ of all closed subsets of G. This space carries a natural topology, known as the *Fell topology*, defined in [28]. The fact that G is locally compact ensures that the Fell topology is compact and Hausdorff.

Definition 3.1.1 ([68])**.** Let G be an étale groupoid. Let $\mathfrak{C}(G)$ be the space of closed subsets of G equipped with the Fell topology, and let $\iota\colon G \to \mathfrak{C}(G)$ be the canonical inclusion given by $\iota(g) := \{g\}$. The *Hausdorff cover* $\tilde{G}$ is defined to be the closure of $\iota(G)$ in $\mathfrak{C}(G) \setminus \{\emptyset\}$.

Since $\mathfrak{C}(G)$ is compact and Hausdorff ([28, Theorem 1]), $\tilde{G}$ is locally compact and Hausdorff. Before describing the groupoid operations on G, we will describe the Fell topology. Let us first describe its basic open sets. Given a compact set $K \subseteq G$ and a finite family of open sets $U_1, \ldots, U_n \subseteq G$, define

$$\mathcal{U}(U_1, \ldots, U_n; K) := \{C \in \mathfrak{C}(G)\colon C \cap U_i \neq \emptyset \ \forall 1 \leq i \leq n, C \cap K = \emptyset\}.$$

The sets $\mathcal{U}(U_1, \ldots, U_n; K)$ form a basis for the Fell topology, where the U_i run through all open subsets of G, and K runs through all compact subsets of G. Since the open bisections form a basis for the topology on G, it then follows that the sets $\mathcal{U}(U; K)$ form a sub-basis for the Fell topology, where U runs through all open bisections and K runs through all compact bisections contained in an open bisection.

Let us now describe the elements of $\tilde{G}$, as well as its topology, in terms of nets. In the language of [28], a net (g_α) in G is *primitive* if any limit point of any subnet is also a limit point of (g_α). Then, a non-empty subset $\boldsymbol{g} \subseteq G$ belongs to $\tilde{G}$ if and only if there exists a primitive net (g_α) in G whose set of limit points is precisely $\boldsymbol{g}$. Next, given a net (g_α) in G, the image net $\iota(g_\alpha)$ is convergent in $\tilde{G}$ if and only if (g_α) is primitive, in which case the limit $\boldsymbol{g} \in \tilde{G}$ is equal to the set of limit points of (g_α) in G. By the axiom of choice, any net has a primitive subnet, therefore any net (g_α) that converges in G has a subnet (g_β) such that $\iota(g_\beta)$ converges in $\tilde{G}$.

We also need to introduce the following notation associated with open bisections. Given an open bisection $U \subseteq G$, let $\tilde{U} \subseteq \tilde{G}$ be the basic open set defined as

$$\tilde{U} := \mathcal{U}(U; \emptyset) = \{\boldsymbol{g} \in \tilde{G}\colon \boldsymbol{g} \cap U \neq \emptyset\}. \tag{3.1.1}$$

Since open bisections are Hausdorff, each element $\boldsymbol{g} \in \tilde{U}$ has a unique point of intersection with U. Let $\pi_U\colon \tilde{U} \to U$ be the surjective map satisfying

$$\{\pi_U(\boldsymbol{g})\} = \boldsymbol{g} \cap U \tag{3.1.2}$$

for each $\boldsymbol{g} \in \tilde{U}$. Since U is open, the set $\iota(U) \subseteq \tilde{U}$ is dense in $\tilde{U}$. It is then simple to check that π_U is continuous (using the description of convergence in $\tilde{G}$ via primitive nets). In the proof of Lemma 3.1.4(i) we will see that π_U is in fact proper.

For the special case $U = G^{(0)}$, the open set $\tilde{U} \subseteq \tilde{G}$ will be equal to the unit space of $\tilde{G}$ once we have equipped it with its natural groupoid structure. We let $\tilde{G}^{(0)}$ denote this unit space, and let $\pi := \pi_{G^{(0)}}$ denote the (proper) continuous surjection $\tilde{G}^{(0)} \to \tilde{G}$.

Notation. Elements of $\tilde{G}$ will be denoted by either $\boldsymbol{g}$ or $\boldsymbol{h}$. Elements of the unit space $\tilde{G}^{(0)}$ will be denoted by X or Y. This is to ensure clarity in Chapter 4.

The following observation is worth keeping in mind. Whenever G is Hausdorff, the inclusion $\iota\colon G \to \tilde{G}$ is a homeomorphism, and therefore G and $\tilde{G}$ are the same groupoid.

With definitions and notation fixed, we now begin to study the structure of the Hausdorff cover. The following lemma describes the relationship between compact sets in G and $\tilde{G}$. It will be used frequently.

Lemma 3.1.2. *We have the following relationship between compact sets in G and $\tilde{G}$.*

(i) For every compact subset $L \subseteq \tilde{G}^{(0)}$ there exists a compact subset $K \subseteq G^{(0)}$ such that $L \subseteq \overline{\iota(K)}$.

(ii) For every compact subset $L \subseteq \tilde{G}$ there exists a compact subset $K \subseteq G$ such that $L \subseteq \overline{\iota(K)}$, and then $L \cap \iota(G) \subseteq \iota(K)$.

(iii) For every compact subset $K \subseteq G$, the set $\overline{\iota(K)} \subseteq \tilde{G}$ is compact.

Proof. (i): Take a compact subset $L \subseteq \tilde{G}^{(0)}$. The surjection $\pi\colon \tilde{G}^{(0)} \to G^{(0)}$ is continuous, and therefore $\pi(L)$ is a compact subset of $G^{(0)}$. Choose a compact subset $K \subseteq G^{(0)}$ such that $\pi(L) \subseteq \operatorname{int}(K)$. We show that $L \subseteq \overline{\iota(K)}$. Take $X \in L$. As $\iota(G^{(0)})$ is dense in $\tilde{G}^{(0)}$, there exists a net (x_α) in $G^{(0)}$ such that $\iota(x_\alpha)$ converges to X. Since $x_\alpha = \pi(\iota(x_\alpha))$, continuity of π ensures that the net (x_α) converges to $\pi(X)$. Hence x_α eventually lies in K, and thus $X = \lim_\alpha \iota(x_\alpha) \in \overline{\iota(K)}$.

(ii): Take a compact subset $L \subseteq \tilde{G}$. Since the open sets $\tilde{U}$ cover $\tilde{G}$, compactness of L ensures that there exist open bisections $U_1, \ldots, U_n \subseteq G$ and compact sets $M_i \subseteq \tilde{U}_i$

satisfying $L \subseteq \bigcup_{i=1}^n M_i$. Now take compact subsets $K_i \subseteq U_i$ with $\pi_{U_i}(M_i) \subseteq \operatorname{int}(K_i)$. The same argument as in part (i) shows that $M_i \subseteq \overline{\iota(K_i)}$. Hence $L \subseteq \bigcup_{i=1}^n M_i \subseteq \bigcup_{i=1}^n \overline{\iota(K_i)} = \overline{\iota(\bigcup_{i=1}^n K_i)}$. Set $K := \bigcup_{i=1}^n K_i$.

For the second part, let $g \in G$ be such that $\iota(g) \in L$. Since $L \subseteq \overline{\iota(K)}$, there exists a net (g_α) in K such that $\iota(g_\alpha)$ converges to $\iota(g)$. By definition of the Fell topology, this means that (g_α) is a primitive net converging to g in the topology of G, and such that g is the unique limit. Compactness of K implies that (g_α) has a subnet converging in K. But since (g_α) is primitive, the limit of this subnet must be g, so that $g \in K$.

(iii): We show that $\iota(K)$ is contained in a compact subset of $\tilde{G}$. By [28, Theorem 1], the Fell topology is compact. Therefore, the closure of $\iota(K)$ in $\mathfrak{C}(G)$ is compact. Note that the empty set does not belong to this closure. Indeed, the basic open set $\mathcal{U}(\emptyset; K) \subseteq \mathfrak{C}(G)$ contains the empty set and is disjoint from $\iota(K)$. It follows that the (compact) closure of $\iota(K)$ in $\mathfrak{C}(G)$ is contained in $\tilde{G}$. □

The Hausdorff cover $\tilde{G}$ admits a third description, this time as the spectrum of a commutative C^*-algebra (see also [68, Proposition 10]). In the sequel, we let $B_c(G)$ denote the set of bounded Borel functions on G that have strict support contained in a compact set, and let $B_0(G)$ denote the completion of $B_c(G)$ with respect to the supremum norm $\|\cdot\|_\infty$. Let $C^*(\mathscr{C}_c(G)) \subseteq B_0(G)$ be the smallest C^*-subalgebra of $B_0(G)$ containing $\mathscr{C}_c(G)$, where involution and multiplication are defined pointwise.

Given a character χ on $C^*(\mathscr{C}_c(G))$, define

$$\boldsymbol{g}_\chi := \{g \in G \colon \exists\, U \text{ open bisection with } g \in U \text{ and } \chi|_{C_c(U)} = \mathrm{ev}_g\},$$

where ev_g is the "evaluation" character given by $C_c(U) \to \mathbb{C}$, $f \mapsto f(g)$. Given $\boldsymbol{g} \in \tilde{G}$, define $\chi_{\boldsymbol{g}}(f) = \sum_{g \in \boldsymbol{g}} f(g)$ for all $f \in \mathscr{C}_c(G)$. Each $\chi_{\boldsymbol{g}}$ extends (uniquely) to a character on $C^*(\mathscr{C}_c(G))$ by the following lemma.

The dual isomorphism $\mathfrak{U} \colon C^*(\mathscr{C}_c(G)) \xrightarrow{\cong} C_0(\tilde{G})$ can be described as follows with the aid of the canonical inclusion $\iota \colon G \hookrightarrow \tilde{G}$. By viewing $f \in C^*(\mathscr{C}_c(G))$ as a function on $\iota(G) \subseteq \tilde{G}$, the function $\mathfrak{U}(f) \in C_0(\tilde{G})$ is the unique continuous extension of f to $\tilde{G}$. The inverse isomorphism $\mathfrak{U}^{-1} \colon C_0(\tilde{G}) \xrightarrow{\cong} C^*(\mathscr{C}_c(G))$ is witnessed by the map $\tilde{f} \mapsto \tilde{f} \circ \iota$. In other words, the inverse $\mathfrak{U}^{-1}$ is given by restriction from $\tilde{G}$ to G.

Lemma 3.1.3. *The map* $\tilde{G} \to \operatorname{Spec}\Big(C^*(\mathscr{C}_c(G))\Big); \boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}$ *is a homeomorphism with inverse* $\chi \mapsto \boldsymbol{g}_\chi$.

The dual isomorphism $\mathfrak{U} \colon C^*(\mathscr{C}_c(G)) \xrightarrow{\cong} C_0(\tilde{G})$ *restricts to an embedding* $\mathfrak{i} \colon \mathscr{C}_c(G) \hookrightarrow C_c(\tilde{G})$ *that is a* **-algebra homomorphism with respect to convolution and the canonical* *involution. Moreover,* $\mathfrak{i}$ *satisfies*

$$\mathfrak{i}(f)(\boldsymbol{g}) = \sum_{g \in \boldsymbol{g}} f(g) \tag{3.1.3}$$

for all $f \in \mathscr{C}_c(G)$ *and* $\boldsymbol{g} \in \tilde{G}$.

Proof. Firstly, we show that the map $\mathfrak{U}\colon C^*(\mathscr{C}_c(G)) \to C_0(\tilde{G})$ described above is indeed a *-isomorphism with respect to pointwise operations. Take an open bisection $U \subseteq G$ and $f \in C_c(U)$. Upon viewing G as a dense subset of $\tilde{G}$ via the inclusion $\iota\colon G \to \tilde{G}$, one sees that $\boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}(f)$ is the unique continuous extension of f to $\tilde{G}$. By linearity, it then follows that $\boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}(f)$ is continuous on $\tilde{G}$ for any $f \in \mathscr{C}_c(G)$. In particular, any function $f \in \mathscr{C}_c(G)$ has a unique continuous extension $\mathfrak{U}(f)$ to $\tilde{G}$, and $\mathfrak{U}(f)$ belongs to $C_c(\tilde{G})$ by Lemma 3.1.2 (iii). Any function in $C^*(\mathscr{C}_c(G))$ can be approximated uniformly by finite products of functions in $\mathscr{C}_c(G)$, and therefore any $f \in C^*(\mathscr{C}_c(G))$ has a unique continuous extension $\mathfrak{U}(f) \in C_0(\tilde{G})$. This proves that the map $\mathfrak{U}\colon C^*(\mathscr{C}_c(G)) \to C_0(\tilde{G})$ is well-defined. By construction, it is clear that this map is an injective *-homomorphism with respect to pointwise operations. In order to show that $\mathfrak{U}$ is surjective, we show that its image separates points in $\tilde{G}$. Since the image of $\mathfrak{U}$ vanishes nowhere on $\tilde{G}$, surjectivity then follows by the Stone-Weierstrass theorem. Let $\boldsymbol{g}, \boldsymbol{h} \in \tilde{G}$ and suppose that $g \in \boldsymbol{g} \setminus \boldsymbol{h}$. There exists an open bisection U that contains g and is disjoint from $\boldsymbol{h}$. Taking $f \in C_c(U)$ with $f(g) = 1$, one sees that $\mathfrak{U}(f)(\boldsymbol{g}) = 1$ and $\mathfrak{U}(f)(\boldsymbol{h}) = 0$. Therefore, the image of $\mathfrak{U}$ separates points in $\tilde{G}$. This completes the proof that $\mathfrak{U}$ is an isomorphism.

Next, we show that the maps $\boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}$ and $\chi \mapsto \boldsymbol{g}_\chi$ are well-defined homeomorphisms that are inverse to each other. Given $\boldsymbol{g} \in \tilde{G}$, it is immediate by construction that the dual of $\mathfrak{U}$ evaluated at $\boldsymbol{g}$ agrees with $\chi_{\boldsymbol{g}}$ on $\mathscr{C}_c(G)$. Since $\mathscr{C}_c(G)$ generates $C^*(\mathscr{C}_c(G))$, this is the unique character that agrees with $\chi_{\boldsymbol{g}}$ on $C^*(\mathscr{C}_c(G))$. Letting $\chi_{\boldsymbol{g}}$ denote this character, we have that the map $\boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}$ is dual to $\mathfrak{U}$, and is therefore a homeomorphism from $\tilde{G}$ to $\operatorname{Spec}\Big(C^*(\mathscr{C}_c(G))\Big)$. It is straightforward to verify that $\boldsymbol{g}_{\chi_{\boldsymbol{g}}} = \boldsymbol{g}$ for any $\boldsymbol{g} \in \tilde{G}$, and therefore $\chi \mapsto \boldsymbol{g}_\chi$ is inverse to the homeomorphism $\boldsymbol{g} \mapsto \chi_{\boldsymbol{g}}$.

For the last part, note that equation (3.1.3) is automatic by definition of $\chi_{\boldsymbol{g}}$. Operations on the groupoid $\tilde{G}$ are yet to be defined, but in order to show that $\mathfrak{i}$ is *-preserving and multiplicative with respect to convolution, it suffices to know that $\iota\colon G \hookrightarrow \tilde{G}$ is a groupoid homomorphism whose image $\iota(G)$ is an invariant subgroupoid of $\tilde{G}$. Given $f_1, f_2 \in \mathscr{C}_c(G)$, we observe that $\mathfrak{U}^{-1}\Big(\mathfrak{i}(f_1)^*\Big)(g) = \overline{\mathfrak{i}(f_1)(\iota(g)^{-1})} = \overline{f_1(g^{-1})} = f_1^*(g)$ and

$$\mathfrak{U}^{-1}\Big(\mathfrak{i}(f_1) * \mathfrak{i}(f_2)\Big)(g) = \mathfrak{i}(f_1) * \mathfrak{i}(f_2)\Big(\iota(g)\Big) = \sum_{h \in G^{r(g)}} \mathfrak{i}(f_1)(\iota(h))\mathfrak{i}(f_2)(\iota(h^{-1}g)) = f_1(g) * f_2(g).$$

This completes the proof of the lemma because $\mathfrak{i}$ is injective with left inverse $\mathfrak{U}^{-1}$. $\square$

Let us now describe the groupoid structure on the Hausdorff cover $\tilde{G}$. Take $\boldsymbol{g} \in \tilde{G}$. It is clear that $s(g_1) = s(g_2)$ and $r(g_1) = r(g_2)$ for all $g_1, g_2 \in \boldsymbol{g}$. We then define

$$\tilde{s}(\boldsymbol{g}) \coloneqq \{g_1^{-1}g_2 \colon g_1, g_2 \in \boldsymbol{g}\}$$
$$\tilde{r}(\boldsymbol{g}) \coloneqq \{g_1g_2^{-1} \colon g_1, g_2 \in \boldsymbol{g}\}$$

Given $\dot{g} \in \boldsymbol{g}$, we have that $\tilde{s}(\boldsymbol{g})$ is a subgroup of the isotropy group $G_{s(\dot{g})}^{s(\dot{g})} \subseteq G$, and $\tilde{r}(\boldsymbol{g})$ is a subgroup of the isotropy group $G_{r(\dot{g})}^{r(\dot{g})}$. If (g_i) is a primitive net in G whose set of limit points is equal to $\boldsymbol{g}$, then $s(g_i)$ and $r(g_i)$ are primitive nets in $G^{(0)}$ whose sets of limits points are equal to $\tilde{s}(\boldsymbol{g})$ and $\tilde{r}(\boldsymbol{g})$ respectively. Therefore, $\tilde{s}$ and $\tilde{r}$ are well-defined projections $\tilde{G} \to \tilde{G}^{(0)}$. They will function as the range and source maps in $\tilde{G}$. Note also that the element $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \dot{g} \cdot \tilde{s}(\boldsymbol{g}) = \tilde{r}(\boldsymbol{g}) \cdot \dot{g}$. Here, we denote by $g \cdot X = \{gh \colon h \in X\}$ and $Y \cdot g = \{hg \colon h \in Y\}$.

Write $\boldsymbol{g} = \dot{g} \cdot X$ where $X = \tilde{s}(\boldsymbol{g})$. The inverse $\boldsymbol{g}^{-1}$ is defined as $\boldsymbol{g}^{-1} \coloneqq X \cdot \dot{g}^{-1}$. Explicitly, we have $\boldsymbol{g}^{-1} = \{g^{-1} \colon g \in \boldsymbol{g}\}$. Next, take two elements $\boldsymbol{g}, \boldsymbol{h} \in \tilde{G}$. Note that $\tilde{s}(\boldsymbol{g}) = \tilde{r}(\boldsymbol{h})$ if and only if $\boldsymbol{g} = g \cdot X$ and $\boldsymbol{h} = X \cdot h$, for some composable $g, h \in G$ and $X \in \tilde{G}^{(0)}$ (necessarily, we have $X = \tilde{s}(\boldsymbol{g}) = \tilde{r}(\boldsymbol{h})$). In this case, the product $\boldsymbol{gh}$ is defined as $\boldsymbol{gh} \coloneqq (gh) \cdot Y$, where $Y = \tilde{s}(\boldsymbol{h})$. Alternatively, $\boldsymbol{gh} = Z \cdot (gh)$ where $Z = \tilde{r}(\boldsymbol{g})$. Explicitly, we have $\boldsymbol{gh} = \{gh \colon g \in \boldsymbol{g}, h \in \boldsymbol{h}\}$. It is straightforward to show that the inverse and product maps are continuous. Therefore, $\tilde{G}$ is a Hausdorff locally compact groupoid.

Lemma 3.1.4. *Let G be an étale groupoid.*

(i) The Hausdorff cover $\tilde{G}$ is étale.

(ii) If G is ample, then so is $\tilde{G}$.

Proof. (i): We claim that the open sets $\tilde{U}$ as defined in (3.1.1) are open bisections. Since G is covered by its open bisections, it is clear that $\tilde{G}$ is covered by the open sets $\tilde{U}$, and hence it will then follow that $\tilde{G}$ is étale.

For an open bisection U, consider the ideal $C^*(\mathscr{C}_c(G))_U \coloneqq \overline{\operatorname{span}\Big(C_c(U) \cdot C^*(\mathscr{C}_c(G))\Big)} \subseteq C^*(\mathscr{C}_c(G))$. The homeomorphism $\tilde{G} \cong \operatorname{Spec}(C^*(\mathscr{C}_c(G)))$ from Lemma 3.1.3 sends $\tilde{U}$ to the open set $\operatorname{Spec}(C^*(\mathscr{C}_c(G))_U) \subseteq \operatorname{Spec}(C^*(\mathscr{C}_c(G)))$. The inclusion $C_0(U) \subseteq C^*(\mathscr{C}_c(G))_U$ is dual to the (proper) continuous surjection $\pi_U \colon \tilde{U} \twoheadrightarrow U$ from (3.1.2). Moreover, the inclusion $\iota \colon G \hookrightarrow \tilde{G}$ restricts to $U \hookrightarrow \tilde{U}$. Then, note that the restriction $\tilde{r}|_{\tilde{U}} \colon \tilde{U} \to \tilde{r}(\tilde{U}) = \widetilde{r(U)}$ is dual under the Gelfand transform to the isomorphism $C^*(\mathscr{C}_c(G))_{r(U)} \cong C^*(\mathscr{C}_c(G))_U$, $f \mapsto f \circ r$, so that $\tilde{r}|_{\tilde{U}}$ is in fact a homeomorphism. A similar argument holds for the source map. It follows that $\tilde{U}$ is an open bisection.

(ii): Assume that $\tilde{G}$ is ample. By definition, we have that

$$C^*(\mathscr{C}_c(G)) = \overline{\operatorname{span}}(\{f_1 \cdots f_n \colon n \in \mathbb{N},\ f_m \in \mathscr{C}_c(G),\ 1 \leq m \leq n\}),$$

where the closure is taken in $B_0(G)$ with respect to $\|\cdot\|_\infty$. Now, if G is an ample groupoid, the compact open bisections form a basis for the topology on G. In particular, $\operatorname{span}(\{\mathbb{1}_U \colon U \subseteq G$ compact open bisection$\})$ is dense in $\mathscr{C}_c(G)$, which in turn implies that

$$C^*(\mathscr{C}_c(G)) = \overline{\operatorname{span}}(\{\mathbb{1}_{U_1} \cdots \mathbb{1}_{U_n} \colon n \in \mathbb{N},\, U_1, \ldots, U_n \subseteq G \text{ compact open bisections }\}).$$

In particular, this shows that the commutative C^*-algebra $C^*(\mathscr{C}_c(G))_{G^{(0)}}$ defined in the proof of part (i) is equal to

$$\overline{\operatorname{span}}(\{\mathbb{1}_{U_0} \cdot \mathbb{1}_{U_1} \cdots \mathbb{1}_{U_n} \colon n \in \mathbb{N},\, U_0, \ldots, U_n \subseteq G \text{ compact open bisections}, U_0 \subseteq G^{(0)}\})$$

and thus contains a dense *-algebra of projections. The isomorphism $C_0(\tilde{G}^{(0)}) \cong C^*(\mathscr{C}_c(G))_{G^{(0)}}$ then implies that $\tilde{G}^{(0)}$ is totally disconnected. □

Taking $U = G^{(0)}$ in the proof of part (i) of the preceding lemma gives a canonical homeomorphism between the unit space $\tilde{G}^{(0)}$ and $\operatorname{Spec}(C^*(\mathscr{C}_c(G))_{G^{(0)}})$.

Lemma 3.1.5. *Let $X \in \tilde{G}^{(0)}$ and write $x := \pi(X) \in G^{(0)}$. The isotropy group $\tilde{G}^X_X$ is equal to the quotient N_X/X, where $N_X = \{g \in G^x_x \colon g \cdot X \cdot g^{-1} = X\}$ is the normaliser of X in G^x_x.*

Proof. Let us first prove the inclusion $\tilde{G}^X_X \subseteq N_X/X$. Recall that X is a subgroup of the isotropy group G^x_x. From the definitions of $\tilde{s}$ and $\tilde{r}$ it is then clear that any $\boldsymbol{g} \in \tilde{G}^X_X$ is a subset of G^x_x. Now take $\boldsymbol{g} \in \tilde{G}^X_X$ and choose an element $\dot{g} \in \boldsymbol{g}$. Since $\tilde{s}(\boldsymbol{g}) = \tilde{r}(\boldsymbol{g}) = X$ we have $\boldsymbol{g} = \dot{g} \cdot X = X \cdot \dot{g}$. In particular, $\dot{g}$ is an element of the normaliser N_X. It is then clear that $\boldsymbol{g}$ is an element of the quotient N_X/X.

Let us now prove the opposite inclusion. Take $g \in N_X$, and let $U \subseteq G$ be an open bisection containing g. Let (x_i) be a primitive net in $G^{(0)}$ whose set of limit points is equal to X and, without loss of generality, assume that $x_i \in s(U)$ for all i. Let g_i be the unique elements of U such that $s(g_i) = x_i$. Since $s|_U \colon U \to s(U)$ is a homeomorphism, the net (g_i) is primitive and has set of limit points equal to $g \cdot X$. Therefore $\boldsymbol{g} := g \cdot X$ is an element of $\tilde{G}$. Since g belongs to the normaliser N_X we have $\boldsymbol{g} = X \cdot g$. Therefore, $\tilde{s}(\boldsymbol{g}) = X = \tilde{r}(\boldsymbol{g})$, and hence $\boldsymbol{g} \in \tilde{G}^X_X$. □

The following proposition is the first of several results we will see relating the C^*-algebras of G and $\tilde{G}$.

Proposition 3.1.6. *The embedding* $\mathfrak{i}\colon \mathscr{C}_c(G) \hookrightarrow C_c(\tilde{G})$ *from Lemma 3.1.3 induces an injective* **-homomorphism* $\mathfrak{i}_r\colon C_r^*(G) \hookrightarrow C_r^*(\tilde{G})$.

Viewing elements of $C_r^*(G)$ and $C_r^*(\tilde{G})$ as functions on G and $\tilde{G}$ respectively, the *-homomorphism $\mathfrak{i}_r$ is the restriction of the isomorphism $\mathfrak{U}\colon C^*(\mathscr{C}_c(G)) \xrightarrow{\cong} C_0(\tilde{G})$ from Lemma 3.1.3 to $C_r^*(G)$. Therefore, we have $a = \mathfrak{i}_r(a) \circ \iota$ for all $a \in C_r^*(G)$.

Proof. Given $x \in G^{(0)}$, let $\pi_x\colon \mathscr{C}_c(G) \to \mathcal{L}(\ell^2(G_x))$ denote the left regular representation corresponding to x, and for $X \in \tilde{G}^{(0)}$, let $\tilde{\pi}_X\colon C_c(\tilde{G}) \to \mathcal{L}(\ell^2(\tilde{G}_X))$ denote the left regular representation corresponding to X. For every $x \in G^{(0)}$, it is clear that the representation π_x is canonically unitarily equivalent to the representation $\mathscr{C}_c(G) \xrightarrow{\mathfrak{i}} C_c(\tilde{G}) \xrightarrow{\tilde{\pi}_{\iota(x)}} \mathcal{L}(\ell^2(\tilde{G}_{\iota(x)}))$. Therefore, we have $\|\pi_x(f)\| = \|\tilde{\pi}_{\iota(x)}(\mathfrak{i}(f))\|$ for every $x \in G^{(0)}$ and $f \in \mathscr{C}_c(G)$. We conclude that

$$\|f\|_{C_r^*(G)} = \sup_{x \in G^{(0)}} \|\pi_x(f)\| = \sup_{x \in G^{(0)}} \|\tilde{\pi}_{\iota(x)}(\mathfrak{i}(f))\| = \|\mathfrak{i}(f)\|_{C_r^*(\tilde{G})}.$$

The last equality used that $\iota(G^{(0)})$ is dense in $\tilde{G}^{(0)}$ and that, since $\tilde{G}$ is Hausdorff, it suffices to take the supremum over a dense subset of the unit space to obtain the norm in $C_r^*(\tilde{G})$ (see [42, Corollary 2.4] for instance). $\square$

Later in Proposition 3.2.8 we will see that $\mathfrak{i}$ also induces an injective *-homomorphism at the level of full groupoid C^*-algebras.

Remark 3.1.7. For an ample groupoid G, Proposition 3.1.6 has an analogue in the context of Steinberg algebras (see Definition 1.3.6). Recall that the Hausdorff cover $\tilde{G}$ is ample by Lemma 3.1.4. For any ring R, we have a canonical inclusion of Steinberg algebras $\mathfrak{i}_R\colon RG \hookrightarrow R\tilde{G}$ given by $r_U \mapsto r_{\tilde{U}}$.

§ 3.1.2 | Essential and singular parts of the Hausdorff cover

For the following definition, we assume that G is covered by countably many open bisections. Under this hypothesis, the set of dangerous points $D \subseteq G^{(0)}$ has empty interior by Lemma 1.1.12. Moreover, the Hausdorff cover $\tilde{G}$ neatly decomposes into two components: an "essential" part and a "singular" part.

Definition 3.1.8. Let G be an étale groupoid that is covered by countably many open bisections, and let $C \subseteq G^{(0)}$ denote the set of Hausdorff units. Define

$$\tilde{G}^{(0)}_{\text{ess}} := \overline{\iota(C)} \subseteq \tilde{G}^{(0)} \qquad \text{and} \qquad \tilde{G}^{(0)}_{\text{sing}} := \tilde{G}^{(0)} \setminus \tilde{G}^{(0)}_{\text{ess}}.$$

Both are invariant subsets of $\tilde{G}^{(0)}$ by Lemma 1.1.10. We define the *essential* and *singular* parts of the Hausdorff cover as follows.

$$\tilde{G}_{\text{ess}} := \tilde{s}^{-1}(\tilde{G}^{(0)}_{\text{ess}}) \qquad \text{and} \qquad \tilde{G}_{\text{sing}} := \tilde{s}^{-1}(\tilde{G}^{(0)}_{\text{sing}}).$$

Equivalently, we could have defined $\tilde{G}^{(0)}_{\text{sing}}$ to be the set $\operatorname{int}(\pi^{-1}(D))$. Note that $\tilde{G}_{\text{ess}}$ and $\tilde{G}_{\text{sing}}$ are invariant subgroupoids that are closed and open respectively. Their disjoint union is $\tilde{G}$. Therefore, we can, and will, view $C^*_r(\tilde{G}_{\text{sing}})$ as an ideal in $C^*_r(\tilde{G})$. This gives us the following sequence of C^*-algebras:

$$0 \to C^*_r(\tilde{G}_{\text{sing}}) \to C^*_r(\tilde{G}) \to C^*_r(\tilde{G}_{\text{ess}}) \to 0. \tag{3.1.4}$$

This sequence may not be exact in general (see Remark 3.1.15). On the other hand, the following is exact:

$$0 \to \mathfrak{J} \to C^*_r(\tilde{G}) \to C^*_r(\tilde{G}_{\text{ess}}) \to 0 \tag{3.1.5}$$

where $\mathfrak{J} = \{a \in C^*_r(\tilde{G}) \colon \operatorname{supp}^\circ(a) \subseteq \tilde{G}_{\text{sing}}\}$. It is clear that $C^*_r(\tilde{G}_{\text{sing}}) \subseteq \mathfrak{J}$, and that $C^*_r(\tilde{G}_{\text{sing}}) = \mathfrak{J}$ if and only if (3.1.4) is exact. If $\tilde{G}_{\text{sing}}$ is closed, then $C^*_r(\tilde{G})$ decomposes as a direct sum of $C^*_r(\tilde{G}_{\text{sing}})$ and $C^*_r(\tilde{G}_{\text{ess}})$, so that (3.1.4) is exact. If the groupoid $\tilde{G}$ is *inner-exact* ([2, Definition 8.5]) then the sequence (3.1.4) is exact. In particular, this is guaranteed whenever $\tilde{G}$ is amenable.

By definition, the inclusion $\iota\colon G \to \tilde{G}$ maps all Hausdorff units $x \in C$ into $\tilde{G}^{(0)}_{\text{ess}}$. Dangerous units can get mapped to either $\tilde{G}^{(0)}_{\text{ess}}$ or $\tilde{G}^{(0)}_{\text{sing}}$. On the other hand, extremely dangerous units are precisely those that get mapped to $\tilde{G}^{(0)}_{\text{sing}}$.

Lemma 3.1.9. *Let G be an étale groupoid that is covered by countably many open bisections. A unit $x \in G^{(0)}$ is extremely dangerous if and only if $\iota(x) \in \tilde{G}^{(0)}_{\text{sing}}$.*

In the notation of Definition 1.1.8, we can write $\iota(D_0) = \iota(G^{(0)}) \cap \tilde{G}^{(0)}_{\text{sing}}$.

Proof. Let $x \in G^{(0)}$ be an extremely dangerous unit. Take an open neighbourhood $U \subseteq G^{(0)}$ of x, isotropy elements $g_1, \dots, g_n \in G^x_x \setminus \{x\}$, and open bisections $U_1, \dots, U_n$ with $g_i \in U_i$ for each i such that $U \setminus \bigcup_{i=1}^n U_i$ has empty interior. In particular, this implies that $U \setminus \bigcup_{i=1}^n U_i \subseteq D$. For each $i = 1, \dots, n$, let $K_i \subseteq U_i$ be a compact neighbourhood of g_i. Letting $V \coloneqq U \cap \bigcap_{i=1}^n s(\operatorname{int}(K_i))$ and $K \coloneqq \bigcup_{i=1}^n K_i$, we have that $V \setminus K \subseteq D$. Consider the open set $\mathcal{U}(V;K)$ in the Hausdorff cover. It is an open neighbourhood of $\iota(x)$, and since $V \setminus K \subseteq D$, it is clear that $\mathcal{U}(V;K) \cap \iota(G) \subseteq \iota(D)$. In particular, $\mathcal{U}(V;K)$ is disjoint from $\iota(C)$, and therefore $\mathcal{U}(V;K) \subseteq \tilde{G}_{\text{sing}}$.

Conversely, let $x \in G^{(0)}$ be such that $\iota(x) \in \tilde{G}^{(0)}_{\text{sing}}$. By definition of the Fell topology, there exists an open neighbourhood $V \subseteq G^{(0)}$ of x and a compact set $K \subseteq G$ such that $\mathcal{U}(V;K) \subseteq \tilde{G}_{\text{sing}}$. In particular, this implies that $V \setminus K \subseteq D$. Let $g_1, \dots, g_n \in G^x_x \setminus \{x\}$ denote all isotropy group elements in K, and choose open bisections $U_1, \dots, U_n$ with $g_i \in U_i$ for each i. We claim that there exists an open set $U \subseteq V$ containing x and such that $U \setminus \bigcup_{i=1}^n U_i$ has empty interior, thus completing the proof. If no such U exists, then density of C in $G^{(0)}$ ensures that there exists a net (y_α) in $\left(C \cap V \setminus \bigcup_{i=1}^n U_i\right) \setminus \{x\}$

converging to x. By assumption, the net belongs to the compact set K, and therefore has a convergent subnet, that we call (y_β). However, the limit of (y_β) in K must belong to the set $\{g_1, \ldots, g_n\}$. It follows that (y_β) must eventually belong to one of the open bisections $U_1, \ldots, U_n$ – a contradiction. □

Remark 3.1.10. The definition of an extremely dangerous point (Definition 1.1.8) did not require that the ambient groupoid G is covered by countably many open bisections. It follows from Lemma 3.1.9 that a unit $x \in G^{(0)}$ is extremely dangerous if and only if there exists an open subgroupoid $H \subseteq G$ that *is* covered by countably many open bisections, contains x, and satisfies $\iota(x) \in \tilde{H}^{(0)}_{\text{sing}}$.

It is now necessary for us to define the singular ideal and essential C^*-algebra for a non-Hausdorff groupoid. The singular ideal will be the central object of study in Chapter 4.

Definition 3.1.11 ([18, 27, 43])**.** Let G be an étale groupoid. The *singular ideal* J is defined as

$$J := \{a \in C^*_r(G) \colon \operatorname{supp}^\circ(a) \text{ has empty interior in } G\}.$$

The *essential groupoid C^*-algebra* $C^*_{\text{ess}}(G)$ is defined to be the quotient $C^*_r(G)/J$.

For a groupoid G that is covered by countably many open bisections, the singular ideal can equivalently be defined using dangerous points. It follows from this lemma that J is indeed a closed ideal in $C^*_r(G)$.

Lemma 3.1.12 ([43, Proposition 7.18])**.** *Let G be an étale groupoid that is covered by countably many open bisections. An element $a \in C^*_r(G)$ belongs to J if and only if $s(\operatorname{supp}^\circ(a)) \subseteq D$.*

Proof. Since G is covered by countably many bisections, the set D of dangerous units has empty interior by Lemma 1.1.12. Hence $G|_D$ has empty interior also. It is then clear that a belongs to J whenever $\operatorname{supp}^\circ(a) \subseteq G|_D$.

Conversely, assume that $a \in J$, and let $g \in G|_C$. By Lemma 1.1.11, any function $f \in \mathscr{C}_c(G)$ is continuous at g. Since a is a uniform limit of functions in $\mathscr{C}_c(G)$, it is also continuous at g. It follows that $a(g) = 0$, otherwise $\operatorname{supp}^\circ(a)$ would contain an open neighbourhood of g. □

Remark 3.1.13. For an étale groupoid H that is covered by countably many open bisections, it follows from this lemma that J is a closed ideal in $C^*_r(H)$. For the general case, let G be an arbitrary étale groupoid, write $J_G \subseteq C_r^*(G)$ for its singular ideal, and take $a \in C_r^*(G)$, $(b_n) \subset J_G$ such that $b_n \to b \in C_r^*(G)$. By elementary arguments, there exists an open subgroupoid $H \subseteq G$ that is covered by countably many open bisections such that $a, b_1, b_2, \dots \in C_r^*(H)$. Write $J_H \subseteq C_r^*(H)$ for its singular ideal. It is clear that $J_H = J_G \cap C_r^*(H)$ and therefore the lemma implies that $b_1^*, b_1 a, a b_1, b \in J_H \subseteq J_G$. It follows that J_G is a closed ideal in $C_r^*(G)$.

The following proposition is our reason for introducing the singular ideal J here. It states that J embeds canonically into the ideal $\mathfrak{J} = \{a \in C_r^*(\tilde{G}) \colon \operatorname{supp}^\circ(a) \subseteq \tilde{G}_{\mathrm{sing}})\}$ from (3.1.5). It will be useful throughout Chapter 4.

Proposition 3.1.14. *Let $\tilde{G}$ be an étale groupoid that is covered by countably many open bisections, and let $\mathfrak{i}_r \colon C_r^*(G) \hookrightarrow C_r^*(\tilde{G})$ be the canonical embedding from Proposition 3.1.6.*

The image of J under $\mathfrak{i}_r$ is equal to $\mathfrak{J} \cap \operatorname{im}(\mathfrak{i}_r)$. In particular, there is an induced embedding $\mathfrak{i}_{\mathrm{ess}} \colon C_{\mathrm{ess}}^*(G) \hookrightarrow C_r^*(\tilde{G}_{\mathrm{ess}})$.

Proof. Let $a \in J$, and write $\tilde{a} := \mathfrak{i}_r(a)$. By Lemma 3.1.3, we have $\tilde{a} \circ \iota = a$, and therefore $\tilde{a}(\iota(g)) = 0$ for all $g \in G|_C$ by Lemma 3.1.12. However, $\iota(G|_C)$ is dense in $\tilde{G}_{\mathrm{ess}}$, and hence continuity of $\tilde{a}$ implies that $\tilde{a} \equiv 0$ on all of $\tilde{G}_{\mathrm{ess}}$. Therefore $\tilde{a} \in \mathfrak{J}$.

To prove the opposite inclusion, let $a \in C_r^*(G)$ be such that $\tilde{a} := \mathfrak{i}_r(a) \in \mathfrak{J}$. Again, by Lemma 3.1.3, we have $\tilde{a} \circ \iota = a$. Then, the inclusion $\iota(G|_C) \subseteq \tilde{G}_{\mathrm{ess}}$ implies that $a(g) = 0$ for all $g \in G|_C$, and hence $a \in J$ by Lemma 3.1.12.

Finally, since $\mathfrak{i}_r(J) \subseteq \mathfrak{J}$, the embedding $\mathfrak{i}_r \colon C_r^*(G) \hookrightarrow C_r^*(\tilde{G})$ descends to an embedding $\mathfrak{i}_{\mathrm{ess}} \colon C_r^*(G)/J \hookrightarrow C_r^*(\tilde{G})/\mathfrak{J}$. The result follows by (3.1.5). □

Remark 3.1.15. If $\tilde{G}$ is inner-exact (or if $\tilde{G}_{\mathrm{sing}}$ is closed) then $\mathfrak{J} = C_r^*(\tilde{G}_{\mathrm{sing}})$, in which case $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\mathrm{sing}})$. However, there exist groupoids for which $\mathfrak{i}_r(J)$ is not contained in $C_r^*(\tilde{G}_{\mathrm{sing}})$. An example of such a groupoid can be seen by combining the example from [48, § 4.1] with a HLS groupoid from [70] (or [37]). To be a little more precise, let $\mathcal{G}$, $\epsilon \in \mathcal{G}^{(0)}$ and $Y \subseteq \mathcal{G}^{(0)}$ be as in [48, § 4.1]. Keep $\mathcal{G} \setminus \mathcal{G}|_{Y \cup \{\epsilon\}}$ the same, and replace $\mathcal{G}|_{Y \cup \{\epsilon\}}$ with $\mathbb{Z}/2\mathbb{Z} \times G$, where G is as defined in [70, Definition 2.1] (with $\Gamma = \mathbb{F}_2$).

Remark 3.1.16. For an ample groupoid G, Proposition 3.1.14 has an analogue in the context of Steinberg algebras. Let R be a ring, and let RG denote the associated Steinberg algebra. There is a notion of singular ideal $J_R \subseteq RG$ that will be introduced in the final chapter (see Definition 4.1.2). Recall from Proposition 3.1.7 that there is a canonical embedding of Steinberg algebras $\mathfrak{i}_R \colon RG \hookrightarrow R\tilde{G}$. The image of J_R under $\mathfrak{i}_R$ is equal to $R\tilde{G}_{\mathrm{sing}} \cap \operatorname{im}(\mathfrak{i}_R)$.

§ 3.2 | Amenability and nuclearity

In this section, we discuss the relationship between amenability of an étale groupoid and nuclearity of its C^*-algebras. For étale groupoids that are Hausdorff, it is well known that amenability of such a groupoid is equivalent to nuclearity of its reduced (or full) groupoid C^*-algebra (see [3, Chapter 6] and also [11, Theorem 5.6.18]). We generalise this result to the non-Hausdorff setting: in Corollary 3.2.10 it is shown that an étale groupoid G is amenable if and only if $C_r^*(G)$ is nuclear. We do this with the aid of the Hausdorff cover groupoid, and by appealing to known results in the Hausdorff case. In particular, we show that an étale groupoid G is amenable if and only if its Hausdorff cover $\tilde{G}$ is amenable (Theorem 3.2.2).

Throughout this section, let G be an étale groupoid with Hausdorff cover $\tilde{G}$, and let $\iota\colon G \hookrightarrow \tilde{G}$ be the canonical inclusion. We equip G with the Haar system of counting measures. Hence, given $x \in G^{(0)}$ and a function f on G, we have $\lambda^x(f) = \sum_{g \in G^x} f(g)$. Here we only consider f for which the sum on the right hand side converges absolutely. Lastly, recall that the map $x \mapsto \lambda^x(f)$ is continuous on $G^{(0)}$ whenever $f \in \mathscr{C}_c(G)$.

Let us recall the definition of amenability from Chapter 2. We restate it here for convenience. Since we are now dealing with étale groupoids, the definition involves sums rather than integrals.

Definition 3.2.1. Let G be an étale groupoid. We say that G is *amenable* if there exists a net (f_i) of functions in $\mathscr{C}_c(G)^+$ satisfying the following conditions.

(i) $\sum_{g \in G^x} f_i(g) \leq 1$ for all $x \in G^{(0)}$ and all i.

(ii) $\sum_{g \in G^x} f_i(g) \to 1$ uniformly on compact subsets of $G^{(0)}$ (with respect to x).

(iii) $\sum_{h \in G^{r(g)}} |f_i(g^{-1}h) - f_i(h)| \to 0$ uniformly on compact subsets of G (with respect to g).

Our first result compares amenability of G with amenability of $\tilde{G}$. By Renault's observation that topological amenability is a Borel property [61], the two properties are equivalent.

Theorem 3.2.2. *Let G be a non-Hausdorff étale groupoid. Then G is amenable if and only if $\tilde{G}$ is amenable.*

Proof. Assume that G is amenable, and let (f_i) be a net in $\mathscr{C}_c(G)^+$ satisfying the conditions of Definition 3.2.1. Then, the images $\mathfrak{i}(f_i)$ in $C_c(\tilde{G})$ form a net witnessing amenability of $\tilde{G}$ by Lemma 3.1.2 (i) and (ii).

Now assume that $\tilde{G}$ is amenable. Let $(\tilde{f}_i)$ be a net in $C_c(\tilde{G})^+$ witnessing amenability of $\tilde{G}$ by satisfying the conditions of Definition 3.2.1. By Lemma 3.1.3, the functions $f_i := \tilde{f}_i \circ \iota$ are Borel on G. Then, by Lemma 3.1.2 the net (f_i) satisfies the conditions of Corollary 2.3.7. It follows that G is amenable. □

Remark 3.2.3. In [13, Definition 5.2], an alternative definition of topological amenability is given. Their definition uses a net of functions (f_i) that belong to the *-subalgebra of $B_c(G)$ generated by $\mathscr{C}_c(G)$ equipped with pointwise operations. By Lemmas 3.1.3 and 3.1.2 and Proposition 2.1.3, it is easily seen that any non-Hausdorff groupoid G is topologically amenable in the sense of [13, Definition 5.2] if and only if the Hausdorff cover $\tilde{G}$ is amenable. This does not require the abstract results of [61].

§ 3.2.1 | Nuclearity implies amenability

In this subsection and the next, we study the relationship between amenability and nuclearity. Firstly, we show that nuclearity of $C_r^*(G)$ implies amenability of the groupoid $\tilde{G}$. The following result generalises [11, Theorem 5.6.18] to the setting of non-Hausdorff groupoids. We follow the same proof, now adapted to the non-Hausdorff setting.

Theorem 3.2.4. *Let G be an étale groupoid.*

(i) If $C_r^*(G)$ is nuclear, then $\tilde{G}$ is amenable.

(ii) If G is covered by countably many open bisections and $C_{\mathrm{ess}}^*(G)$ is nuclear, then $\tilde{G}_{\mathrm{ess}}$ is amenable.

Before proving this theorem, let us establish some notation. Let ℓ_n^2 denote the Hilbert space $\mathbb{C}^n$, and let $\{u_i\}_{i=1}^n$ denote the standard basis. Equip ℓ_n^2 with the canonical inner product, and use the convention that this inner product is linear in the second variable. For fixed $i,j \in \{1,\dots,n\}$ let $e_{i,j} \in M_n(\mathbb{C})$ be the matrix which has value 1 in the (i,j)-th entry and is 0 elsewhere. Given $a_{i,j} \in C_r^*(G)$ for all $i,j \in \{1,\dots,n\}$, let $[a_{i,j}] \in M_n(C_r^*(G))$ denote the matrix with (i,j)-th entry $a_{i,j}$. Equip the algebraic tensor product $\ell_n^2 \odot \ell_n^2 \odot C_r^*(G)$ with the structure of a right pre-Hilbert $C_r^*(G)$-module equipped with the canonical inner product $\langle \cdot,\cdot \rangle$. Explicitly, we have

$$(\xi_1 \otimes \xi_2 \otimes b_1) \cdot b_2 := \xi_1 \otimes \xi_2 \otimes (b_1 * b_2)$$
$$\langle \xi_1 \otimes \xi_2 \otimes b_1,\ \xi_3 \otimes \xi_4 \otimes b_2 \rangle := \langle \xi_1, \xi_3 \rangle \langle \xi_2, \xi_4 \rangle b_1^* * b_2$$

where $*$ denotes multiplication in $C_r^*(G)$. Any element of $\ell_n^2 \odot \ell_n^2 \odot C_r^*(G)$ can be viewed as a function from G to the Hilbert space $\ell_n^2 \otimes \ell_n^2$ as follows: $(\xi_1 \otimes \xi_2 \otimes b)(g) := b(g)(\xi_1 \otimes \xi_2)$. Lastly, given $a_1, a_2 \in M_n(\mathbb{C})$, let $a_1 \otimes a_2 \otimes 1$ denote the operator defined on $\ell_n^2 \odot \ell_n^2 \odot C_r^*(G)$ via $(a_1 \otimes a_2 \otimes 1)(\xi_1 \otimes \xi_2 \otimes b) := a_1(\xi_1) \otimes a_2(\xi_2) \otimes b$.

Proof. To prove (i), we construct a net (ζ_i) of non-negative functions on G satisfying the following conditions: each function ζ_i belongs to the subalgebra of $B_0(G)$ generated by $\mathscr{C}_c(G)$; $\|\zeta_i\|^2_{\ell^2(G_x)} \leq 1$ for all $x \in G^{(0)}$; and $(\zeta_i^* * \zeta_i)(g) \to 1$ uniformly on compact subsets of G (with respect to $g \in G$). Let $\tilde{\zeta}_i$ denote the image of ζ_i in $C_c(\tilde{G})$ under the canonical isomorphism $C^*(\mathscr{C}_c(G)) \cong C_0(\tilde{G})$. Defining $\tilde{f}_i(\boldsymbol{g}) := \tilde{\zeta}_i(\boldsymbol{g}^{-1})$ for all $\boldsymbol{g} \in \tilde{G}$, the net $(\tilde{f}_i)$ satisfies the conditions of Proposition 2.1.3 (using Lemma 3.1.2(ii)), and therefore $\tilde{G}$ is amenable.

Fix a compact set $K \subseteq G$ and $\varepsilon > 0$. Let $U_1, \ldots, U_m$ be open bisections covering K such that, for every l, there exists an open bisection V_l containing a compact superset of U_l. For each l, let $f_l \in C_c(V_l) \subseteq \mathscr{C}_c(G)$ be such that $0 \leq f_l \leq 1$ and $f_l \equiv 1$ on U_l. Then $\|f_l\|_{C_r^*(G)} \leq 1$ and $(f_l^* * f_l)(x) = 1$ for $x \in s(U_l)$. Since $C_r^*(G)$ is nuclear, there exists a positive integer n and c.c.p. maps $\psi\colon C_r^*(G) \to M_n(\mathbb{C})$ and $\varphi\colon M_n(\mathbb{C}) \to C_r^*(G)$ such that $\|(\varphi \circ \psi)(f_l) - f_l\|_{C_r^*(G)} < \varepsilon$ for every l.

Let A be the positive matrix $[\varphi(e_{i,j})] \in M_n(C_r^*(G))$ and define $B = [b_{i,j}] := A^{\frac{1}{2}} \in M_n(C_r^*(G))$. Define $\eta_\varphi := \sum_{j,k} u_j \otimes u_k \otimes b_{k,j} \in \ell^2_n \odot \ell^2_n \odot C_r^*(G)$, and note that for any $a \in M_n(\mathbb{C})$ we have $\varphi(a) = \langle \eta_\varphi, (a \otimes 1 \otimes 1)\eta_\varphi \rangle$. Note also that η_φ has norm at most 1 with respect to the $C_r^*(G)$-valued inner product. Approximate the $b_{k,j} \in C_r^*(G)$ by functions $\zeta_{k,j} \in \mathscr{C}_c(G)$ such that $\eta'_\varphi := \sum_{j,k} u_j \otimes u_k \otimes \zeta_{k,j} \in \ell^2_n \odot \ell^2_n \odot \mathscr{C}_c(G)$ has norm at most 1, and is within ε of η_φ (with respect to the $C_r^*(G)$-valued inner product). Define the non-negative function $\zeta \in B_0(G)$ by $\zeta(g)^2 := \sum_{j,k} |\zeta_{k,j}(g)|^2$ for $g \in G$. Note that ζ lies in the subalgebra of $B_0(G)$ generated by $\mathscr{C}_c(G)$. Also, for every $x \in G^{(0)}$ we have

$$\|\zeta\|^2_{\ell^2(G_x)} = \sum_{g \in G_x} |\zeta(g)|^2 = \sum_{j,k} (\zeta_{k,j}^* * \zeta_{k,j})(x) = \langle \eta'_\varphi, \eta'_\varphi \rangle(x) \leq 1 \tag{3.2.1}$$

since the supremum norm is dominated by the norm $\|\cdot\|_{C_r^*(G)}$ on $\mathscr{C}_c(G)$. To complete the proof of part (i), we show that $\zeta^* * \zeta$ is uniformly close to 1 on K.

For each l, write $a_l = \psi(f_l) \in M_n(\mathbb{C})$, and note that $\|a_l\| \leq 1$. Using the right pre-Hilbert $C_r^*(G)$-module structure of $\ell^2_n \odot \ell^2_n \odot C_r^*(G)$ we have

$$\begin{aligned} &\langle (a_l \otimes 1 \otimes 1)\eta'_\varphi,\ \eta'_\varphi \cdot f_l \rangle = \langle (a_l \otimes 1 \otimes 1)\eta'_\varphi,\ \eta'_\varphi \rangle * f_l \\ =&\langle \eta'_\varphi,\ (a_l \otimes 1 \otimes 1)\eta'_\varphi \rangle^* * f_l \approx_{2\varepsilon} \ \langle \eta_\varphi,\ (a_l \otimes 1 \otimes 1)\eta_\varphi \rangle^* * f_l \\ =&\varphi(a_l)^* * f_l \approx_\varepsilon f_l^* * f_l \end{aligned}$$

where the symbol $\approx_\varepsilon$ should be understood as meaning "within ε with respect to the norm $\|\cdot\|_{C_r^*(G)}$". Therefore, for any $x \in s(U_l)$ we have

$$|1 - \langle (a_l \otimes 1 \otimes 1)\eta'_\varphi, \eta'_\varphi \cdot f_l \rangle(x)| = |(f_l^* * f_l)(x) - \langle (a_l \otimes 1 \otimes 1)\eta'_\varphi, \eta'_\varphi \cdot f_l \rangle(x)| \leq 3\varepsilon.$$

since the supremum norm is dominated by the norm $\|\cdot\|_{C_r^*(G)}$. Fix arbitrary $g \in K$.

Assume that $g \in U_l$, and write $x = s(g)$. View elements of $\ell^2_n \odot \ell^2_n \odot \mathscr{C}_c(G)$ as functions from G to $\ell^2_n \otimes \ell^2_n$ in the manner described earlier. For any $h \in G_x$ we have

$$(\eta'_\varphi \cdot f_l)(h) = \sum_{\tilde{g} \in G_x} \eta'_\varphi(h\tilde{g}^{-1}) f_l(\tilde{g}) = \eta'_\varphi(hg^{-1}) \in \ell^2_n \otimes \ell^2_n.$$

This implies

$$\begin{aligned} 1 - 3\varepsilon \leq |\langle (a_l \otimes 1 \otimes 1)\eta'_\varphi, \eta'_\varphi \cdot f_l \rangle(x)| &= \left| \sum_{h \in G_x} \langle (a_l \otimes 1)\eta'_\varphi(h), (\eta'_\varphi \cdot f_l)(h) \rangle_{\ell^2_n \otimes \ell^2_n} \right| \\ &\leq \sum_{h \in G_x} \left\| \eta'_\varphi(h) \right\|_{\ell^2_n \otimes \ell^2_n} \left\| \eta'_\varphi(hg^{-1}) \right\|_{\ell^2_n \otimes \ell^2_n} = \sum_{h \in G_x} \zeta(h)\zeta(hg^{-1}) = \sum_{\tilde{h} \in G_{r(g)}} \zeta(\tilde{h}g)\zeta(\tilde{h}) \\ &= (\zeta^* * \zeta)(g). \end{aligned}$$

By (3.2.1) we have $\|\zeta\|^2_{\ell^2(G_y)} \leq 1$ for all $y \in G^{(0)}$, and therefore the Cauchy-Schwarz inequality implies that $1 - 3\varepsilon \leq (\zeta^* * \zeta)(g) \leq 1$. This proves part (i).

The proof of part (ii) is similar so we will be brief. Let K and f_l be as above, and let $\Psi \colon C^*_r(G) \to C^*_{\mathrm{ess}}(G)$ denote the canonical surjection. Given arbitrary $\varepsilon > 0$, nuclearity of $C^*_{\mathrm{ess}}(G)$ implies that there exist c.c.p. maps $\dot{\psi} \colon C^*_{\mathrm{ess}}(G) \to M_n(\mathbb{C})$ and $\dot{\varphi} \colon M_n(\mathbb{C}) \to C^*_{\mathrm{ess}}(G)$ such that

$$\left\| (\dot{\varphi} \circ \dot{\psi})(\Psi(f_l)) - \Psi(f_l) \right\|_{C^*_{\mathrm{ess}}(G)} < \varepsilon$$

for each l. Note that for any $f \in \mathscr{C}_c(G)$ and $g \in G|_C$ we have $|f(g)| \leq \|\Psi(f)\|_{C^*_{\mathrm{ess}}(G)}$ by Proposition 3.1.14. By an argument identical to the one above, there exists a non-negative function ζ in the subalgebra of $B_0(G)$ generated by $\mathscr{C}_c(G)$ such that $\|\zeta\|^2_{\ell^2(G_x)} \leq 1$ for all $x \in C$ and $1 - 3\varepsilon \leq (\zeta^* * \zeta)(g) \leq 1$ for all $g \in (G|_C) \cap K$. It follows that $\tilde{G}_{\mathrm{ess}}$ is amenable. $\square$

Remark 3.2.5. Let us comment on the connection between this result and [13, Theorem 5.12]. Given a groupoid G that is covered by countably many open bisections, the essential Hausdorff cover $\tilde{G}_{\mathrm{ess}}$ is amenable if and only if G is "essentially amenable" in the sense of [13, Definition 5.4]. This can be seen by applying Lemmas 3.1.2 and 3.1.3 as in Remark 3.2.3. Therefore, statement (ii) of Theorem 3.2.4 can be deduced from [13, Theorem 5.12].

§ 3.2.2 | Amenability implies nuclearity

In this subsection, we prove the converse to Theorem 3.2.4(i). That is, we show that whenever $\tilde{G}$ is amenable, the C^*-algebra $C^*_r(G)$ is nuclear. We do this in several steps. The findings of this section are then summarised in Corollary 3.2.10.

We begin with a technical lemma that generalises a result from [13, § 4] about extending representations. Note that similar arguments appear in the proof of Renault's disintegration theorem (see [59] and [51, Appendix B]). Our proof follows [13, Lemma 4.7]. We include it for completeness, and because we do not impose the constraint that G can be covered by countably many open bisections. Recall that $B_c(G)$ denotes the set of bounded Borel functions on G that have strict support contained in some compact set.

Lemma 3.2.6. *Let G be an étale groupoid. Any *-representation $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ extends to a *-representation $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$. Moreover, the commutators of the ranges of π and $\tilde{\pi}$ are equal.*

Before proving the lemma we introduce some terminology and notation. Let U be a Hausdorff locally compact space. From Section 1.2, recall that a *positive Radon measure* on U is a positive Borel measure that is inner regular on open sets, outer regular on all Borel sets, and finite on compacts. A *complex Radon measure* on U is a complex Borel measure of the form $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ for some finite positive Radon measures μ_j. Integration with respect to a complex Radon measure is defined in the obvious way. Any complex Radon measure ν on U has an associated linear functional $\overline{\nu}$ on $C_c(U)$ given by $\overline{\nu}(f) = \int_U f \,\mathrm{d}\nu$ for $f \in C_c(U)$. By the Riesz-Markov-Kakutani representation theorem (see [29, Theorem 7.17]), the mapping $\nu \mapsto \overline{\nu}$ is a one-to-one correspondence between complex Radon measures on U and linear functionals $C_c(U) \to \mathbb{C}$ that are bounded with respect to the supremum norm. Moreover, $\|\overline{\nu}\| = \|\nu\|_1$, where $\|\nu\|_1 := |\nu|(U)$, and $|\nu|$ denotes the total variation of ν. For further details, see [29, § 7.3].

We state a simple lemma about extending complex Radon measures to finite unions of open bisections.

Lemma 3.2.7. *Let G be an étale groupoid, and let $U_1, \ldots, U_n \subseteq G$ be open bisections. For each $i = 1, \ldots, n$ let ν_i be complex Radon measure on U_i.*

Suppose that for any $i, j \in \{1, \ldots, n\}$ the complex Radon measures ν_i and ν_j agree on $U_i \cap U_j$. That is, $\nu_i(B) = \nu_j(B)$ for any Borel set $B \subseteq U_i \cap U_j$. Then, there exists a unique complex measure ν on $U_1 \cup \cdots \cup U_n$ whose restriction to each U_i agrees with ν_i.

Proof. Define $V_1 := U_1$, $V_2 := U_2 \setminus U_1$, $\ldots$, $V_n := U_n \setminus (U_1 \cup \cdots \cup U_{n-1})$. For a Borel set $B \subseteq U_1 \cup \cdots \cup U_n$ define

$$\nu(B) := \sum_{i=1}^{n} \nu_i(B \cap V_i).$$

It is simple to check that ν is a well-defined complex measure satisfying the required conditions. Uniqueness is clear. $\square$

In the following proof, the inner product $\langle \cdot, \cdot \rangle$ on H is linear in the second variable.

Proof of Lemma 3.2.6. Let $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ be a *-algebra representation. Fix $\xi, \eta \in H$ and define the linear functional $\pi_{\xi,\eta}\colon \mathscr{C}_c(G) \to \mathbb{C}$ by $\dot{f} \mapsto \left\langle \xi, \pi(\dot{f})\eta \right\rangle$. For any open bisection $U \subseteq G$, the restriction of π to $C_c(U)$ is bounded with respect to the supremum norm. Therefore, the restriction of $\pi_{\xi,\eta}$ to $C_c(U)$ is bounded with norm $\left\| \pi_{\xi,\eta}|_{C_c(U)} \right\| \leq \|\xi\| \, \|\eta\|$. By the Riesz-Markov-Kakutani representation theorem, there exists a unique complex Radon measure $\nu^U_{\xi,\eta}$ on U such that $\pi_{\xi,\eta}(\dot{f}) = \int_U \dot{f} \, \mathrm{d}\nu^U_{\xi,\eta}$ for all $\dot{f} \in C_c(U)$. It will be useful for later to note that $\left\| \nu^U_{\xi,\eta} \right\|_1 \leq \|\xi\| \, \|\eta\|$. Next, let $\mathcal{F} = \{U_1, \ldots, U_n\}$ be any finite set of open bisections. For any $i, j \in \{1, \ldots, n\}$, uniqueness in the Riesz-Markov-Kakutani representation theorem ensures that the complex Radon measures $\nu^{U_i}_{\xi,\eta}$ and $\nu^{U_j}_{\xi,\eta}$ agree on the intersection $U_i \cap U_j$. Therefore, by Lemma 3.2.7, there exists a unique complex measure $\nu^{\mathcal{F}}_{\xi,\eta}$ on $\bigcup_{i=1}^n U_i$ whose restriction to any U_i agrees with $\nu^{U_i}_{\xi,\eta}$.

We now define an extension $\tilde{\pi}_{\xi,\eta}\colon B_c(G) \to \mathbb{C}$. Given $f \in B_c(G)$, choose a finite collection of open bisections $\mathcal{F} = \{U_1, \ldots, U_n\}$ that cover $\operatorname{supp}^\circ(f)$, and define

$$\tilde{\pi}_{\xi,\eta}(f) \coloneqq \int_{\bigcup_{i=1}^n U_i} f \, \mathrm{d}\nu^{\mathcal{F}}_{\xi,\eta}.$$

To see that this is well-defined, simply observe that the complex Borel measures $\nu^{\mathcal{F}}_{\xi,\eta}$ agree on intersections for different choices of finite set $\mathcal{F}$. It is also clear that the map $\tilde{\pi}_{\xi,\eta}$ is linear. Next, consider the maps $\xi \mapsto \tilde{\pi}_{\xi,\eta}$ and $\eta \mapsto \tilde{\pi}_{\xi,\eta}$ defined on the Hilbert space H. We claim that for fixed $\eta \in H$, the map $\xi \mapsto \tilde{\pi}_{\xi,\eta}$ is anti-linear. To see this, note that the map $\xi \mapsto \pi_{\xi,\eta}$ is anti-linear. Then, since complex Radon measures are determined by their action on compactly supported continuous functions, the map $\xi \mapsto \nu^U_{\xi,\eta}$ is anti-linear for any open bisection U. It follows that the map $\xi \mapsto \tilde{\pi}_{\xi,\eta}$ is anti-linear as claimed. Similarly, for fixed $\xi \in H$, the map $\eta \mapsto \tilde{\pi}_{\xi,\eta}$ is linear. We also need a boundedness condition. From earlier, we have $\left\| \nu^U_{\xi,\eta} \right\|_1 \leq \|\xi\| \, \|\eta\|$ for any open bisection U. Therefore, given any finite set $\mathcal{F} = \{U_1, \ldots, U_n\}$ of open bisections, we have $\left\| \nu^{\mathcal{F}}_{\xi,\eta} \right\|_1 \leq n \, \|\xi\| \, \|\eta\|$. It follows that for any fixed $f \in B_c(G)$, the maps $\eta \mapsto \pi_{\xi,\eta}(f)$ and $\xi \mapsto \pi_{\xi,\eta}(f)$ are bounded. We are now ready to define the desired extension $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$ of π.

Given $f \in B_c(G)$ let $\tilde{\pi}(f) \in \mathcal{L}(H)$ be the unique operator such that $\tilde{\pi}_{\xi,\eta}(f) = \left\langle \xi, \tilde{\pi}(f)\eta \right\rangle$ for all $\xi, \eta \in H$. The linearity and boundedness conditions described above ensure that such an operator always exists. It is clear that $\tilde{\pi}_{\xi,\eta}$ extends $\pi_{\xi,\eta}$ for each $\xi, \eta \in H$, and hence $\tilde{\pi}(f) = \pi(f)$ for all $f \in \mathscr{C}_c(G)$. We must therefore show that the map $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$ is a *-algebra representation. Linearity of $\tilde{\pi}$ is clear, and this will allow us to simplify the subsequent computations. Indeed, any function in $B_c(G)$ can be written as a finite linear combination of functions in $B_c(G)$ that are each supported in some open bisection. So, it suffices to prove that $\tilde{\pi}$ is multiplicative and *-preserving for functions in $B_c(G)$ whose strict support is contained in some open bisection. Throughout the remainder of the proof, we will only work with such functions.

Let us show that $\tilde{\pi}(f^*) = \tilde{\pi}(f)^*$ for any $f \in B_c(G)$ whose strict support is contained in some open bisection. Fix an open bisection $U \subseteq G$ and let $\xi, \eta \in H$. In the following calculation, let $\overline{\left(\nu_{\xi,\eta}^{U^{-1}}\right)^{-1}}$ be the complex Radon measure on U defined by $\overline{\left(\nu_{\xi,\eta}^{U^{-1}}\right)^{-1}}(B) := \overline{\nu_{\xi,\eta}^{U^{-1}}(B^{-1})}$ for Borel $B \subseteq U$. For any $\dot{f} \in C_c(U)$ we have $\tilde{\pi}(\dot{f}^*) = \pi(\dot{f}^*) = \pi(\dot{f})^* = \tilde{\pi}(\dot{f})^*$, and thus

$$\begin{aligned}\int_U \dot{f}(g)\, \mathrm{d}\nu_{\eta,\xi}^U(g) = \left\langle \eta, \pi(\dot{f})\xi \right\rangle = \overline{\left\langle \xi, \pi(\dot{f}^*)\eta \right\rangle} = \overline{\int_{U^{-1}} \dot{f}^*(g)\, \mathrm{d}\nu_{\xi,\eta}^{U^{-1}}(g)} \\ = \int_{U^{-1}} \dot{f}(g^{-1})\, \mathrm{d}\overline{\nu_{\xi,\eta}^{U^{-1}}}(g) = \int_U \dot{f}(g)\, \mathrm{d}\overline{\left(\nu_{\xi,\eta}^{U^{-1}}\right)^{-1}}(g).\end{aligned}$$

Since $\dot{f} \in C_c(U)$ was arbitrary, it follows that the complex Radon measures $\nu_{\eta,\xi}^U$ and $\overline{\left(\nu_{\xi,\eta}^{U^{-1}}\right)^{-1}}$ are equal on U. Therefore, replacing $\dot{f}$ in the calculations above with some $f \in B_c(G)$ satisfying $\mathrm{supp}^\circ(f) \subseteq U$ shows that $\left\langle \eta, \tilde{\pi}(f)\xi \right\rangle = \overline{\left\langle \xi, \tilde{\pi}(f^*)\eta \right\rangle}$. The Hilbert space elements $\xi, \eta \in H$ were arbitrary, hence $\tilde{\pi}(f^*) = \tilde{\pi}(f)^*$, as required.

Next we show that $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$ is multiplicative. Fix open bisections $U_1, U_2 \subseteq G$. Whenever $f_1, f_2 \in B_c(G)$ satisfy $\mathrm{supp}^\circ(f_1) \subseteq U_1$ and $\mathrm{supp}^\circ(f_2) \subseteq U_2$, the convolution $f_1 * f_2$ has strict support contained in the open bisection $V := U_1U_2$ and satisfies $(f_1 * f_2)(g) = f_1(gU_2^{-1})f_2(U_1^{-1}g)$ for $g \in V$. Here, gU_2^{-1} and $U_1^{-1}g$ denote the unique elements of the sets $\{gg_2^{-1} \colon g_2 \in U_2, s(g_2) = s(g)\}$ and $\{g_1^{-1}g \colon g_1 \in U_1, r(g_1) = r(g)\}$ respectively. Fix $\dot{f}_1 \in C_c(U_1)$. Multiplicativity of the map $\pi\colon \mathscr{C}_c(G) \to \mathcal{L}(H)$ ensures the following for any $\dot{f}_2 \in C_c(U_2)$ and $\xi, \eta \in H$:

$$\begin{aligned}\int_{U_2} \dot{f}_2(g)\, \mathrm{d}\nu_{\pi(\dot{f}_1)^*\xi,\eta}^{U_2}(g) &= \left\langle \pi(\dot{f}_1)^*\xi, \pi(\dot{f}_2)\eta \right\rangle = \left\langle \xi, \pi(\dot{f}_1)\pi(\dot{f}_2)\eta \right\rangle \\ &= \left\langle \xi, \pi(\dot{f}_1 * \dot{f}_2)\eta \right\rangle = \int_V (\dot{f}_1 * \dot{f}_2)(g)\, \mathrm{d}\nu_{\xi,\eta}^V(g) \\ &= \int_V \dot{f}_1(gU_2^{-1})\dot{f}_2(U_1^{-1}g)\, \mathrm{d}\nu_{\xi,\eta}^V(g) = \int_{U_2} \dot{f}_2(g)\, \mathrm{d}\mu(g).\end{aligned}$$

Here, the complex measure μ on $U_1^{-1}V \subseteq U_2$ is the pushforward of the complex measure $\dot{f}_1\left((\cdot)U_2^{-1}\right)\, \mathrm{d}\nu_{\xi,\eta}^V$ on V under the homeomorphism $V \to U_1^{-1}V; g \mapsto U_1^{-1}g$. The set of complex Radon measures is invariant under multiplication by bounded Borel functions, so μ is a complex Radon measure. Extend μ to U_2 by 0. This extension is still a complex Radon measure because $U_1^{-1}V$ is open. Then, since $\dot{f}_2 \in C_c(U_2)$ was arbitrary, the equations above ensure that the complex Radon measures μ and $\nu_{\pi(\dot{f}_1)^*\xi,\eta}^{U_2}$ are equal on U_2. Therefore, the function $\dot{f}_2$ can be replaced in the calculations above by any $f_2 \in B_c(G)$ satisfying $\mathrm{supp}^\circ(f_2) \subseteq U_2$. This proves the equality $\left\langle \xi, \tilde{\pi}(\dot{f}_1)\tilde{\pi}(f_2)\eta \right\rangle = \left\langle \xi, \tilde{\pi}(\dot{f}_1 * f_2)\eta \right\rangle$ for any $f_2 \in B_c(G)$ satisfying $\mathrm{supp}^\circ(f_2) \subseteq U_2$. The Hilbert space elements $\xi, \eta \in H$ were arbitrary, hence $\tilde{\pi}(\dot{f}_1 * f_2) = \tilde{\pi}(\dot{f}_1)\tilde{\pi}(f_2)$. By taking adjoints and reversing the roles of U_1 and U_2, we also have $\tilde{\pi}(f_1 * \dot{f}_2) = \tilde{\pi}(f_1)\tilde{\pi}(\dot{f}_2)$ whenever $\dot{f}_2 \in C_c(U_2)$ and $f_1 \in B_c(G)$ satisfies $\operatorname{supp}^\circ(f_1) \subseteq U_1$. Then, by repeating an argument identical to the one above with $\dot{f}_1$ replaced by f_1 gives $\tilde{\pi}(f_1 * f_2) = \tilde{\pi}(f_1)\tilde{\pi}(f_2)$ for any $f_2 \in B_c(G)$ satisfying $\operatorname{supp}^\circ(f_2) \subseteq U_2$. This completes the proof that the extension $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$ is a *-algebra representation.

We now address the "moreover" part of the lemma. Let $T \in \mathcal{L}(H)$ be any operator that commutes with the range of π, and let $\xi, \eta \in H$. For an open bisection $U \subseteq G$ and $\dot{f} \in C_c(U)$ observe that

$$\begin{aligned} \int_U \dot{f}(g) \,\mathrm{d}\nu^U_{T^*(\xi),\eta}(g) &= \left\langle T^*(\xi), \pi(\dot{f})(\eta)\right\rangle = \left\langle \xi, \left(T\pi(\dot{f})\right)(\eta)\right\rangle = \left\langle \xi, \left(\pi(\dot{f})T\right)(\eta)\right\rangle \\ &= \overline{\left\langle T(\eta), \pi(\dot{f}^*)(\xi)\right\rangle} = \int_{U^{-1}} \dot{f}(g^{-1}) \,\mathrm{d}\overline{\nu^{U^{-1}}_{T(\eta),\xi}}(g) \\ &= \int_U \dot{f}(g) \,\mathrm{d}\overline{\left(\nu^{U^{-1}}_{T(\eta),\xi}\right)^{-1}}(g). \end{aligned}$$

The function $\dot{f} \in C_c(U)$ is arbitrary. Since any complex Radon measure on U is determined by its action on $C_c(U)$, similar arguments to earlier parts of the proof show that

$$\left\langle \xi, \left(T\tilde{\pi}(f)\right)(\eta)\right\rangle = \left\langle \xi, \left(\tilde{\pi}(f)T\right)(\eta)\right\rangle$$

for any $\xi, \eta \in H$ and any $f \in B_c(G)$ satisfying $\operatorname{supp}^\circ(f) \subseteq U$. The Hilbert space elements $\xi, \eta \in H$ were arbitrary, hence T commutes with $\tilde{\pi}(f)$. By linearity, it follows that T commutes with the range of $\tilde{\pi}$ as desired. □

With the main technical lemma of this subsection out of the way, we can now prove the desired results regarding amenability and nuclearity. The following proposition tells us that the embedding $\mathfrak{i}\colon \mathscr{C}_c(G) \hookrightarrow C_c(\tilde{G})$ from Lemma 3.1.3 induces an embedding $\mathfrak{i}_{\max}\colon C^*(G) \hookrightarrow C^*(\tilde{G})$ of full C^*-algebras that is well behaved with respect to taking maximal tensor products.

Proposition 3.2.8. *Let G be an étale groupoid, and let A be a C^*-algebra. The map $\mathfrak{i} \odot \mathrm{id}_A$ induces an isometric *-homomorphism $\mathfrak{i}_{\max} \otimes \mathrm{id}_A\colon C^*(G) \otimes_{\max} A \hookrightarrow C^*(\tilde{G}) \otimes_{\max} A$. In particular, $\mathfrak{i}$ induces an isometric *-homomorphism $\mathfrak{i}_{\max}\colon C^*(G) \hookrightarrow C^*(\tilde{G})$.*

In the proof of this proposition, we abuse notation slightly and identify $\mathscr{C}_c(G)$ with its image under $\mathfrak{i}$. In particular, we view $\mathscr{C}_c(G) \odot A$ as a subset of both $C^*(G) \odot A$ and $C^*(\tilde{G}) \odot A$.

We also need to define products of *-representations. Given C^*-algebras A and B, let $\pi\colon B \to \mathcal{L}(H)$ and $\sigma\colon A \to \mathcal{L}(H)$ be *-representations with commuting ranges. We let $\pi \times \sigma\colon B \odot A \to \mathcal{L}(H)$ be the *-representation given by $(\pi \times \sigma)(b \otimes a) \coloneqq \pi(b)\sigma(a)$ for $b \in$

B, $a \in A$. It is simple to check that $\pi \times \sigma$ is well-defined (see [11, Proposition 3.1.17]). Moreover, by [11, Theorem 3.2.6], any non-degenerate *-representation of $B \odot A$ is of this form.

Proof. By universality of the full groupoid C^*- algebra there exists a *-homomorphism $\mathfrak{i}_{\max}\colon C^*(G) \to C^*(\tilde{G})$ extending $\mathfrak{i}$. Then, by universality of the maximal norm on the algebraic tensor product $C^*(G) \odot A$, the map $\mathfrak{i}_{\max} \odot \mathrm{id}_A$ induces a *-homomorphism $\mathfrak{i}_{\max} \otimes \mathrm{id}_A\colon C^*(G) \otimes_{\max} A \to C^*(\tilde{G}) \otimes_{\max} A$. In order to show that this map is an isometry, it suffices to prove the inequality $\|\theta\|_{C^*(G)\otimes_{\max} A} \leq \|\theta\|_{C^*(\tilde{G})\otimes_{\max} A}$ for all $\theta \in \mathscr{C}_c(G) \odot A$. The result then follows.

Let $\varphi\colon C^*(G) \odot A \to \mathcal{L}(H)$ be a non-degenerate *-representation. We claim that there exists a *-representation $\tilde{\varphi}\colon C^*(\tilde{G}) \odot A \to \mathcal{L}(H)$ that agrees with φ on $\mathscr{C}_c(G) \odot A$. By [11, Theorem 3.2.6], φ is of the form $\pi \times \sigma$ for some non-degenerate *-representations $\pi\colon C^*(G) \to \mathcal{L}(H)$ and $\sigma\colon A \to \mathcal{L}(H)$ with commuting ranges. By Lemma 3.2.6 there exists a *-representation $\tilde{\pi}\colon B_c(G) \to \mathcal{L}(H)$ extending π, and whose range commutes with the range of σ. By Lemma 3.1.2(ii), the image of $C_c(\tilde{G})$ under the canonical isomorphism $\mathfrak{U}^{-1}\colon C_0(\tilde{G}) \xrightarrow{\cong} C^*(\mathscr{C}_c(G))$ is contained in $B_c(G)$. Moreover, it is obvious that the restriction of $\mathfrak{U}^{-1}$ to $C_c(\tilde{G})$ is *-preserving and multiplicative with respect to the involution and convolution product. Therefore, $C_c(\tilde{G})$ can be viewed as a subalgebra of $B_c(G)$, and $\tilde{\pi}$ restricts to a *-representation of $C_c(\tilde{G})$. Let $\tilde{\pi}_{\max}\colon C^*(\tilde{G}) \to \mathcal{L}(H)$ denote the induced *-representation. The range of $\tilde{\pi}_{\max}$ commutes with the range of σ, and therefore $\tilde{\varphi} \coloneqq \tilde{\pi}_{\max} \times \sigma\colon C^*(\tilde{G}) \odot A \to \mathcal{L}(H)$ is a well-defined *-representation. By construction, $\tilde{\varphi}$ agrees with $\varphi = \pi \times \sigma$ on $\mathscr{C}_c(G) \odot A$, so this proves the claim.

Take arbitrary $\theta \in \mathscr{C}_c(G) \odot A$. Let $\varphi\colon C^*(G) \odot A \to \mathcal{L}(H)$ be a non-degenerate *-representation, and, by applying the claim, let $\tilde{\varphi}\colon C^*(\tilde{G}) \odot A \to \mathcal{L}(H)$ be a *-representation that agrees with φ on $\mathscr{C}_c(G) \odot A$. The inequality $\|\varphi(\theta)\|_{\mathcal{L}(H)} = \|\tilde{\varphi}(\theta)\|_{\mathcal{L}(H)} \leq \|\theta\|_{C^*(\tilde{G})\otimes_{\max} A}$ holds by definition of the maximal tensor product. Taking the supremum over all non-degenerate *-representations φ yields the inequality $\|\theta\|_{C^*(G)\otimes_{\max} A} \leq \|\theta\|_{C^*(\tilde{G})\otimes_{\max} A}$, as desired. □

The following theorem generalises results regarding amenability and nuclearity for Hausdorff groupoids to the non-Hausdorff setting. The proof is quick, since the embeddings from the previous proposition allow us to reduce the problem to the Hausdorff setting. We then appeal to known results.

Theorem 3.2.9. *Let G be an étale groupoid.*

(i) If the canonical surjection $C^*(\tilde{G}) \twoheadrightarrow C^*_r(\tilde{G})$ is an isomorphism, then so is the canonical surjection $C^*(G) \twoheadrightarrow C^*_r(G)$.

(ii) If $\tilde{G}$ is amenable, then the canonical surjection $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism, and both $C^*(G)$ and $C_r^*(G)$ are nuclear.

Proof. (i): For any $f \in \mathscr{C}_c(G)$ we have $\|f\|_{C_r^*(G)} = \|\mathfrak{i}(f)\|_{C_r^*(\tilde{G})}$ and $\|f\|_{C^*(G)} = \|\mathfrak{i}(f)\|_{C^*(\tilde{G})}$ by Propositions 3.1.6 and 3.2.8 respectively. Part (i) then follows.

(ii): Let A be an arbitrary C^*-algebra. We have the following commutative diagram

$$\begin{tikzcd} C^*(G) \otimes_{\max} A \arrow[r, "\mathfrak{i}_{\max}\otimes \mathrm{id}_A"] \arrow[d] & C^*(\tilde{G}) \otimes_{\max} A \arrow[d] \\ C_r^*(G) \otimes_{\min} A \arrow[r, "\mathfrak{i}_r \otimes \mathrm{id}_A"'] & C_r^*(\tilde{G}) \otimes_{\min} A \end{tikzcd}$$

where the vertical maps are the canonical projections. If the Hausdorff groupoid $\tilde{G}$ is amenable, then the canonical projection $C^*(\tilde{G}) \twoheadrightarrow C_r^*(\tilde{G})$ is an isomorphism by [11, Corollary 5.6.17]. Also, amenability of $\tilde{G}$ implies that $C_r^*(\tilde{G})$ is nuclear by [11, Theorem 5.6.18], and therefore the vertical map on the right is an isomorphism. The horizontal map $\mathfrak{i}_{\max} \otimes \mathrm{id}_A$ is injective by Proposition 3.2.8, and therefore the vertical map on the left is also injective. Since A is an arbitrary C^*-algebra, this proves that $C^*(G)$ is nuclear, as required. $\square$

We now summarise all the findings of this section. The following corollary is an immediate consequence of Theorems 3.2.2, 3.2.4, 3.2.9, [11, Theorem 5.6.18] and [11, Corollary 5.6.17].

Corollary 3.2.10. *Let G be an étale groupoid. Consider the following statements.*

(i) G is amenable.

(ii) $\tilde{G}$ is amenable.

(iii) $C_r^*(G)$ is nuclear.

(iv) $C_r^*(\tilde{G})$ is nuclear.

(v) $C^*(G)$ is nuclear.

(vi) $C^*(\tilde{G})$ is nuclear.

(vii) The canonical surjection $C^*(G) \twoheadrightarrow C_r^*(G)$ is an isomorphism.

(viii) The canonical surjection $C^*(\tilde{G}) \twoheadrightarrow C_r^*(\tilde{G})$ is an isomorphism.

We have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Rightarrow$ (viii) $\Rightarrow$ (vii).

Remark 3.2.11. Even without using the abstract results of [61], this chapter proves that statements (ii) to (vi) of the previous corollary are all equivalent to topological amenability in the sense of [13, Definition 5.2] (see Remark 3.2.3). Note also that the implications (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) $\Rightarrow$ (vii) in Corollary 3.2.10 can be deduced from [13, Theorem 5.12] whenever G is covered by countably many open bisections.

In the previous corollary, neither implication "(viii) $\Rightarrow$ (vi)" nor "(vii) $\Rightarrow$ (viii)" holds in general. For a counterexample to the implication "(viii) $\Rightarrow$ (vi)", see [70]. A counterexample to the implication "(vii) $\Rightarrow$ (viii)" will be provided in Example 4.8.5 of the following chapter.

§ 3.3 | Ideals via the Hausdorff cover

The goal of this section is to generalise [17, Theorem 2.10] to the setting of non-Hausdorff groupoids. Precisely, given an amenable second-countable (non-Hausdorff) étale groupoid G, we prove that ideals in $C^*(G)$ are determined by their isotropy fibres (see Section 1.5). The theorem itself only applies to amenable groupoids, though the methods used to prove it apply to any étale groupoid.

Let G be an étale groupoid and let $\tilde{G}$ denote its Hausdorff cover. Given an ideal I in the full groupoid C^*-algebra $C^*(G)$, denote by $\tilde{I}$ the ideal in $C^*(\tilde{G})$ generated by $\mathfrak{i}_{\max}(I)$. Here, $\mathfrak{i}_{\max}\colon C^*(G) \hookrightarrow C^*(\tilde{G})$ is the embedding from Proposition 3.2.8.

We will prove two properties of the assignment $I \mapsto \tilde{I}$. Firstly, we show the assignment is injective. Secondly, we describe the isotropy fibres of $\tilde{I}$ in terms of the isotropy fibres of I. Theorem 3.3.3 then follows immediately.

Lemma 3.3.1. *Let I be an ideal in $C^*(G)$. Then $\mathfrak{i}_{\max}^{-1}(\tilde{I}) = I$.*

Proof. The inclusion $I \subseteq \mathfrak{i}_{\max}^{-1}(\tilde{I})$ holds trivially. To prove the reverse inclusion, fix a *-representation $\rho\colon C^*(G) \to \mathcal{L}(H)$ with $\ker(\rho) = I$. By Lemma 3.2.6, there exists a *-representation $\tilde{\rho}\colon C^*(\tilde{G}) \to \mathcal{L}(H)$ such that $\tilde{\rho} \circ \mathfrak{i}_{\max} = \rho$. Since $\mathfrak{i}_{\max}(I) = \mathfrak{i}_{\max}(\ker(\rho)) \subseteq \ker(\tilde{\rho})$, we have $\tilde{I} \subseteq \ker(\tilde{\rho})$, and therefore $\mathfrak{i}_{\max}^{-1}(\tilde{I}) \subseteq \mathfrak{i}_{\max}^{-1}(\ker(\tilde{\rho})) = \ker(\rho) = I$. □

Let us now describe the isotropy fibres of $\tilde{I}$ in terms of the isotropy fibres of I. Given $x \in G^{(0)}$, recall that the isotropy fibre of I at x is defined in Definition 1.5.1 by $I_x \coloneqq \eta_x(I)$ where $\eta_x\colon C^*(G) \to C^*(G_x^x)$ is the canonical map. Fix $X \in \tilde{G}^{(0)}$. Write $x \coloneqq \pi(X) \in G^{(0)}$, and recall that X is equal to a subgroup of G_x^x. By Lemma 3.1.5 the isotropy group $\tilde{G}_X^X \subseteq \tilde{G}$ is equal to the quotient N_X/X, where N_X is the normaliser of X in G_x^x. Denote by $E_{N_X}\colon C^*(G_x^x) \to C^*(N_X)$ the canonical conditional expectation and by $Q_X\colon C^*(N_X) \to C^*(N_X/X)$ the *-homomorphism induced by the quotient map $N_X \to N_X/X$. Then, let $P_X\colon C^*(G_x^x) \to C^*(\tilde{G}_X^X)$ denote the composition.

Proposition 3.3.2. *Let I be an ideal in $C^*(G)$. Let $X \in \tilde{G}^{(0)}$ and write $x := \pi(X) \in G^{(0)}$. Then*

$$\tilde{I}_X = \overline{P_X(I_x)} \trianglelefteq C^*(\tilde{G}_X^X).$$

Proof. The map P_X is a conditional expectation, and therefore $K_X := \overline{P_X(I_x)}$ is a closed ideal in $C^*(\tilde{G}_X^X)$. We claim that $\mathcal{K} := (K_X)_{X\in\tilde{G}^{(0)}}$ is an invariant family of ideals. Take arbitrary $\boldsymbol{g} \in \tilde{G}$, and let $\Psi_{\boldsymbol{g}}$ be as in Subsection 1.5. Since $\pi\Big(\tilde{r}(\boldsymbol{g})\Big) = r(g_0)$ and $\pi\Big(\tilde{s}(\boldsymbol{g})\Big) = s(g_0)$, we have that $\Psi_{\boldsymbol{g}} \circ P_{\tilde{s}(\boldsymbol{g})} = P_{\tilde{r}(\boldsymbol{g})} \circ \Psi_{g_0}$ for any $g_0 \in \boldsymbol{g}$. Therefore, using that $\Psi_{\boldsymbol{g}}$ is a homeomorphism and that $(I_x)_{x\in G^{(0)}}$ is an invariant family of ideals, it follows that

$$\Psi_{\boldsymbol{g}}\Big(K_{s(\boldsymbol{g})}\Big) = \overline{\Psi_{\boldsymbol{g}} \circ P_{\tilde{s}(\boldsymbol{g})}\Big(I_{s(g_0)}\Big)} = \overline{P_{\tilde{r}(\boldsymbol{g})} \circ \Psi_{g_0}\Big(I_{s(g_0)}\Big)} = \overline{P_{\tilde{r}(\boldsymbol{g})}\Big(I_{r(g_0)}\Big)} = K_{r(\boldsymbol{g})}.$$

Hence, $\mathcal{K} = (K_X)_{X\in\tilde{G}^{(0)}}$ is an invariant family of ideals as claimed. Next, consider the following diagram.

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\mathfrak{i}_{\max}} & C^*(\tilde{G}) \\ \Big\downarrow{\scriptstyle \eta_x} & & \Big\downarrow{\scriptstyle \eta_X} \\ C^*(G_x^x) & \xrightarrow{P_X} & C^*(\tilde{G}_X^X) \end{array}$$

It is clear that the diagram commutes on the dense subalgebra $\mathscr{C}_c(G)$, and therefore commutes on the whole of $C^*(G)$ by continuity. This implies that $K_X = \overline{\eta_X\Big(\mathfrak{i}_{\max}(I)\Big)}$. The restriction map η_X sends closed ideals to closed ideals, and therefore $K_X \subseteq \tilde{I}_X$. The equality $K_X = \overline{\eta_X\Big(\mathfrak{i}_{\max}(I)\Big)}$ also implies that $\mathfrak{i}_{\max}(I) \subseteq I(\mathcal{K})$ by Lemma 1.5.3. Therefore, we have $\tilde{I} \subseteq I(\mathcal{K})$, from which it follows that $\tilde{I}_X \subseteq K_X$ by another application of Lemma 1.5.3. $\square$

We can now prove the main theorem of this section. The proof makes use of [17, Theorem 2.10], which in turn relies on known cases of the Effros-Hahn conjecture [39]. It is therefore necessary for us to only consider amenable and second-countable groupoids. In fact, the theorem fails for non-amenable groupoids (see [17, Examples 2.11 & 2.12]).

Theorem 3.3.3. *Let G be an amenable and second-countable (non-Hausdorff) étale groupoid. Let I and K be ideals in $C^*(G)$. Then, $I = K$ if and only if $I_x = K_x$ for all $x \in G^{(0)}$.*

Proof. The forward implication is trivial. Conversely, if $I_x = K_x$ for all $x \in G^{(0)}$ then $\tilde{I}_X = \tilde{K}_X$ for all $X \in \tilde{G}^{(0)}$ by Proposition 3.3.2. The Hausdorff cover groupoid $\tilde{G}$ is Hausdorff, second-countable, and also amenable by Theorem 3.2.2. Therefore

[17, Theorem 2.10] implies that $\tilde{I} = \tilde{K}$. Finally, we have $I = \mathfrak{i}_{\max}^{-1}(\tilde{I}) = \mathfrak{i}_{\max}^{-1}(\tilde{K}) = K$ by Lemma 3.3.1. □

Remark 3.3.4. Let us briefly describe how the ideals constructed in this subsection allow us to build more C^*-algebras on $\tilde{G}$. Let I be any ideal in the full C^*-algebra $C^*(G)$, and let $B \coloneqq C^*(G)/I$ denote the quotient. Then, letting $\tilde{B} \coloneqq C^*(\tilde{G})/\tilde{I}$, we have a canonical embedding $B \hookrightarrow \tilde{B}$. More generally, an argument analogous to Proposition 3.2.8 yields embeddings $B \otimes_{\max} A \hookrightarrow \tilde{B} \otimes_{\max} A$ for any C^*-algebra A. It follows that nuclearity of $\tilde{B}$ implies nuclearity of B. In general, however, it is not easy to compute $\tilde{B}$. For example, if $B = C^*_{\mathrm{ess}}(G)$, then $\tilde{B}$ need not equal $C^*_r(\tilde{G}_{\mathrm{ess}})$ (take G as in Example 4.8.3). If $B = C^*_r(G)$, then $\tilde{B}$ need not equal $C^*_r(\tilde{G})$ (see Example 4.8.5).

§ 3.4 | Examples

In this section, we compute the Hausdorff cover for the examples appearing in Subsection 1.6. In each case we bring attention to the singular part of the Hausdorff cover $\tilde{G}_{\mathrm{sing}}$. In particular, we comment on whether its unit space $\tilde{G}^{(0)}_{\mathrm{sing}}$ is empty, non-empty and discrete, or non-empty and non-discrete. We do this for the following reasons: in the first case, the singular ideal J vanishes by Proposition 3.1.14, and in the second case, the singular ideal can be computed by the formula in Theorem 4.7.2. In general, one should expect $\tilde{G}_{\mathrm{sing}}$ to be non-empty and non-discrete, and in this case the singular ideal is harder to describe – see Chapter 4 for more details.

§ 3.4.1 | Bundle of groups examples

In the first five examples, G is a bundle of groups, and therefore $\tilde{G}$ will also be a bundle of groups.

Example 3.4.1. Let G be the groupoid from Example 1.6.1, and write $G = X \setminus \{x_0\} \cup \Gamma$. The essential part of the Hausdorff cover $\tilde{G}_{\mathrm{ess}}$ contains no non-trivial elements – that is, $\tilde{G}_{\mathrm{ess}}$ is equal to its unit space $\tilde{G}^{(0)}_{\mathrm{ess}}$. As a set, $\tilde{G}^{(0)}_{\mathrm{ess}} = \{\iota(x) \colon x \in X \setminus \{x_0\}\} \cup \{\Gamma\}$ and is homeomorphic to $X \cong G^{(0)}$ via the map $\pi \colon \tilde{G}^{(0)}_{\mathrm{ess}} \to G^{(0)}$. The singular part $\tilde{G}_{\mathrm{sing}}$ contains a single unit, namely $\iota(x_0)$, and its isotropy group is isomorphic to Γ.

Example 3.4.2. Let G be the groupoid appearing in Example 1.6.2. Since G contains no extremely dangerous points, $\tilde{G}_{\mathrm{sing}}$ is empty by Lemma 3.1.9. The unit space of $\tilde{G}_{\mathrm{ess}}$ has two components, which we call X_1 and X_2. The first is given by $X_1 = \{\iota(x) \colon x \in [0,1)\} \cup \{\Gamma \times \{1\}\}$ and is homeomorphic to $[0,1]$ via the obvious map. The second is $X_2 = \{\iota(x) \colon x \in [1,2]\}$ and is homeomorphic to $[1,2]$ via the obvious map. Each element of X_1 has trivial isotropy group, and each element of X_2 has isotropy group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Overall, we have $\tilde{G} \cong X_1 \sqcup (\mathbb{Z}/2\mathbb{Z} \times X_2)$.

Example 3.4.3. Let G be the groupoid from Example 1.6.3. The unit space of $\tilde{G}_{\text{ess}}$ splits into three components, each homeomorphic to the interval $[0,1]$. These correspond to each of the three intervals $I \times \{A\}$, $I \times \{B\}$ and $I \times \{C\}$. Any unit in $\tilde{G}^{(0)}_{\text{ess}}$ has isotropy group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The singular part $\tilde{G}_{\text{sing}}$ has a single unit, namely $\iota(0)$, and its isotropy group is isomorphic to the Klein four-group, V_4.

Example 3.4.4. Let x_n be the sequence given by $x_n = 1 - \frac{1}{n}$, $n \geq 1$, and let G be the corresponding groupoid from Example 1.6.4. The essential part of the Hausdorff cover $\tilde{G}_{\text{ess}}$ is equal to its unit space, and is homeomorphic to the unit interval. Explicitly, $\tilde{G}^{(0)}_{\text{ess}}$ is equal to the set

$$\left(\bigcup_{n=1}^{\infty} \{\mathbb{Z}/2\mathbb{Z} \times \{x_n\}\} \cup \{\iota(x) \colon x_n < x < x_{n+1}\}\right) \cup \{V_4 \ \times \{1\}\}$$

and is homeomorphic to $[0,1]$ via the obvious map. Next, the singular part $\tilde{G}_{\text{sing}}$ has unit space equal to the set $\{\iota(x_n) \colon n \geq 1\} \cup \Big\{\{e, A\} \times \{1\}\Big\} \cup \{\iota(1)\}$. The unit $\{\iota(1)\}$ is isolated in $\tilde{G}_{\text{sing}}$, and the remaining part is homeomorphic to the set $\{x_n \colon n \geq 1\} \cup \{1\}$ via the obvious map. In particular, $\tilde{G}^{(0)}_{\text{sing}}$ is not discrete. The unit $\{\iota(1)\}$ has isotropy group equal to V_4, and all other units in $\tilde{G}^{(0)}_{\text{sing}}$ have isotropy group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Example 3.4.5. Let G be the groupoid from Example 1.6.5. The unit space of $\tilde{G}_{\text{ess}}$ splits into three clopen components each homeomorphic to $Y \coloneqq \mathbb{N} \cup \{\infty\}$, the one-point compactification of the natural numbers. These correspond to each of the three copies of Y labelled as $Y \times \{A\}$, $Y \times \{B\}$, $Y \times \{C\}$. For $g \in \{A, B, C\}$, Lemma 3.1.5 implies that the isotropy group at (∞, g) is equal to the quotient $(\mathbb{F} \times V_4)/X_g$, where $X_g = \{(1, e), (1, g)\}$. The three restrictions $\tilde{G}_{\text{ess}}|_Y$ are all isomorphic to the product $G_{\text{HLS}} \times \mathbb{Z}/2\mathbb{Z}$, where G_{HLS} is the groupoid from (1.6.2). It follows from [70, Lemmas 2.7 and 2.8] that the canonical surjection $C^*(\tilde{G}_{\text{ess}}) \twoheadrightarrow C^*_r(\tilde{G}_{\text{ess}})$ is an isomorphism. The singular part $\tilde{G}_{\text{sing}}$ has a single unit, namely $\iota(\infty)$, and its isotropy group is isomorphic to $\mathbb{F}_2 \times V_4$. This unit is isolated, and $\mathbb{F}_2$ is non-amenable, so therefore the surjection $C^*(\tilde{G}) \twoheadrightarrow C^*_r(\tilde{G})$ is not an isomorphism.

In Example 4.8.5 we will see that the singular ideal J vanishes for the previous example. It then follows from Proposition 3.1.14 that there exists a canonical embedding $C^*_r(G) \hookrightarrow C^*_r(\tilde{G}_{\text{ess}})$. Moreover, Example 4.8.5 shows that there is also a canonical embedding $C^*(G) \hookrightarrow C^*(\tilde{G}_{\text{ess}})$. This then implies that $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism.

§ 3.4.2 | Groupoids from self-similar groups

In this subsection we describe the Hausdorff cover $\tilde{G}$ for the groupoids associated with the Grigorchuk and Grigorchuk-Erschler groups (see Subsection 1.6.4). In each case, we do this by describing the unit space $\tilde{G}^{(0)}$.

Let T denote the tree of finite words in the alphabet $\{0,1\}$, and let ∂T denote its boundary. We write $1^m \in T$ for the word of length m containing only 1, and 1^∞ for the infinite word in ∂T containing only 1.

Example 3.4.6. Let $N = \{e, a, b, c, d\}$ denote the nucleus of the Grigorchuk group, and let G be the associated groupoid from Example 1.6.13. We describe the essential and singular parts of $\tilde{G}$ separately. The singular part $\tilde{G}_{\mathrm{sing}}$ is discrete, and its unit space $\tilde{G}^{(0)}_{\mathrm{sing}}$ is equal to the set $\{\iota(u1^\infty) : u \in T\}$. These units form a single orbit. We now describe the essential part $\tilde{G}_{\mathrm{ess}}$. As always, the inclusion $\iota|_C : C \to \tilde{G}^{(0)}_{\mathrm{ess}}$ is a homeomorphism onto its image, where $C \subseteq G^{(0)}$ denotes the set of Hausdorff units. For $g = b, c, d$ write $X_g \coloneqq \{[e, 1^\infty], [g, 1^\infty]\}$. Each X_g belongs to $\tilde{G}^{(0)}_{\mathrm{ess}}$, and we have $\pi^{-1}(\{1^\infty\}) = \{\iota(1^\infty), X_b, X_c, X_d\}$. We clearly also have $\pi^{-1}(\{u1^\infty\}) = \{\iota(u1^\infty), uX_bu^*, uX_cu^*, uX_du^*\}$ for any $u \in T$. Given $u \in T$ and a net of infinite words $(\omega_i) \subset \partial T$, the net $\iota(\omega_i)$ converges to uX_du^* in $\tilde{G}$ if and only if ω_i converges to $u1^\infty$ pointwise, and eventually, each ω_i is either equal to $u1^\infty$ or is of the form $u1^{3m}0\omega'$ for some $m \geq 1$ and $\omega' \in \partial T$. Convergence to uX_cu^* and uX_bu^* can be characterised in a similar fashion, by replacing $3m$ with $3m+1$ and $3m+2$ respectively. Overall, $\tilde{G}_{\mathrm{ess}} = \iota(C) \cup \{uX_bu^*, uX_cu^*, uX_du^* : u \in T\}$ with the topology described above.

Example 3.4.7. Let $N = \{e, h, \alpha, \beta, \gamma\}$ denote the nucleus of the Grigorchuk-Erschler group, and let G be the associated groupoid from Example 1.6.14. The singular part $\tilde{G}_{\mathrm{sing}}$ of the Hausdorff cover is discrete, and its unit space $\tilde{G}^{(0)}_{\mathrm{sing}}$ is equal to the set $\{\iota(u1^\infty) : u \in T\}$. These units form a single orbit, and have isotropy group $V_4 \rtimes_\varphi \mathbb{Z}$ (see Lemma 1.6.15). We now describe the essential part $\tilde{G}_{\mathrm{ess}}$. For $g = \alpha, \beta$ write $X_g \coloneqq \{[e, 1^\infty], [g, 1^\infty]\}$. Each X_g belongs to $\tilde{G}^{(0)}_{\mathrm{ess}}$, and we have $\pi^{-1}(\{1^\infty\}) = \{\iota(1^\infty), X_\alpha, X_\beta\}$. It is then clear that $\pi^{-1}(\{u1^\infty\}) = \{\iota(u1^\infty), uX_\alpha u^*, uX_\beta u^*\}$ for any $u \in T$. Given $u \in T$ and a net of infinite words $(\omega_i) \subset \partial T$, the net $\iota(\omega_i)$ converges to $uX_\alpha u^*$ in $\tilde{G}$ if and only if ω_i converges to $u1^\infty$ pointwise, and eventually, each ω_i is either equal to $u1^\infty u^*$ or is of the form $u1^{2m}0\omega'$ for some $m \geq 1$ and $\omega' \in \partial T$. Convergence to $uX_\beta u^*$ can be characterised in a similar fashion, by replacing $2m$ by $2m+1$. Overall, $\tilde{G}_{\mathrm{ess}} = \iota(C) \cup \{uX_\alpha u^*, uX_\beta u^* : u \in T\}$ with the topology described above, and where $C \subseteq G^{(0)}$ denotes the set of Hausdorff units.

§ 3.4.3 | Groupoids of germs from Bieri-Strebel groups

As in Subsection 1.6.3, let P be a non-trivial subgroup of the multiplicative group $\mathbb{R}_{>0}$, and let A be a $\mathbb{Z}[P]$-submodule of the additive group $\mathbb{R}$ containing P. Additionally, we now ask that both A and P are countable. Let $\mathrm{Germ}\Big(\Gamma(I; A, P) \curvearrowright I\Big)$ and $\mathrm{Germ}\Big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\Big)$ denote the groupoids from Examples 1.6.10 and 1.6.11 respectively.

Lemma 3.4.8. *Let A and P be as above. Then the groupoids* $\mathrm{Germ}\big(\Gamma(I; A, P) \curvearrowright I\big)$ *and* $\mathrm{Germ}\big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\big)$ *are both second-countable.*

Proof. Since the interval I and circle $\mathbb{T}$ are second-countable, it suffices to prove that each groupoid is covered by countably many open bisections. Let us prove this for the groupoid $G := \mathrm{Germ}\big(\Gamma(I; A, P) \curvearrowright I\big)$. The proof for $\mathrm{Germ}\big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\big)$ is similar. Let Ω denote the set of all tuples $(x, y, p, q) \in A \times A \times P \times P$ for which there exists $f \in \Gamma(I; A, P)$ with $f(x) = y$, slope p to the left of x, and slope q to the right of x. Given $\omega := (x, y, p, q) \in \Omega$, choose a sequence $(f_n^\omega)_{n \geq 1} \subset \Gamma(I; A, P)$ such that each f_n^ω satisfies $f_n^\omega(x) = y$, has slope p to the left of x, slope q to the right of x, and such that the interval to the left of x on which the f_n^ω have slope p approaches maximal length as $n \to \infty$. For each f_n^ω note, by definition, that the set of germs $U_n^\omega := \{[f_n^\omega, x] \colon x \in [0, 1]\}$ is an open bisection in G. Since both A and P are countable, the set $\{U_n^\omega \colon \omega \in \Omega, n \geq 1\}$ is countable. Density of $A \cap (0, 1)$ in $[0, 1]$ then ensures that these open bisections cover all of G. □

Example 3.4.9. Let $G := \mathrm{Germ}\big(\Gamma(I; A, P) \curvearrowright I\big)$ be the groupoid from Example 1.6.10. Let us first describe the singular part of the Hausdorff cover, $\tilde{G}_{\mathrm{sing}}$. The groupoid $\tilde{G}_{\mathrm{sing}}$ is discrete, and its unit space, $\tilde{G}^{(0)}_{\mathrm{sing}}$, is equal to the set $\{\iota(x) \colon x \in A \cap (0, 1)\}$. The isotropy groups of units in $\tilde{G}^{(0)}_{\mathrm{sing}}$ are isomorphic to P^2, and the orbits are as described in Example 1.6.10. Despite being discrete, $\tilde{G}_{\mathrm{sing}}$ is in fact dense in $\tilde{G}$. Indeed, density of $A \cap (0, 1)$ in $[0, 1]$ ensures that whenever $x \in C$, there exists a net (a_α) in $A \cap (0, 1)$ converging to x. This net is automatically primitive in G, and therefore $\iota(a_\alpha) \to \iota(x)$ in $\tilde{G}$. Next, let us describe the essential part $\tilde{G}_{\mathrm{ess}}$. Its unit space $\tilde{G}^{(0)}_{\mathrm{ess}}$ is homeomorphic to the Cantor space. We can write $\tilde{G}^{(0)}_{\mathrm{ess}} \cong K \cup L$ where $K := [0, 1] \setminus (A \cap (0, 1))$ and $L := \{a_-, a_+ \colon a \in A \cap (0, 1)\}$. In the union $K \cup L$, the space K is equipped with the usual topology, and a net (x_α) in K converges to a_- if and only if it converges to a in $[0, 1]$ and eventually belongs to $[0, a)$. Similarly, a net (x_α) in K converges to a_+ if and only if it converges to a in $[0, 1]$ and eventually belongs to $(a, 1]$. Concretely, under the homeomorphism $\tilde{G}^{(0)}_{\mathrm{ess}} \cong K \cup L$, the units a_- and a_+ in L correspond to the closed sets of germs $\{[f, a] \colon f(a) = a, f \text{ has slope } 1 \text{ to the left of } a\}$ and $\{[f, a] \colon f(a) = a, f \text{ has slope } 1 \text{ to the right of } a\}$ respectively. The isotropy group at each $k \in K$ is trivial, and the units a_- and a_+ have isotropy groups isomorphic to $\mathbb{Z}$. Overall, $\tilde{G}^{(0)}$ is compact and totally disconnected, so that $\tilde{G}$ is an ample groupoid.

In the special case where $F = \Gamma(I; \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ is Thompson's group F, the singular part $\tilde{G}^{(0)}_{\mathrm{sing}}$ is equal to the set $\{\iota(x) \colon x \in \mathbb{Z}[\frac{1}{2}] \cap (0, 1)\}$. Moreover, $\tilde{G}_{\mathrm{sing}}$ consists of a single orbit, and its isotropy groups are isomorphic to $\mathbb{Z}^2$.

Example 3.4.10. Let $G := \mathrm{Germ}\big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\big)$ be the groupoid from Example 1.6.11. The Hausdorff cover $\tilde{G}$ is similar to that in the previous example, so we will be brief. The singular part of the Hausdorff cover $\tilde{G}_{\text{sing}}$ is discrete. Its unit space, $\tilde{G}^{(0)}_{\text{sing}}$, is equal to the set $\{\iota(t)\colon t \in A/\mathbb{Z}\}$. All isotropy groups are isomorphic to P^2, and the orbits are as described in Example 1.6.10. As in the previous example, $\tilde{G}_{\text{sing}}$ is dense in $\tilde{G}$. Next, let us describe the essential part $\tilde{G}_{\text{ess}}$. Its unit space $\tilde{G}^{(0)}_{\text{ess}}$ is homeomorphic to the Cantor space. Explicitly, we can write $\tilde{G}^{(0)}_{\text{ess}} \cong K \cup L$ where $K \coloneqq \mathbb{T} \setminus (A/\mathbb{Z})$ and $L \coloneqq \{a_-, a_+\colon a \in A/\mathbb{Z}\}$. The topology on $K \cup L$ is similar to that in the previous example. Each $k \in K$ has trivial isotropy group, and both a_- and a_+ have isotropy groups isomorphic to $\mathbb{Z}$ for any $a \in A/\mathbb{Z}$. The groupoid $\tilde{G}_{\text{ess}}$ is minimal.

In the special case where $T = T(\mathbb{T}; \mathbb{Z}[\frac{1}{2}], \langle 2\rangle)$ is Thompson's group T, the singular part $\tilde{G}^{(0)}_{\text{sing}}$ is equal to the set $\{\iota(t)\colon t \in \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}\}$. Moreover, $\tilde{G}_{\text{sing}}$ consists of a single orbit, and its isotropy groups are isomorphic to $\mathbb{Z}^2$.

In general, if H is an open subgroupoid of G, it need not be the case that $\tilde{H}$ is an open subgroupoid of $\tilde{G}$. Nevertheless, if $H \coloneqq \operatorname{Germ}\big(\Gamma(I; A, P) \curvearrowright I\big)$ and $G \coloneqq \operatorname{Germ}\big(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T}\big)$, then $\tilde{H}$ *is* an open subgroupoid of $\tilde{G}$. To see this, let $j\colon H \hookrightarrow G$ be the inclusion described in Subsection 1.6.3. For each $\boldsymbol{h} \in \tilde{H}$ with $0 \neq \tilde{s}(\boldsymbol{h}) \neq 1$, define $\tilde{j}(\boldsymbol{h}) \coloneqq \{j(h)\colon h \in \boldsymbol{h}\} \in \tilde{G}$. The remaining elements of $\tilde{H}$ are of the form $\iota([f,0])$ or $\iota([f,1])$ for some $f \in \Gamma(I; A, P)$. Define $\tilde{j}(\iota([f,0]))$ to be the set of germs $[\tilde{f}, 0] \in G$ where $\tilde{f} \in T(\mathbb{T}; A, P)$ satisfies $\tilde{f}(0) = 0$ and has the same slope as f to the right of 0. Similarly, define $\tilde{j}(\iota([f,1]))$ to be the set of germs $[\tilde{f}, 0] \in G$ where $\tilde{f} \in T(\mathbb{T}; A, P)$ satisfies $\tilde{f}(0) = 0$ and has the same slope as f to the left of 0. Note that we can also write $\tilde{j}(\iota([f,0])) = j([f,0]) \cdot 0_+$ and $\tilde{j}(\iota([f,1])) = j([f,1]) \cdot 0_-$ in the notation of Example 3.4.10. Then, $\tilde{j}\colon \tilde{H} \to \tilde{G}$ is a continuous open inclusion of groupoids.

To finish the chapter, we show that a particular class of Bieri-Strebel groups have amenable groupoids of germs. The class in question is the countable family of groups $T_n \coloneqq T(\mathbb{T}; \mathbb{Z}[\frac{1}{n}], \langle n\rangle)$ for $n \geq 2$. The argument is similar to the one given in [63, Theorem 3.3] (see Example 2.5.1). However, this time we will not need to consider the underlying Borel structure: we use the Hausdorff cover instead.

Example 3.4.11. Let $G \coloneqq \operatorname{Germ}(T_n \curvearrowright \mathbb{T})$. We will show that the Hausdorff cover $\tilde{G}$ is amenable. It then follows that G is amenable by Theorem 3.2.2. We consider the essential and singular parts of the Hausdorff cover separately. The singular part $\tilde{G}_{\text{sing}}$ is a discrete groupoid with isotropy group $\mathbb{Z}^2$, and with $n-1$ orbits. Since $\mathbb{Z}^2$ is amenable, it is clear that $\tilde{G}_{\text{sing}}$ is also amenable. Next, consider the essential part $\tilde{G}_{\text{ess}}$. Write its unit space as $\tilde{G}^{(0)}_{\text{ess}} = K \cup L$ as in Example 3.4.10, where $K = \mathbb{T} \setminus (\mathbb{Z}[\frac{1}{n}]/\mathbb{Z})$ and $L = \{a_-, a_+\colon a \in \mathbb{Z}[\frac{1}{n}]/\mathbb{Z}\}$. Each $k \in K$ has a unique n-ary expansion $k = \sum_{i=1}^{\infty} \alpha_i n^{-i}$ for $\alpha_i \in \{0, \dots, n-1\}$. Each $a \in \mathbb{Z}[\frac{1}{n}]/\mathbb{Z}$ has two possible n-ary expansions. It has one that terminates $a = \sum_{i=1}^{\infty} \alpha_i^+ n^{-i}$, and one that doesn't $a = \sum_{i=1}^{\infty} \alpha_i^- n^{-i}$. Define $\psi\colon \tilde{G}_{\mathrm{ess}}^{(0)} \to \{0, \ldots, n-1\}^{\mathbb{N}}$ by $\psi(k) := (\alpha_i)_{i=1}^{\infty}$ for $k \in K$, and $\psi(a_-) := (\alpha_i^-)_{i=1}^{\infty}$, $\psi(a_+) := (\alpha_i^+)_{i=1}^{\infty}$ for $a_-, a_+ \in L$. The map ψ is a homeomorphism. Moreover, ψ induces an isomorphism between $\tilde{G}_{\mathrm{ess}}$ and the Cuntz groupoid O_n from [58]. Therefore, $\tilde{G}_{\mathrm{ess}}$ is amenable by [58, Proposition III.2.5]. Overall, $\tilde{G}$ decomposes into two invariant subgroupoids that are open and closed respectively. Both subgroupoids are amenable, and hence $\tilde{G}$ is amenable also.

Amenability of $\tilde{G}$ ensures that $C_r^*(\tilde{G})$ fits into the following short exact sequence of C^*-algebras. This is because amenability implies inner-exactness, from which it follows that (3.1.4) is exact.

$$0 \to \bigoplus_{i=1}^{n-1} C(\mathbb{T}^2) \otimes \mathcal{K} \to C_r^*(\tilde{G}) \to \mathcal{O}_n \to 0$$

By Proposition 3.1.14, the essential C^*-algebra $C_{\mathrm{ess}}^*(G)$ embeds into $\mathcal{O}_n$, and the singular ideal J embeds into $\bigoplus_{i=1}^{n-1} C(\mathbb{T}^2) \otimes \mathcal{K}$. Later, in Corollary 4.8.8, we will see that $J \cong \bigoplus_{i=1}^{n-1} C_0(\mathbb{R}^2) \otimes \mathcal{K}$.

Let us briefly comment on the generalisations of Thompson's group F given by $F_n := \Gamma(I; \mathbb{Z}[\frac{1}{2}], \langle n \rangle)$. Write $H := \mathrm{Germ}\big(F_n \curvearrowright I\big)$ and $G := \mathrm{Germ}\big(T_n \curvearrowright \mathbb{T}\big)$. By the earlier comment, $\tilde{H}$ is an open subgroupoid of the Hausdorff groupoid $\tilde{G}$. Example 3.4.11 shows that G is amenable, so it follows that $\tilde{H}$ and hence H are also amenable.

Chapter 4

The singular ideal

The final chapter of this thesis discusses the singular ideal of a non-Hausdorff étale groupoid. The singular ideal J is an ideal in the reduced C^*-algebra $C_r^*(G)$, and its existence is one of the main reasons why non-Hausdorff groupoid C^*-algebras are poorly understood. In a notable recent development ([38]), a deep connection is established relating the singular ideal and quasi-regular representations of isotropy groups. This connection proved to be very powerful, and allowed [38] to characterise vanishing of J and $J \cap \mathscr{C}_c(G)$. In this chapter, we will use quasi-regular representations of isotropy groups to study the structure of $J \cap \mathscr{C}_c(G)$, and to study when $J \cap \mathscr{C}_c(G)$ is dense in J.

Let us outline the structure of this chapter. In Section 4.1, we reconcile the different definitions of singular ideal appearing in the literature. Section 4.2 includes background results from [38] regarding the relationship between the singular ideal and quasi-regular representations of isotropy groups. In Section 4.3, we give explicit instructions for building elements of $J \cap \mathscr{C}_c(G)$, and show that elements constructed in this way have dense linear span in $J \cap \mathscr{C}_c(G)$. Section 4.4 includes similar methods, this time adapted to ample groupoids and the associated algebraic singular ideal $J_{\mathbb{C}}$. The next two sections then study when $J \cap \mathscr{C}_c(G)$ is dense in J. In particular, a characterisation is given for when $J \cap \mathscr{C}_c(G)$ is dense in J for amenable second-countable étale groupoids (Theorem 4.5.4). Overall, for certain classes of groupoids, the results of Sections 4.3 through 4.6 combine to give explicit dense spanning sets for the singular ideal.

The final sections consider the singular ideal of groupoids for which $\tilde{G}_{\text{sing}}$ is a discrete subset of the Hausdorff cover. Theorem 4.7.2 provides a formula for computing the singular ideal of such a groupoid. This generalises a computation in [31], where it was shown that the groupoid $\mathrm{Germ}(T \curvearrowright \mathbb{T})$ from Subsection 1.6.3 has singular ideal isomorphic to $C_0(\mathbb{R}^2) \otimes \mathcal{K}$. Finally, Subsection 4.8 applies Theorem 4.7.2 to the examples of non-Hausdorff groupoids appearing earlier in Section 1.6.

The result in Section 4.1 is taken from [9]. Sections 4.3 through 4.6 are based on the main results of [31]. This thesis further develops the techniques of [31] in Section 4.7, and these techniques are then applied to selected examples in Section 4.8.

§ 4.1 | Equivalent definitions

Let us recall the definition of the singular ideal, seen previously in Subsection 3.1.2. Given a Steinberg algebra RG as defined in Definition 1.3.6, we also state the definition of the singular ideal in RG.

Definition 4.1.1 ([18, 27, 43])**.** Let G be an étale groupoid. The *singular ideal* J is defined as

$$J \coloneqq \{a \in C_r^*(G) \colon \operatorname{supp}^\circ(a) \text{ has empty interior in } G\}.$$

Definition 4.1.2 ([18,66,67])**.** Let G be an ample groupoid and R a ring. The *singular ideal of* RG is defined as

$$J_R \coloneqq \{f \in RG \colon \operatorname{supp}^\circ(f) \text{ has empty interior in } G\}.$$

When $R = \mathbb{C}$, we call $J_{\mathbb{C}}$ the *algebraic singular ideal.*

For an explanation as to why J is a closed ideal, see Remark 3.1.13. A similar argument works for J_R. In Lemma 3.1.9, an alternative description of the singular ideal is given using dangerous points. Let us discuss a third (equivalent) definition that appears in the literature.

It is often convenient to consider the set $s(\operatorname{supp}^\circ(a))$ instead of $\operatorname{supp}^\circ(a)$. Indeed, the set J is sometimes defined to be those elements $a \in C_r^*(G)$ for which $s(\operatorname{supp}^\circ(a))$ has empty interior. Let us reconcile these two seemingly different definitions. Note that the contents of Lemma 4.1.3 can also be deduced from [5, Proposition 4.6] for a class of functions larger than $C_r^*(G)$.

In the following lemma, we write $S_\varepsilon(f) \coloneqq \{g \in G \colon |f(g)| > \varepsilon\}$ for a function $f \colon G \to \mathbb{C}$. Recall that a subset of $G^{(0)}$ is *nowhere dense* if its closure in $G^{(0)}$ has empty interior.

Lemma 4.1.3. *Let G be an étale groupoid.*

(i) Let $f \in \mathscr{C}_c(G)$ and $\varepsilon \geq 0$ be such that $S_\varepsilon(f)$ has empty interior. Then $s(S_\varepsilon(f))$ is nowhere dense.

(ii) If $a \in J$, then $s(S_\varepsilon(a))$ is nowhere dense for any $\varepsilon > 0$.

(iii) An element $a \in C_r^*(G)$ belongs to J if and only if $s(\operatorname{supp}^\circ(a))$ has empty interior in $G^{(0)}$.

(iv) *Let G be an ample groupoid and R a ring. An element $f \in RG$ belongs to J_R if and only if $s(\operatorname{supp}^\circ(f))$ is nowhere dense.*

Proof. Let us prove (i). Assume for a contradiction that $s(S_\varepsilon(f))$ is not nowhere dense, so that there is a non-empty open set $U \subseteq \overline{s(S_\varepsilon(f))}^{G^{(0)}}$. Let n be the smallest positive integer for which there exist open bisections $U_1, \dots, U_n \subseteq G$, $f_i \in C_c(U_i)$, and a function $\chi \in C_c(U)$ satisfying $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on some non-empty open subset $V \subseteq U$, such that $f * \chi = \sum_{i=1}^n f_i$. Note also that $f * \chi \neq 0$. For $i = 1, \dots, n$ define open sets $S_i := \operatorname{supp}^\circ(f_i)$. Our first claim is that, by minimality of n, we have $|(f * \chi)(g)| \leq \varepsilon$ whenever $g \in S_i \cap S_j$ for $i \neq j$. Indeed, suppose that there are $i \neq j$ and $g \in S_i \cap S_j$ with $|(f * \chi)(g)| > \varepsilon$. Since $f * \chi(g) = f(g)\chi(s(g))$ this implies that $|f(g)| > \varepsilon$ and $\chi(s(g)) > 0$. Choose $\chi' \in C_c(s(S_i \cap S_j) \cap \operatorname{supp}^\circ(\chi))$ such that $0 \leq \chi' \leq 1$ and $\chi' \equiv 1$ on some open set W containing $s(g)$. Then $f * \chi' = (f * \chi) * (\chi^{-1}\chi') = \sum_{i=1}^n f_i * (\chi^{-1}\chi')$. But both $f_i * (\chi^{-1}\chi')$ and $f_j * (\chi^{-1}\chi')$ belong to $C_c(S_i \cap S_j)$, meaning that we reduced n, thus contradicting minimality. This proves the first claim.

Our second claim is that $s(S_\varepsilon(f * \chi))$ is nowhere dense in $G^{(0)}$. For $i = 1, \dots, n$ define open sets $W_i := S_\varepsilon(f_i) \cap (\bigcup_{j \neq i} S_j)$. Observe that $S_\varepsilon(f_i) \setminus W_i \subseteq S_\varepsilon(f)$. Then, since $S_\varepsilon(f)$ has empty interior and $S_\varepsilon(f_i)$ is open, this implies that $S_\varepsilon(f_i) \subseteq \overline{W_i}^{U_i}$. Hence $S_\varepsilon(f_i) \setminus W_i \subseteq \partial_{U_i} W_i := \overline{W_i}^{U_i} \setminus W_i$. Since each W_i is open and contained in a compact subset of U_i, $\partial_{U_i} W_i$ is compact and has empty interior. Also, by the first claim, we have $s(S_\varepsilon(f * \chi)) = \bigcup_{i=1}^n s(S_\varepsilon(f_i) \setminus W_i) \subseteq \bigcup_{i=1}^n s(\partial_{U_i} W_i)$. Thus $s(S_\varepsilon(f * \chi))$ is nowhere dense, because it is contained in a finite union of closed sets with empty interior. This proves the second claim.

To complete the proof of (i), observe that V satisfies $V \subseteq U \subseteq \overline{s(S_\varepsilon(f))}^{G^{(0)}}$ and $V \cap s(S_\varepsilon(f)) = V \cap s(S_\varepsilon(f * \chi))$. Since V is open, we have $V \subseteq \overline{s(S_\varepsilon(f * \chi))}^{G^{(0)}}$. However, the second claim then implies that $V = \emptyset$, contradicting the assumption that V was non-empty. This completes the proof of (i). Taking $\varepsilon = 0$, an identical argument for $f \in RG$ yields (iv).

We now prove (ii). Let $a \in J$ and take $f_{\varepsilon/2} \in \mathscr{C}_c(G)$ with $\left\|a - f_{\varepsilon/2}\right\|_\infty < \varepsilon/2$. Then $S_{\varepsilon/2}(f_{\varepsilon/2}) \subseteq \operatorname{supp}^\circ(a)$ has empty interior. By part (i), $s\big(S_{\varepsilon/2}(f_{\varepsilon/2})\big)$ is nowhere dense in $G^{(0)}$. However, $S_\varepsilon(a) \subseteq S_{\varepsilon/2}(f_{\varepsilon/2})$, and hence $s(S_\varepsilon(a))$ is nowhere dense in $G^{(0)}$.

Finally, we prove (iii). The backwards implication is clear, so, given $a \in J$, it suffices to prove that $s(\operatorname{supp}^\circ(a))$ has empty interior. As $s(\operatorname{supp}^\circ(a)) = \bigcup_{n=1}^\infty s(S_{1/n}(a))$, part (ii) implies that $s(\operatorname{supp}^\circ(a))$ is the countable union of nowhere dense sets, and thus has empty interior by the Baire category theorem. $\square$

Remark 4.1.4. Taking $\varepsilon = 0$ in Lemma 4.1.3(i) implies that $s(\operatorname{supp}^\circ(a))$ is nowhere dense whenever $a \in J \cap \mathscr{C}_c(G)$. This property does *not* hold for arbitrary $a \in J$. Also, the proof of the lemma implies that whenever $f\colon G\to\mathbb{C}$ is a uniform limit of functions belonging to $\mathscr{C}_c(G)$, then $\operatorname{supp}^\circ(f)$ has empty interior in G if and only if $s(\operatorname{supp}^\circ(f))$ has empty interior in $G^{(0)}$. Such functions are also considered in [5, Proposition 4.6].

§ 4.2 | Background on quasi-regular representations

In this section we record necessary results from [38]. The first subsection recalls the relevant definitions associated with quasi-regular representations, and establishes notation that will feature throughout the remainder of the chapter. The second includes two lemmas that show how elements of $J\cap\mathscr{C}_c(G)$ can be built using kernels of quasi-regular representations. Finally, the third subsection shows how the isotropy fibres of the singular ideal are described by kernels of quasi-regular representations – see Theorem 4.2.6 in particular.

§ 4.2.1 | Quasi-regular representations

We begin by recalling the definition of the Chabauty topology on the set of subgroups of a discrete group. Fix a discrete group Γ, and let $\operatorname{Sub}(\Gamma)$ denote the set of all subgroups of Γ. Consider the map

$$\begin{aligned}\operatorname{Sub}(\Gamma) &\longrightarrow \{0,1\}^\Gamma\\ X &\longmapsto \mathbb{1}_X\end{aligned}$$

where $\mathbb{1}_X\colon\Gamma\to\{0,1\}$ is the indicator function. Equip $\{0,1\}^\Gamma$ with the product topology. The *Chabauty topology* on $\operatorname{Sub}(\Gamma)$ is the weakest topology for which the map above is continuous. This topology agrees with the Fell topology of [28] discussed in Chapter 3. Note in particular that $\operatorname{Sub}(\Gamma)$ is a compact totally disconnected space.

We now recall the definition of a quasi-regular representation. Given a subgroup $X\in\operatorname{Sub}(\Gamma)$, write Γ/X for the set of left-cosets. The associated *quasi-regular representation* $\lambda_{\Gamma/X}\colon\Gamma\to B\Big(\ell^2(\Gamma/X)\Big)$ is given by

$$\lambda_{\Gamma/X}(g)\delta_{hX}\coloneqq\delta_{ghX}$$

for $g\in\Gamma$ and $hX\in\Gamma/X$, where $\delta_{hX}\in\ell^2(\Gamma/X)$ denotes a standard basis vector. We also write $\lambda_{\Gamma/X}$ for the associated *-representation on the group algebra $\mathbb{C}[\Gamma]$. Moreover, when Γ is amenable, we also use $\lambda_{\Gamma/X}$ to denote the associated *-representation on the C^*-algebra $C^*_r(\Gamma)$.

Notation. Given our decision to abuse in the manner described above, let us make clear that whenever $\mathcal{X}\subseteq\operatorname{Sub}(\Gamma)$, we write $\bigcap\limits_{X\in\mathcal{X}}\ker(\lambda_{\Gamma/X})\cap\mathbb{C}[\Gamma]$ to denote the set of $b\in\mathbb{C}[\Gamma]$ such that $\lambda_{\Gamma/X}(b)=0$ for all $X\in\mathcal{X}$. This does not require Γ to be amenable.

Definition 4.2.1 ([38, Definition 5.1]). Let Γ be an amenable discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a set of subgroups. Define the ideal $J_{\Gamma,\mathcal{X}} \trianglelefteq C_r^*(\Gamma)$ as

$$J_{\Gamma,\mathcal{X}} \coloneqq \bigcap_{X \in \mathcal{X}} \ker(\lambda_{\Gamma/X}).$$

We will primarily be interested in the case where $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ is closed in the Chabauty topology and invariant under conjugation by elements of Γ. Indeed, by [7, Corollary 3.4(i)], if $\mathcal{Y} \subseteq \mathrm{Sub}(\Gamma)$ denotes the smallest closed and conjugation-invariant set of subgroups containing $\mathcal{X}$, then $J_{\Gamma,\mathcal{Y}} = J_{\Gamma,\mathcal{X}}$.

We finish our discussion on quasi-regular representations with the following lemma. It describes the relationship between the ideals of Definition 4.2.1 when passing from Γ to a (normal) subgroup $\Lambda \subseteq \Gamma$. The lemma combines results in [38], and we follow the same proofs.

For a subgroup $\Lambda \subseteq \Gamma$, we write $I_\Lambda \colon C_r^*(\Lambda) \to C_r^*(\Gamma)$ for the canonical inclusion and $E_\Lambda \colon C_r^*(\Gamma) \to C_r^*(\Lambda)$ for the canonical conditional expectation. Given a closed and conjugation-invariant set of subgroups $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$, we define $\Lambda \cap \mathcal{X} \coloneqq \{\Lambda \cap X \colon X \in \mathcal{X}\}$. It is simple to check that this is a closed and conjugation-invariant set of subgroups of Λ.

Lemma 4.2.2 ([38, Propositions 7.4 & 7.7]). *Let Γ be an amenable discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a closed and conjugation-invariant set of subgroups.*

(i) For any subgroup $\Lambda \subseteq \Gamma$ we have

$$I_\Lambda\Big(J_{\Lambda,\Lambda\cap\mathcal{X}}\Big) = J_{\Gamma,\mathcal{X}} \cap I_\Lambda\Big(C_r^*(\Lambda)\Big) \quad \textit{and} \quad I_\Lambda\Big(J_{\Lambda,\Lambda\cap\mathcal{X}} \cap \mathbb{C}[\Lambda]\Big) = J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Lambda].$$

(ii) Let $N \subseteq \Gamma$ be the (normal) subgroup generated by all $X \in \mathcal{X}$. Then,

$$E_N\Big(J_{\Gamma,\mathcal{X}}\Big) = J_{N,\mathcal{X}} \quad \textit{and} \quad E_N\Big(J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]\Big) = J_{N,\mathcal{X}} \cap \mathbb{C}[N].$$

In [38], a more general version of this lemma is proven for discrete groups Γ that are not necessarily amenable. We will only need to consider the case when Γ is amenable.

Proof. (i): Given $X \in \mathcal{X}$, it is clear that the restriction of $\lambda_{\Gamma/X}$ to $\mathbb{C}[\Lambda]$ is unitarily equivalent to the representation $\lambda_{\Lambda/\Lambda\cap X}$. Therefore $\ker(\lambda_{\Gamma/X}) \cap I_\Lambda\Big(C_r^*(\Lambda)\Big) = I_\Lambda\Big(\ker(\lambda_{\Lambda/\Lambda\cap X})\Big)$. The two equalities of part (i) then follow from the definitions.

(ii): Note that $E_N \circ I_N$ is the identity map on $C_r^*(\Lambda)$. Since N contains all the subgroups $X \in \mathcal{X}$, the inclusions $J_{N,\mathcal{X}} \subseteq E_N(J_{\Gamma,\mathcal{X}})$ and $J_{N,\mathcal{X}} \cap \mathbb{C}[N] \subseteq E_N(J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma])$ then follow from part (i).

It remains to prove that $E_N(J_{\Gamma,\mathcal{X}}) \subseteq J_{N,\mathcal{X}}$. Fix $a \in J_{\Gamma,\mathcal{X}}$ and $X \in \mathcal{X}$. We will show that $E_N(a) \in \ker(\lambda_{N/X})$, and in order to show this, we must prove that $\xi := \lambda_{N/X}\Big(E_N(a)\Big)(\delta_{gX}) = 0$ for all $g \in N$.

Let (a_n) be a sequence in $\mathbb{C}[\Gamma]$ converging to a with respect to the reduced norm. Define the sequences $\xi_n := \lambda_{N/X}\Big(E_N(a_n)\Big)(\delta_{gX}) \in \ell^2(N/X)$ and $\eta_n := \lambda_{\Gamma/X}(a_n)(\delta_{gX}) \in \ell^2(\Gamma/X)$ for each n. By continuity of $\lambda_{\Gamma/X}$ the sequence (η_n) converges to 0 as $n \to \infty$. We claim that the sequence (ξ_n) also converges to 0 as $n \to \infty$. By continuity of the maps E_N and $\lambda_{N/X}$, the sequence (ξ_n) converges to ξ in $\ell^2(N/X)$, so the lemma follows from the claim. In order to prove the claim, it suffices to show that $\langle \xi_n, \delta_{hX}\rangle \to 0$ for all $h \in N$. For each $h \in N$ compute

$$\begin{aligned}\langle \xi_n, \delta_{hX}\rangle &= \sum_{k\in N} E_N(a_n)(k)\langle \delta_{kgX}, \delta_{hX}\rangle = \sum_{k\in hXg^{-1}} E_N(a_n)(k) = \sum_{k\in hXg^{-1}} a_n(k)\\ &= \sum_{k\in\Gamma} a_n(k)\langle \delta_{kgX}, \delta_{hX}\rangle = \langle \eta_n, \delta_{hX}\rangle \xrightarrow{n\to\infty} 0\end{aligned}$$

since $hXg^{-1} \subseteq N$. This completes the proof of the claim, and thus the lemma. □

§ 4.2.2 | Constructing singular functions

This subsection records two results that isolate and generalise observations from [38]. In Section 4.3, they will be used to provide explicit instructions for building elements of $J \cap \mathscr{C}_c(G)$. In order to state these results, we need the following definition. This definition itself will play a key role throughout the remainder of the chapter.

Definition 4.2.3 ([38])**.** Let G be an étale groupoid that is covered by countably many open bisections. For a unit $x \in G^{(0)}$ define

$$\mathcal{X}(x) := \pi^{-1}\Big(\{x\}\Big) \cap \tilde{G}^{(0)}_{\text{ess}}.$$

Elements of $\mathcal{X}(x)$ are subgroups of the isotropy group G^x_x. By definition of the Fell topology, a subgroup $X \subseteq G^x_x$ belongs to $\mathcal{X}(x)$ if and only if there exists a primitive net $(x_\alpha) \subset C$ of (Hausdorff) units whose set of limit points is precisely X (see Section 3.1). Note, by Lemma 3.1.9, that a unit $x \in G^{(0)}$ is extremely dangerous if and only if the singleton $\iota(x)$ does *not* belong to $\mathcal{X}(x)$ ([38, Corollary 5.6]).

It is noted in [38] that $\mathcal{X}(x)$ is a conjugation-invariant collection of subgroups of G^x_x that is closed in the Chabauty topology. To see conjugation-invariance, take $X \in \mathcal{X}(x)$ and $g \in G^x_x$. The set $\boldsymbol{g} := g \cdot X$ belongs to $\tilde{G}$ and has source X, so belongs to $\tilde{G}_{\text{ess}}$. As a set, the subgroup $g \cdot X \cdot g^{-1}$ is equal to the product $\boldsymbol{g}X\boldsymbol{g}^{-1}$ in $\tilde{G}_{\text{ess}}$, and therefore belongs to $\mathcal{X}(x)$. The fact that $\mathcal{X}(x)$ is closed in the Chabauty topology follows from continuity of π, together with the observation that the Fell and Chabauty topologies agree on G_x^x.

The following results isolate observations from [38, Proposition 5.19]. We follow the same proofs (see also [9, Theorem 4.7]).

Lemma 4.2.4. *Let G be an étale groupoid that is covered by countably many open bisections. Fix $x \in G^{(0)}$, and let $g_1, \dots, g_n \in G_x$ be distinct points.*

There exist open bisections $U_1, \dots, U_n \subseteq G$ satisfying the following conditions.

(i) $g_i \in U_i$ for all $i \in \{1, \dots, n\}$.

(ii) $s(U_i) = s(U_j)$ for all $i, j \in \{1, \dots, n\}$.

(iii) $W_I = \emptyset$ whenever $I \subseteq \{1, \dots, n\}$ is not of the form

$$\{g_i \colon i \in I\} = kX \cap \{g_1, \dots, g_n\} \tag{4.2.1}$$

for some $k \in G_x$ and $X \in \mathcal{X}(x)$.

Here $W_I \coloneqq s\Big(\bigcap_{i \in I} U_i \setminus \bigcup_{j \notin I} U_j\Big) \cap C$.

The conditions of the lemma are satisfied for all "sufficiently small" open bisections. Precisely, if $U_1, \dots, U_n$ satisfy (i) – (iii), then for any open set $U \subseteq G^{(0)}$ containing x, the cutdowns $U_i' \coloneqq U_i \cap s^{-1}(U)$ also satisfy (i) – (iii).

Proof. It is clear that there exist open bisections $V_1, \dots, V_n \subseteq G$ satisfying the conditions (i) and (ii). Write $V \coloneqq s(V_1)$ and $V_I \coloneqq s\Big(\bigcap_{i \in I} V_i \setminus \bigcup_{j \notin I} V_j\Big) \cap C$ for subsets $I \subseteq \{1, \dots, n\}$. Choose an open subset $W \subseteq G^{(0)}$ such that $x \in W \subseteq V$ and $W \cap V_I = \emptyset$ whenever $x \notin \overline{V_I}^{G^{(0)}}$. Define $U_i \coloneqq V_i \cap s^{-1}(W)$ for $i \in \{1, \dots, n\}$. It is clear that the open bisections $U_1, \dots, U_n \subseteq G$ satisfy conditions (i) and (ii). We show that they also satisfy condition (iii).

Let $I \subseteq \{1, \dots, n\}$ be such that $W_I \neq \emptyset$, where W_I is as in the statement of the lemma. In order to complete the proof, we show that I is of the form in (4.2.1). It is clear that $W_I = W \cap V_I$ and hence $x \in \overline{W_I}^{G^{(0)}}$ by the construction of W. Let (x_α) be a primitive net in W_I that converges to x, and let $X \subseteq G_x^x$ denote the set of limit points. By definition of the Fell topology, the net $\iota(g_\alpha)$ converges to X in $\tilde{G}$, and therefore $X \in \tilde{G}^{(0)}_{\mathrm{ess}}$. For each α, let $g_\alpha \in \bigcap_{i \in I} U_i \setminus \bigcup_{j \notin I} U_j$ be the unique element such that $s(g_\alpha) = x_\alpha$. Continuity of multiplication on the groupoid implies that the net (g_α) has set of limit points equal to $g_{i_0}X$, where i_0 is any element of I. The unit x belongs to X, and hence $\{g_i \colon i \in I\} \subseteq g_{i_0}X$ for any $i_0 \in I$. On the other hand, if $j \notin I$ then $g_\alpha \notin U_j$ and hence the net (g_α) does not converge to g_j. Therefore $g_j \notin g_{i_0}X$, and it follows that $\{g_i \colon i \in I\} = g_{i_0}X \cap \{g_1, \dots, g_n\}$ for any $i_0 \in I$. This completes the proof. $\square$

The next proposition is a generalisation of the construction of singular elements found in [38, Proposition 5.19] (see also [9, Theorem 4.7]). Below, $\tilde{G}_{\text{sing}}$ denotes the singular part of the Hausdorff cover as defined in Definition 3.1.8, and $\mathfrak{i}\colon \mathscr{C}_c(G) \hookrightarrow C_c(\tilde{G})$ denotes the canonical embedding from Lemma 3.1.3.

Proposition 4.2.5. *Let G be an étale groupoid that is covered by countably many open bisections.*

Fix $x \in G^{(0)}$, and suppose that $b \in \mathbb{C}[G_x^x]$ belongs to $\ker(\lambda_{G_x^x/X})$ for all $X \in \mathcal{X}(x)$. Let $\{g_1, \ldots, g_n\} \subseteq G_x^x$ denote the support of b, let $h \in G_x$, and let $U_1, \ldots, U_n$ be open bisections containing $hg_1, \ldots, hg_n$ and satisfying the conditions of Lemma 4.2.4. Then

$$f^\psi \coloneqq \sum_{i=1}^{n} b(g_i)(\psi \circ s|_{U_i}) \tag{4.2.2}$$

belongs to $J \cap \mathscr{C}_c(G)$ for any $\psi \in C_c\big(s(U_i)\big)$ (note that this does not depend on i). Moreover, $\mathfrak{i}(f^\psi)$ belongs to $C_c(\tilde{G}_{\text{sing}})$.

Note that $f^\psi = 0$ if and only if $b = 0$.

Proof. For $I \subseteq \{1, \ldots, n\}$ set $\check{S}_I \coloneqq (\bigcap_{i \in I} U_i) \setminus (\bigcup_{j \notin I} U_j)$ and $W_I \coloneqq s(\check{S}_I) \cap C$, as in Lemma 4.2.4. Take arbitrary $\psi \in C_c\big(s(U_i)\big)$, and let f^ψ be as in (4.2.2). We first show that $s(\operatorname{supp}^\circ(f^\psi)) \subseteq D$, from which it follows that $f^\psi \in J \cap \mathscr{C}_c(G)$ by Lemma 3.1.12.

Take $g \in \operatorname{supp}^\circ(f^\psi)$, and let $I \subseteq \{1, \ldots, n\}$ be such that $g \in \check{S}_I$. If $W_I \neq \emptyset$, then condition (iii) of Lemma 4.2.4 ensures that $\{hg_i : i \in I\} = kX \cap \{hg_1, \ldots, hg_n\}$ for some $k \in G_x$ and $X \in \mathcal{X}(x)$. Since $b \in \ker(\lambda_{G_x^x/X})$, this implies that

$$f^\psi(g) = \sum_{i \in I} b(g_i)\psi(s(g)) = \sum_{t \in h^{-1}kX} b(t)\psi(s(g)) = 0,$$

a contradiction. Therefore, we must have $W_I = \emptyset$. It follows that $s(\check{S}_I) \subseteq D$, and hence $s(g) \in D$, as desired.

Next, we show that $\mathfrak{i}(f^\psi) \in C_c(\tilde{G}_{\text{sing}})$. Let $K \coloneqq \operatorname{supp}(\psi) \subseteq s(U_i)$, and for each i let $K_i \coloneqq U_i \cap s^{-1}(K)$. Each K_i is compact. The first part of the proof shows that

$$\operatorname{supp}^\circ(f^\psi) \subseteq \bigcup_{W_I = \emptyset} \left(\bigcap_{i \in I} K_i \setminus \bigcup_{j \notin I} U_j \right)$$

where the union is taken over all nonempty $I \subseteq \{1, \ldots, n\}$ for which $W_I = \emptyset$. The goal is to show that $\mathfrak{i}(f^\psi)$ has support contained in $\tilde{G}_{\text{sing}}$. Let $\boldsymbol{g} \in \operatorname{supp}\big(\mathfrak{i}(f^\psi)\big) \subseteq \tilde{G}$. Since $\iota(G)$ is dense in $\tilde{G}$, we know that $\operatorname{supp}\big(\mathfrak{i}(f^\psi)\big)$ is the closure of $\iota\big(\operatorname{supp}^\circ(f^\psi)\big)$ in $\tilde{G}$. Therefore, by definition of the Fell topology, there exists $I \subseteq \{1, \ldots, n\}$ with $W_I = \emptyset$

and a primitive net $(g_\alpha) \subset \bigcap_{i\in I} K_i \setminus \bigcup_{j\notin I} U_j$ whose set of limit points is precisely $\boldsymbol{g}$. We claim that $\boldsymbol{g} \cap K_i \neq \emptyset$ for each $i \in I$, and $\boldsymbol{g} \cap U_j = \emptyset$ for each $j \notin I$. Given $i \in I$, the net (g_α) belongs to the compact set K_i, thus admits a subnet that converges in K_i. But (g_α) is primitive, and therefore has a limit in K_i itself. This proves that $\boldsymbol{g} \cap K_i \neq \emptyset$. For any $j \notin I$, the net (g_α) is disjoint from the open set U_j, and hence $\boldsymbol{g} \cap U_j = \emptyset$.

It remains to prove that $\boldsymbol{g} \in \tilde{G}_{\mathrm{sing}}$. Let $(h_\beta) \subset G$ be a primitive net whose set of limit points is precisely $\boldsymbol{g}$. We must prove that $h_\beta \notin G|_C$ eventually. Let $V, L \subseteq G^{(0)}$ be an open and compact set respectively such that $K \subseteq V \subseteq L \subseteq s(U_i)$. For each i write $V_i := U_i \cap s^{-1}(V)$ and $L_i := U_i \cap s^{-1}(L)$. The claim above proves that $\boldsymbol{g} \cap V_i \neq \emptyset$ for each $i \in I$, and $\boldsymbol{g} \cap L_j = \emptyset$ for each $j \notin I$. It follows that $h_\beta \in \bigcap_{i\in I} V_i$ eventually. Also, for each $j \notin I$, primitivity of (h_β) and compactness of L_j ensure that $h_\beta \notin L_j$ eventually. Now, $s(V_i) \subseteq s(L_j)$ for any $i, j \in \{1, \ldots, n\}$. Using that each U_j is a bisection, we make the crucial final deduction that $h_\beta \notin U_j$ eventually for any $j \notin I$. Overall, we have that $h_\beta \in \check{S}_I$ eventually. By assumption $W_I = \emptyset$, and therefore $h_\beta \notin G|_C$ eventually. This completes the proof of the proposition. □

§ 4.2.3 | Isotropy fibres of the singular ideal

We have reached the main theorem of this section. This theorem explicitly describes the isotropy fibres of the singular ideal in terms of kernels of quasi-regular representations, and will be key throughout the remainder of this chapter. As stated here, it combines [38, Theorem 5.5] and [38, Proposition 5.19].

Given a unit $x \in G^{(0)}$, let $\eta_x \colon \mathscr{C}_c(G) \to \mathbb{C}[G_x^x]$ and $\eta_{x,r} \colon C_r^*(G) \to C_r^*(G_x^x)$ denote the restriction maps from Subsection 1.5. When G_x^x is amenable, write $J_x := \eta_{x,r}(J)$ (see Definition 1.5.1). For a unit $x \in G^{(0)}$, let $\mathcal{X}(x)$ be as in Definition 4.2.3.

Theorem 4.2.6 ([38, Theorem 5.5] & [38, Proposition 5.19])**.** *Let G be an étale groupoid that is covered by countable many open bisections.*

(i) For any $x \in G^{(0)}$ we have

$$\eta_x\Big(J \cap \mathscr{C}_c(G)\Big) = \bigcap_{X \in \mathcal{X}(x)} \ker(\lambda_{G_x^x/X}) \cap \mathbb{C}[G_x^x]. \tag{4.2.3}$$

(ii) Let $x \in G^{(0)}$ be such that the isotropy group G_x^x is amenable. Then

$$J_x = J_{G_x^x, \mathcal{X}(x)}. \tag{4.2.4}$$

When the isotropy group G_x^x is amenable, the right-hand side of (4.2.3) can be replaced with $J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. Note that this only works since all quasi-regular representations of G_x^x are weakly contained in the left-regular representation of G_x^x.

Remark 4.2.7. An analogue of the ideal $J_{\Gamma,\mathcal{X}}$ from Definition 4.2.1 can be defined for groups Γ that are non-amenable. This level of generality is discussed in [38]. Moreover, this ideal always embeds canonically into the reduced C^*-algebra $C_r^*(\Gamma)$ (see the proof of [38, Proposition 7.2]). Let us call the image $J^r_{\Gamma,\mathcal{X}} \trianglelefteq C_r^*(\Gamma)$. With this (temporary) notation, Equation (4.2.4) can be upgraded to the equality $\eta_{x,r}(J) = J^r_{G_x^x,\mathcal{X}(x)}$ for any $x \in G^{(0)}$, regardless of whether or not G_x^x is amenable (see [38, Theorem 5.5]). On the other hand, care must be taken in part (i) of the preceding theorem: the right-hand side of Equation (4.2.3) must <u>not</u> be replaced with the intersection $J^r_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. The inclusion from left to right will hold, but the opposite inclusion can fail. Indeed, for a non-amenable discrete group Γ, the groupoid from Example 1.6.1 provides a counterexample to the inclusion $J^r_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x] \subseteq \eta_x\Big(J \cap \mathscr{C}_c(G)\Big)$.

We include a (partial) proof of Theorem 4.2.6. We prove (i) with the aid of the two lemmas appearing in the previous subsection. As for part (ii), we prove the inclusion $J_x \subseteq J_{G_x^x,\mathcal{X}(x)}$ but omit the proof of the opposite inclusion. This omission is justified, since the equality of (4.2.4) is used just once in the thesis, namely in Theorem 4.5.1. Any result that invokes Theorem 4.5.1 relies only on the inclusion $J_x \subseteq J_{G_x^x,\mathcal{X}(x)}$.

Proof. First, consider equation (4.2.3). The inclusion $\eta_x(J \cap \mathscr{C}_c(G)) \subseteq J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ follows immediately from Proposition 4.2.5. We omit the proof of the reverse inclusion, since it follows an argument similar to, but easier than the one below.

Assume that G_x^x is amenable. We prove the inclusion $J_x \subseteq J_{G_x^x,\mathcal{X}(x)}$ of (4.2.4). Fix $a \in J$ and $X \in \mathcal{X}(x)$. We must prove that $\eta_{x,r}(a) \in \ker(\lambda_{G_x^x/X})$, and in order to show this, it suffices to show that $\xi \coloneqq \lambda_{G_x^x/X}\Big(\eta_{x,r}(a)\Big)(\delta_{gX}) = 0$ for all $g \in G_x^x$.

Let $g \in G_x^x$, and let $(f_n) \subset \mathscr{C}_c(G)$ be a sequence converging to a in the reduced norm. Define $\xi_n \coloneqq \lambda_{G_x^x/X}\Big(\eta_x(f_n)\Big)(\delta_{gX}) \in \ell^2\Big(G_x^x/X\Big)$ for each n. We claim that $\langle \xi_n, \delta_{hX}\rangle \to 0$ as $n \to \infty$ for any $h \in G_x^x$. By definition of the quasi-regular representation $\lambda_{G_x^x/X}$, and since $\eta_x(f_n)$ is finitely supported on G_x^x, we have that $\xi_n = \sum_{k \in G_x^x} f_n(k)\delta_{kgX}$. Writing $\boldsymbol{g} \coloneqq g \cdot X$ and $\boldsymbol{h} \coloneqq h \cdot X$ for elements of the Hausdorff cover $\tilde{G}$, we compute

$$\langle \xi_n, \delta_{hX}\rangle = \sum_{k \in G_x^x} f_n(k)\langle \delta_{kgX}, \delta_{hX}\rangle = \sum_{k \in \boldsymbol{h}\boldsymbol{g}^{-1}} f_n(k) = \mathfrak{i}(f_n)(\boldsymbol{h}\boldsymbol{g}^{-1}) \xrightarrow{n\to\infty} \mathfrak{i}_r(a)(\boldsymbol{h}\boldsymbol{g}^{-1})$$

where the penultimate step uses (3.1.3). Since $X \in \tilde{G}^{(0)}_{\text{ess}}$, the elements $\boldsymbol{g}$, $\boldsymbol{h}$ and therefore $\boldsymbol{h}\boldsymbol{g}^{-1}$ also belong to $\tilde{G}_{\text{ess}}$. Proposition 3.1.14 then implies that $\mathfrak{i}_r(a)(\boldsymbol{h}\boldsymbol{g}^{-1}) = 0$. This proves the claim.

Continuity of the maps $\eta_{x,r}$ and $\lambda_{G_x^x/X}$ ensures that $\xi_n \to \xi$ in $\ell^2\Big(G_x^x/X\Big)$. By the claim, it follows that $\langle \xi, \delta_{hX}\rangle = 0$ for every $h \in G_x^x$. However, the vectors δ_{hX} form an orthonormal basis for $\ell^2\Big(G_x^x/X\Big)$, so we must have $\xi = 0$. This completes the proof. $\square$

As mentioned earlier, [38] uses Theorem 4.2.6 to produce a condition on an étale groupoid that characterises when the implication "$J \neq \{0\} \implies J \cap \mathscr{C}_c(G) \neq \{0\}$" holds. This condition is then verified for many classes of groupoids. We note that there is still no known groupoid for which this implication fails. In this spirit, and as a simple application of the methods just introduced, we give an explicit class of groupoids for which the implication holds. In particular, we recover [9, Theorem 4.7]. Note that a much more general version of the following theorem is proved in [38, Corollary 7.12].

Theorem 4.2.8. *Let G be an étale groupoid.*

Assume that $G_x^x \cap \overline{G^{(0)}}$ is finite for any $x \in G^{(0)}$. Then $J = \{0\}$ if and only if $J \cap \mathscr{C}_c(G) = \{0\}$.

We follow the original proof of [9, Theorem 4.7], but now phrased using the language of quasi-regular representations. We only sketch the details, since the core arguments already appear earlier in the proofs of Proposition 4.2.5 and Theorem 4.2.6. This result can also be deduced from the "non-amenable" versions of Lemma 4.2.2 and Theorem 4.2.6(ii) that appear in [38].

Proof. We sketch the proof. The forward implication is clear. So, assume that $J \neq \{0\}$ and let $a \in J \setminus \{0\}$. Since a is the limit of a sequence of functions in $\mathscr{C}_c(G)$, we can therefore assume that G is covered by countably many open bisections.

Next, since J is an ideal, we can assume that $a(x) \neq 0$ for some unit $x \in G^{(0)}$. Let N denote the (normal) subgroup of G_x^x generated by the subgroups in $\mathcal{X}(x)$. The set $\mathcal{X}(x)$ is a finite conjugation-invariant collection of finite subgroups, so [38, Lemma 7.9] ensures that N itself is finite. Let $b := a|_N \in \mathbb{C}[N]$ denote the restriction of a to the finite subgroup N. By following the same argument as in the proof of Theorem 4.2.6 above, one can show that $b \in \ker(\lambda_{N/X})$ for every $X \in \mathcal{X}(x)$. Then, by following the same argument as in Proposition 4.2.5, we obtain an element $f \in J \cap \mathscr{C}_c(G)$ with $f|_N = a$. In particular, f is non-zero. □

The following corollary from [38] is now immediate by combining Theorems 4.2.6 and 4.2.8.

Corollary 4.2.9 ([38, Corollary 5.21])**.** *Let G be an étale groupoid that is covered by countably many open bisections.*

(i) $J \cap \mathscr{C}_c(G) = 0$ if and only if

$$\bigcap_{X \in \mathcal{X}(x)} \ker(\lambda_{G_x^x/X}) \cap \mathbb{C}[G_x^x] = \{0\} \tag{4.2.5}$$

for all $x \in G^{(0)}$.

(ii) Assume that $G_x^x \cap \overline{G^{(0)}}$ *is finite for all* $x \in G^{(0)}$*. Then* $J = \{0\}$ *if and only if* (4.2.5) *holds for all* $x \in G^{(0)}$*.*

In the previous corollary, one can replace G_x^x with the finite normal subgroup $N \subseteq G_x^x$ generated by the subgroups $X \in \mathcal{X}(x)$.

Remark 4.2.10. Let us comment on the analogous results for the algebraic singular ideal $J_{\mathbb{C}}$ in the complex Steinberg algebra $\mathbb{C}G$. Given an ample groupoid G that is covered by countably many open bisections, one can use Theorem 4.2.6 and Proposition 4.2.5 to show that $J_{\mathbb{C}} = \{0\}$ if and only if Equation (4.2.5) holds for all $x \in G^{(0)}$. This can also be seen by combining Theorem 4.2.6(i) with a result we prove later, stating that $J_{\mathbb{C}}$ is always dense in $J \cap \mathscr{C}_c(G)$ (Theorem 4.4.4).

§ 4.3 | Structure of $J \cap \mathscr{C}_c(G)$

In the remainder of this chapter, we will use quasi-regular representations to study the structure of the singular ideal. In this section, we provide explicit instructions for building "elementary" elements of $J \cap \mathscr{C}_c(G)$, and show that these elementary elements have dense linear span in $J \cap \mathscr{C}_c(G)$.

Let us describe the approach. Begin with an (extremely dangerous) unit $x \in G^{(0)}$ and a finitely supported function $b \colon G_x^x \to \mathbb{C}$ that satisfies the following linear equations:

$$\sum_{g \in kX} b(g) = 0 \text{ for all } k \in G_x^x \text{ and } X \in \mathcal{X}(x). \tag{4.3.1}$$

Given $h \in G_x$, translate the support of b by left-multiplication, and choose open bisections $V_1, \ldots, V_n$ containing these translates. Lastly, choose a sufficiently small open neighbourhood $W \subseteq \bigcap_{i=1}^n s(V_i)$ of x so that the open bisections $U_i := V_i \cap s^{-1}(W)$ satisfy the conditions of Lemma 4.2.4. Then, for any $\psi \in C_c(W)$, the function f^ψ (from (4.2.2)) belongs to $J \cap \mathscr{C}_c(G)$ (see Proposition 4.2.5). Overall, the function f^ψ is determined by data in the tuple $(x, h, b, U_1, \ldots, U_n, \psi)$. By ranging over all valid tuples, one obtains a family of functions whose linear span is dense in $J \cap \mathscr{C}_c(G)$ (see Theorem 4.3.5).

An element $b \in \mathbb{C}[G_x^x]$ satisfies the equations (4.3.1) if and only if, for every subgroup $X \in \mathcal{X}(x)$, b belongs to the kernel of the associated quasi-regular representation $\lambda_{G_x^x/X}$ (this uses that $\mathcal{X}(x)$ is invariant under conjugation). In order to find $b \in \mathbb{C}[G_x^x]$ satisfying (4.3.1), one should therefore compute the kernels $\mathbb{C}[G_x^x] \cap \ker(\lambda_{G_x^x/X})$.

The construction of singular elements in this section will use Lemma 4.2.4 and Proposition 4.2.5. In addition to these, we will need one more lemma.

Lemma 4.3.1. *Let G be an étale groupoid. Fix $f \in \mathscr{C}_c(G)$, $x \in G^{(0)}$, and assume that $f(g) = 0$ for all $g \in G_x$. Then f is continuous at each $g \in G_x$.*

Proof. Let $g \in G_x$ and let (g_α) be a net in G converging to g. In order to prove that $\lim_\alpha f(g_\alpha) = 0$, it suffices to find a subnet (g_β) satisfying $\lim_\beta f(g_\beta) = 0$ (since (g_α) is arbitrary). In view of this, let us choose a primitive subnet (g_β) so that $\iota(g_\beta)$ converges in $\tilde{G}$. Let $\boldsymbol{g} \in \tilde{G}$ denote the limit. Since $\boldsymbol{g} \subseteq G_x$, we have $\mathfrak{i}(f)(\boldsymbol{g}) = \sum_{g\in\boldsymbol{g}} f(g) = 0$ by equation (3.1.3). Moreover, $f(g_\beta) = \mathfrak{i}(f)\big(\iota(g_\beta)\big) \to \mathfrak{i}(f)(\boldsymbol{g})$ by continuity of $\mathfrak{i}(f)$. This completes the proof. □

Remark 4.3.2. The previous lemma can be generalised in the following way. Let a be an arbitrary element of $C_r^*(G)$. If the isotropy group G_x^x is amenable, then a is continuous at each point of G_x provided that $a(g) = 0$ for all $g \in G_x$. This result fails if the isotropy group G_x^x is not amenable. To see this, consider the groupoid in Example 1.6.1 when Γ is non-amenable.

§ 4.3.1 | Elementary singular functions in $J \cap \mathscr{C}_c(G)$

Let G be an étale groupoid that is covered by countably many open bisections. Each "elementary" element of $J \cap \mathscr{C}_c(G)$ is assembled from the data in a tuple of the form $\Xi := (x, h, b, U_1, \ldots, U_n, \psi)$ where:

- $x \in G^{(0)}$;
- $h \in G_x$;
- $b \in \mathbb{C}[G_x^x]$ belongs to $\ker(\lambda_{G_x^x/X})$ for all $X \in \mathcal{X}(x)$, and has support $\{g_1, \ldots, g_n\}$;
- $U_1, \ldots, U_n \subseteq G$ are open bisections satisfying the conditions of Lemma 4.2.4. That is, they have equal source, contain the elements $hg_1, \ldots, hg_n \in G_x$, and are such that $W_I = \emptyset$ whenever $I \subseteq \{1, \ldots, n\}$ is not of the form

  $$\{g_i \colon i \in I\} = kX \cap \{g_1, \ldots, g_n\} \qquad \text{for some } k \in G_x^x, X \in \mathcal{X}(x).$$

  Here $W_I := s\Big(\bigcap_{i\in I} U_i \setminus \bigcup_{j\notin I} U_j\Big) \cap C$.
- $\psi \in C_c\big(s(U_i)\big)$ (note that this does not depend on i).

Definition 4.3.3. Let G be an étale groupoid that is covered by countably many open bisections. Let Ξ be a tuple as described above, and define

$$f^\Xi := \sum_{i=1}^n b(g_i)(\psi \circ s|_{U_i}). \tag{4.3.2}$$

A function f^{Ξ} of this form will be called an *elementary* singular function. Let $\mathcal{E} \subseteq J \cap \mathscr{C}_c(G)$ denote the linear span of all such elementary functions.

By Proposition 4.2.5, any elementary singular function f^{Ξ} belongs to $J \cap \mathscr{C}_c(G)$, and hence $\mathcal{E} \subseteq J \cap \mathscr{C}_c(G)$. Moreover, Proposition 4.2.5 ensures that $\mathfrak{i}(\mathcal{E}) \subseteq C_c(\tilde{G}_{\text{sing}})$. In general, $\mathcal{E}$ is a proper subset of $J \cap \mathscr{C}_c(G)$. Nevertheless, in the following subsection we will see that $\mathcal{E}$ is always dense in $J \cap \mathscr{C}_c(G)$.

§ 4.3.2 | Elementary singular functions have dense linear span in $J \cap \mathscr{C}_c(G)$

Before proving that $\mathcal{E}$ is dense in $J \cap \mathscr{C}_c(G)$, we need a technical lemma. Given any $f \in J \cap \mathscr{C}_c(G)$, Theorem 4.2.6(i) ensures that the restriction to any source fibre G_x is a (finite) sum of translates of functions in $\bigcap_{X \in \mathcal{X}(x)} \ker(\lambda_{G_x^x/X}) \cap \mathbb{C}[G_x^x]$. By choosing small enough open bisections around the support of $f|_{G_x}$, one can construct an element of $\mathcal{E}$. The following lemma makes this precise.

Lemma 4.3.4. *Let G be an étale groupoid that is covered by countably many open bisections. Fix $f \in J \cap \mathscr{C}_c(G)$, $x \in G^{(0)}$, and let $g_1, \dots, g_n \in G_x$ be all elements of G_x on which f is non-zero.*

There exist open bisections $U_1, \dots, U_n$ with $g_i \in U_i$ for each i, and with equal source such that

$$f^{\psi} := \sum_{i=1}^{n} f(g_i)(\psi \circ s|_{U_i}) \tag{4.3.3}$$

belongs to $\mathcal{E}$ for any $\psi \in C_c\Big(s(U_i)\Big)$.

Proof. Let $U_1, \dots, U_n$ be open bisections satisfying the conditions of Lemma 4.2.4 and such that U_{i_1} and U_{i_2} are disjoint whenever $r(g_{i_1}) \neq r(g_{i_2})$. In particular, $U_1, \dots, U_n$ have equal source and satisfy $g_i \in U_i$ for each i. We show that $f^{\psi} \in \mathcal{E}$ for any $\psi \in C_c(s(U_i))$.

Partition the set $\{g_1, \dots, g_n\}$ according to its images under the range map r and write $\{g_1, \dots, g_n\} = \bigsqcup_{j=1}^{m} \{g_1^j, \dots, g_{k_j}^j\}$. Then, write $\{U_1, \dots, U_n\} = \bigsqcup_{j=1}^{m} \{U_1^j, \dots, U_{k_j}^j\}$ where $g_i^j \in U_i^j$. Fix $\psi \in C_c(s(U_i))$ and define

$$f^j := \sum_{i=1}^{k_j} f\Big(g_i^j\Big)\Big(\psi \circ s|_{U_i^j}\Big)$$

for each $j \in \{1, \dots, m\}$. We claim that each f^j is elementary in the sense of Definition 4.3.3. Given $j \in \{1, \dots, m\}$, let V be an open bisection containing $(g_1^j)^{-1}$, and let $\varphi \in C_c(V)$ be such that $\varphi\Big(\big(g_1^j\big)^{-1}\Big) = 1$. Define $b_j := \eta_x(\varphi * f)$, where $\eta_x \colon \mathscr{C}_c(G) \to \mathbb{C}[G_x^x]$ is the restriction map. Since $\varphi * f \in J \cap \mathscr{C}_c(G)$, Theorem 4.2.6(i) implies that b_j

belongs to $\ker(\lambda_{G^x_x/X})$ for all $X \in \mathcal{X}(x)$. Moreover, b_j has support $(g^j_1)^{-1}\{g^j_1, \dots, g^j_{k_j}\}$, and the initial assumptions on the bisections $U_1, \dots, U_n$ ensure that $U^j_1, \dots, U^j_{k_j}$ satisfy the conditions of Lemma 4.2.4. Therefore, $\Xi_j := \left(x, g^j_1, b_j, U^j_1, \dots, U^j_{k_j}, \psi\right)$ is a valid tuple in the sense of Subsection 4.3.1. By construction, we have $f^j = f^{\Xi_j}$, and therefore f^j is elementary, as claimed. To complete the proof of the lemma, simply observe that $f^\psi = \sum_{j=1}^m f^j$, where f^ψ is as in (4.3.3). □

Recall from Definition 1.3.1 a net $(f_\alpha) \subset \mathscr{C}_c(G)$ is said to converge to $f \in \mathscr{C}_c(G)$ in the inductive limit topology if $f_\alpha \to f$ uniformly, and there exists a compact set $K \subseteq G$ such that $\operatorname{supp}^\circ(f_\alpha) \subseteq K$ eventually.

Theorem 4.3.5. *Let G be an étale groupoid that is covered by countably many open bisections. Then $\mathcal{E}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to the inductive limit topology.*

By definition (see Definition 1.3.4 and Subsection 1.3.2), the inductive limit topology is stronger than the topology induced by the full norm $\|\cdot\|_{C^*(G)}$ (and also the norm $\|\cdot\|_I$). Therefore, $\mathcal{E}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to any C^*-norm.

Proof. Let $f \in J \cap \mathscr{C}_c(G)$, and let $K \subseteq G$ be a compact set with $\operatorname{supp}^\circ(f)$ contained in its interior. It suffices to find, for each $\varepsilon > 0$, a function $f' \in \mathcal{E}$ satisfying $\operatorname{supp}^\circ(f') \subseteq K$ and $\|f - f'\|_\infty \leq \varepsilon$.

Fix $\varepsilon > 0$. Take $x \in s(K)$, and let $g_1, \dots, g_n \in G_x$ be all elements of G_x on which f is non-zero. Let $U_1, \dots, U_n \subseteq K$ be open bisections as in Lemma 4.3.4 so that, in particular, $g_i \in U_i$ for each i. Note that this is possible since $g_1, \dots, g_n$ all lie in the interior of K. Write $U_x := s(U_i)$ (note that this does not depend on i). If $\{g_1, \dots, g_n\}$ is empty, let $U_x = G^{(0)}$. By Lemma 4.3.4, the function

$$f^\psi := \sum_{i=1}^n f(g_i)(\psi \circ s|_{U_i}) \tag{4.3.4}$$

belongs to $\mathcal{E}$ for any $\psi \in C_c(U_x)$. Moreover, $\operatorname{supp}^\circ(f^\psi) \subseteq K$. Fix some $\psi_x \in C_c(U_x)$ with $\psi_x(x) = 1$. Since $f|_{G_x} = f^{\psi_x}|_{G_x}$ and $\operatorname{supp}^\circ(f - f^{\psi_x}) \subseteq K$, Lemma 4.3.1 ensures that there exists an open neighbourhood $W_x \subseteq U_x$ of x such that $\left|f(g) - f^{\psi_x}(g)\right| \leq \varepsilon$ whenever $s(g) \in W_x$.

Now, the sets $\{W_x\}_{x \in s(K)}$ form an open cover for the compact set $s(K)$, so we may select a finite subcover $W_{x_1}, \dots, W_{x_m}$. Let $\varphi_1, \dots, \varphi_m \in C_c(G^{(0)})$ be a partition of unity for $s(K)$ subordinate to this cover, and define

$$f' := \sum_{j=1}^m (\varphi_j \circ s) f^{\psi_{x_j}} = \sum_{j=1}^m f^{\varphi_j \cdot \psi_{x_j}}.$$

From (4.3.4), it is clear that each $f^{\varphi_j \cdot \psi_{x_j}}$ belongs to $\mathcal{E}$, and therefore f' also belongs to $\mathcal{E}$. We also have $\operatorname{supp}^\circ(f') \subseteq K$ and $\|f - f'\|_\infty \leq \varepsilon$ by construction. This completes the proof of the theorem. $\square$

Remark 4.3.6. Theorem 4.3.5 extends to groupoids G that are not covered by countably many open bisections. For an open subgroupoid $H \subseteq G$ that *is* covered by countably many open bisections, define $\mathcal{E}_H$ to be the linear span of elementary singular functions f^Ξ as in (4.3.2) for tuples Ξ in H. Then, define $\mathcal{E} \subseteq J \cap \mathscr{C}_c(G)$ to be the subspace generated by all the $\mathcal{E}_H$. With this modification, the subspace $\mathcal{E}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to the inductive limit topology.

§ 4.4 | Ample groupoids

In this section, we describe an explicit spanning set for the algebraic singular ideal $J_{\mathbb{C}}$ of an ample groupoid. The spanning set consists of "elementary" functions that are constructed in a similar manner to those in Subsection 4.3.1. In Subsection 4.4.2 we prove that the algebraic singular ideal $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ for any ample groupoid.

Let us first recall the necessary definitions (see Definitions 1.1.4, 1.3.6 and 4.1.2). An *ample groupoid* is an étale groupoid with a basis of compact open bisections. If G is an ample groupoid, the *Steinberg algebra* $\mathbb{C}G$ over the complex numbers is defined in [65] as

$$\mathbb{C}G \coloneqq \operatorname{span}_{\mathbb{C}} \{\mathbb{1}_U \colon U \subseteq G \text{ a compact open bisection}\}.$$

The *algebraic singular ideal* can then be defined as the intersection $J_{\mathbb{C}} \coloneqq J \cap \mathbb{C}G$. Since $\mathbb{C}G \subseteq \mathscr{C}_c(G)$, we have $J_{\mathbb{C}} \subseteq J \cap \mathscr{C}_c(G)$.

§ 4.4.1 | $J_{\mathbb{C}}$ is linearly spanned by algebraic elementary singular functions

Let G be an ample groupoid that is covered by countably many open bisections. In analogy with Subsection 4.3.1, each "elementary" element of $J_{\mathbb{C}}$ is assembled from the data in a tuple $\Theta \coloneqq (x, h, b, U_1, \ldots, U_n)$ where:

- $x \in G^{(0)}$;
- $h \in G_x$;
- $b \in \mathbb{C}[G_x^x]$ belongs to $\ker(\lambda_{G_x^x/X})$ for all $X \in \mathcal{X}(x)$, and has support $\{g_1, \ldots, g_n\}$;
- $U_1, \ldots, U_n \subseteq G$ are compact open bisections satisfying the conditions of Lemma 4.2.4. That is, they have equal source, contain the elements $hg_1, \ldots, hg_n \in G_x$, and are such that $W_I = \emptyset$ whenever $I \subseteq \{1, \ldots, n\}$ is <u>not</u> of the form

$$\{g_i \colon i \in I\} = kX \cap \{g_1, \ldots, g_n\} \qquad \text{for some } k \in G_x^x, X \in \mathcal{X}(x).$$

Here $W_I := s\Big(\bigcap_{i\in I} U_i \setminus \bigcup_{j\notin I} U_j\Big) \cap C$.

Definition 4.4.1. Let G be an ample groupoid that is covered by countably many open bisections. Let Θ be a tuple as described above, and define

$$f^\Theta := \sum_{i=1}^{n} b(g_i)\mathbb{1}_{U_i}. \tag{4.4.1}$$

A function f^Θ of this form will be called an algebraic *elementary* singular function.

By Proposition 4.2.5, any elementary function f^Θ belongs to $J_{\mathbb{C}}$.

Theorem 4.4.2. *Let G be an ample groupoid that is covered by countably many open bisections. The algebraic singular ideal $J_{\mathbb{C}}$ is linearly spanned by elementary functions f^Θ as in* (4.4.1)*.*

Proof. Let $f \in J_{\mathbb{C}}$, and take a compact set $K \subseteq G$ with $\mathrm{supp}^\circ(f) \subseteq K$. Fix $x \in s(K)$, and let $g_1, \ldots, g_n$ be all elements of G_x on which f is non-zero. An argument identical to that in Lemma 4.3.4 yields compact open bisections $U_1, \ldots, U_n$ with $g_i \in U_i$ for each i, and with equal source such that

$$f^x := \sum_{i=1}^{n} f(g_i)\mathbb{1}_{U_i} \tag{4.4.2}$$

is a finite sum of functions of elementary functions. Precisely, the sum $\sum_{r(g_i)=y} f(g_i)\mathbb{1}_{U_i}$ is elementary for each $y \in G^{(0)}$. Write $U_x := s(U_i)$ (note that this does not depend on i). If $\{g_1, \ldots, g_n\}$ is empty let $U_x := G^{(0)}$. We have $f|_{G_x} = f^x|_{G_x}$, so, by Lemma 4.3.1, $f - f^x$ is continuous at each $g \in G_x$. However, both functions f and f^x belong to $\mathbb{C}G$, hence only take finitely many values. Therefore, there exists an open neighbourhood $V_x \subseteq U_x$ of x such that $f(g) = f^x(g)$ whenever $s(g) \in V_x$. Replace each U_i by the cutdown $U_i \cap s^{-1}(V_x)$ so that $s(U_i) = V_x$ for all i. Note that the function f^x from (4.4.2) is still a finite sum of elementary functions.

Now, the sets $\{V_x\}_{x\in s(K)}$ form an open cover for the compact set $s(K)$, so we may select a finite subcover $V_{x_1}, \ldots, V_{x_m}$. The sets V_{x_j} are clopen in the Hausdorff space $G^{(0)}$, so, by removing intersections, we can assume that they are pairwise disjoint. Note that the functions f^{x_j} are still finite sums of elementary functions. Then, observe that

$$f = \sum_{j=1}^{m} f^{x_j}.$$

It is now clear that f itself is a finite sum of elementary functions, as desired. □

Remark 4.4.3. Theorem 4.4.2 extends to ample groupoids G that are not covered by countably many open bisections. After a modification analogous to that in Remark 4.3.6, one obtains, for any ample groupoid, an explicit set of elementary functions that linearly span $J_{\mathbb{C}}$.

§ 4.4.2 | $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$

We now prove that the algebraic singular ideal $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ for any ample groupoid – no countability assumptions are needed.

Theorem 4.4.4. *Let G be an ample groupoid. The algebraic singular ideal $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to the inductive limit topology.*

By definition (see also Subsection 1.3.2), the inductive limit topology is stronger than the topology induced by the full norm $\|\cdot\|_{C^*(G)}$ (and also the norm $\|\cdot\|_I$). Therefore, $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ with respect to any C^*-norm.

Proof. Let us first prove the theorem in the case where G is covered by countably many open bisections. By Theorem 4.3.5, it suffices to prove that any elementary function in $J \cap \mathscr{C}_c(G)$ can be approximated by functions in $J_{\mathbb{C}}$ with respect to the inductive limit topology. Let $f = f^{\Xi} \in J \cap \mathscr{C}_c(G)$ be an elementary singular function as in Definition 4.3.3 for some tuple $\Xi = (x, h, b, U_1, \dots, U_n, \psi)$. Explicitly, we have

$$f^{\Xi} = \sum_{i=1}^{n} b(g_i)(\psi \circ s|_{U_i}),$$

where $\{g_1, \dots, g_n\}$ is the support of b. Take a compact set $K \subseteq G$ containing every U_i. By definition of the inductive limit topology it suffices to find, for each $\varepsilon > 0$, a function $f' \in J_{\mathbb{C}}$ satisfying $\operatorname{supp}^{\circ}(f') \subseteq K$ and $\left\|f^{\Xi} - f'\right\|_{\infty} \leq \varepsilon$. Fix $\varepsilon > 0$, and define $U := s(U_i)$ (note that this does not depend on i). Since U is totally disconnected and $\psi \in C_c(U)$, there exists a locally constant function $\varphi \in C_c(U)$ satisfying $\|\psi - \varphi\|_{\infty} \leq \frac{\varepsilon}{n \max_i(|b(g_i)|)}$. Define

$$f' := \sum_{i=1}^{n} b(g_i)(\varphi \circ s|_{U_i}).$$

It is clear that f' belongs to the Steinberg algebra $\mathbb{C}G$, and by Proposition 4.2.5, f' belongs to the singular ideal J, hence also to $J_{\mathbb{C}}$. Moreover, it is clear that $\operatorname{supp}^{\circ}(f') \subseteq K$ and $\left\|f^{\Xi} - f'\right\|_{\infty} \leq \varepsilon$. This completes the proof of the theorem in the case where G is covered by countably many open bisections.

Now, let G be an arbitrary ample groupoid, and let $f \in J \cap \mathscr{C}_c(G)$. Write $f = \sum_{j=1}^{m} f_j$ for some $f_j \in C_c(V_j)$ and open bisections V_j, and let $H \subseteq G$ be the subgroupoid generated by the open bisections $V_1, \dots, V_m$. Then H is an open subgroupoid that is covered by countably many open bisections. By the argument above, f can be approximated in the inductive limit topology by functions in the algebraic singular ideal of H. However, since H is open in G, the algebraic singular ideal of H is contained in the algebraic singular ideal of G. The result follows. $\square$

§ 4.5 | Characterisation of density

In Section 4.3, we described an explicit dense subspace of the subalgebra $J \cap \mathscr{C}_c(G)$. Then in Section 4.4, we proved that the algebraic singular ideal $J_{\mathbb{C}}$ is dense in $J \cap \mathscr{C}_c(G)$ for any ample groupoid G. In order to describe the structure of J, it is therefore natural to ask the following question: for which groupoids is $J \cap \mathscr{C}_c(G)$ dense in J?

The main result of this section characterises density of $J \cap \mathscr{C}_c(G)$ in J for a certain class of groupoids (Theorem 4.5.1). This characterisation is derived from Theorems 3.3.3 and 4.2.6 and, therefore, it is necessary for us to restrict our attention to étale groupoids G that are amenable and second-countable. In this case, recall by Corollary 3.2.10 that we have a canonical isomorphism $C^*(G) \cong C_r^*(G)$. Then, the restriction maps η_x and $\eta_{x,r}$ from Subsection 1.5 agree, and become maps between reduced C^*-algebras. We denote this common restriction map as $\eta_x \colon G_r^*(G) \to C_r^*(G_x^x)$. Write $J_x \coloneqq \eta_x(J)$ as in Theorem 4.2.6.

Proposition 4.5.1. *Let G be an amenable and second-countable étale groupoid. Then, $J \cap \mathscr{C}_c(G)$ is dense in J if and only if $J_x \cap \mathbb{C}[G_x^x]$ is dense in J_x for all $x \in G^{(0)}$.*

Proof. Define $I \coloneqq \overline{J \cap \mathscr{C}_c(G)}$. Since $\mathscr{C}_c(G)$ is dense in $C_r^*(G)$, I is an ideal and $I \subseteq J$. By Theorem 4.2.6 we have $\eta_x(J \cap \mathscr{C}_c(G)) = J_x \cap \mathbb{C}[G_x^x]$. Continuity of η_x then implies that $J_x \cap \mathbb{C}[G_x^x] \subseteq \eta_x(I) \subseteq \overline{J_x \cap \mathbb{C}[G_x^x]}$. Since $\eta_x(I) = I_x$ is closed, it follows that $I_x = \overline{J_x \cap \mathbb{C}[G_x^x]}$. Therefore, Theorem 3.3.3 shows that $I = J$ if and only if $\overline{J_x \cap \mathbb{C}[G_x^x]} = J_x$ for all $x \in G^{(0)}$. $\square$

Remark 4.5.2. With appropriate modifications to our definitions, the forward implication of this theorem will hold for any étale groupoid. Recall from Remark 1.5.2 that in the absence of amenability, $C_r^*(G_x^x)$ is not always the "correct" codomain for the restriction map $\eta_{x,r}$; one should replace $C_r^*(G_x^x)$ with an exotic C^*-algebra. Let $C_e^*(G_x^x)$ denote this exotic C^*-algebra, and let $\eta_{x,e} \colon C_r^*(G) \to C_e^*(G_x^x)$ denote the associated map. By [17, Lemma 2.1], the map $\eta_{x,e} \colon C_r^*(G) \to C_e^*(G_x^x)$ sends closed ideals to closed ideals. Writing $J_x \coloneqq \eta_{x,e}(J) \trianglelefteq C_e^*(G_x^x)$, the inclusion $\eta_{x,e}(J \cap \mathscr{C}_c(G)) \subseteq J_x \cap \mathbb{C}[G_x^x]$ holds trivially. We have shown the following: whenever $J \cap \mathscr{C}_c(G)$ is dense in J, the intersection $J_x \cap \mathbb{C}[G_x^x]$ is dense in J_x.

Overall, the contents of this remark give a necessary criterion for the density of $J \cap \mathscr{C}_c(G)$ in J. Let us demonstrate how this criterion can be applied in practice. In [48], examples of groupoids are provided for which $J \cap \mathscr{C}_c(G)$ fails to be dense in J. Let $\mathcal{G}$ denote the groupoid from [48, § 4.1]. Using the methods of [38, Theorem 5.5], it can be shown that the isotropy fibre J_ϵ is non-zero, yet that intersection $J_\epsilon \cap \mathbb{C}[\mathcal{G}_\epsilon^\epsilon]$ vanishes. This gives an alternative proof of the fact that $J \cap \mathscr{C}_c(G)$ is not dense in J for this groupoid.

Using Theorem 4.2.6, we can now reduce the density problem to one about kernels of quasi-regular representations in group C^*-algebras.

Definition 4.5.3. Let Γ be an amenable discrete group. Define $\mathcal{D}_\Gamma$ to be the set of all closed conjugation-invariant sets of subgroups $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ for which $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\mathcal{X}}$. We say that Γ satisfies the *density property* if $\mathcal{D}_\Gamma$ contains all closed conjugation-invariant sets of subgroups.

The following characterisation of density is immediate by Proposition 4.5.1 and Theorem 4.2.6.

Theorem 4.5.4. *Let G be an amenable and second-countable étale groupoid. Then $J \cap \mathscr{C}_c(G)$ is dense in J if and only if $\mathcal{X}(x) \in \mathcal{D}_{G_x^x}$ for all $x \in G^{(0)}$. In particular, if all the isotropy groups G_x^x satisfy the density property, then $J \cap \mathscr{C}_c(G)$ is dense in J.*

In order to check whether $J \cap \mathscr{C}_c(G)$ is dense in J using this theorem, it suffices to check $\mathcal{X}(x) \in \mathcal{D}_{G_x^x}$ for extremely dangerous points $x \in G^{(0)}$. To see why this is true, note that $\iota(x)$ belongs to $\mathcal{X}(x)$ whenever x is not extremely dangerous. In this case, $J_{G_x^x,\mathcal{X}(x)}$ is equal to the kernel of the left-regular representation, and is therefore trivial in $C_r^*(G_x^x)$. It is then certainly the case that $\mathcal{X}(x) \in \mathcal{D}_{G_x^x}$.

It is clear that finite groups satisfy the density property, and hence $J \cap \mathscr{C}_c(G)$ is dense in J for any amenable second-countable étale groupoid with finite isotropy groups. In the next section, we will find more classes of groupoids that satisfy the density property. Let us ask the following question.

Question 4.5.5. *Which discrete amenable groups satisfy the density property?*

Remark 4.5.6. Currently, we do not know of any amenable discrete group that fails to have the density property. It follows from the construction in [38, § 6] that there exists a discrete amenable group failing the density property *if and only if* there exists an amenable and second-countable étale groupoid for which $J \cap \mathscr{C}_c(G)$ is not dense in J.

§ 4.6 | Groups with the density property

The main goal of this section is to prove that finite-by-nilpotent groups satisfy the density property (Theorem 4.6.13). Along the way, we prove various properties of the set $\mathcal{D}_\Gamma$, and it is shown that the density property passes to quotients and direct limits.

The section finishes by applying the results of the previous section to deduce that $J \cap \mathscr{C}_c(G)$ is dense in J for certain concrete classes of groupoids. In particular, it is shown that $J_{\mathbb{C}}$ is dense in J for any groupoid associated with a contracting self-similar group action (see Corollary 4.6.15).

This section is split into three parts. Subsection 4.6.1 contains some reduction properties that simplify the conditions that need to be checked for the finding elements of the set $\mathcal{D}_\Gamma$. Subsection 4.6.2 proves some permanence results for $\mathcal{D}_\Gamma$ and for the density property. These observations are then put into practice in Subsection 4.6.3 to prove Theorem 4.6.13.

Throughout this section, Γ denotes an amenable discrete group. We work with the reduced group C^*-algebra $C^*_r(\Gamma)$ since this is canonically isomorphic to the full C^*-algebra. The set of subgroups $\mathrm{Sub}(\Gamma)$ will always be equipped with the Chabauty topology, and given a subset $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ we write $J_{\Gamma,\mathcal{X}} := \bigcap_{X\in\mathcal{X}} \ker(\lambda_{\Gamma/X}) \trianglelefteq C^*_r(\Gamma)$ (see Definition 4.2.1). For a subgroup $\Lambda \subseteq \Gamma$, let $I_\Lambda \colon C^*_r(\Lambda) \to C^*_r(\Gamma)$ denote the canonical inclusion and $E_\Lambda \colon C^*_r(\Gamma) \to C^*_r(\Lambda)$ the canonical conditional expectation (see Lemma 4.2.2).

Let us record two preliminary lemmas about ideals and dense *-subalgebras inside general C^*-algebras. They will be useful throughout this section.

Lemma 4.6.1. *Let A be a C^*-algebra, $\mathcal{A} \subseteq A$ a dense *-subalgebra, and $I \trianglelefteq A$ an ideal. Then $I \cap \mathcal{A}$ is dense in I if and only if $I \cap \mathcal{A}$ contains a bounded net (u_β) such that* $\lim_\beta \|a - au_\beta\| = 0$ *for all $a \in I$. Moreover, the u_β can be chosen to be positive contractions.*

A bounded net (u_β) satisfying $\lim_\beta \|a - au_\beta\| = 0$ for all $a \in I$ is said to *converge strongly to the identity* in I. The lemma has the following consequence: given $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$, the subalgebra $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\mathcal{X}}$ if and only if $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ contains a net of positive contractions converging strongly to the identity in $J_{\Gamma,\mathcal{X}}$.

Proof. Assume that $I \cap \mathcal{A}$ is dense in I. Since I is a C^*-algebra, it contains an approximate unit (u_β). Then $I \cap \mathcal{A}$ is a dense *-subalgebra, so each u_β can be approximated by a sequence of positive elements in the unit ball of $I \cap \mathcal{A}$. By a diagonalisation argument, $I \cap \mathcal{A}$ contains a net of positive contractions in $I \cap \mathcal{A}$ converging strongly to the identity in I. Conversely, assume that $I \cap \mathcal{A}$ contains a net of elements of norm at most $M > 0$ converging strongly to the identity in I. Let $a \in I$ and $\varepsilon > 0$. There exist $u \in I \cap \mathcal{A}$ and $b \in \mathcal{A}$ such that $\|u\| \leq M$, $\|a - au\| < \frac{\varepsilon}{2}$ and $\|a - b\| < \frac{\varepsilon}{2M}$. Then $bu \in I \cap \mathcal{A}$ and

$$\|a - bu\| \leq \|a - au\| + \|(a - b)u\| < \varepsilon.$$

$\square$

Given an ideal and a subalgebra of a C^*-algebra, the following simple lemma says that the subalgebra is dense whenever it is dense in the quotient, and has dense intersection with the ideal.

Lemma 4.6.2. *Let A be a C^*-algebra, $\mathcal{A} \subseteq A$ a *-subalgebra and $I \trianglelefteq A$ a closed ideal. Let $Q\colon A \to A/I$ denote the quotient map. If $Q(\mathcal{A})$ is dense in $Q(A)$ and $I \cap \mathcal{A}$ is dense in I, then $\mathcal{A}$ is dense in A.*

Proof. Let $a \in A$ and $\varepsilon > 0$. Since $Q(\mathcal{A})$ is dense in $Q(A)$, we may choose $b \in \mathcal{A}$ such that $\|Q(a) - Q(b)\| < \frac{\varepsilon}{2}$. By definition of the quotient norm, there exists $\tilde{c} \in I$ such that $\|a - b - \tilde{c}\| < \frac{\varepsilon}{2}$. Since $I \cap \mathcal{A}$ is dense in I, there exists $c \in I \cap \mathcal{A}$ such that $\|c - \tilde{c}\| < \frac{\varepsilon}{2}$. Hence, $d := b + c \in \mathcal{A}$ satisfies $\|a - d\| < \varepsilon$. □

§ 4.6.1 | Reduction properties

Given a closed conjugation-invariant set $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$, recall from Definition 4.5.3 that $\mathcal{X}$ belongs to $\mathcal{D}_\Gamma$ if and only if $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\mathcal{X}}$. Our goal is to find elements of $\mathcal{D}_\Gamma$; however, in practice, the defining condition is difficult to check. In this subsection we show that it suffices to assume that Γ is generated by the subgroups $X \in \mathcal{X}$ (Theorem 4.6.4), and that $\mathcal{X}$ is minimal in a certain sense (Theorem 4.6.6).

The following lemma is well-known approximation result for C^*-algebras of amenable groups. Given an amenable discrete group Γ and a normal subgroup N, define $a_{gN} := \delta_g \cdot E_N(\delta_{g^{-1}} \cdot a) \in C^*_r(\Gamma)$ for $a \in C^*_r(\Gamma)$ and $gN \in \Gamma/N$. Note that this is independent of the choice of representative g. If $a_{gN} = 0$ for all but finitely many cosets gN, then a is equal to the (finite) sum $\sum_{gN} a_{gN}$.

Lemma 4.6.3. *Let Γ be an amenable discrete group and N a normal subgroup. For any $a \in C^*_r(\Gamma)$, there is a net (a_i) of finite linear combinations of a_{gN} (for $gN \in \Gamma/N$) that converges to a.*

In practice, such a net (a_i) can be built from a Følner net in Γ/N.

Proof. The group Γ/N is amenable, so [11, Theorem 2.6.8] ensures that there exists a net $(\tilde{\varphi}_i)$ of finitely supported positive definite functions $\Gamma/N \to \mathbb{C}$ that converge pointwise to 1. Let $q_N\colon \Gamma \to \Gamma/N$ denote the quotient map, and define the functions $\varphi_i := \tilde{\varphi}_i \circ q_N$ for each i. The φ_i are positive definite, and converge pointwise to 1 on Γ. Without loss of generality we may assume $\varphi_i(e) = 1$ for all i. Define multiplier maps $m_i\colon C^*_r(\Gamma) \to C^*_r(\Gamma)$ for $a = \sum_{g\in\Gamma} a_g\delta_g \in \mathbb{C}[\Gamma]$ as $m_i(a) = \sum_{g\in\Gamma} \varphi_i(g)a_g\delta_g$. By [11, Theorem 2.5.11(4)] the multiplier maps m_i are unital and completely positive, and are therefore contractions.

For $a \in C_r^*(\Gamma)$, observe that $m_i(a)_{gN} = \tilde{\varphi}_i(gN)a_{gN}$ for each $gN \in \Gamma/N$. Since $\tilde{\varphi}_i$ is finitely support, it follows that $m_i(a)$ is a finite linear combination of a_{gN}. By definition, it is clear that $m_i(a)$ converges to a whenever $a \in \mathbb{C}[\Gamma]$. Since the m_i are contractions, $m_i(a)$ converges to a for all $a \in C_r^*(\Gamma)$. To complete the proof of the lemma, define $a_i := m_i(a)$ for each i. $\square$

Using the previous lemma, we now prove the first of two reduction properties.

Theorem 4.6.4. *Let Γ be an amenable discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a closed and conjugation-invariant set of subgroups.*

Let $\Lambda \subseteq \Gamma$ be a subgroup such that $X \subseteq \Lambda$ for all $X \in \mathcal{X}$. Then, $\mathcal{X} \in \mathcal{D}_\Gamma$ if and only if $\mathcal{X} \in \mathcal{D}_\Lambda$.

Proof. Let $N \subseteq \Gamma$ be the (normal) subgroup generated by all $X \in \mathcal{X}$. It suffices to prove the theorem in the case $\Lambda = N$, since N is independent of the ambient group.

Assume that $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\mathcal{X}}$. Continuity of E_N ensures that $E_N(J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma])$ is dense in $E_N(J_{\Gamma,\mathcal{X}})$, and by Lemma 4.2.2(ii) we have $J_{N,\mathcal{X}} \cap \mathbb{C}[N] = E_N(J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma])$ and $J_{N,\mathcal{X}} = E_N(J_{\Gamma,\mathcal{X}})$.

Conversely, assume that $J_{N,\mathcal{X}} \cap \mathbb{C}[N]$ is dense in $J_{N,\mathcal{X}}$. By Lemma 4.6.1 there exists a net of contractions $(u_\beta) \subset J_{N,\mathcal{X}} \cap \mathbb{C}[N]$ converging strongly to the identity in $J_{N,\mathcal{X}}$. Since $I_N\big(J_{N,\mathcal{X}}\big) \subseteq J_{\Gamma,\mathcal{X}}$ by Lemma 4.2.2(i), we can treat (u_β) as a net in $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$. Fix $a \in J_{\Gamma,\mathcal{X}}$. For every coset $gN \in \Gamma/N$, observe that $\delta_{g^{-1}} \cdot a_{gN} = E_N(\delta_{g^{-1}} \cdot a)$ belongs to $J_{N,\mathcal{X}}$ by Lemma 4.2.2(ii). It follows that $\lim_\beta \|a_{gN} \cdot u_\beta - a_{gN}\| = 0$ for all $gN \in \Gamma/N$. Now, let (a_i) be a net converging to a as in Lemma 4.6.3. Since each a_i is a finite linear combination of terms a_{gN}, we have $\lim_\beta \|a_i \cdot u_\beta - a_i\| = 0$ for any fixed i. This implies that $\lim_\beta \|a \cdot u_\beta - a\| = 0$, as $\|u_\beta\| \leq 1$ for all β. The element $a \in J_{\Gamma,\mathcal{X}}$ was arbitrary, so Lemma 4.6.1 ensures that $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\mathcal{X}}$ as desired. $\square$

The following corollary describes the first concrete elements of $\mathcal{D}_\Gamma$. This corollary will be used in the proofs of Theorems 4.6.8 and 4.6.13 to describe many more elements of $\mathcal{D}_\Gamma$.

Corollary 4.6.5. *Let Γ be an amenable discrete group. For any normal subgroup $N \subseteq \Gamma$, we have $\{N\} \in \mathcal{D}_\Gamma$.*

Proof. By Theorem 4.6.4, we can assume that $N = \Gamma$. Note that the ideal $J_{\Gamma,\{\Gamma\}}$ is equal to the kernel of the trivial representation $\epsilon\colon C_r^*(\Gamma) \to \mathbb{C}$. Let (F_i) be a Følner net for Γ and set $v_i = \frac{1}{|F_i|}\sum_{g\in F_i} \delta_g$ for each i. By definition, we have $\lim_i \|\delta_g \cdot v_i - v_i\| = 0$ for any $g \in \Gamma$. The vectors δ_g have dense linear span in $C^*_r(\Gamma)$, and therefore $\lim_i \|a \cdot v_i - \epsilon(a)v_i\| = 0$ for any $a \in C^*_r(\Gamma)$. By Kesten's criterion ([11, Theorem 2.6.8(8)]) we have $\|v_i\| = 1$ for all i, so the previous limit implies that $\lim_i \|a \cdot v_i\| = |\epsilon(a)|$ for all $a \in C^*_r(\Gamma)$. Set $u_i = \delta_e - v_i$ for each i, where $e \in \Gamma$ denotes the unit. Then (u_i) is a net of elements in $J_{\Gamma,\{\Gamma\}} \cap \mathbb{C}[\Gamma]$ of norm at most 2 satisfying $\lim_i \|a \cdot u_i - a\| = \lim_i \|a \cdot v_i\| = |\epsilon(a)| = 0$ for any $a \in J_{\Gamma,\{\Gamma\}}$. It follows from Lemma 4.6.1 that $J_{\Gamma,\{\Gamma\}} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,\{\Gamma\}}$, as desired. □

Take an arbitrary closed and conjugation-invariant subset $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$. In order to determine whether $\mathcal{X}$ belongs to $\mathcal{D}_\Gamma$, Theorem 4.6.4 allows us to reduce the size of the ambient group Γ. The second reduction property, below, allows us to reduce the size of $\mathcal{X}$, and assume that it is minimal in a certain sense.

Define $\mathcal{X}_{\min} \subseteq \mathcal{X}$ to be the collection of subgroups $X \in \mathcal{X}$ for which $X' \in \mathcal{X}$ and $X' \subseteq X$ implies $X = X'$. The set $\mathcal{X}_{\min}$ is conjugation-invariant but not necessarily closed.

Theorem 4.6.6. *Let Γ be an amenable discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a closed and conjugation-invariant set of subgroups.*

For every $X \in \mathcal{X}$, there exists $Y \in \mathcal{X}_{\min}$ such that $Y \subseteq X$. Therefore, $J_{\Gamma,\mathcal{X}} = J_{\Gamma,\overline{\mathcal{X}}_{\min}}$, so that $\mathcal{X} \in \mathcal{D}_\Gamma$ if and only if $\overline{\mathcal{X}}_{\min} \in \mathcal{D}_\Gamma$.

Proof. Fix $X \in \mathcal{X}$. Define $\mathcal{Z}_X := \{Y \in \mathcal{X} \colon Y \subseteq X\}$ and (partially) order $\mathcal{Z}_X$ by inclusion. Take an arbitrary chain $\mathcal{C} \subseteq \mathcal{Z}_X$ in $\mathcal{Z}_X$, and view $\mathcal{C}$ as a net ordered by inclusion. The net $\mathcal{C}$ converges to $Y_* := \bigcap_{Y \in \mathcal{C}} Y$ by definition of the Chabauty topology. Since $\mathcal{X}$ is closed, Y_* belongs to $\mathcal{X}$ and hence to $\mathcal{Z}_X$. Therefore, every chain in $\mathcal{Z}_X$ has a lower bound. Zorn's lemma implies that $\mathcal{Z}_X$ contains a minimal element $Y \in \mathcal{Z}_X$. By definition of $\mathcal{Z}_X$, it is clear that $Y \in \mathcal{X}_{\min}$. This proves the first part of the theorem.

For the next part of the theorem, note that $\ker(\lambda_{\Gamma/X_1}) \subseteq \ker(\lambda_{\Gamma/X_2})$ whenever $X_1 \subseteq X_2$ are subgroups of Γ (see [6, Appendix E and F]). Also, for any subset $\mathcal{Y} \subseteq \mathrm{Sub}(\Gamma)$, [7, Proposition 3.3] ensures that $\bigcap_{X \in \mathcal{Y}} \ker(\lambda_{\Gamma/X}) = \bigcap_{X \in \overline{\mathcal{Y}}} \ker(\lambda_{\Gamma/X})$. Therefore

$$J_{\Gamma,\mathcal{X}} = \bigcap_{X \in \mathcal{X}} \ker(\lambda_{\Gamma/X}) = \bigcap_{X \in \mathcal{X}_{\min}} \ker(\lambda_{\Gamma/X}) = \bigcap_{X \in \overline{\mathcal{X}}_{\min}} \ker(\lambda_{\Gamma/X}) = J_{\Gamma,\overline{\mathcal{X}}_{\min}}.$$

The statement "$\mathcal{X} \in \mathcal{D}_\Gamma$ if and only if $\overline{\mathcal{X}}_{\min} \in \mathcal{D}_\Gamma$" then holds by definition. □

§ 4.6.2 | Permanence properties

This subsection proves some permanence results for $\mathcal{D}_\Gamma$ and for the density property. It is shown that $\mathcal{D}_\Gamma$ is "preserved" under quotients (Theorem 4.6.8) and countable increasing unions (Proposition 4.6.9) of the ambient group. As a consequence, the density property passes to quotients and direct limits.

Firstly, we prove that $\mathcal{D}_\Gamma$ is closed under finite unions.

Proposition 4.6.7. *Let Γ be an amenable discrete group. Let $\mathcal{X}_1, \dots, \mathcal{X}_n \subseteq \mathrm{Sub}(\Gamma)$ be a closed and conjugation-invariant sets of subgroups belonging to $\mathcal{D}_\Gamma$. Then the union $\mathcal{X} := \bigcup_{i=1}^{n} \mathcal{X}_i$ also belongs to $\mathcal{D}_\Gamma$.*

Proof. By definition, we have $J_{\Gamma,\mathcal{X}} = \bigcap_{i=1}^n J_{\Gamma,\mathcal{X}_i}$. For each $i = 1, \dots, n$ there exists, by Lemma 4.6.1, a net $(u_{i,\alpha})_{\alpha \in A_i}$ of positive contractions in $J_{\Gamma,\mathcal{X}_i} \cap \mathbb{C}[\Gamma]$ converging strongly to the identity in $J_{\Gamma,\mathcal{X}_i}$. Given $\alpha_i \in A_i$, define $u_{\alpha_1,\dots,\alpha_n} := u_{1,\alpha_1} \cdot \ldots \cdot u_{n,\alpha_n} \in J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma]$. Treat $(u_{\alpha_1,\dots,\alpha_n})$ as a net where $A_1 \times \cdots \times A_n$ is equipped with the lexicographic partial ordering. By a diagonalisation argument, this net has a subnet (u_β) that converges strongly to the identity in $J_{\Gamma,\mathcal{X}}$, and therefore $\mathcal{X} \in \mathcal{D}_\Gamma$ by Lemma 4.6.1. □

We now show that the density property is preserved under passing to quotients. For an amenable discrete group Γ and a normal subgroup N, let $Q_N : C^*_r(\Gamma) \to C^*_r(\Gamma/N)$ be the *-homomorphism induced by the quotient map $q_N : \Gamma \to \Gamma/N$. Define

$$q_N^* \colon \mathrm{Sub}(\Gamma/N) \to \mathrm{Sub}(\Gamma);\ Y \mapsto q_N^{-1}(Y).$$

This map is a continuous embedding satisfying $\mathrm{Conj}_g \circ q_N^* = q_N^* \circ \mathrm{Conj}_{q_N(g)}$ for all $g \in \Gamma$, where Conj_g denotes conjugation by g. Therefore, if $\mathcal{Y}$ is a closed and conjugation-invariant set of subgroups of Γ/N, then the image $q_N^*(\mathcal{Y})$ is a closed and conjugation-invariant set of subgroups of Γ.

Theorem 4.6.8. *Let Γ be an amenable discrete group and N a normal subgroup. If $\mathcal{Y} \subseteq \mathrm{Sub}(\Gamma/N)$ is a closed and conjugation-invariant set of subgroups of Γ/N, then*

$$J_{\Gamma,q_N^*(\mathcal{Y})} = Q_N^{-1}(J_{\Gamma/N,\mathcal{Y}}) \quad \text{and} \quad Q_N\Big(J_{\Gamma,q_N^*(\mathcal{Y})} \cap \mathbb{C}[\Gamma]\Big) = J_{\Gamma/N,\mathcal{Y}} \cap \mathbb{C}[\Gamma/N]. \tag{4.6.1}$$

Moreover, we have $\mathcal{Y} \in \mathcal{D}_{\Gamma/N}$ if and only if $q_N^*(\mathcal{Y}) \in \mathcal{D}_\Gamma$. Consequently, if Γ satisfies the density property, then so does Γ/N.

Proof. Given $Y \in \mathcal{Y}$, it is easy to see that $\lambda_{\Gamma/q_N^*(Y)} = \lambda_{(\Gamma/N)/Y} \circ Q_N$ and therefore

$\ker\left(\lambda_{\Gamma/q_N^*(Y)}\right) = Q_N^{-1}\Big(\ker(\lambda_{(\Gamma/N)/Y})\Big)$. Hence,

$$J_{\Gamma,q_N^*(\mathcal{Y})} = \bigcap_{Y\in\mathcal{Y}} \ker\left(\lambda_{\Gamma/q_N^*(Y)}\right) = Q_N^{-1}\Big(\bigcap_{Y\in\mathcal{Y}} \ker(\lambda_{(\Gamma/N)/Y})\Big) = Q_N^{-1}(J_{\Gamma/N,\mathcal{Y}}).$$

It follows that $Q_N\Big(J_{\Gamma,q_N^*(\mathcal{Y})} \cap \mathbb{C}[\Gamma]\Big) = J_{\Gamma/N,\mathcal{Y}} \cap \mathbb{C}[\Gamma/N]$, where the inclusion from right to left uses that $\mathbb{C}[\Gamma/N] = Q_N\Big(\mathbb{C}[\Gamma]\Big)$. This proves (4.6.1).

To prove the second part of the theorem suppose $q_N^*(\mathcal{Y}) \in \mathcal{D}_\Gamma$, so that $J_{\Gamma,q_N^*(\mathcal{Y})} \cap \mathbb{C}[\Gamma]$ is dense in $J_{\Gamma,q_N^*(\mathcal{Y})}$. It follows from (4.6.1), as well as continuity and surjectivity of Q_N that $J_{\Gamma/N,\mathcal{Y}} \cap \mathbb{C}[\Gamma/N]$ is dense in $J_{\Gamma/N,\mathcal{Y}}$. Therefore $\mathcal{Y} \in \mathcal{D}_{\Gamma/N}$, as required.

Conversely, assume that $\mathcal{Y} \in \mathcal{D}_{\Gamma/N}$. Write $A = J_{\Gamma,q_N^*(\mathcal{Y})}$, $\mathcal{A} = A \cap \mathbb{C}[\Gamma]$, and $I = J_{\Gamma,\{N\}}$. Each $X \in q_N^*(\mathcal{Y})$ contains the normal subgroup N, hence $I \subseteq A$. Since $I = \ker(Q_N)$, we can naturally identify the restriction $Q_N|_A$ with the quotient map $A \to A/I$. By the assumption $\mathcal{Y} \in \mathcal{D}_{\Gamma/N}$, the latter half of (4.6.1) implies that $Q_N(\mathcal{A})$ is dense in $Q_N(A) = J_{\Gamma/N,\mathcal{Y}}$. Also, $I \cap \mathcal{A} = J_{\Gamma,\{N\}} \cap \mathbb{C}[\Gamma]$ is dense in I by Corollary 4.6.5. Therefore, Lemma 4.6.2 implies that $\mathcal{A}$ is dense in A. This is precisely the condition $q_N^*(\mathcal{Y}) \in \mathcal{D}_\Gamma$, as desired. □

Our second permanence property shows that the density property is preserved under countable increasing unions. Given a subgroup $\Lambda \subseteq \Gamma$ and a closed and conjugation-invariant set of subgroups $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$, write $\Lambda \cap \mathcal{X} := \{\Lambda \cap X \colon X \in \mathcal{X}\}$ (see Lemma 4.2.2).

Proposition 4.6.9. *Suppose that an amenable discrete group Γ is a countable increasing union of subgroups $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Gamma$. Let $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ be closed and conjugation-invariant.*

If $\Lambda_n \cap \mathcal{X} \in \mathcal{D}_{\Lambda_n}$ for all $n \in \mathbb{N}$, then $\mathcal{X} \in \mathcal{D}_\Gamma$. In particular, if all Λ_n have the density property, then so does Γ.

Proof. For each $n \in \mathbb{N}$, write $\mathcal{X}_n := \Lambda_n \cap \mathcal{X}$. By assumption, $J_{\Lambda_n,\mathcal{X}_n} \cap \mathbb{C}[\Lambda_n]$ is dense in $J_{\Lambda_n,\mathcal{X}_n}$ for each n, and therefore Lemma 4.2.2 (i) implies that $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Lambda_n]$ is dense in $J_{\Gamma,\mathcal{X}} \cap I_{\Lambda_n}\Big(C_r^*(\Lambda_n)\Big)$ for every n. Hence, $J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Gamma] = \bigcup_n J_{\Gamma,\mathcal{X}} \cap \mathbb{C}[\Lambda_n]$ is dense in $\bigcup_n J_{\Gamma,\mathcal{X}} \cap I_{\Lambda_n}\Big(C_r^*(\Lambda_n)\Big)$. The countable increasing union $\bigcup_{n=1}^\infty I_{\Lambda_n}\Big(C_r^*(\Lambda_n)\Big)$ is dense in $C_r^*(\Gamma)$, so $\bigcup_n J_{\Gamma,\mathcal{X}} \cap I_{\Lambda_n}\Big(C_r^*(\Lambda_n)\Big)$ is dense in $J_{\Gamma,\mathcal{X}}$ by [20, Lemma III.4.1]. This completes the proof. □

It follows from Theorem 4.6.8 and Proposition 4.6.9 that the density property is preserved under taking direct limits.

Corollary 4.6.10. *If $(\varphi_n : \Gamma_n \to \Gamma_{n+1})_n$ is a sequence of homomorphisms between groups with the density property, then the direct limit $\Gamma := \varinjlim \Gamma_n$ has the density property.*

Proof. Let $(\mu_n : \Gamma_n \to \Gamma)_n$ be the limit homomorphisms. By Theorem 4.6.8, the images $\mu_n(\Gamma_n)$ have the density property for each $n \in \mathbb{N}$. The direct limit Γ is the countable increasing union of the subgroups $\mu_n(\Gamma)$, and therefore Γ has the density property by Proposition 4.6.9. □

§ 4.6.3 | Applications

We will now show that finite-by-nilpotent groups satisfy the density property. We also show that any closed conjugation-invariant collection $\mathcal{X}$ of finite (or co-finite) subgroups of an amenable group Γ belongs to $\mathcal{D}_\Gamma$. Let us establish a key lemma that unifies the various arguments.

Lemma 4.6.11. *Let Γ be a discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a closed and conjugation-invariant set of subgroups. If every $X \in \mathcal{X}_{\min}$ is finitely generated, then $\mathcal{X}_{\min}$ is finite.*

Proof. Given $X \in \mathcal{X}_{\min}$, let $\{x_1, ..., x_n\}$ be a finite generating set for X. Note that $\mathcal{U}(X) := \{X' \in \mathcal{X} \colon X \subseteq X'\} = \bigcap_{i=1}^{n} \{X' \in \mathcal{X} \colon x_i \in X'\}$. Therefore, $\mathcal{U}(X)$ is open in $\mathcal{X}$ with respect to the Chabauty topology. By Theorem 4.6.6, $\{\mathcal{U}(X) \colon X \in \mathcal{X}_{\min}\}$ forms an open cover for $\mathcal{X}$. Since $\mathcal{U}(X) \cap \mathcal{X}_{\min} = \{X\}$ for each $X \in \mathcal{X}_{\min}$, the cover $\{\mathcal{U}(X) \colon X \in \mathcal{X}_{\min}\}$ admits no proper subcover. Therefore, compactness of $\mathcal{X}$ ensures that $\mathcal{X}_{\min}$ is finite. □

A group Γ is *nilpotent* if there is a finite sequence $\{e\} = \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_n = \Gamma$ of subgroups that are all normal in Γ for which the quotients Λ_{i+1}/Λ_i are contained in the centre of Γ/Λ_i for all i. We say that Γ is *finite-by-nilpotent* if there is a finite normal subgroup $N \trianglelefteq \Gamma$ such that Γ/N is nilpotent.

A subgroup $X \subseteq \Gamma$ is called *subnormal* if there is a sequence $X = \Lambda_0 \trianglelefteq \Lambda_1 \trianglelefteq \cdots \trianglelefteq \Lambda_k = \Gamma$ such that Λ_i is normal in Λ_{i+1} for all $i < k$. Such a sequence is called a *subnormal series*. In a nilpotent group, every subgroup is subnormal. For every subgroup X of a finite-by-nilpotent group Γ, there exists a subnormal subgroup Λ of Γ that contains X as a finite index subgroup ([45, Section 6.3]).

The following known lemma about finite-by-nilpotent groups immediately justifies the consideration of Lemma 4.6.11.

Lemma 4.6.12. *Let Γ be a finitely generated finite-by-nilpotent group. Then every subgroup of Γ is finitely generated.*

Proof. Every finitely generated finite-by-nilpotent group is finite-by-polycyclic. Every subgroup of a finite-by-polycyclic group is finitely generated ([36]). □

The proof of the following theorem combines all of the reduction and permanence results from Subsections 4.6.1 and 4.6.2.

Theorem 4.6.13. *Let Γ be an amenable discrete group and $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ a closed and conjugation-invariant set of subgroups. Assume that one of the following holds.*

(I) Every $X \in \mathcal{X}$ is finite.

(II) Every $X \in \mathcal{X}$ is co-finite in Γ.

(III) Γ is countable and finite-by-nilpotent.

Then $\mathcal{X} \in \mathcal{D}_\Gamma$.

Proof. (I): Assume that every $X \in \mathcal{X}$ is finite. By Lemma 4.6.11 the set $\mathcal{X}_{\min}$ is finite. Let $N \subseteq \Gamma$ be the subgroup generated by all $X \in \mathcal{X}_{\min}$. Then N is generated by a finite conjugation-invariant family of finite subgroups, and is therefore finite by [38, Lemma 7.9]. Since $C_r^*(N) = \mathbb{C}[N]$, it holds trivially that $\mathcal{X}_{\min} \in \mathcal{D}_N$, so Theorems 4.6.4 and 4.6.6 imply that $\mathcal{X} \in \mathcal{D}_\Gamma$.

(II): Assume that every $X \in \mathcal{X}$ is co-finite in Γ. For any subgroup $\Lambda \subseteq \Gamma$, the intersection $\Lambda \cap X$ is co-finite in Λ for every $X \in \mathcal{X}$. By Proposition 4.6.9 we can therefore assume that Γ is finitely generated. Next, since every co-finite subgroup of a finitely generated group is finitely generated, Lemma 4.6.11 implies that $\mathcal{X}_{\min}$ is finite. In particular, the normal subgroup $N := \bigcap_{X \in \mathcal{X}_{\min}} X$ is co-finite. Write $\mathcal{Y} = \{X/N \colon X \in \mathcal{X}_{\min}\}$ and let $q_N^*(\mathcal{Y})$ be as in Theorem 4.6.8. Then, $\mathcal{Y}$ is a conjugation-invariant family of subgroups in the finite group Γ/N with $q_N^*(\mathcal{Y}) = \mathcal{X}_{\min}$. Since $C_r^*(\Gamma/N) = \mathbb{C}[\Gamma/N]$ it holds trivially that $\mathcal{Y} \in \mathcal{D}_{\Gamma/N}$, and therefore $\mathcal{X}_{\min} = q_N^*(\mathcal{Y}) \in \mathcal{D}_\Gamma$ by Theorem 4.6.8. Theorem 4.6.6 then implies that $\mathcal{X} \in \mathcal{D}_\Gamma$.

(III): Let Γ be a countable and finite-by-nilpotent group. The goal is to show that any closed conjugation-invariant family $\mathcal{X} \subseteq \mathrm{Sub}(\Gamma)$ belongs to $\mathcal{D}_\Gamma$. It suffices to assume that Γ is finite-generated by Proposition 4.6.9, as every subgroup of a finite-by-nilpotent group is still finite-by-nilpotent. Lemma 4.6.12 ensures that every subgroup of Γ is finitely generated, and therefore $\mathcal{X}_{\min}$ is finite by Lemma 4.6.11. By Theorem 4.6.6, it therefore suffices to assume that $\mathcal{X}$ is finite.

We prove that $\mathcal{X} \in \mathcal{D}_\Gamma$ by induction on $|\mathcal{X}|$. The base case $|\mathcal{X}| = 1$ is covered by Corollary 4.6.5. Suppose that $|\mathcal{X}| \geq 2$, and assume for any finite-by-nilpotent group Λ that the set $\mathcal{D}_\Lambda$ contains all conjugation-invariant families of subgroups of cardinality less than $|\mathcal{X}|$. Let $N \subseteq \Gamma$ be the subgroup generated by all $X \in \mathcal{X}$. If N contains every $X \in \mathcal{X}$ as a finite index subgroup, then $\mathcal{X} \in \mathcal{D}_N$ by (II) above, and hence $\mathcal{X} \in \mathcal{D}_\Gamma$ by Theorem 4.6.4. Suppose otherwise, and choose $X_0 \in \mathcal{X}$ with infinite index in N. Since Γ is finite-by-nilpotent, there exists a subnormal subgroup $\Lambda \subseteq \Gamma$ containing X_0 as a finite-index subgroup. Note that $N \nsubseteq \Lambda$, by assumption. Let $\Gamma = \Lambda_0 \trianglerighteq \Lambda_1 \trianglerighteq \cdots \trianglerighteq \Lambda_n = \Lambda$ be a subnormal series, and let $k \in \{1, \ldots, n\}$ be minimal such that $N \nsubseteq \Lambda_k$. Define $\mathcal{X}_1 \coloneqq \{X \in \mathcal{X} \colon X \subseteq \Lambda_k\}$ and $\mathcal{X}_2 = \mathcal{X} \setminus \mathcal{X}_1$. Both are non-empty, and since Λ_k is normal in Λ_{k-1}, both $\mathcal{X}_1$ and $\mathcal{X}_2$ are conjugation-invariant families of subgroups in the finite-by-nilpotent group Λ_{k-1}. By the inductive hypothesis, both $\mathcal{X}_1$ and $\mathcal{X}_2$ belong to $\mathcal{D}_{\Lambda_{k-1}}$. Proposition 4.6.7 ensures that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \in \mathcal{D}_{\Lambda_{k-1}}$, and therefore $\mathcal{X} \in \mathcal{D}_\Gamma$ by Theorem 4.6.4. This completes the proof of the theorem. $\square$

Using the results of Section 4.5, the previous theorem is now applied to the singular ideal of an étale groupoid. The following corollary is immediate by Theorems 4.5.4 and 4.6.13, where $\mathcal{X}(x)$ is as defined in Definition 4.2.3.

Corollary 4.6.14. *Let G be an amenable and second-countable étale groupoid. For each extremely dangerous unit $x \in G^{(0)}$, suppose that one of the following holds.*

(I) The isotropy group G_x^x is finite-by-nilpotent.

(II) The subgroups $X \in \mathcal{X}(x)$ are all finite.

(III) The subgroups $X \in \mathcal{X}(x)$ are all co-finite in G_x^x.

Then $J \cap \mathscr{C}_c(G)$ is dense in J.

We apply the previous corollary to the class of groupoids arising from contracting self-similar groups (see Subsection 1.6.4).

Corollary 4.6.15. *Let G be the groupoid associated with a contracting self-similar group action. Then $J_\mathbb{C}$ is dense in J.*

Proof. By Theorem 4.4.4 it suffices to show that $J \cap \mathscr{C}_c(G)$ is dense in J. By [49, Corollary 2.19] the groupoid G is second-countable and amenable. Lemma 1.6.12 implies that every element of $\tilde{G}^{(0)}$ is finite. The result follows by Corollary 4.6.14 (II). $\square$

Remark 4.6.16. The previous result extends to groupoids G arising from contracting self-similar *groupoid* actions on finite graphs without sources. Such a groupoid is amenable by [49, Corollary 2.19], and it follows from either [44, Corollary 6.23] or [1, Proposition 9.1] that $G_x^x \cap \overline{G^{(0)}}$ is finite for all $x \in G^{(0)}$.

§ 4.7 | Computing singular ideals

Our attention now turns to étale groupoids for which $\tilde{G}_{\mathrm{sing}}$ is discrete. The main result of this section (Theorem 4.7.2) gives an explicit formula capable of computing the singular ideal for such groupoids. This discreteness assumption is fairly strong, but is nevertheless satisfied in many natural examples. Indeed, in Subsection 3.4 we saw that $\tilde{G}_{\mathrm{sing}}$ is discrete for the groupoids associated with the Grigorchuk and Grigorchuk-Erschler groups, and whenever G is the groupoid of germs associated with the canonical action of a Bieri-Strebel group. In Section 4.8 we will use Theorem 4.7.2 to compute the singular ideals of these groupoids.

The proof of the theorem uses the elementary observation that a discrete groupoid has C^*-algebra that decomposes as a direct sum (over orbits) of the isotropy group C^*-algebras tensored with the compacts. Given this description of $C^*_r(\tilde{G}_{\mathrm{sing}})$, the proof then amounts to checking that isotropy fibres of the singular ideal (see Theorem 4.2.6) embed as C^*-subalgebras of J. This is the content of the following lemma.

As usual, we will view elements of reduced group and groupoid C^*-algebras as functions on the ambient group or groupoid. In particular, we will view any $a \in J_{G_x^x,\mathcal{X}(x)}$ as a function on the isotropy group $G_x^x \subseteq G$.

Lemma 4.7.1. *Let G be an étale groupoid that is covered by countably many open bisections. Let $x \in G^{(0)}$ be an extremely dangerous unit such that $\iota(x)$ is isolated in the Hausdorff cover $\tilde{G}$, $\{\iota(x)\} = \pi^{-1}(\{x\}) \cap \tilde{G}^{(0)}_{\mathrm{sing}}$, and for which the isotropy group G_x^x is amenable and satisfies the density property.*

(i) Given $a \in J_{G_x^x,\mathcal{X}(x)}$, let $\Phi_x(a)\colon G \to \mathbb{C}$ denote the extension of a to G by 0. The map $\Phi_x\colon J_{G_x^x,\mathcal{X}(x)} \to C^*_r(G)$ is a well-defined isometric *-homomorphism with range in J.

(ii) Now assume that G is ample. The map Φ_x restricts to a *-algebra homomorphism $J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x] \to J_{\mathbb{C}}$.

Proof. To prove part (i), let us first consider $\Phi_x(b)$ for $b \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. We will use the methods of Subsection 4.3.1 to prove that $\Phi_x(b)$ belongs to $J \cap \mathscr{C}_c(G)$. Let $\{g_1, \ldots, g_n\} \subseteq G_x^x$ denote the support of b, and let $U_1, \ldots, U_n$ be open bisections with $g_i \in U_i$ for each i satisfying the conditions of Lemma 4.2.4. In particular, $s(U_i) = s(U_j)$ for all $i, j \in \{1, \ldots, n\}$. Let $\psi \in C_c(U_i)$ be such that $\psi \equiv 1$ on an open neighbourhood W of x. Define Ξ to be the tuple $(x, x, b, U_1, \ldots, U_n, \psi)$, and let $f^{\Xi} \in J \cap \mathscr{C}_c(G)$ be an (elementary) function as defined in (4.3.2). Explicitly, we define

$$f^{\Xi} := \sum_{i=1}^{n} b(g_i)(\psi \circ s|_{U_i}). \tag{4.7.1}$$

We claim that there exists an open neighbourhood $V \subseteq W$ of x such that $f(g) = 0$ whenever $s(g) \in V \setminus \{x\}$. Indeed, take any net (g_α) in $U_1 \cup ... \cup U_n$ such that $s(g_\alpha)$ converges to x. By passing to a subnet if necessary, assume that the net (g_α) is primitive, and that there exists $i_0 \in \{1, \dots, n\}$ such that $g_\alpha \in U_{i_0}$ for all α. Let $\boldsymbol{g} \in \tilde{G}$ denote the set of limit points, so that $\iota(g_\alpha)$ converges to $\boldsymbol{g}$ in $\tilde{G}$. Note in particular that $g_{i_0} \in \boldsymbol{g}$. Since $\iota(x)$ is isolated in $\tilde{G}$, the singleton $\iota(g)$ is also isolated in $\tilde{G}$, and therefore $\boldsymbol{g} \neq \iota(g)$. The condition $\{\iota(x)\} = \pi^{-1}(\{x\}) \cap \tilde{G}^{(0)}_{\text{sing}}$ ensures that $\boldsymbol{g} \in \tilde{G}_{\text{ess}}$, and therefore $\mathfrak{i}(f)(\boldsymbol{g}) = 0$ by Proposition 3.1.14. Since $\mathfrak{i}(f)$ is continuous on $\tilde{G}$, this implies that $f(g_\alpha)$ converges to 0. Now, the net (g_α) belongs to $s^{-1}(W)$ eventually, and since $\psi \equiv 1$ on W, it is clear by (4.7.1) that f^Ξ only takes finitely many values on the set $s^{-1}(W)$. Therefore, we must have $f(g_\alpha) = 0$ eventually. We have shown that any net (g_α) in G with $s(g_\alpha)$ converging to x has a subnet (g_β) such that $f(g_\beta) = 0$ eventually. The claim then follows. Next, take any $\chi \in C_v(V)$ with $\chi(x) = 1$. It is simple to check that $f^\Xi * \chi = \Phi_x(b)$ so that $\Phi_x(b)$ does indeed belong to $J \cap \mathscr{C}_c(G)$.

We have shown that the map $\Phi_x \colon J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x] \to J \cap \mathscr{C}_c(G)$ is well-defined. This map is isometric with respect to the reduced norm. Since G_x^x has the density property, $J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ is dense in $J_{G_x^x, \mathcal{X}(x)}$. It follows that Φ_x extends to an embedding $J_{G_x^x, \mathcal{X}(x)} \to J$. Convergence with respect to the reduced norm implies pointwise convergence, so it is clear that this extension agrees with the definition of Φ_x given in the statement of the lemma. This completes the proof of part (i).

Now assume that G is ample. We can assume that the open bisections $U_1, \dots, U_n$ from earlier are compact. Then, the function f^Ξ from (4.7.1) belongs to $J_{\mathbb{C}}$. Using the argument above, let $V \subseteq G^{(0)}$ be a compact open neighbourhood of x such that $f(g) = 0$ whenever $s(g) \in V \setminus \{x\}$. Then $f^\Xi * \mathbb{1}_V = \Phi_x(b)$ belongs to $J_{\mathbb{C}}$. □

We can now prove the main theorem of this section. In what follows, let $\mathfrak{i}_r \colon C_r^*(G) \hookrightarrow C_r^*(\tilde{G})$ denote the embedding from Proposition 3.1.6. Throughout, we will use the fact that $\mathfrak{i}_r(a) \circ \iota = a$ for all $a \in C_r^*(G)$ (Lemma 3.1.3). In the following theorem, it is necessary to assume that $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\text{sing}})$. Recall from Subsection 3.1.2 that this holds in many cases, for example whenever $\tilde{G}$ is inner-exact or whenever $\tilde{G}_{\text{sing}}$ is closed.

Theorem 4.7.2. *Let G be an étale groupoid that is covered by countably many open bisections. Assume that $\tilde{G}_{\text{sing}}$ is discrete.*

(i) For each orbit $Q \in D_0/G$ choose a representative $x_Q \in D_0$ with $G(x_Q) = Q$. The subalgebra $J \cap \mathscr{C}_c(G)$ contains a dense subspace linearly spanned by elements of the form $\delta_k \Phi_x(b) \delta_h^*$ for representatives $x = x_Q$, $b \in J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ and $h, k \in G_x$.

(ii) If G is an ample groupoid, then the algebraic singular ideal $J_{\mathbb{C}}$ is linearly spanned by elements of the form $\delta_k \Phi_x(b) \delta_h^*$ as above.

In addition to discreteness of $\tilde{G}_{\text{sing}}$, assume that $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\text{sing}})$ and that $\tilde{G}_{\text{sing}}$ has amenable isotropy groups that satisfy the density property.

(iii) The subalgebra $J \cap \mathscr{C}_c(G)$ is dense in J, and

$$J \cong \bigoplus_{G(x) \in D_0/G} J_{G_x^x, \mathcal{X}(x)} \otimes \mathcal{K}_x$$

where the direct sum is taken over the orbits in D_0. Moreover, the images of $J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ under Φ_x (for $x \in D_0$) generate J as an ideal.

Here, $\mathcal{K}_x$ denotes the C^*-algebra of compact operators on the Hilbert space $\ell^2\Big(G(x)\Big)$. The theorem includes dependence on the Hilbert space since it could be finite or infinite-dimensional. In part (i), δ_k represents the function on G that is equal to 1 at k and is 0 elsewhere, and δ_h^* represents the function on G that is equal to 1 at h^{-1} and is 0 elsewhere. Then, $\delta_k \Phi_x(b) \delta_h^*$ is defined to be the convolution product $\delta_k * \Phi_x(b) * \delta_h^*$. Explicitly, $\delta_k \Phi_x(b) \delta_h^*(g) = \Phi_x(b)(k^{-1}gh)$ for $g \in G_{r(h)}^{r(k)}$ and is 0 otherwise. Lastly, $D_0 \subseteq G^{(0)}$ denotes the set of extremely dangerous points (Definition 1.1.8). Recall that $\iota(D_0) = \iota(G^{(0)}) \cap \tilde{G}_{\text{sing}}^{(0)}$ by Lemma 3.1.9.

Proof. Let us prove (i). Given $b \in J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ and $h, k \in G_x$, let U_h and U_k be open bisections containing h and k respectively, and let $\chi_h \in C_c(U_h)$ and $\chi_k \in C_c(U_k)$ be such that $\chi_h(h) = \chi_k(k) = 1$. It is simple to check that $\delta_k \Phi_x(b) \delta_h^*$ is equal to the convolution product $\chi_k * \Phi_x(b) * \chi_h^*$, and thus belongs to $J \cap \mathscr{C}_c(G)$ by Lemma 4.7.1.

We claim that any $f \in J \cap \mathscr{C}_c(G)$ with $\mathfrak{i}(f) \in C_c(\tilde{G}_{\text{sing}})$ is equal to a finite sum of elements of the form $\delta_k \Phi_x(b) \delta_h^*$. Take arbitrary $f \in J \cap \mathscr{C}_c(G)$ with $\mathfrak{i}(f) \in C_c(\tilde{G}_{\text{sing}})$. The groupoid $\tilde{G}_{\text{sing}}$ is discrete, and therefore f has finite support. In particular, f has support contained in a finite union of sets $G_y^z := r^{-1}(\{z\}) \cap s^{-1}(\{y\})$ for $y, z \in D_0$ belonging to same orbit Q of D_0/G. Given such a set G_y^z, let $x = x_Q$ denote the representative of the orbit Q, and choose $h \in G_x^y$ and $k \in G_x^z$. We prove that the restriction of f to G_y^z agrees with a product $\delta_k \Phi_x(b) \delta_h^*$ for some $b \in J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. The claim will then follow, since $\delta_k \Phi_x(b) \delta_h^*$ is supported on the set G_y^z.

Let U_h and U_k be open bisections containing h and k respectively, and let $\chi_h \in C_c(U_h)$ and $\chi_k \in C_c(U_k)$ be such that $\chi_h(h) = \chi_k(k) = 1$. The product $\chi_k^* * f * \chi_h$ belongs to $J \cap \mathscr{C}_c(G)$ and therefore, by Theorem 4.2.6, its restriction to G_x^x belongs to $J_{G_x^x, \mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. Let b denote this restriction. It is then simple to check that $\delta_k \Phi_x(b) \delta_h^*$ agrees with f on G_y^z, as desired. This completes the proof the claim.

To complete the proof of part (i), let $\mathcal{E}$ be as in Definition 4.3.3. Proposition 4.2.5 ensures that $\mathfrak{i}(f) \in C_c(\tilde{G}_{\text{sing}})$ for any $f \in \mathcal{E}$. Also, $\mathcal{E}$ is dense in $J \cap \mathscr{C}_c(G)$ by Theorem 4.3.5. Part (i) of the theorem now follows from the claim.

To prove part (ii), assume that G is an ample groupoid. By Lemma 4.7.1(ii) the element $\Phi_x(b)$ belongs to $J_{\mathbb{C}}$ for any $x \in D_0$ and $b \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. Given $h, k \in G_x$, let U_h and U_k be compact open bisections containing h and k respectively. Then $\delta_k \Phi_x(b) \delta_h^*$ is equal to the convolution product $\mathbb{1}_{U_k} * \Phi_x(b) * \mathbb{1}_{U_h}^*$, thus belongs to $J_{\mathbb{C}}$. The proof of part (i) shows that $\mathcal{E}$ is linearly spanned by elements of the form $\delta_k \Phi_x(b) \delta_h^*$, and $J_{\mathbb{C}} \subseteq \mathcal{E}$ by Theorem 4.4.2. Part (ii) then follows.

Before proving (iii), we construct an isometric *-homomorphism $\varphi\colon C_r^*(G_x^x) \to C_r^*(\tilde{G}_{\text{sing}})$ satisfying $\varphi(a) \circ \iota = \Phi_x(a)$ for any $a \in J_{G_x^x,\mathcal{X}(x)}$. Given $b \in \mathbb{C}[G_x^x]$, view it as a function on $\iota(G_x^x)$, and let $\varphi(b)$ be the extension to $\tilde{G}_{\text{sing}}$ by 0. We have $\varphi(b) \in C_c(\tilde{G}_{\text{sing}})$ since $\iota(x)$ is isolated. By definition of the reduced norm, this map extends to an isometric *-homomorphism $C_r^*(G_x^x) \to C_r^*(\tilde{G}_{\text{sing}})$, which we will also denote φ. The equality $\varphi(a) \circ \iota = \Phi_x(a)$ holds for any $a \in J_{G_x^x,\mathcal{X}(x)}$ by construction.

We now prove (iii). By definition, the complement $\tilde{G}_{\text{ess}} = \tilde{G} \setminus \tilde{G}_{\text{sing}}$ is closed and $\iota(G)$ is dense in $\tilde{G}$, so clearly $\tilde{G}_{\text{sing}} \subseteq \iota(G)$. Discreteness of $\tilde{G}_{\text{sing}}$ ensures that $C_r^*(\tilde{G}_{\text{sing}})$ decomposes as the direct sum of the C^*-algebras of each orbit. By assumption, J embeds into $C_r^*(\tilde{G}_{\text{sing}})$,and it therefore suffices to show that the restriction of J to the C^*-algebra of each orbit $\tilde{G}_{\text{sing}}\big(\iota(x)\big)$ is isomorphic to $J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}_x$. Without loss of generality, we can assume that $\tilde{G}_{\text{sing}}$ has a single orbit. In this case, $D_0 = G^{(0)} \cap \iota^{-1}(\tilde{G}_{\text{sing}})$ is a single orbit in $G^{(0)}$.

Fix $x \in D_0$. We will prove that $J \cong J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$, where $\mathcal{K} := \mathcal{K}\big(\ell^2(D_0)\big)$. For each $y \in D_0$, choose $h_y \in G_x^y$. For any $y, z \in D_0$, let $e_{z,y} \in \mathcal{K}$ be the partial isometry that maps δ_y to δ_z, and let $\mathcal{F}_0 \subseteq \mathcal{K}$ be the space of finite rank operators spanned by the operators $e_{z,y}$. Define the map

$$\begin{aligned} \psi\colon C_r^*(G_x^x) \odot \mathcal{F}_0 &\to C_r^*(\tilde{G}_{\text{sing}}) \\ a \otimes e_{z,y} &\mapsto \delta_{\iota(h_z)} \varphi(a) \delta_{\iota(h_y)}^* \end{aligned} \tag{4.7.2}$$

where $\odot$ denotes the algebraic tensor product, and where φ is as defined earlier. The map ψ is a *-homomorphism. It is clear that the range of ψ contains $C_c(\tilde{G}_{\text{sing}})$. Since φ is isometric, it follows from the definition of the reduced norm that ψ is also isometric with respect to the minimal tensor product. Therefore, ψ extends to a *-isomorphism

$$\Psi\colon C_r^*(G_x^x) \otimes \mathcal{K} \to C_r^*(\tilde{G}_{\text{sing}}).$$

We show that the image of $J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$ under Ψ is equal to $\mathfrak{i}_r(J)$. Let us first prove the inclusion from left to right. Take $h, k \in G_x$ and $b \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. Let $y = r(h)$ and $z = r(k)$. Since $J_{G_x^x,\mathcal{X}(x)}$ is an ideal, the product $\delta_k \Phi_x(b) \delta_h^*$ is equal to $\delta_{h_z} \Phi_x(\tilde{b}) \delta_{h_y}^*$, for some $\tilde{b} \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. By part (i), the subalgebra $J \cap \mathscr{C}_c(G)$ therefore contains a dense subspace that is linearly spanned by elements of the form $\delta_{h_z} \Phi_x(b) \delta_{h_y}^*$, for

$b \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ and $y, z \in D_0$. It follows that the image of $\left(J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]\right) \odot \mathcal{F}_0$ under Ψ is dense in $\mathfrak{i}\left(J \cap \mathscr{C}_c(G)\right)$. Since G_x^x satisfies the density property, $J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$ is dense in $J_{G_x^x,\mathcal{X}(x)}$, and therefore continuity of Ψ ensures that it maps $J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$ into the closure of $\mathfrak{i}(J \cap \mathscr{C}_c(G))$ – a subset of $\mathfrak{i}_r(J)$.

Let us now prove the opposite inclusion: that $\mathfrak{i}_r(J)$ is contained in the image of $J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$ under Ψ. In particular, this will also show that $\mathfrak{i}_r(J)$ is equal to the closure of $\mathfrak{i}(J \cap \mathscr{C}_c(G))$. Take arbitrary $c \in J$. For any $y, z \in D_0$, Theorem 4.2.6 ensures that $\delta_{h_z}^* c\, \delta_{h_y}$ is equal to $\Phi_x(a)$ for some $a \in J_{G_x^x,\mathcal{X}(x)}$, so that $\delta_{\iota(h_z)}^* \mathfrak{i}_r(c) \delta_{\iota(h_y)} = \Psi(a \otimes e_{z,y})$. Applying Ψ^{-1} gives

$$(\delta_e \otimes e_{z,z}) \Psi^{-1}\left(\mathfrak{i}_r(c)\right)(\delta_e \otimes e_{y,y}) = a \otimes e_{z,y} \in J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}. \tag{4.7.3}$$

The net $\left(\delta_e \otimes \sum_{y \in F} e_{y,y}\right)_F$ over all finite subsets $F \subseteq D_0$ is an approximate unit for $C_r^*(G_x^x) \otimes \mathcal{K}$. It follows from (4.7.3) that $\Psi^{-1}\left(\mathfrak{i}_r(c)\right) \in J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$. This proves that $J \cong J_{G_x^x,\mathcal{X}(x)} \otimes \mathcal{K}$.

The "moreover" part of the theorem follows from part (i) and the fact that $J \cap \mathscr{C}_c(G)$ is dense in J. □

Remark 4.7.3. In the previous theorem, amenability of the isotropy groups was only needed to use the definition of the "density property" from Definition 4.5.3. There exists a modification of Definition 4.5.3 that is appropriate for non-amenable groups (see also Remark 4.2.7). With this altered definition, the assumption that G has amenable isotropy groups can be dropped in Theorem 4.7.2.

We record an interesting corollary of Theorem 4.7.2. This corollary can be also be proven directly (see Remark 4.7.5), and yields an alternative proof of Theorem 4.7.2 with the aid of Proposition 3.3.2.

Corollary 4.7.4. *Let G be an étale groupoid that is covered by countably many open bisections. Assume that $\tilde{G}_{\mathrm{sing}}$ is discrete, that $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\mathrm{sing}})$, and that G has amenable isotropy groups that satisfy the density property.*

The image $\mathfrak{i}_r(J)$ is an ideal in $C_r^*(\tilde{G})$.

Proof. Given $\iota(x) \in \tilde{G}_{\mathrm{sing}}^{(0)}$, let $\tilde{\Phi}_x \colon J_{G_x^x,\mathcal{X}(x)} \to C_r^*(\tilde{G})$ be the composition of Φ_x with the canonical inclusion $\mathfrak{i}_r \colon C_r^*(G) \hookrightarrow C_r^*(\tilde{G})$. By the final comment of Theorem 4.7.2, it suffices to prove that $\tilde{f} * \tilde{\Phi}_x(b) \in \mathfrak{i}_r(J)$ for any $\tilde{f} \in C_c(\tilde{G})$ and $b \in J_{G_x^x,\mathcal{X}(x)} \cap \mathbb{C}[G_x^x]$. Given such b and $\tilde{f}$, define $a := \tilde{f} * \tilde{\Phi}_x(b)$, so that

$$a(\boldsymbol{g}) = \sum_{\boldsymbol{h} \in \tilde{G}^{\tilde{r}(\boldsymbol{g})}} \tilde{f}(\boldsymbol{h}) \tilde{\Phi}_x(b)(\boldsymbol{h}^{-1}\boldsymbol{g})$$

for all $\boldsymbol{g} \in \tilde{G}$. Since $\tilde{\Phi}_x(b)$ is supported in $\iota(G_x^x)$, it is clear that $a(\boldsymbol{g})$ is only non-zero when $\boldsymbol{g} \in \iota(G_x)$. Take any $f \in \mathscr{C}_c(G)$ satisfying $f(g) = \tilde{f}(\iota(g))$ for all $g \in G_x$. Then $a = \mathfrak{i}(f) * \tilde{\Phi}_x(b) = \mathfrak{i}(f * \Phi(b)) \in \mathfrak{i}_r(J)$. This completes the proof. $\square$

Remark 4.7.5. This corollary can be proven directly by applying [9, Theorem 3.13] instead of Theorem 4.7.2. Moreover, it suffices to assume that $J \cap \mathscr{C}_c(G)$ is dense in J, and that $\tilde{G}_{\text{sing}} \subseteq \iota(G)$. The condition $\tilde{G}_{\text{sing}} \subseteq \iota(G)$ is crucial – without this, $\mathfrak{i}_r(J)$ can fail to be an ideal in $C_r^*(\tilde{G})$ even for compact étale groupoids (see Example 4.8.4).

§ 4.8 | Examples

In this section we compute the singular ideal for the groupoids appearing in Section 1.6. This includes the groupoids associated with the Grigorchuk and Grigorchuk-Erschler groups (see Examples 1.6.13 and 1.6.14), as well as the groupoids of germs associated with the canonical actions of Bieri-Strebel groups on $[0, 1]$ and $\mathbb{T}$ (see Examples 1.6.10 and 1.6.11).

Groupoids associated with self-similar group actions are minimal and effective, and therefore their essential C^*-algebras are simple by [43, Theorem 7.26]. In general, the singular ideal J is non-vanishing, and can be quite mysterious. However, in Corollary 4.6.15 we proved that the algebraic singular ideal $J_{\mathbb{C}}$ is dense in J for any contracting self-similar group. Much of the structure of the singular ideal J can therefore be understood through $J_{\mathbb{C}}$. In some cases, the Hausdorff covers of these groupoids have discrete singular part $\tilde{G}_{\text{sing}}$, and therefore the techniques of Section 4.7 can be applied. Indeed, this is the case for the Grigorchuk and Grigorchuk-Erschler groups.

The groupoid of germs $\mathrm{Germ}(T \curvearrowright \mathbb{T})$ associated with the canonical action of Thompson's group T on the circle $\mathbb{T}$ is discussed in [40, Example 7.8], and it is deduced that its essential C^*-algebra is simple. More generally, given any Bieri-Strebel group $T(\mathbb{T}; A, P)$ as defined in Definition 1.6.9, the groupoid of germs $G \coloneqq \mathrm{Germ}(T(\mathbb{T}; A, P) \curvearrowright \mathbb{T})$ is minimal and effective, and thus $C_{\text{ess}}^*(G)$ is simple. In Corollary 4.8.8, we show that the singular ideal J contains many ideals, so that $C_r^*(G)$ is far from being simple.

§ 4.8.1 | Bundle of groups examples

Example 4.8.1. Given a locally compact Hausdorff space X with a non-isolated point $x_0 \in X$, together with a non-trivial discrete group Γ, let G be the groupoid constructed in Example 1.6.1. The only extremely dangerous point in $G^{(0)} \cong X$ is the unit x_0, and its isotropy group can be identified with Γ. From the description of the Hausdorff cover $\tilde{G}$ in Example 3.4.1 it is easy to see that $\mathcal{X}(x_0)$ is equal to the singleton set $\{\Gamma\}$. When Γ is amenable, we have $J_{G_{x_0}^{x_0}, \mathcal{X}(x_0)} = \ker(\epsilon)$, where $\epsilon\colon C_r^*(\Gamma) \to \mathbb{C}$ is the trivial representation. Therefore we have a canonical isomorphism $J \cong \ker(\epsilon)$. When Γ is non-amenable, the embedding $\mathfrak{i}_r\colon C^*_r(G) \hookrightarrow C^*_r(\tilde{G})$ is surjective. Also, $\tilde{G}_{\mathrm{sing}}$ is closed in $\tilde{G}$ so that (3.1.4) is exact. Therefore, Proposition 3.1.14 ensures that the singular ideal is isomorphic to $C^*_r(\tilde{G}_{\mathrm{sing}}) \cong C^*_r(\Gamma)$.

Example 4.8.2. The groupoid G from Example 1.6.2 has no extremely dangerous points, and therefore its singular ideal vanishes by Lemma 3.1.9 and Proposition 3.1.14.

Example 4.8.3. Let G be the groupoid appearing in Example 1.6.3. Recall that $G^{(0)}$ contains a single extremely dangerous point that we call 0. In order to show that the singular ideal vanishes, we will prove that $J_{G_0^0,\mathcal{X}(0)}$ is zero. Let V_4 denote the Klein four-group, and write $V_4 = \{e, A, B, C\}$. The unit 0 has isotropy group isomorphic to V_4 and, by the description of the Hausdorff cover $G^{(0)}$ in Example 3.4.3, the set $\mathcal{X}(0)$ contains all three 2-element subgroups of V_4, namely $\{e, A\}$, $\{e, B\}$ and $\{e, C\}$. A simple exercise in linear algebra then reveals that $J_{V_4,\mathcal{X}(0)}$ is zero.

Example 4.8.4. Let G be the groupoid from Example 1.6.4. In order to describe the singular ideal J, we invoke Proposition 3.1.14 and instead describe its isomorphic image $\mathfrak{i}_r(J)$ in $C^*_r(\tilde{G}_{\mathrm{sing}})$. By the argument in Example 3.4.4, we can write $\tilde{G}^{(0)}_{\mathrm{sing}} = \{1, x_1, x_2, \dots\} \sqcup \{*\}$. The groupoid $\tilde{G}_{\mathrm{sing}}$ is a bundle of groups over this set with isotropy group V_4 at $*$ (written $V_4 = \{*, A, B, C\}$), and isotropy group $\mathbb{Z}/2\mathbb{Z}$ for the other $x \in \tilde{G}^{(0)}_{\mathrm{sing}}$ (written $\mathbb{Z}/2\mathbb{Z} = \{e_x, a_x\}$). The C^*-algebra $C^*_r(\tilde{G}_{\mathrm{sing}})$ is then simply the set of functions $\tilde{f}$ on this set satisfying $\tilde{f}(e_1) = \lim_n \tilde{f}(e_{x_n})$ and $\tilde{f}(a_1) = \lim_n \tilde{f}(a_{x_n})$. However, given any $f \in \mathscr{C}_c(G)$, the image $\mathfrak{i}(f) \in C_c(\tilde{G})$ satisfies the additional conditions $\mathfrak{i}(f)(e_1) = \mathfrak{i}(f)(*) + \mathfrak{i}(f)(A)$ and $\mathfrak{i}(f)(a_1) = \mathfrak{i}(f)(B) + \mathfrak{i}(f)(C)$. Using the methods of Subsection 4.3.1, it is then easy to verify that $\mathfrak{i}_r(J)$ is equal to the set of functions on $\tilde{G}^{(0)}_{\mathrm{sing}}$ satisfying these constraints. In particular, $\mathfrak{i}(J)$ is not an ideal in $C^*_r(\tilde{G}^{(0)}_{\mathrm{sing}})$.

Example 4.8.5. Let G be the groupoid from Example 1.6.5. We show that the canonical surjection $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism. Note that the embedding $\mathfrak{i}\colon \mathscr{C}_c(G) \hookrightarrow C_c(\tilde{G})$ and restriction map $C_c(\tilde{G}) \twoheadrightarrow C_c(\tilde{G}_{\mathrm{ess}})$ compose to induce *-homomorphisms $C^*(G) \to C^*(\tilde{G}_{\mathrm{ess}})$ and $C^*_r(G) \to C^*_r(\tilde{G}_{\mathrm{ess}})$. In Example 3.4.5 we saw that $C^*(\tilde{G}_{\mathrm{ess}}) = C^*_r(\tilde{G}_{\mathrm{ess}})$. Therefore, it suffices to prove that the map $C^*(G) \to C^*(\tilde{G}_{\mathrm{ess}})$ is an injective. In particular, this implies injectivity of the map $C^*_r(G) \to C^*_r(\tilde{G}_{\mathrm{ess}})$, and therefore the singular ideal J vanishes by Proposition 3.1.14.

Let us show that the canonical map $C^*(G) \to C^*(\tilde{G}_{\mathrm{ess}})$ is injective. Take arbitrary $f \in \mathscr{C}_c(G)$, and let $\tilde{f} \in C_c(\tilde{G}_{\mathrm{ess}})$ denote the restriction of $\mathfrak{i}(f) \in C_c(\tilde{G})$ to $\tilde{G}_{\mathrm{ess}}$. By Example 3.4.5, the singular part $\tilde{G}_{\mathrm{sing}}$ is isomorphic to the group $\Gamma := \mathbb{F}_2 \times V_4$. Write f_Γ for the restriction $\mathfrak{i}(f)|_{\tilde{G}_{\mathrm{sing}}}$. The Hausdorff cover $\tilde{G}$ is the disjoint union of the clopen subgroupoids $\tilde{G}_{\mathrm{ess}}$ and $\tilde{G}_{\mathrm{sing}}$, and therefore $\|\mathfrak{i}(f)\|_{C^*(\tilde{G})} = \max\left\{\left\|\tilde{f}\right\|_{C^*(\tilde{G}_{\mathrm{ess}})}, \|f_\Gamma\|_{C^*(\Gamma)}\right\}$. In order to show that $C^*(G) \to C^*(\tilde{G}_{\mathrm{ess}})$ is injective, it suffices to show that $\left\|\tilde{f}\right\|_{C^*(\tilde{G}_{\mathrm{ess}})} \geq \|f_\Gamma\|_{C^*(\Gamma)}$.

Let X_A, X_B, X_C denote the three 2-element subgroups of V_4, and view them as normal subgroups of $\Gamma = \mathbb{F}_2 \times V_4$. Let $\lambda_{\Gamma/X_g} : C^*(\Gamma) \to C^*(\Gamma/X_g)$ be the (quasi-regular) quotient maps for $g = A, B, C$. The kernel is $C^*(\mathbb{F}_2) \otimes \ker(\lambda_{V_4/X_g})$, where $\lambda_{V_4/X_g} : C^*(V_4) \to C^*(V_4/X_g)$ is another quasi-regular map. Since $\bigcap_{g=A,B,C} \ker(\lambda_{V_4/X_g}) = \{0\}$ we also have $\bigcap_{g=A,B,C} \ker(\lambda_{\Gamma/X_g}) = \{0\}$. It follows that the direct sum of maps $\bigoplus_{g=A,B,C} \lambda_{\Gamma/X_g} : C^*(\Gamma) \to \bigoplus_{g=A,B,C} C^*(\Gamma/X_g)$ is injective. Following Example 3.4.5 let $(\infty, A), (\infty, B), (\infty, C)$ denote the three units in $\tilde{G}_{\text{ess}}$ with isotropy groups Γ/X_g. Let $\Lambda_A, \Lambda_B, \Lambda_C$ denote these isotropy groups. For any $f \in \mathscr{C}_c(G)$ we have

$$\left\|\tilde{f}\right\|_{C^*(\tilde{G}_{\text{ess}})} \geq \max_{g=A,B,C} \left\{ \left\| \tilde{f}|_{\Lambda_g} \right\|_{C^*(\Lambda_g)} \right\} = \max_{g=A,B,C} \left\{ \left\| \lambda_{\Gamma/X_g}(f_\Gamma) \right\|_{C^*(\Gamma/X_g)} \right\} = \|f_\Gamma\|_{C^*(\Gamma)}$$

where the equality $\tilde{f}|_{\Lambda_g} = \lambda_{\Gamma/X_g}(f_\Gamma)$ uses (3.1.3) from Lemma 3.1.3. It follows from this inequality that $C^*(G) \to C^*(\tilde{G}_{\text{ess}})$ is injective, as required.

As noted in Example 3.4.5, the canonical surjection $C^*(\tilde{G}_{\text{ess}}) \twoheadrightarrow C_r^*(\tilde{G}_{\text{ess}})$ is an isomorphism, and therefore it follows that $C^*(G) \twoheadrightarrow C_r^*(G)$ is also an isomorphism. It was observed in Example 3.4.5 that $C^*(\tilde{G}) \twoheadrightarrow C_r^*(\tilde{G})$ is not an isomorphism, and therefore this groupoid is a counterexample to implication "(vii) $\Rightarrow$ (viii)" of Corollary 3.2.10.

§ 4.8.2 | Groupoids from self-similar groups

We now discuss the groupoids associated with the Grigorchuk and Grigorchuk-Erschler groups. The groupoid associated with the Grigorchuk group has vanishing singular ideal J (see [18, Theorem 5.22]), whereas the groupoid associated with the Grigorchuk-Erschler group has non-vanishing singular ideal (see [67, § 7.5] for instance). In fact, the singular ideal J_R is non-vanishing over any ring R. Let us study these two examples using the methods introduced earlier in Section 4.7. Note that the assumption $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\text{sing}})$ in Theorem 4.7.2 is satisfied automatically by Proposition 3.1.14, and using that the groupoids are amenable (see (3.1.4)).

In each of the following examples, let $1^\infty \in \{0,1\}^{\mathbb{N}}$ denote the infinite word containing only 1.

Example 4.8.6. Let $\{e, a, b, c, d\}$ denote the nucleus of the Grigorchuk group (see Example 1.6.13), and let G be the groupoid associated with its self-similar action. As discussed in earlier examples, the extremely dangerous units form a single orbit. Therefore, in order to understand the singular ideal, it suffices to study just one extremely dangerous unit. We consider $1^\infty \in G^{(0)}$. For $g = b, c, d$ write $X_g = \{[e, 1^\infty], [g, 1^\infty]\}$. By Example 3.4.6, we have $\mathcal{X}(1^\infty) = \{X_b, X_c, X_d\}$. The subgroups in $\mathcal{X}(1^\infty)$ generate a copy of the Klein four-group $V_4 \cong \{e, b, c, d\}$, and $\mathcal{X}(1^\infty)$ contains all order two subgroups. Note that we are in the same situation as Example 4.8.3, and therefore it is simple to show that the ideal $J_{V_4,\mathcal{X}(1^\infty)}$ vanishes. By Lemma 4.2.2, it then follows that $J_{G_{1^\infty}^{1^\infty},\mathcal{X}(1^\infty)} = 0$, and hence the singular ideal vanishes. This recovers [18, Theorem 5.22].

Example 4.8.7. Let $\{e, h, \alpha, \beta, \gamma\}$ denote the nucleus of the Grigorchuk-Erschler group (see Example 1.6.14), and let G be the groupoid associated with its self-similar action. By Lemma 1.6.15, we have $G_{1^\infty}^{1^\infty} = \{[e, 1^\infty], [\alpha, 1^\infty], [\beta, 1^\infty], [\gamma, 1^\infty]\} \rtimes_\varphi \mathbb{Z} \cong V_4 \rtimes_\varphi \mathbb{Z}$. Define the element $b \in \mathbb{C}[G_{1^\infty}^{1^\infty}]$ by $b(1^\infty) = 1 = b([\gamma, 1^\infty])$ and $b([\alpha, 1^\infty]) = -1 = b([\beta, 1^\infty])$. It has been observed in [67, Example 7.7] and [30, Example 6.11] that the extension of b to G by zero belongs to the algebraic singular ideal $J_\mathbb{C}$. Using the methods of Section 4.7, we will see that this element b generates both $J_\mathbb{C}$ and J as ideals in their respective algebras.

For $g = \alpha, \beta$ write $X_g = \{[e, 1^\infty], [g, 1^\infty]\}$. By Example 3.4.7 we have $\mathcal{X}(1^\infty) = \{X_\alpha, X_\beta\}$. The subgroups in $\mathcal{X}(1^\infty)$ generate a copy of the Klein four-group $V_4 \cong \{e, \alpha, \beta, \gamma\}$. It is then simple to show that the ideal $J_{V_4,\mathcal{X}(1^\infty)} \subseteq C_r^*(V_4)$ is the one-dimensional subspace spanned by b (defined above). Now, by the semidirect product decomposition $G_{1^\infty}^{1^\infty} = V_4 \rtimes_\varphi \mathbb{Z}$ from Lemma 1.6.15, we also have a crossed product decomposition at the level of C^*-algebras: $C_r^*(G_{1^\infty}^{1^\infty}) = C_r^*(V_4) \rtimes_\varphi \mathbb{Z}$. Under this identification, we have $J_{G_{1^\infty}^{1^\infty},\mathcal{X}(1^\infty)} = J_{V_4,\mathcal{X}(1^\infty)} \rtimes_\varphi \mathbb{Z}$ (apply Lemmas 4.2.2 and 4.6.3 for example). Since $J_{V_4,\mathcal{X}(1^\infty)}$ is one-dimensional, we therefore have $J_{G_{1^\infty}^{1^\infty},\mathcal{X}(1^\infty)} \cong C(\mathbb{T})$. Also, the group $G_{1^\infty}^{1^\infty}$ is finite-by-nilpotent and therefore satisfies the density property (Theorem 4.6.13). We know that $\tilde{G}_{\text{sing}}$ is discrete, and hence we can apply Theorem 4.7.2(iii) to deduce that

$$J \cong C(\mathbb{T}) \otimes \mathcal{K}.$$

We now discuss the the algebraic singular ideal $J_\mathbb{C}$. Let $\Phi_{1^\infty}(b) \colon G \to \mathbb{C}$ denote the extension of b to G by zero. Lemma 4.7.1 ensures that $\Phi_{1^\infty}(b)$ belongs to $J_\mathbb{C}$, and by Theorem 4.7.2(ii) the element $\Phi_{1^\infty}(b)$ generates $J_\mathbb{C}$ as an ideal. It is then simple to check that

$$J_\mathbb{C} \cong \mathbb{C}[\mathbb{Z}] \odot M_\infty$$

where M_∞ denotes the algebraic union $\bigcup_{n=1}^\infty M_n(\mathbb{C})$ of matrix algebras, and $\odot$ denotes the algebraic tensor product.

§ 4.8.3 | Groupoids of germs associated with Bieri-Strebel groups

In this final subsection, we apply Theorem 4.7.2 to the groupoids of germs associated with Bieri-Strebel groups. Most assumptions of this theorem are satisfied by the results of Subsection 3.4.3: these groupoids are second-countable, have abelian isotropy groups, and are such that the singular part of the Hausdorff cover $\tilde{G}_{\text{sing}}$ is discrete. We state our findings in the form of a corollary.

In what follows, we use P to denote a countable non-trivial subgroup of the multiplicative group $\mathbb{R}_{>0}$, and use A to denote a countable $\mathbb{Z}[P]$-submodule of the additive group $\mathbb{R}$ such that $P \subseteq A$. As in Subsection 1.6.3, we let $IP \cdot A$ be the submodule of A generated by elements of the form $(p-1)a$ for $p \in P$, $a \in A$. Lastly, let $\hat{P}$ denote the Pontryagin dual of P, where P is equipped with the discrete topology. We denote the unit in $\hat{P}$ by $1_{\hat{P}}$.

Corollary 4.8.8. *Let P and A be as above, and let Γ equal either $\Gamma(I; A, P)$ or $T(\mathbb{T}; A, P)$. Let G be the groupoid of germs associated with the canonical action of Γ on I or $\mathbb{T}$.*

(i) The subalgebra $J \cap \mathscr{C}_c(G)$ has a dense subspace linearly spanned by elements of the form $\delta_{[f,x]} b \delta^*_{[\dot{f},x]}$ where $x \in A \cap (0,1)$, $f, \dot{f} \in \Gamma$ and b is a finitely supported function on $G_x^x \cong P^2$ satisfying the linear equations

$$\sum_{p \in P} b(p,q) = 0 = \sum_{p \in P} b(q,p) \qquad \text{for all } q \in P. \tag{4.8.1}$$

(ii) Suppose that $\mathfrak{i}_r(J) \subseteq C_r^*(\tilde{G}_{\mathrm{sing}})$. Then $J \cap \mathscr{C}_c(G)$ is dense in the singular ideal J, and J satisfies the formula

$$J \cong \bigoplus_{A/(IP \cdot A)} C_0\Big((\hat{P} \setminus \{1_{\hat{P}}\})^2\Big) \otimes \mathcal{K} \tag{4.8.2}$$

where the direct sum is taken over all additive cosets $A/(IP \cdot A)$.

In particular, if Γ is one of Thompson's groups F or T, then the groupoid G has singular ideal J isomorphic to $C_0(\mathbb{R}^2) \otimes \mathcal{K}$.

By imposing additional constraints on P and A, the formula (4.8.2) can be simplified. For example, if $IP \cdot A$ is equal to A, then the direct sum will have a single summand (indeed, this is the case for Thompson's groups F and T). If P is finitely generated, then it is isomorphic to $\mathbb{Z}^n$ for some positive integer n (by the structure theorem for finitely generated abelian groups). In this case, the Pontryagin dual is easily described by $\hat{P} \cong \mathbb{T}^n$.

Proof. To prove (i), recall from Subsection 1.6.3 that any $x \in D_0$ has isotropy group isomorphic to P^2. By the description of $\tilde{G}_{\mathrm{ess}}$ in Subsection 3.4.3, the set $\mathcal{X}(x)$ consists of the two canonical copies of P in P^2. Therefore, a finitely supported function $b\colon P^2 \to \mathbb{C}$ belongs to $J_{G_x^x, \mathcal{X}(x)}$ if and only if it satisfies the linear equations (4.8.1). Statement (i) then follows from Theorem 4.7.2 (i).

We now prove (ii). By Gelfand theory, there is a canonical isomorphism $C_r^*(P) \cong C(\hat{P})$. The quasi-regular representations induced by the two canonical homomorphisms $P^2 \twoheadrightarrow P$ correspond to the two restriction maps $\pi_1, \pi_2\colon C(\hat{P}^2) \twoheadrightarrow C(\hat{P})$. Concretely, these restriction maps are given by $\pi_1(f)(\gamma) \coloneqq f(\gamma, 1_{\hat{P}})$ and $\pi_2(f)(\gamma) \coloneqq f(1_{\hat{P}}, \gamma)$ for $\gamma \in \hat{P}$, $f \in C(\hat{P}^2)$. By definition, $J_{G_x^x, \mathcal{X}(x)}$ is equal to the intersection of the kernels of π_1 and π_2, and is therefore isomorphic to $C_0\Big((\hat{P} \setminus \{1_{\hat{P}}\})^2\Big)$ for every $x \in A \cap (0,1)$. Statement (ii) then follows from Theorem 4.7.2 (iii), since all orbits in D_0 are infinite.

The final claim of the corollary follows from the observations $\hat{\mathbb{Z}} = \mathbb{T}$ and $\mathbb{T} \setminus \{1\} \cong \mathbb{R}$. Theorem 4.7.2 applies to the groupoids $\mathrm{Germ}(F \curvearrowright I)$ and $\mathrm{Germ}(T \curvearrowright \mathbb{T})$ because they are amenable, and hence their Hausdorff covers are inner-exact. $\square$

Bibliography

[1] J. Aakre, *Simplicity of algebras and C^*-algebras of self-similar groupoids*, J. Math. Anal. Appl. *559* (2026), no. 1, 130446.

[2] C. Anantharaman-Delaroche, *Groupoid exactness and the weak containment problem*, preprint, arXiv:1605.05117.

[3] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Monogr. Enseign. Math. *36*, Enseign. Math., Geneva, 2000. With a foreword by G. Skandalis and Appendix B by E. Germain.

[4] K. Bardadyn, B. K. Kwaśniewski and A. McKee, *Banach algebras associated to twisted étale groupoids: Inverse semigroup disintegration and representations on L^p-spaces*, J. Funct. Anal. *289* (2025), no. 12, 111163.

[5] K. Bardadyn, B. K. Kwaśniewski and A. McKee, *Banach algebras associated to twisted étale groupoids: simplicity and pure infiniteness*, preprint, arXiv:2406.05717.

[6] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, New Math. Monogr., *11*, Cambridge Univ. Press, 2008.

[7] B. Bekka and M. Kalantar, *Quasi-regular representations of discrete groups and associated C^*-algebras*, Trans. Amer. Math. Soc. *373* (2020), no. 3, 2105–2133.

[8] R. Bieri and R. Strebel, *On groups of PL-homeomorphisms of the real line*, Math. Surveys Monogr. *215*, Amer. Math. Soc., Providence, RI, 2016.

[9] K. A. Brix, J. Gonzales, J. B. Hume and X. Li, *On Hausdorff covers for non-Hausdorff groupoids*, preprint, arXiv:2503.23203.

[10] K. A. Brix and E. Scarparo, *C^*-simplicity and representations of topological full groups of groupoids*, J. Funct. Anal. *277* (2019), no. 9, 2981–2996.

[11] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Grad. Stud. Math. *88*, Amer. Math. Soc., Providence, RI, 2008.

[12] N. Brownlowe, A. Buss, D. Gonçalves, J. Hume, A. Sims and M. Whittaker, *KK-duality for self-similar groupoid actions on graphs*, Trans. Amer. Math. Soc. *377* (2024), no. 8, 5513–5560.

[13] A. Buss and D. Martínez, *Essential groupoid amenability and nuclearity of groupoid C^*-algebras*, J. Funct. Anal. *290* (2026), no. 11, 111427.

[14] J. W. Cannon, W. J. Floyd and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) *42* (1996), no. 3-4, 215–256.

[15] M.-D. Choi and E. Effros, *Separable nuclear C^*-algebras and injectivity*, Duke Math. J. *43* (1976), 309–332.

[16] J. Christensen and S. Neshveyev, *(Non)exotic completions of the group algebras of isotropy groups*, Int. Math. Res. Not. IMRN *19* (2022), 15155–15186.

[17] J. Christensen and S. Neshveyev, *Isotropy fibers of ideals in groupoid C^*-algebras*, Adv. Math. *447* (2024), Paper No. 109696.

[18] L. O. Clark, R. Exel, E. Pardo, A. Sims and C. Starling, *Simplicity of algebras associated to non-Hausdorff groupoids*, Trans. Amer. Math. Soc. *372* (2019), no. 5, 3669–3712.

[19] A. Connes, *A survey of foliations and operator algebras*, Operator algebras and applications, part 1 (Kingston, Ont., 1980), 521–628, Proc. Sympos. Pure Math. *38*, Amer. Math. Soc., Providence, RI, 1982.

[20] K. R. Davidson, *C^*-algebras by example*, Fields Inst. Monogr., Amer. Math. Soc., Providence, RI, 1996.

[21] M. Dokuchaev, R. Exel and H. Pinedo *Twisted Steinberg algebras, regular inclusions and induction*, preprint, arXiv:2504.11639.

[22] J. Dixmier, *von Neumann algebras*, North-Holland Math. Libr. *27*, North-Holland Publishing Co., Amsterdam, 1981.

[23] E. G. Effros and F. Hahn, *Locally compact transformation groups and C^*-algebras*, Mem. Amer. Math. Soc., No. 75, Amer. Math. Soc., Providence, R.I., 1967.

[24] A. Erschler, *Boundary behavior for groups of subexponential growth*, Ann. of Math. *160* (2004), 1183–1210.

[25] R. Exel, *Inverse semigroups and combinatorial C^*-algebras*, Bull. Braz. Math. Soc. (N. S.) *39* (2008), 191–313.

[26] R. Exel, *Non-Hausdorff étale groupoids*, Proc. Amer. Math. Soc. *139* (2011), no.3, 897–907.

[27] R. Exel and D. R. Pitts, *Characterizing groupoid C^*-algebras of non-Hausdorff étale groupoids*, Lecture Notes in Math. *2306*, Springer, Cham, 2022.

[28] J. M. G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. *13* (1962), 472–476.

[29] G. B. Folland, *Real Analysis*, Modern techniques and their applications, Second edition, Pure Appl. Math. (N. Y.), Wiley-Intersci. Publ., John Wiley & Sons, Inc., New York, 1999.

[30] E. Gardella, V. Nekrashevych, B. Steinberg and A. Vdovina, *Simplicity of C^*-algebras of contracting self-similar groups*, Commun. Math. Phys. *406* (2025), 251.

[31] J. Gonzales and J. B. Hume, *On dense subalgebras of the singular ideal in groupoid C^*-algebras*, preprint, arXiv:2511.04391.

[32] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Funct. Anal. Appl. *14* (1980), 41–43.

[33] R. I. Grigorchuk, *Degrees of growth of finitely generated groups, and the theory of invariant means*, Math. USSR Izv. *25* (1985), no. 2, 259–300.

[34] P. Hahn, *Haar Measure for Measure Groupoids*, Trans. Amer. Math. Soc. *242* (1978), 1–33.

[35] P. Hahn, *The regular representations of measure groupoids*, Trans. Amer. Math. Soc. *242* (1978), 35–72.

[36] P. Hall, *Finiteness conditions for soluble groups*, Proc. Lond. Math. Soc. (3) *4* (1954), 419–436.

[37] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. *12* (2002), 330–354.

[38] J. B. Hume, *Characterization of zero singular ideal in étale groupoid C^*-algebras via compressible maps*, preprint, arXiv:2509.07262.

[39] M. Ionescu and D. P. Williams, *The generalized Effros-Hahn conjecture for groupoids*, Indiana Univ. Math. J. *58* (2009), 2489–2508.

[40] M. Kalantar and E. Scarparo, *Boundary maps, germs and quasi-regular representations*, Adv. Math. *394* (2022), Paper No. 108130.

[41] M. Kennedy, S. J. Kim, X. Li, S. Raum and D. Ursu, *The ideal intersection property for essential groupoid C^*-algebras*, preprint, arXiv:2107.03980.

[42] M. Khoshkam and G. Skandalis, *Regular representation of groupoid C^*-algebras and applications to inverse semigroups*, J. Reine Angew. Math. *546* (2002), 47–72.

[43] B. K. Kwaśniewski and R. Meyer, *Essential crossed products for inverse semigroup actions: simplicity and pure infiniteness*, Doc. Math. *26* (2021), 271–335.

[44] B. K. Kwaśniewski and A. Mundey, *Twisted operator algebras of self-similar groupoid actions on arbitrary graphs*, preprint, arXiv:2511.07906.

[45] J. C. Lennox and S. E. Stonehewer, *Subnormal subgroups of groups*, Oxford Univ. Press, Oxford, 1987.

[46] X. Li, *Every classifiable simple C^*-algebra has a Cartan subalgebra*, Invent. Math. *219* (2020), no. 2, 653–699.

[47] G. W. Mackey, *Ergodic theory and virtual groups*, Math. Ann. *166* (1966), 187–207.

[48] D. Martínez and N. Szakács, *Algebraic singular functions are not always dense in the ideal of C^*-singular functions*, Bull. Lond. Math. Soc. *58* (2026), no. 5, e70381.

[49] A. Miller and B. Steinberg, *Homology and K-theory for self-similar actions of groups and groupoids*, preprint, arXiv:2409.02359v3.

[50] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Stud. Adv. Math. *91*, Cambridge Univ. Press, Cambridge, 2003.

[51] P. Muhly and D. P. Williams, *Renault's equivalence theorem for groupoid crossed products*, NYJM Monogr. *3*, State University of New York, University at Albany, Albany, NY, 2008.

[52] V. Nekrashevych, *Self-similar groups*, Math. Surveys Monogr. *117*, Amer. Math. Soc., Providence, RI, 2005.

[53] V. Nekrashevych, *C^*-algebras and self-similar groups*, J. Reine Agnew. Math. *630* (2009), 59–123.

[54] V. Nekrashevych, *Palindromic subshifts and simple periodic groups of intermediate growth*, Ann. of Math. (2) *187* (2018), no. 3, 667–719.

[55] S. Neshveyev and G. Schwartz, *Non-Hausdorff étale groupoids and C^*-algebras of left cancellative monoids*, Münster J. Math. *16* (2023), no. 1, 147–175.

[56] P. Nyland and E. Ortega, *Topological full groups of ample groupoids with applications to graph algebras*, Int. J. Math. *30* (2019), no. 4, 1950018.

[57] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1999.

[58] J. Renault, *A groupoid approach to C^*-algebras*, Lect. Notes in Math. *793*, Springer, Berlin, 1980.

[59] J. Renault, *Représentation des produits croisés d'algèbres de groupoïdes*, J. Oper. Theory *18* (1987), no. 1, 67–97.

[60] J. Renault, *The Fourier algebra of a measured groupoid and its multipliers*, J. Funct. Anal. *145* (1997), no. 2, 455–490.

[61] J. Renault, *Topological amenability is a Borel property*, Math. Scand. *117* (2015), no. 1, 5–30.

[62] M. Rørdam, F. Larsen and N. Laustsen, *An Introduction to K-Theory for C^*-Algebras.* Cambridge Univ. Press, 2000.

[63] E. Scarparo, *Relatively amenable actions of Thompson's groups*, Bull. Aust. Math. Soc. *106* (2022), 144–150.

[64] M. Stein, *Groups of piecewise linear homeomorphisms*, Trans. Am. Math. Soc. *332* (1992), no. 2, 477–514.

[65] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. *223* (2010), no. 2, 689–727.

[66] B. Steinberg and N. Szakács, *Simplicity of inverse semigroup and étale groupoid algebras*, Adv. Math. *380* (2021), Paper No. 107611, 55 pp.

[67] B. Steinberg and N. Szakács, *On the simplicity of Nekrashevych algebras of contracting self-similar groups*, Math. Ann. *386* (2023), no. 3-4, 1391–1428.

[68] T. Timmermann, *The Fell compactification and non-Hausdorff groupoids*, Math. Z. *269* (2011), no. 3-4, 1105–1111.

[69] H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. *19* (1952), no. 2, 289–292.

[70] R. Willett, *A non-amenable groupoid whose maximal and reduced C^*-algebras are the same*, Münster J. Math. *8* (2015), no. 1, 241–252.

[71] D. Williams, *A Tool Kit for Groupoid C^*-Algebras*, Amer. Math. Soc., Providence, RI, 2019.

[72] K. Yoshida, *On the Simplicity of C^*-algebras Associated to Multispinal Groups*, preprint, arXiv:2102.02199.

[73] R. J. Zimmer, *Hyperfinite factors and amenable ergodic actions*, Invent. Math. *41* (1977), 23–31.