# Affine Quantum Bruhat Graphs and Double Affine Demazure Products

by

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# Abstract

In recent work, Muthiah and Puskás give a conjectural definition of a Demazure product for a double affine Weyl semigroup $W_{\mathcal{T}}$ by considering its associated Kac-Moody affine Hecke algebra. Motivated by this, we generalise work by Schremmer, who gave a formula for the single affine Demazure product using the quantum Bruhat graph. We focus on type $\widehat{\mathrm{SL}}_2$, where we prove that the corresponding affine quantum Bruhat graph has a well-defined weight function, and satisfies nice properties involving length posivite sets, allowing us to construct a well-defined Demazure product for $W_{\mathcal{T}}$ of this type. We go on to show that this product is associative for level greater than one, as well as linking it to the single affine Demazure product.

Furthermore, assuming that our double affine Demazure product is well-defined, we then give specific results connecting this product with Muthiah and Orr's length function on $W_{\mathcal{T}}$ in general type, verifying that it matches examples computed by Muthiah and Puskás in the Kac-Moody affine Hecke algebra. We also discuss the semi-infinite Bruhat order and its link to the quantum Bruhat graph, giving a candidate definition for the double affine semi-infinite Bruhat order, and obtaining some preliminary results.

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# Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

# Contents

# Chapter 1

# Introduction

*Weyl groups* are fundamental objects within representation theory, arising from their associated Lie groups, which have seen use throughout many topics within both mathematics and physics ([Wey50], [Kac83], [Lus85], [LP07]). In essence, a Weyl group describes how reflections through various walls, or hyperplanes, interact amongst each other. By introducing certain shifted hyperplanes to our system, we obtain an *affine* Weyl group, which describes both reflections and translations. Both finite and affine Weyl groups are examples of Coxeter groups, which carry a very rich and useful structure ([Cox34], [Bro89], [Hum90], [Bre02], [BB05], [Dav12]).

A generalisation of the Lie group, known as the *Kac-Moody group*, has been a topic of great interest since its introduction ([KP85], [Tit92], [Kum12], [AMRW19]). Whereas Lie groups are traditionally finite dimensional, Kac-Moody groups relax this condition and allow us to consider infinite dimensional objects. Each Kac-Moody group also has an underlying Weyl group $W$, potentially of either finite or affine type, along with an associated *Hecke algebra* $\mathcal{H}$, a certain deformation of the group algebra of $W$. By considering affinisations of Kac-Moody groups that are already of affine type, we recover the *Kac-Moody affine Hecke algebra* $\widehat{\mathcal{H}}$, first constructed in [BKP16]. This algebra is indexed by the *double affine Weyl semigroup* $W_{\mathcal{T}}$, our main object of study for this thesis.

Unlike its finite and single (non-extended) affine counterparts, $W_{\mathcal{T}}$ fails to be a Coxeter group, but much progress has been made in giving it Coxeter-like structures, such as a length function and a Bruhat order ([BKP16], [BPGR16], [Mut18], [MO19], [Wel19], [Phi24]). One structure of interest that has *a priori* not been generalised to our double affine setting is the *Demazure product*, which describes the 'leading term' in the expansion of the product of two basis elements of the corresponding Hecke algebra, or equivalently describing multiplication in the 0-Hecke algebra ([He15], [HW15], [HN21], [Sch24]).

Motivated by a conjecture of Muthiah and Puskás [MP26] that this 'leading term' should exist in the double affine setting (see Conjecture 1.1.1 below), we seek to define a *double affine Demazure product* on $W_{\mathcal{T}}$. In this thesis, we do so by generalising work by Schremmer, who used the *finite quantum Bruhat graph* to calculate single affine Demazure products [Sch24]. In doing so, we define the *affine* quantum Bruhat graph, and with it a *generalised* Demazure product. We focus in particular on the case of type $\widehat{\mathrm{SL}}_2$, studying its quantum Bruhat graph in order to obtain a well-defined, associative product on $W_{\mathcal{T}}$, which satisfies properties expected of the Demazure product, presenting a positive indication that this product will be defined in general and helping to give us a deeper understanding of the Kac-Moody affine Hecke algebra itself. Furthermore, assuming that our double affine Demazure product is well-defined in general type, we give results connecting this product to the length function on $W_{\mathcal{T}}$, showing that it satisfies other Demazure-like properties. We finish by discussing the links between this product and a generalisation of the *semi-infinite Bruhat order* to the double affine setting, which has the potential for applications in affine generalisations of several structures in representation theory ([Ish22], [Kat25]).

We note that most of this work is comprised of work published by the author [Dea25]. In this thesis, we provide a more in depth and concrete background, as well as going on to give further results relating our double affine Demazure product to the Coxeter-theoretic single affine version, and giving results on the double affine semi-infinite Bruhat order. We aim to produce further papers based on these new results, along with ongoing work on general-type affine quantum Bruhat graphs.

## § 1.1 Background

In this section, we discuss the main objects of focus of this thesis in some more detail.

### § 1.1.1 Demazure Products

Consider a Coxeter group $W$ of either finite or affine type, with simple reflections $S$, length function $\ell(\cdot)$, and Bruhat order $\leq$ (see section 2.2.1 for details). The *Demazure product* of $x, y$, denoted $x * y$, is defined to be the maximal element, with respect to the Bruhat order, of the form $x'y'$, where $x' \leq x$ and $y' \leq y$. More concretely, it is the associative product defined by:

$$x * s := \begin{cases} xs & \text{if } \ell(xs) > \ell(x), \\ x & \text{if } \ell(xs) < \ell(x), \end{cases} \tag{1.1}$$

where $x \in W$ and $s \in S$. The Demazure product can also be defined by considering the $q = 0$ specialisation of a suitable normalisation of the Hecke algebra. Fixing a basis $\{T_x\}_{x \in W}$ of the Hecke algebra, we have:

$$T_x T_y \equiv (-1)^{\ell(x)+\ell(y)-\ell(x*y)} T_{x*y} \pmod{q}, \quad x, y \in W. \tag{1.2}$$

We can therefore view $x * y$ as describing the leading term (or constant term, with respect to $q$) of the product $T_xT_y$. In this way, understanding the Demazure product is an initial step towards understanding the corresponding Hecke algebra.

In the literature, the $q = 0$ specialisation of this algebra is referred to as the 0-Hecke algebra, for which the product of two basis elements is simply $T_xT_y = T_{x*y}$.

In the affine case, this basis arises from the realisation of $W$ as a Coxeter group for some affine root system. However, we can also write $W = W_{\mathrm{fin}} \ltimes \mathbb{Z}\Phi^\vee$, a semidirect product of the corresponding finite Weyl group $W_{\mathrm{fin}}$ with the coroot lattice $\mathbb{Z}\Phi^\vee$. This alternative presentation will later allow us to affinise a (Kac-Moody) Weyl group which is already itself affine.

### § 1.1.2 Kac-Moody Affine Hecke Algebras

In the last few years, there has been significant progress in constructing affinisations of Kac-Moody algebras arising from considering the Kac-Moody group $G$ over $p$-adic fields. Let $G$ be an (untwisted) affine Kac-Moody group over a $p$-adic field, with Weyl group $W$, now fixed to be of affine type. Let $G^+$ be the semigroup of $G$ where the Cartan decomposition holds. The full Kac-Moody affine Hecke algebra $\widehat{\mathcal{H}}$ is the convolution algebra of $\mathbb{C}$-valued functions of $I\backslash G^+/I$ with finite support, defined by Braverman, Kazhdan and Patnaik [BKP16] for $G$ untwisted affine, and by Bardy-Panse, Gaussent and Rousseau [BPGR16] in the general case.

The Kac-Moody affine Hecke algebra has a basis $\{T_x\}_{x\in W_{\mathcal{T}}}$. Here, the indexing set is the double affine Weyl semigroup $W_{\mathcal{T}} = W \ltimes \mathcal{T}$, where $\mathcal{T}$ is the *Tits cone*, consisting of the set of $W$-translates of the dominant coweights of $G$. This Hecke algebra can be thought of as a version of Cherednik's double affine Hecke algebra, which itself can be constructed by instead taking the full coroot lattice of $G$ rather than the Tits cone $\mathcal{T}$ in the semidirect product [Che95].

Note that, since $G$ is already affine, the Kac-Moody affine Hecke algebra is in fact double affine. We use the terms Kac-Moody affine and double affine synonymously when $G$ is itself affine. If instead $G$ was of finite type, then $\mathcal{T}$ is the full set of coweights and so $W_{\mathcal{T}}$ is exactly the (extended) affine Weyl group.

For $T_x, T_y \in \hat{\mathcal{H}}$, we can write their product as:

$$T_xT_y = \sum_{z\in W_{\mathcal{T}}} c^z_{x,y}T_z, \quad c^z_{x,y} \in \mathbb{Z}[q]. \tag{1.3}$$

The sum here will have only finitely many non-zero terms, as we have specified that the functions which make up the algebra have finite support. Note also that this product is typically arduous to calculate; an algorithm for calculation such products is detailed in [MP26, Sec. 2.5.4].

Bardy-Panse, Gaussent and Rousseau [BPGR16], and (independently) Muthiah

[Mut18], proved a conjecture by Braverman, Kazhdan and Patnaik [BKP16] that the structure constants $c^z_{x,y}$ are integer-coefficient polynomials in $q$. Muthiah and Puskás further conjecture the following, a generalisation of Equation (1.2).

**Conjecture 1.1.1** (Muthiah, Puskás [MP26, Conjecture 7.3])**.** *There is exactly one coefficient $c^z_{x,y} \in \mathbb{Z}[q]$ in the product $T_xT_y$ which is non-zero modulo $q$. In particular, $T_xT_y \equiv (-1)^{\ell(x)+\ell(y)-\ell(z)}T_z \pmod q$ for some $z \in W_{\mathcal{T}}$.*

### § 1.1.3 Quantum Bruhat Graphs

Conjecture 1.1.1 gives one candidate for the Demazure product, but we instead take another approach, generalising work by Schremmer [Sch24], who gave a more explicit formula for calculating the single affine Demazure product via the finite quantum Bruhat graph $\mathrm{QBG}(W)$. This concept is much easier to work with than the Kac-Moody affine Hecke algebra, and generalises very naturally to a double-affine setting.

The quantum Bruhat graph is a directed, weighted graph, which has vertices in $W$ and weights in $\Phi^\vee \cup \{0\}$, the set of coroots (with zero). The 'classical' edges align with the Hasse diagram for the Bruhat order of $W$, with extra 'quantum' edges introduced. The (finite) quantum Bruhat graph was first studied for its use in quantum cohomology [Pos05], but has since found several applications within algebraic combinatorics and representation theory (e.g. [Ish22], [Sad23]). Affine quantum Bruhat graphs were first considered by Mare and Mihalcea [MM18], which they refer to as the 'moment graph', focusing on type $\tilde{A}_1$. These graphs were then later studied in more detail by Welch [Wel19].

It is known that $\mathrm{QBG}(W)$ is strongly connected for both finite and affine types, but in the finite case, we also know that any two shortest paths within $\mathrm{QBG}(W)$ between two fixed vertices have the same weight, giving us a well-defined weight function. In the affine case, the existence of a well-defined weight function is an open question for general type, as the current proofs rely on results that only apply in finite type, one of which is the Diamond lemma (Proposition 2.5.6). We make a conjecture that the Diamond lemma fails in a specific set of circumstances (Conjecture 2.5.7).

Schremmer [Sch24] relates the single affine Demazure product to the quantum Bruhat graph as follows (Theorem 2.5.15 in the main text).

**Theorem 1.1.2.** *Fix a (finite) Weyl group $W$ with coweight lattice $P^\vee$. Construct the extended affine Weyl group $\widehat{W} = W \ltimes P^\vee = \{w\varepsilon^\mu \mid w \in W, \mu \in P^\vee\}$, and let $x = w_x\varepsilon^{\mu_x}, y = w_y\varepsilon^{\mu_y} \in \widehat{W}$, where $w_x, w_y \in W$, $\mu_x, \mu_y \in P^\vee$. Choose a pair of length positive elements $(u, v) \in W \times W$ such that the distance $u \Rightarrow w_yv$ in the quantum Bruhat graph $\mathrm{QBG}(W)$ is minimal amongst all such pairs, and let $p$ be any shortest* *path* $u \Rightarrow w_y v$ *in* $\mathrm{QBG}(W)$*. Then the Demazure product* $x * y$ *is given as:*

$$x * y := w_x uv^{-1}\varepsilon^{vu^{-1}\mu_x+\mu_y-v\mathrm{wt}(p)}. \tag{1.4}$$

This definition relies on the notion of *length positivity.* Given $x \in W$ of affine type, consider the length functional $\ell(x, \alpha)$ (see Definition 2.5.11). We say that $u \in W$ is *length positive* if $\ell(x, u\alpha)$ is non-negative for every positive root $\alpha$. The set of length positive elements is then some subset of $W$.

Note that the right hand side of Equation 1.4 appears to depend on the choice of pair $(u, v)$, which is not necessarily unique, as well as the choice of path $p$. When considering the double-affine case, we will need to show that the generalised product we define is independent of these choices.

## § 1.2 Main Results

The role of this thesis is to extend Schremmer's work to the double affine case, particularly in type $\widehat{\mathrm{SL}}_2$, providing insight into the nature of double affine Demazure products in general. We go on to give results and links to other mathematical objects in general type further on.

To begin with, we provide a classification of length positive elements corresponding to $W_{\mathcal{T}}$ of type $\widehat{\mathrm{SL}}_2$. Let $\mathrm{LP}(x)$ denote the set of length positive elements of $x \in W_{\mathcal{T}}$.

**Theorem 1.2.1.** *Let* $x \in W_{\mathcal{T}}$ *be an element of the double affine Weyl semigroup of type* $\widehat{\mathrm{SL}}_2$ *with non-zero level. Then* $\mathrm{LP}(x)$ *has size at most 3, and can be computed algorithmically.*

We then extend results regarding the finite quantum Bruhat graph to type $\widehat{\mathrm{SL}}_2$, obtaining the following theorem.

**Theorem 1.2.2** (Thm. 3.2.4, Thm. 3.2.7)**.** *Let* $W$ *be the Weyl group for* $\widehat{SL}_2$ *with quantum Bruhat graph* $\mathrm{QBG}(W)$*. Let* $u, v \in W$*. Then:*

*(i) Any two shortest paths from* $u$ *to* $v$ *have the same total weight.*

*(ii) The length of a path between* $u$ *and an element of* $\mathrm{LP}(x) \subseteq W$ *for some* $x \in W_{\mathcal{T}}$*, in either direction, is minimised by a unique element of* $\mathrm{LP}(x)$*.*

This theorem, amongst others, leads us to the main result of this thesis.

**Theorem 1.2.3** (Thm. 3.2.8, Thm. 3.3.8)**.** *In type* $\widehat{SL}_2$*, Schremmer's formula for the Demazure product in* $W_{\mathcal{T}}$ *is well-defined, and for level greater than one, is associative.*

We then further conjecture that the formula is associative for the entirety of $W_{\mathcal{T}}$; see Conjecture 3.3.10 in the main text.

Our hope is that a proof of well-definedness in arbitrary type could be developed by using the root systems of rank 2, including type $\tilde{A}_1$, as a basis.

In general type, we prove that this product satisfies other important properties that we expect of a Demazure product in the general case, assuming well-definedness. In particular, we have the following results.

**Theorem 1.2.4** (Thm. 4.1.1, Thm. 4.1.2)**.** *Consider any Kac-Moody root datum, and let $x, y \in W_{\mathcal{T}}$. Assume that the Demazure product $x * y$ is well-defined. Then:*

*(i)* $\ell(x * y) = \ell(x) + \ell(y) \iff \ell(xy) = \ell(x) + \ell(y)$*, in which case* $x * y = xy$.

*(ii) For any shortest path $p$ in* $\mathrm{QBG}(W)$ *between two explicitly described elements,* $\ell(x * y) = \ell(x) + \ell(y) - \ell(p)$.

We go on to calculate examples using our definition of the double affine Demazure product, matching up with those computed by Muthiah and Puskás in the Hecke algebra.

## § 1.3 Structure of the Thesis

In Chapter 2, we introduce the preliminaries and notation used throughout. We begin by discussing general root systems and Weyl groups, including their affinisations, before discussing Kac-Moody root systems in Section 2.1.3, which we fix for the remainder of the thesis. We then discuss Coxeter groups in general and the Coxeter theoretic notions we will use, before constructing the Hecke algebra and its Kac-Moody affine variant. In Section 2.4, we formally define the double affine Weyl semigroup, and then move on to define the quantum Bruhat graph, which we use to construct a double affine Demazure product in Section 2.5.2.

Chapter 3 is concerned with results in type $\widehat{\mathrm{SL}}_2$. We begin by classifying length positive elements, before giving an algorithm for computing $\mathrm{LP}(x)$ for a general element $x \in W_{\mathcal{T}}$. We then discuss the affine quantum Bruhat graph of type $\widehat{\mathrm{SL}}_2$, which we use to prove well-definedness of our double affine Demazure product, and then show how by restricting $x \in W$, we recover the single affine Demazure product. Afterwards, we prove that this product is associative. To do this, we first give a method of determining the length positive set of the Demazure product of two elements. We then use this to split the conditions for associativity into a Weyl component and a lattice component, which are dealt with separately, but ultimately proving that the product is in fact associative.

Chapter 4 then focuses on results in general type. In Section 4.1 and Section 4.2, we assume that the double affine Demazure product is well-defined, and obtain results regarding the length of the Demazure product, and its relation to the single affine case. We also compute examples first computed by Muthiah and Puskás in the Hecke algebra, verifying that our results align with theirs. Finally, in Section 4.3, we discuss the notion of the semi-infinite Bruhat order, and its link to the quantum Bruhat graph. We finish by giving some new definitions and preliminary results regarding a candidate for the double affine semi-infinite Bruhat order in Section 4.3.3.

# Chapter 2

# Preliminaries

This chapter aims to introduce the preliminaries and notation surrounding concepts we use within Kac-Moody theory. We begin by introducing finite root systems and Weyl groups, along with their affinisations, before considering the more general notion of a Kac-Moody root system. We then discuss the Coxeter theory of Weyl groups, and link them to the Hecke algebra. Finally, we discuss the notion of a double affine Weyl semigroup, and go on to define the quantum Bruhat graph in order to give a definition of a double affine Demazure product, the main focus of this thesis.

In this chapter, we generally follow (and refer the reader to) [Hum90] for results on root systems, Weyl groups and Coxeter groups, [Kac90] for results on Kac-Moody root systems, and [CMHL03] for results on Iwahori-Hecke algebras.

## § 2.1 Root Systems and Weyl Groups

Root systems arise from the study of the decomposition of Lie algebras. We can consider the vector space $V$ spanned by the set of roots $\Phi$, and look at reflections generated by hyperplanes normal to these roots. This reflection subgroup of $\mathrm{GL}(V)$ is known as the Weyl group of $\Phi$, denoted $W$.

### § 2.1.1 Finite Type

For this section, we let $V$ be a finite-dimensional vector space over $\mathbb{R}$, with inner product $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{R}$. For a given $\alpha \in V\backslash\{0\}$, let $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$.

**Definition 2.1.1.** *Let $\alpha \in V$. The **reflection** associated to $\alpha$ is the map*

$$s_\alpha(\beta) := \beta - \langle\alpha^\vee,\beta\rangle\alpha = \beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

We also define $H_\alpha := \{\lambda \in V \mid \langle\alpha^\vee,\lambda\rangle = 0\}$ to be the hyperplane in $V$ orthogonal to $\alpha$, with $s_\alpha$ geometrically describing the reflection of $V$ through $H_\alpha$. Note that $s_\alpha^2$ is simply the identity map, as would be expected of a reflection.

**Definition 2.1.2.** *A (finite) **root system** $\Phi$ over $V$ is a finite spanning set of non-zero* *vectors, called roots, such that for all* $\alpha \in V$*:*

(i) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$.

(ii) $s_\alpha(\Phi) = \Phi$.

*Furthermore, we say that* $\Phi$ *is* ***crystallographic*** *if for all* $\alpha, \beta \in \Phi$, $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.

For the rest of this section, we assume that $\Phi$ is crystallographic. We call a root system $\Phi$ *irreducible* if it cannot be decomposed into two non-empty subsystems $\Phi = \Phi_1 \sqcup \Phi_2$ where $\langle \alpha^\vee, \beta \rangle = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$. The *rank* of $\Phi$ is $\dim V$. The set of *coroots* is $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. Note that $\langle \alpha^\vee, \alpha \rangle = 2$ for all $\alpha \in \Phi$.

Every root system can be decomposed as a disjoint union of positive roots and negative roots.

**Definition 2.1.3.** *A subset* $\Phi^+ \subset \Phi$ *is a set of* ***positive roots*** *for* $\Phi$ *if* $\Phi = \Phi^+ \sqcup -\Phi^+$*, and if for all* $\alpha, \beta \in \Phi^+$ *such that* $\alpha + \beta \in \Phi$*, we have* $\alpha + \beta \in \Phi^+$*. We call* $\Phi^- := -\Phi^+$ *the set of* ***negative roots****.*

The choice of positive roots is not unique, however it is always possible to construct an isometry of $V$ from one choice to any other, hence we usually refer to $\Phi^+$ as *the* set of positive roots for $\Phi$.

**Definition 2.1.4.** *A positive root* $\alpha \in \Phi^+$ *is a* ***simple root*** *if it cannot be written as the sum of two elements of* $\Phi^+$*. We write* $\Delta$ *for the set of simple roots of* $\Phi$*.*

Note that the set of simple roots forms a basis for $V$, and hence the rank of $\Phi$ is equal to $|\Delta|$. If we fix a positive system, then the set of simple roots is unique. Otherwise, a choice of simple roots corresponds to a choice of a positive system. Unless otherwise stated, we write $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ for the simple roots of a root system of rank $n$, reserving $\alpha_i$ for elements of $\Delta$.

**Example 2.1.5.** The only possible root system (up to scaling) of rank 1 is $\Phi = \{\pm\alpha\}$, known as type $A_1$. One example of a rank 2 system is the $A_2$ root system, where $\Delta = \{\alpha_1, \alpha_2\}$ and $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. These examples are depicted in Figure 2.1.

**Proposition 2.1.6.** *Every positive (respectively negative) root can be written as a linear combination of simple roots with non-negative (respectively non-positive) integer coefficients. In particular,* $\Phi^+ \subset \mathbb{Z}_{\geq 0}\Delta$*.*

**Definition 2.1.7.** *Let* $\alpha \in \Phi$ *and write* $\alpha = \sum_{i=1}^n c_i\alpha_i$*. The* ***height*** *of* $\alpha$ *is defined as* $\mathrm{ht}(\alpha) = \sum_{i=1}^n c_i$*.*

**Definition 2.1.8.** *Fix a root system* $\Phi$*. We say that* $\lambda \in V$ *is a* ***weight*** *for* $\Phi$ *if* $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$ *for every* $\alpha \in \Phi$*.*

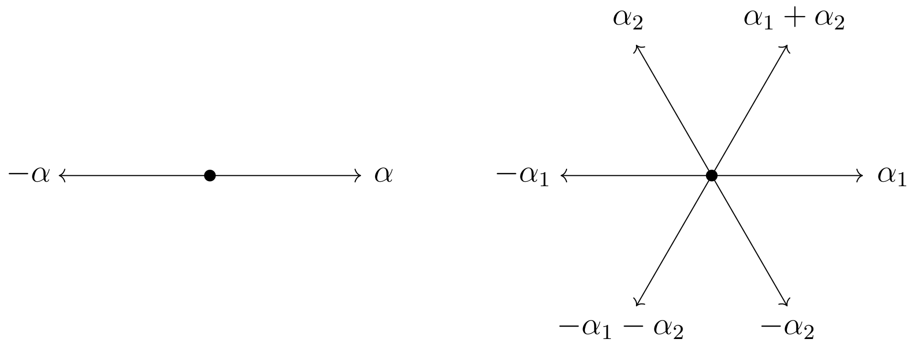


Figure 2.1: The root systems of type $A_1$ and $A_2$ respectively.

Note that if $\lambda_1$ and $\lambda_2$ are weights, then so is $c_1\lambda_1 + c_2\lambda_2$ for any $c_1, c_2 \in \mathbb{Z}$, and hence the set of weights forms a lattice.

Given a weight $\lambda$, we call $\lambda^\vee$ a *coweight*. We write $P$ for the weight lattice, and $P^\vee = \{\lambda^\vee \mid \lambda \in P\}$ for the coweight lattice. We also denote $\mathbb{Z}\Phi$ (respectively $\mathbb{Z}\Phi^\vee$) for the root lattice (respectively coroot lattice). Note that we have $\mathbb{Z}\Phi \subseteq P$ and $\mathbb{Z}\Phi^\vee \subseteq P^\vee$, in particular every root is a weight and every coroot is a coweight.

Let $\{\lambda_1, ..., \lambda_n\}$ be the set of *fundamental weights*, which satisfy the conditions $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. Let $\rho$ denote the sum of the fundamental weights, which is equal to the half sum of the positive roots.

**Proposition 2.1.9.** *Let $\alpha \in \Phi$. Then* $\mathrm{ht}(\alpha) = \langle \alpha^\vee, \rho \rangle$.

**Definition 2.1.10.** *The* ***dominance ordering of roots*** *is defined by the condition that for $\alpha, \beta \in \Phi$, $\alpha \geq \beta$ if $\alpha - \beta \in \mathbb{Z}_{\geq 0}\Delta$. That is, $\alpha - \beta$ can be written as a sum of simple roots with non-negative integer coefficients.*

With respect to the dominance order, there is a unique highest root which we denote as $\theta$. We also write $\alpha > 0$ (respectively $\alpha < 0$) if $\alpha$ is a positive (respectively negative) root.

**Definition 2.1.11.** *Given a crystallographic root system $\Phi$, the (finite)* ***Weyl group*** *$W$ is the group generated by the reflections $s_\alpha$ for $\alpha \in \Phi$. That is, $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$.*

Write $S = \{s_i = s_{\alpha_i} \mid i \in \{1, ..., n\}\}$ for the set of *simple reflections*, i.e. those reflections which arise from simple roots. Note that we could equivalently define $W$ as the group generated only by the simple reflections. Throughout, we write $e$ for the identity element of the Weyl group.

**Example 2.1.12.** For the $A_1$ root system, the only non-trivial reflection is $s_\alpha$, and hence $W = \{e, s_\alpha\} \cong S_2$, the symmetric group on 2 elements. With respect to Figure 2.1, the element $s_\alpha$ simply reflects the line through the origin, swapping the roots $\alpha$ and $-\alpha$. For the $A_2$ root system, we find $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \cong S_3$. Note that $s_{\alpha_1+\alpha_2} = s_1s_2s_1 = s_2s_1s_2$. Again with respect to Figure 2.1, the element $s_{\alpha_1} = s_1$

reflects along a vertical line through the origin. For example, $\alpha_2$ will be mapped to $\alpha_1 + \alpha_2$, i.e. $s_1(\alpha_2) = \alpha_1 + \alpha_2$.

### § 2.1.2 Affine Type

Affine Weyl groups are generalisations of finite Weyl groups. Both describe reflections, however for affine Weyl groups, we do not require that the hyperplanes pass through the origin. This alteration introduces translations into our group, elements of infinite order. Consequently, the corresponding affine root system is infinite.

In Section 2.1.3, we will want to affinise a root system which is already itself affine. In light of this, we avoid referring to the $\Phi$ as a *finite* root system, and simply call it a root system, as much of what is defined here will generalise immediately.

Define the sign function $\mathrm{sgn} : \mathbb{Z} \to \{\pm 1\}$ by:

$$\mathrm{sgn}(n) = \begin{cases} 1 & \text{if } n \geq 0, \\ -1 & \text{if } n < 0. \end{cases}$$

**Definition 2.1.13.** *Fix a root system $\Phi$, along with a choice of simple roots $\Delta$ and corresponding positive root system $\Phi^+$.*

(i) *The set of* ***affinised roots*** *is $\Phi_{\text{aff}} = \Phi \oplus \mathbb{Z}\delta = \{\alpha + n\delta \mid \alpha \in \Phi, n \in \mathbb{Z}\}$, where $\delta$ is called the* ***imaginary root****, which satisfies $\langle \delta, \delta \rangle = \langle \alpha^\vee, \delta \rangle = 0$ for all $\alpha \in \Phi$.*

(ii) *The set of* ***positive affinised roots*** *is $\Phi^+_{\text{aff}} = \{\mathrm{sgn}(n)(\alpha + n\delta) \mid \alpha \in \Phi^+, n \in \mathbb{Z}\}$.*

(iii) *The set of* ***negative affinised roots*** *is $\Phi^-_{\text{aff}} = -\Phi^+_{\text{aff}}$, such that $\Phi_{\text{aff}} = \Phi^+_{\text{aff}} \sqcup \Phi^-_{\text{aff}}$.*

(iv) *The set of* ***affinised coroots*** *is $\Phi^\vee_{\text{aff}} = \Phi^\vee \oplus \mathbb{Z}\delta$.*

(v) *The* ***affinised root lattice*** *is $\mathbb{Z}\Phi_{\text{aff}}$, and the* ***affine coroot lattice*** *is $\mathbb{Z}\Phi^\vee_{\text{aff}}$.*

(vi) *For finite $\Phi$ only, the set of* ***simple affine roots*** *is $\Delta_{\text{aff}} = \Delta \sqcup \{-\theta + \delta\}$, where $\theta$ is the highest root of $\Phi$.*

We write $\tilde{\alpha}_0 = -\theta + \delta$ and $\tilde{\alpha}_i = \alpha_i$ for $i > 0$, such that the set of simple affine roots is $\Delta_{\text{aff}} = \{\tilde{\alpha}_0, \tilde{\alpha}_1, ..., \tilde{\alpha}_n\}$. In general, we use the notation $\tilde{\alpha}$ to denote a single affine root. As in the finite case, we have a height function and a partial ordering on $\Phi_{\text{aff}}$.

**Example 2.1.14.** Figure 2.2 depicts portions of the affine root systems of types $\tilde{A}_1$ and $\tilde{A}_2$. In general, we can view an affinised root system as an infinite number of copies of the associated finite root system, each corresponding to a multiple of the imaginary root $\delta$.

In order to obtain an affine generalisation of Proposition 2.1.9, we will need to define an affine weight lattice.

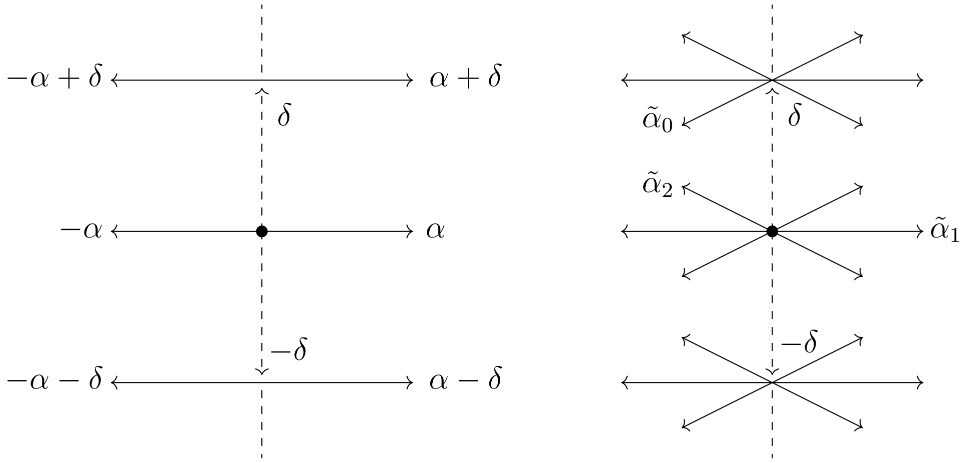


Figure 2.2: The root systems of type $\tilde{A}_1$ and $\tilde{A}_2$ respectively. For type $\tilde{A}_2$, the roots $\tilde{\alpha}_1 = \alpha_1, \tilde{\alpha}_2 = \alpha_2$, and $\tilde{\alpha}_0 = -\alpha_1 - \alpha_2 + \delta$ have each been labelled. In both types, the vertical dashed lines correspond to multiples of the imaginary root $\delta$.

**Definition 2.1.15.** *The **affine weight lattice** is $P_{\text{aff}} = P \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$, where $\Lambda_0$ is chosen such that $\langle \alpha_i^\vee, \Lambda_0 \rangle = 0$ and $\langle \delta, \Lambda_0 \rangle = 1$. The corresponding **affine fundamental weights** are $\{\Lambda_0, ..., \Lambda_n\}$, where $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$. We also define $Q = \mathbb{Z}\Phi \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0 \subseteq P_{\text{aff}}$.*

The choice of $\Lambda_0$ here is not necessarily unique; we assume that we have fixed a choice of $\Lambda_0$ throughout. Denote $\rho_{\text{aff}}$ for the sum of the affine fundamental weights. Note that $\rho_{\text{aff}}$ is not equal to the half sum of the positive affine roots as in the finite case, since we have infinitely many of these.

We use the same notation $\Lambda_0$ (and $\delta$) for its dual when working within the coweight lattice - the relevant choice will be clear from context.

**Proposition 2.1.16.** *Let $\tilde{\alpha} \in \Phi_{\text{aff}}$. Then $\text{ht}(\tilde{\alpha}) = \langle \tilde{\alpha}^\vee, \rho_{\text{aff}} \rangle$.*

We can then construct the affine Weyl group in two different ways, either as the group generated by the reflections of an affine root system, or as a semi-direct product of the underlying Weyl group with the coroot lattice.

**Definition 2.1.17.** *Let $\tilde{\alpha} = \alpha + n\delta \in \Phi^+$ be a positive affine root. Then the **affine reflection** associated to $\tilde{\alpha}$ is the reflection that fixes the affine hyperplane associated to $\tilde{\alpha}$, $H_{\tilde{\alpha}} = \{\lambda \in V \mid \langle \alpha^\vee, \lambda \rangle = -n\}$.*

The set of simple affine reflections is $S_{\text{aff}} = \{s_i = s_{\tilde{\alpha}_i} \mid i \in \{0, ..., n\}\}$.

**Definition 2.1.18.** *The **affine Weyl group** $W_{\text{aff}}$ is the group generated by the simple affine reflections, that is, $W_{\text{aff}} = \langle s_i \mid i \in \{0, ..., n\} \rangle$.*

**Example 2.1.19.** Let $\Phi = \{\pm\alpha\}$ be the root system of type $A_1$. Then $\Phi_{\text{aff}}$ is the affine root system of type $\widetilde{A}_1$. We have $S_{\text{aff}} = \{s_0, s_1\}$, with $\tilde{\alpha}_0 = -\alpha + \delta$ and $\tilde{\alpha}_1 = \alpha$.

Note that $s_0 s_1$ geometrically corresponds to a translation, and in particular has infinite order. Viewing $W_{\text{aff}}$ as being generated by a reflection $s_1$ and a 'rotation' $s_0 s_1$, we see that $W_{\text{aff}} \cong D_\infty$, the infinite dihedral group.

We can construct an action of $W_{\text{aff}}$ on $\Phi_{\text{aff}}$ by considering the simple reflections. In particular, let $\tilde{\alpha} = \alpha + n\delta$ and $\tilde{\beta} = \beta + m\delta$, and define

$$s_{\tilde{\alpha}}(\tilde{\beta}) := \tilde{\beta} - \langle \tilde{\alpha}^\vee, \tilde{\beta} \rangle \tilde{\alpha},$$

where $\tilde{\alpha}^\vee = 2\tilde{\alpha}/\langle \tilde{\alpha}, \tilde{\alpha} \rangle$ as before.

**Proposition 2.1.20.** *Taking the finite coroot lattice as an abelian group under addition, we have $W_{\text{aff}} \cong W \ltimes \mathbb{Z}\Phi^\vee$.*

We write an element $x \in W_{\text{aff}}$ as $x = w\varepsilon^\mu$, where $w \in W$ and $\mu \in \mathbb{Z}\Phi^\vee$. The action of $x$ on $\Phi_{\text{aff}}$ is then given explicitly by

$$w\varepsilon^\mu(\beta + m\delta) = w\beta + (m - \langle \mu, \beta \rangle)\delta, \tag{2.1}$$

for $\beta + m\delta \in \Phi_{\text{aff}}$. We can use the action to determine an expression for $s_{\tilde{\alpha}}$ in this form and go between the presentations $W_{\text{aff}}$ and $W \ltimes \mathbb{Z}\Phi^\vee$. Let $\tilde{\alpha} = \alpha + n\delta, \tilde{\beta} = \beta + m\delta \in \Phi_{\text{aff}}$.

$$\begin{aligned}
s_{\tilde{\alpha}}(\tilde{\beta}) &= \tilde{\beta} - \langle \tilde{\alpha}^\vee, \tilde{\beta} \rangle \tilde{\alpha} \\
&= \beta + m\delta - \langle \alpha^\vee + n\delta, \beta + m\delta \rangle (\alpha + n\delta) \\
&= \beta - \langle \alpha^\vee, \beta \rangle \alpha + (m - n\langle \alpha^\vee, \beta \rangle)\delta \\
&= s_\alpha(\beta) + (m - \langle n\alpha^\vee, \beta \rangle)\delta \\
&= s_\alpha \varepsilon^{n\alpha^\vee}(\tilde{\beta}).
\end{aligned}$$

Hence $s_{\tilde{\alpha}} = s_\alpha \varepsilon^{n\alpha^\vee}$, and in particular, $s_0 = s_{\tilde{\alpha}_0} = s_\theta \varepsilon^{-\theta^\vee}$. The above action on $\mathbb{Z}\Phi^\vee$ extends naturally to an action on the coweight lattice $P^\vee$, which allows us to define another type of Weyl group.

**Definition 2.1.21.** *The* ***extended affine Weyl group*** *is the semidirect product $\widehat{W} = W \ltimes P^\vee$.*

From the definition of weights, it is easy to see that $W_{\text{aff}} \subseteq \widehat{W}$. In fact, $\widehat{W}$ has the presentation $W_{\text{aff}} \rtimes \Omega$, where $\Omega$ is the group of automorphisms of the Dynkin diagram of $W$. Note that, unlike the affine Weyl group, the extended affine Weyl group is typically not generated by simple reflections, however it does have a well-defined Bruhat order, and length function, which can take the value zero for certain non-identity elements. In particular, $\Omega$ is the group of length zero elements of $\widehat{W}$.

The construction of the affine Weyl group as the semidirect product $W \ltimes \mathbb{Z}\Phi^\vee$ is key to the construction of the double affine Weyl semigroup. In particular, both the finite and affine Weyl groups are Coxeter groups, as we detail in Section 2.2, and so we will be able to repeat this construction to affinise again.

### § 2.1.3 Kac-Moody Type

The previous sections detail the classical way to define a finite or affine root system and its Weyl group. However, we can also define the more general concept of a Kac-Moody root system. This is done by encoding the values of $\langle \alpha_i^\vee, \alpha_j \rangle$ as a matrix, from which the root system can be recovered. Once this has been done, we are then able to affinise again. If our underlying Kac-Moody system was itself affine, we obtain a double affine object.

**Definition 2.1.22.** *A **generalised Cartan matrix** is an $n \times n$ matrix $A = (a_{ij})$ such that for all $i, j \in \{1, ..., n\}$:*

i) $a_{ii} = 2$.

ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$.

iii) $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.

We call $A$ *indecomposable* if it cannot be decomposed into a direct sum of smaller generalised Cartan matrices. Following [Kac90], we do not require $A$ to be symmetrisable, unlike some other literature.

**Definition 2.1.23.** *A **Kac-Moody root datum** is a tuple $\mathcal{D} = (I, A, \Lambda, \Pi, \Pi^\vee)$, where:*

i) *$I$ is a finite indexing set.*

ii) *$A = (a_{ij})_{i,j \in I}$ is a generalised Cartan matrix.*

iii) *$\Lambda$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-dual $\Lambda^\vee$. That is, $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$.*

iv) *$\Pi = \{\alpha_i \mid i \in I\} \subset \Lambda$ and $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset \Lambda^\vee$.*

v) *$\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$, where $\langle \cdot, \cdot \rangle : \Lambda^\vee \times \Lambda \to \mathbb{Z}$ is the natural pairing between $\Lambda$ and its dual.*

We can identify $\Pi$ (respectively $\Pi^\vee$) with $\Delta$ (respectively $\Delta^\vee$). Similarly, $\Lambda$ (respectively $\Lambda^\vee$) can be identified with the weight lattice $P$ (respectively the coweight lattice $P^\vee$). Finally, the bracket $\langle \cdot, \cdot \rangle$ can be identified with the inner product.

**Example 2.1.24.** The Kac-Moody root datum for the $A_2$ root system is as follows:

i) $I = \{1, 2\}$.

ii) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

iii) $\Lambda = \mathbb{Z}[\lambda_1, \lambda_2]$, $\Lambda^\vee = \mathbb{Z}[\lambda_1^\vee, \lambda_2^\vee]$.

iv) $\Pi = \{\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = -\lambda_1 + 2\lambda_2\}$, $\Pi^\vee = \{\alpha_1^\vee = 2\lambda_1^\vee - \lambda_2^\vee, \alpha_2^\vee = -\lambda_1^\vee + 2\lambda_2^\vee\}$.

An explicit formula for the pairing can then be recovered by requiring $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$.

The *Kac-Moody root system* associated to $\mathcal{D}$ can be obtained by considering the corresponding Kac-Moody algebra, a particular type of Lie algebra (see [Kac90]), and furthermore each Kac-Moody algebra gives rise to a group known as the *Kac-Moody group* $G$ (see [Kum12]). We often refer to a Kac-Moody root system by its corresponding group rather than explicitly by the root datum. Note that if the generalised Cartan matrix $A$ is indecomposable, then the corresponding root system $\Phi$ is irreducible.

We then define the Weyl group $W$ of a Kac-Moody group as before, by considering the reflections generated by $\{s_i = s_{\alpha_i}\}_{i \in I}$. Note that the pairing $\langle \cdot, \cdot \rangle$ is invariant under the action of the Weyl group.

**Example 2.1.25.** The Kac-Moody group $G = \mathrm{SL}_2(\mathbb{C})$ gives rise to the $A_1$ root system. Similarly, $G = \mathrm{SL}_3(\mathbb{C})$ has root system of type $A_2$. In general, we refer to the Kac-Moody root system associated to $\mathrm{SL}_n(\mathbb{C})$ as being of type $A_{n-1}$, which has Weyl group $S_n$, the symmetric group on $n$ elements.

Note that Kac-Moody root systems are much more general than finite and affine root systems. In order to classify them, we first split them into three possible types.

**Definition 2.1.26.** *Let $A$ be an indecomposable generalised Cartan matrix.*

i) *If the principal minors of $A$ are all positive, then we say $A$ is of **finite type**.*

ii) *If the proper principal minors of $A$ are all positive and* $\det A = 0$*, then we say $A$ is of **affine type**.*

iii) *If neither of the above hold, we say $A$ is of **indefinite type**.*

We often instead refer to the root system $\Phi$, or the Kac-Moody group $G$, as being of finite, affine, or indefinite type rather than the generalised Cartan matrix. The finite and affine type Kac-Moody root systems correspond to those that we have previously defined. Indefinite type matrices are known to be unwieldy; for this thesis, we are mostly concerned with Kac-Moody systems of finite or affine type, which we assume from now on.

**Proposition 2.1.27.** *If $A$ is an indecomposable generalised Cartan matrix of either finite or affine type, then any proper principal sub-matrix of $A$ can be decomposed into generalised Cartan matrices of finite type.*

*Proof.* If the proper principal minors of $A$ are all positive, then taking any sub-matrix $B$ (and decomposing as necessary), we find that all of its principal minors must be positive. $\square$

The above description allows us to think of affine Kac-Moody root systems as being 'one step away' from being affine. Namely, the removal of any index (or equivalently any simple root) results in a direct sum of finite systems. This can be viewed explicitly via a construction known as the Dynkin diagram. In order to define these, we first note the following proposition.

**Proposition 2.1.28.** *Let $A = (a_{ij})_{i,j\in I}$ be a generalised Cartan matrix of finite or affine type. Then $a_{ij}a_{ji} < 4$ for all $i \neq j \in I$, unless $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, the (generalised) Cartan matrix of type $\tilde{A}_1$, or $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ or its transpose.*

*Proof.* From the definition, any proper principal minor of $A$ must be positive. Assuming $A$ is of rank greater than 2, we choose the proper principal minor $a_{ii}a_{jj} - a_{ij}a_{ji} = 4 - a_{ij}a_{ji} > 0$, and hence $a_{ij}a_{ji} < 4$. If $A$ is of rank 2 and of finite type, then similarly we obtain $a_{ij}a_{ji} < 4$. If $A$ is of rank 2 and of affine type, then its determinant is 0 and hence $a_{12}a_{21} = 4$, and so $A$ is one of the matrices listed in the proposition, in which case $a_{ij}a_{ji} = 4$ for $i, j = 1, 2$. $\square$

The Kac-Moody root system associated to the matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ is known as the type $A_2^{(2)}$ root system, and is an example of a *twisted* affine root system (see [Kac90, Ch. 8] for details).

**Definition 2.1.29.** *Let $A = (a_{ij})_{i,j\in I}$ be a generalised Cartan matrix of finite or affine type. The* ***Dynkin diagram*** *associated to $A$ is a graph with vertices labelled by $I$, and $\max\{|a_{ij}|, |a_{ji}|\}$ edges between vertices $i$ and $j$, with an arrow from vertex $i$ to $j$ if $|a_{ij}| > 1$ and $|a_{ij}| \geq |a_{ij}|$.*

From Proposition 2.1.28, we see that every pair of vertices is connected by no more than 3 sets of lines, unless $A$ is of type $A_2^{(2)}$ (or its dual). In the unique case of type $\widetilde{A}_1$, we have an edge which contains arrows going in both directions between two vertices, as pictured below in Figure 2.3.

**Example 2.1.30.** The Dynkin diagrams of type $A_n$ for $n \geq 1$, $\widetilde{A}_1$, and $\widetilde{A}_n$ for $n \geq 2$ are pictured in Figure 2.3.

The removal of a simple root from our system corresponds graphically to the removal of a vertex of the Dynkin diagram. For type $\widetilde{A}_n$, we see that the removal of any simple root yields the respective finite type root system $A_n$.

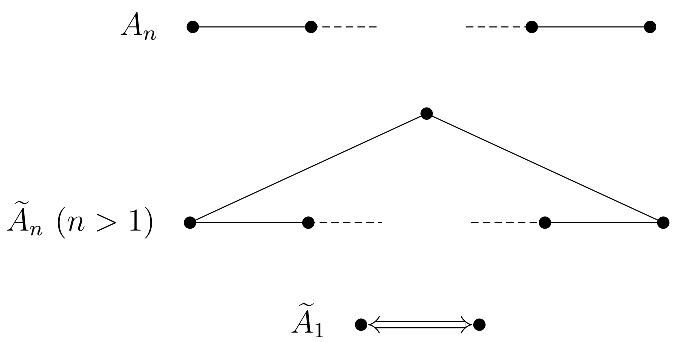


Figure 2.3: The Dynkin diagrams of types $A_n$ for $n \geq 1$, $\widetilde{A}_n$ for $n > 1$, and $\widetilde{A}_1$ respectively.

We will also be interested in considering affinisations of Kac-Moody root systems. In order to obtain these, we can apply an analagous approach to our construction of an affine root system from a finite one.

**Definition 2.1.31.** *Let $\Phi = \Phi^+ \sqcup \Phi^-$ be a Kac-Moody root system. Then the* ***Kac-Moody affine roots*** *associated to $\Phi$ are:*

$$\Phi_{\text{aff}} = \Phi \oplus \mathbb{Z}\pi = \{\hat{\alpha} = \tilde{\alpha} + m\pi \mid \tilde{\alpha} \in \Phi, m \in \mathbb{Z}\}.$$

If $G$ is itself of affine type, then we refer to $\Phi_{\text{aff}}$ as the set of *double affine roots*. In this case, we use the notation $\delta$ and $n$ when referring to a root of $G$, with the symbols $\pi$ and $m$ used to refer to the double affine components. In particular, a double affine root takes the following equivalent forms:

$$\hat{\alpha} = \tilde{\alpha} + m\pi = \alpha + n\delta + m\pi.$$

As before, we can split up the set of Kac-Moody affine roots into two disjoint positive and negative components.

**Definition 2.1.32.** *The set of* ***positive Kac-Moody affine roots*** *is:*

$$\Phi^+_{\text{aff}} := \{\hat{\alpha} = \operatorname{sgn}(m)(\alpha + m\pi) \mid \alpha \in \Phi^+, m \in \mathbb{Z}\}.$$

*The corresponding set of* ***negative Kac-Moody affine roots*** *is $\Phi^-_{\text{aff}} = -\Phi^+_{\text{aff}}$.*

For the rest of this thesis, we use the Kac-Moody root system, and refer to 'the Weyl group' as the Weyl group $W$ associated to the Kac-Moody group $G$. In particular, $W$ may be of either finite or affine type unless specified.

## § 2.2 Coxeter Groups

In order to better understand the structure of the Weyl group, we can consider the Coxeter structure that they hold. Finite and affine Weyl groups have a very rich Coxeter theory surrounding them; our ultimate goal is to develop a Coxeter-like structure for the double affine Weyl semigroup in order to better understand both the (semi)groups themselves, as well as the corresponding Hecke algebras.

### § 2.2.1 General Coxeter Theory

**Definition 2.2.1.** *A* ***Coxeter system*** *is a pair* $(W, S)$*, where* $S = \{s_i \mid i \in I\}$ *is a (finite) generating set for* $W$*, such that* $(s_i s_j)^{m_{i,j}} = e$*, where* $m_{i,j} = m_{j,i} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ *satisfies:*

(i) $m_{i,i} = 1$ *for all* $i \in I$*. That is,* $s_i^2 = e$*.*

(ii) $m_{i,j} \geq 2$ *for all* $i \neq j \in I$*.*

(iii) $m_{i,j} = \infty$ *if and only if* $s_i$ *and* $s_j$ *have no relation between them.*

We refer to $S$ as the set of *simple reflections.* We call $W$ a *Coxeter group* if there exists a generating set $S \subset W$ for which $(W, S)$ is a Coxeter system, in which case we call $|S|$ the *rank* of $W$. When $2 \leq m_{i,j} < \infty$, we refer to the corresponding relation $(s_i s_j)^{m_{i,j}} = e$ as a *braid relation.*

**Proposition 2.2.2.** *Every Weyl group arising from a Kac-Moody root system is a Coxeter group.*

Given a generalised Cartan matrix $A = (a_{ij})$, we can recover the values of $m_{i,j}$ as

$$a_{ij}a_{ji} = 4\cos^2\left(\frac{\pi}{m_{i,j}}\right).$$

In particular, since $0 \leq a_{ij}a_{ji} \leq 4$, the possible values of $m_{i,j}$ for a Weyl group are $2, 3, 4, 6$, and $\infty$, where by convention we set $\cos(\frac{\pi}{\infty}) = 1$. In the literature, Coxeter groups such that $m_{i,j} \in \{2, 3, 4, 6, \infty\}$ are sometimes referred to as *crystallographic Coxeter groups.*

**Example 2.2.3.** The Coxeter group of type $I_2(n)$ is a rank 2 group with generating set $S = \{s_1, s_2\}$ and relations $s_1^2 = s_2^2 = (s_1 s_2)^n = e$, resulting in the Weyl group $W \cong D_n$, the group of symmetries of an $n$-gon. In particular, $W$ is not a Weyl group unless $n \in \{2, 3, 4, 6, \infty\}$.

**Example 2.2.4.** The Coxeter group of type $A_n$ has $m_{i,i} = 1$ for all $i$, $m_{i,i+1} = 3$ for $i \in \{1, ..., n-1\}$, and $m_{i,j} = 2$ otherwise. We can identify the simple reflections $s_i$ with the transposition $(i\ i+1)$ to recover $W \cong S_{n+1}$, as we have previously seen for $n = 1$ and 2. The Coxeter group for $\widetilde{A}_n$ introduces the affine simple reflection $s_0$ which for $n > 1$ braids with both $s_1$ and $s_n$ in the same manner, i.e. $m_{0,1} = m_{0,n} = 3$ and $m_{0,i} = 2$ otherwise.

For each Coxeter group $W$, we can associate a (not necessarily crystallographic) root system on which it acts. If $W$ is a Weyl group, this root system is naturally the Kac-Moody root system. For the rest of this section, we fix a Coxeter group $W$ with root system $\Phi$.

**Theorem 2.2.5** ([Mat64])**.** *Let $\mathbf{w} = (s_{i_1}, s_{i_2}, ..., s_{i_m})$ and $\mathbf{w}' = (s_{j_1}, s_{j_2}, ..., s_{j_{m'}})$, for some non-negative integers $m, m'$. Then $s_{i_1}s_{i_2}\cdots s_{i_m} = s_{j_1}s_{j_2}\cdots s_{j_{m'}}$ if and only if $\mathbf{w}'$ and $\mathbf{w}$ can be obtained from each other by applying braid relations and relations of the form $s^2 = e$.*

**Definition 2.2.6.** *Let $w \in W$. A **reduced expression** $w$ is a list of simple reflections $\mathbf{w} = (s_{i_1}, s_{i_2}, ..., s_{i_m})$ such that $w = s_{i_1}s_{i_2}...s_{i_m}$, with $m$ the minimal possible. The **length** of $w$ is then $\ell(w) = m$.*

We refer to $\ell : W \to \mathbb{Z}_{\geq 0}$ as the *length function* on $W$. If $W$ is finite, then it contains a unique longest element, typically denoted $w_0$ in the literature, otherwise the length function is unbounded. The following definition and proposition give a useful geometric description of the length function.

**Definition 2.2.7.** *Let $w \in W$. Then the **inversion set** of $w$ is the set of positive roots which become negative under the action of $w$. That is,* $\mathrm{Inv}(w) = \{\alpha > 0 \mid w\alpha < 0\}$.

**Proposition 2.2.8.** *For $w \in W$, $\ell(w) = |\mathrm{Inv}(w)|$.*

The longest element $w_0$ (if it exists) is then the unique element such that the inversion set is the entire set of positive roots, i.e. $\mathrm{Inv}(w_0) = \Phi^+$. We now give a partial ordering of the Weyl group.

**Definition 2.2.9.** *Let $u, v \in W$, and fixed a reduced expression $\mathbf{v}$ of $v$. The **(strong) Bruhat order** on $W$ is given by the relations $u \leq v$ whenever there exists a substring of $\mathbf{v}$ which is a reduced expression for $u$.*

The following proposition gives an alternative definition for the Bruhat order in terms of reflections and the length function.

**Proposition 2.2.10.** *The Bruhat order is generated by the following relations for all $w \in W$ and $\alpha \in \Phi^+$:*

(i) *$ws_\alpha > w$ if $\ell(ws_\alpha) > \ell(w)$, in which case $w\alpha > 0$.*

(ii) *$ws_\alpha < w$ if $\ell(ws_\alpha) < \ell(w)$, in which case $w\alpha < 0$.*

We say that $v \in W$ *covers* $u \in W$ if $u < v$ and if there exists no $w \in W$ such that $u < w < v$. The length function then grades the Bruhat order, that is, $v$ covers $u$ if and only if $v = us_\alpha$ for some $\alpha \in \Phi^+$ such that $\ell(v) = \ell(u) + 1$.

We can then construct the *Bruhat graph* as the Hasse diagram of the Bruhat order. Namely, we have an edge $w \to ws_\alpha$ if $\ell(ws_\alpha) = \ell(w) + 1$.

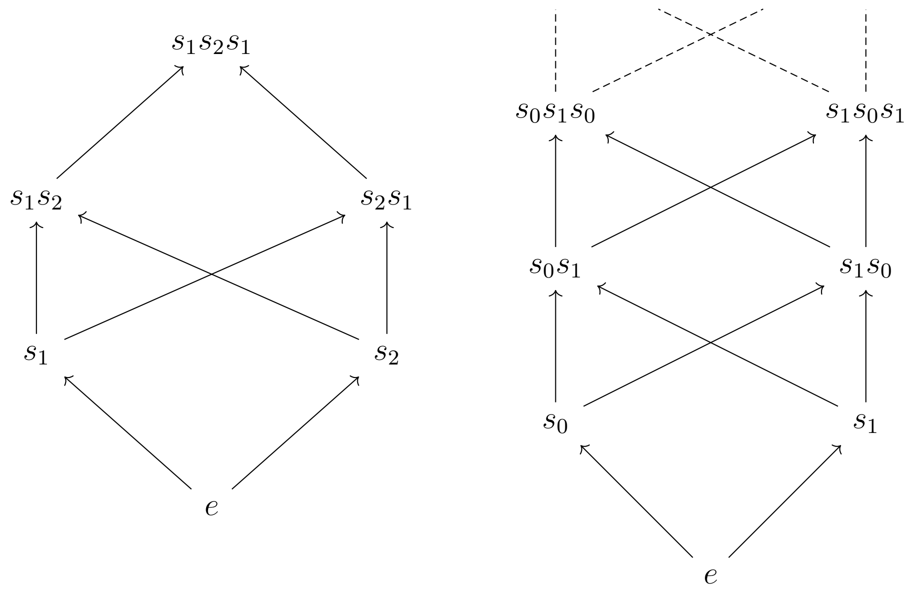


Figure 2.4: The Bruhat graphs for $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = e\rangle$ of type $A_2$, and $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = e\rangle$ of type $\widetilde{A}_1$ respectively.

**Example 2.2.11.** The Bruhat graph of types $A_2$ and $\widetilde{A}_1$ are shown in Figure 2.4.

### § 2.2.2 Demazure Products

We now turn to our main object of study for Coxeter groups.

**Definition 2.2.12** ([He09, Lemma 1])**.** *The* ***Demazure product*** *of $u, v \in W$ is defined as the common maximum of the following sets with respect to the Bruhat order:*

$$u * v := \max\{u'v \mid u' \leq u\} = \max\{uv' \mid v' \leq v\} = \max\{u'v' \mid u' \leq u, v' \leq v\}.$$

It is a non trivial statement that these sets have a unique maximum, as well as having a common maximum. Following this notation and writing $u * v = u'v = uv'$, we have that:

$$\ell(u * v) = \ell(u') + \ell(v) = \ell(u) + \ell(v').$$

For $u, v \in W$, we say that the product $uv$ is *length additive* with respect to $u$ and $v$ if $u * v = uv$, or equivalently $\ell(uv) = \ell(u) + \ell(v)$. We say the *length deficit* of the product $uv$ is $\ell(uv) - \ell(u) - \ell(v)$. The Demazure product is very closely related to the covering relations of the Bruhat order; in particular, we have the following description.

**Proposition 2.2.13.** *The Demazure product on $W$ is the associative product generated by the following relation for $w \in W, s \in S$:*

$$w * s = \begin{cases} ws \text{ if } \ell(ws) > \ell(w), \\ w \quad \text{if } \ell(ws) < \ell(w). \end{cases}$$

Taking this instead as the *definition*, it can be seen how the Demazure product explicitly constructs the 'largest' possible element out of its inputs.

**Example 2.2.14.** Let $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1\rangle$ be the Coxeter group of type $\widetilde{A}_1$, and let $u = s_0s_1, v = s_1s_0s_1 \in W$. We can calculate their Demazure product iteratively using Proposition 2.2.13:

$$\begin{aligned} u * v \;&=\; s_0s_1 * s_1s_0s_1 = s_0s_1 * s_1 * s_0 * s_1 \\ &= s_0s_1 * s_0 * s_1 \\ &= s_0s_1s_0 * s_1 \;=\; s_0s_1s_0s_1. \end{aligned}$$

As we discuss in the following section, the Demazure product is a useful tool for studying multiplication in the Hecke algebra. In particular, the 'leading term' in the multiplication of two basis elements is determined by the Demazure product of their indices.

## § 2.3 Hecke Algebras

In order to understand the nature of the double affine Weyl semigroup, we build up a picture of the construction of various types of Hecke algebra, leading to the Kac-Moody affine Hecke algebra from which the double affine Weyl semigroup arises. For these constructions, we follow the standard Lie-theoretic terminology (see e.g. [Mil17] for a reference). We begin by constructing the finite Hecke algebra, followed by its affinisation. Then we consider the Kac-Moody affine Hecke algebra by applying an analogous process to a Hecke algebra arising from a general Kac-Moody group $G$.

### § 2.3.1 Finite and Affine Hecke Algebras

Fix a field $k$ and let $G$ be a (finite dimensional) reductive Lie group over $k$. Define a *Borel subgroup* of $G$ to be a maximal closed, connected, and solvable algebraic subgroup of $G$, and fix a Borel subgroup $B$. Let $W$ be the Weyl group of $G$.

**Proposition 2.3.1.** *The* ***Bruhat decomposition*** *of $G$ gives a bijection between the double cosets of $G$ with respect to $B$ and the Weyl group, i.e. $B\backslash G/B \longleftrightarrow W$.*

**Example 2.3.2.** Let $G = \mathrm{SL}_n(\mathbb{C})$, the Lie group of type $A_{n+1}$. We can choose the Borel subgroup $B$ to be the subgroup of upper triangular matrices of $G$, giving the permutation matrices as a possible set of double coset representatives for $B\backslash G/B$, which clearly bijects with the Weyl group $W = S_n$.

**Definition 2.3.3.** *The* ***Hecke algebra*** *$\mathcal{H}$ of the Lie group $G$ consists of the set of complex-valued $B$-biinvariant functions of $G$, with multiplication given by convolution.*

Note that we can equivalently view $\mathcal{H}$ as the set of functions $f : B\backslash G/B \to \mathbb{C}$ under convolution. We can construct a basis for $\mathcal{H}$ by considering the indicator functions $\mathbb{I}_w$, such that $\mathbb{I}_w(BgB) = 1$ if $BgB \longleftrightarrow w$ under the bijection given by the Bruhat decomposition, and $\mathbb{I}_w(BgB) = 0$ otherwise.

We refer to $\mathcal{H}$ as being the *finite* Hecke algebra as its underlying Weyl group is finite, but note that since $\mathcal{H}$ is a $k$-vector space, it itself is (typically) not finite. The following alternative construction is typically easier to work with.

**Proposition 2.3.4.** *$\mathcal{H}$ is isomorphic to the associative $k$-algebra with basis $\{T_w \mid w \in W\}$ generated by the relations:*

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

*where $q \in k$ is a generic parameter.*

The isomorphism with the Hecke algebra as defined above can be made explicit by identifying the indicator function $\mathbb{I}_w$ with the element $T_w$.

From this, we see that the Hecke algebra is essentially a deformation of the group algebra of the Weyl group $k[W]$. In particular, setting $q = 1$ gives $\mathcal{H} \cong k[W]$. We are also interested in the specialisation $q = 0$, where the relations of the basis elements align (up to a sign) with our relations for the Demazure product from Proposition 2.2.13, and every product of basis elements returns only one term. Explicitly, we have the following proposition.

**Proposition 2.3.5.** *Let $x, y \in W$. Then:*

$$T_xT_y \equiv (-1)^{\ell(x)+\ell(y)-\ell(x*y)}T_{x*y} \mod q. \tag{2.2}$$

Note that we could instead view this as simply being multiplication in the $q = 0$ specialisation of the Hecke algebra, however this proposition demonstrates how the Demazure product controls the aforementioned 'leading term' of the product of two basis elements.

We now turn our sights on affinising the Hecke algebra. In order to do this, we consider an alternative to the Bruhat decomposition. Let $L = k((t))$ be the field of formal Laurent series in $t$, and let $\mathcal{O} = k[[t]]$ be its ring of integers. We define the *Iwahori subgroup* $I$ of $G$ to be $I = \{g \in G(\mathcal{O}) \mid g \in B(k) \mod t\}$.

**Proposition 2.3.6.** *The* ***Iwahori decomposition*** *of $G$ gives a bijection between the double cosets of $G(L)$ with respect to $I$ and the affine Weyl group, i.e. $I\backslash G/I \longleftrightarrow W_{\text{aff}}$.*

We can then define a new Hecke algebra in the same manner as before.

**Definition 2.3.7.** *The* ***Iwahori-Hecke algebra*** *$\mathcal{H}_{\text{aff}}$ of $G$ consists of the set of complex-valued $I$-biinvariant functions on $G(L)$.*

Propositions 2.3.4 and 2.3.5 generalise to $\mathcal{H}_{\text{aff}}$. In particular, we have an infinite basis $\{T_w \mid w \in W_{\text{aff}}\}$.

Recall that we had two separate constructions for the affine Weyl group - one as the Weyl group associated to an affine root system, and one as the semi-direct product of the finite Weyl group with a translational lattice. We similarly have two separate constructions for a basis of the Hecke algebra. The first is the usual $T$-basis, however the semi-direct product becomes more apparent when we instead consider the following presentation.

**Proposition 2.3.8** ([Lus89]). *The Iwahori-Hecke algebra is isomorphic to the algebra generated by $\{T_w \mid w \in W\}$ and $\{\Theta^\mu \mid \mu \in P^\vee\}$, where the $T_w$ are subject to the relations in Proposition 2.3.4 along with:*

$$\Theta^{\mu_1}\Theta^{\mu_2} = \Theta^{\mu_1+\mu_2}, \tag{2.3}$$

$$T_s\Theta^\mu - \Theta^{s\mu}T_s = (q-1)\frac{\Theta^\mu - \Theta^{s\mu}}{1-\Theta^{-\alpha_s^\vee}}, \tag{2.4}$$

*where $s \in S$ is a simple reflection with associated root $\alpha_s$.*

This alternative construction of $\mathcal{H}_{\text{aff}}$ is referred to as the *Bernstein presentation*. Equation (2.3) demonstrates the embedding of the lattice $P^\vee$ in $\mathcal{H}_{\text{aff}}$, whereas Equation (2.4) shows how $\mathcal{H}_{\text{aff}}$ functions as a deformation of the group algebra of the affine Weyl group.

### § 2.3.2 Kac-Moody Affine Hecke Algebras

To define the Kac-Moody affine Hecke algebra, we want to apply an Iwahori-like construction to a Kac-Moody group $G$ rather than a Lie group. If $G$ is of finite type, then the construction works as before. However, if $G$ is of affine type, then we have to modify the construction.

**Proposition 2.3.9.** *Let $G$ be a Kac-Moody group of (untwisted) affine type over the field $L$. Then the Iwahori decomposition applies only to a sub-semigroup $G^+$ of $G$, for which we obtain the bijection $I\backslash G^+/I \longleftrightarrow W_{\mathcal{T}}$.*

Here, $W_{\mathcal{T}}$ is the double affine Weyl semigroup that will be defined in section 2.4. Using this sub-semigroup instead of the full group, we are finally able to construct our algebra of interest.

**Definition 2.3.10.** *The* ***Kac-Moody affine Hecke algebra*** *$\widehat{\mathcal{H}}$ is the algebra of $I$-biinvariant complex-valued functions on $G^+$ which have finite support.*

The Kac-Moody affine Hecke algebra is essentially a positive version of Cherednik's double affine Hecke algebra [Che95], as shown in [BKP16]. Namely, as discussed in

Section 2.4, the Kac-Moody affine Hecke algebra is defined by using the Tits cone, a subset of the (affine) coroot lattice, and as such we obtain a subalgebra of Cherednik's double affine Hecke algebra.

The Iwahori decomposition explicitly gives $\widehat{\mathcal{H}}$ a $T$-basis $\{T_x \mid x \in W_{\mathcal{T}}\}$, however it is useful to note that $\widehat{\mathcal{H}}$ also has a Bernstein-type presentation, generated by the elements $\{T_w \mid w \in W_{\text{aff}}\}$ and $\{\Theta^\mu \mid \mu \in P^\vee_{\text{aff}}\}$. Each presentation is subject to the same relations as before, that is Proposition 2.3.4 for the $T$-basis and Proposition 2.3.8 for the Bernstein-type presentation.

## § 2.4 Double Affine Weyl Semigroups

Throughout this section, fix an (untwisted) affine Kac-Moody group $G$ with Weyl group $W$, which will be itself of affine type. Let $P^\vee$ be the coweight lattice of $G$, also now of affine type. Henceforth, we write $\Phi$ for the set of roots of $G$, and $\Phi_{\text{fin}}$ for the corresponding finite-type root system. We typically use the notation $\alpha \in \Phi_{\text{fin}}$ and $\tilde{\alpha} \in \Phi$. We say a coweight $\mu \in P^\vee$ is *dominant* if $\langle \mu, \tilde{\alpha} \rangle \geq 0$ for every simple root $\tilde{\alpha} \in \Delta$, and write $P^\vee_{\text{dom}}$ for the set of dominant coweights.

Following [Wel19], we define the set $X = \mathbb{Z}\Phi^\vee_{\text{fin}} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0 \subseteq P^\vee = P^\vee_{\text{fin}} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$, where $P^\vee_{\text{fin}}$ is the coweight lattice of the corresponding finite type Kac-Moody group. Let $X_{\text{dom}}$ be the set of dominant coweights of $X$.

The action of $W$ on $P^\vee$ arises by generalising the action of $w_o\tau^\lambda \in W_{\text{aff}}$ on $\Phi_{\text{aff}}$ given in Eqn. 2.1 [Kac90], namely:

$$w_o\tau^\lambda(\mu_o + m\delta + l\Lambda_0) := w_o\mu_o + lw_o\lambda + \left(m - \langle \mu_o, \lambda^\vee \rangle - \frac{l}{2}\langle \lambda, \lambda^\vee \rangle\right)\delta + l\Lambda_0, \quad (2.5)$$

where $\lambda^\vee$ is the weight associated to the coweight $\lambda$.

With this action, we define the following.

**Definition 2.4.1.** *The* ***Tits cone*** *is* $\mathcal{T} := W(X_{\text{dom}})$*.*

That is, $\mathcal{T}$ is the set of $W$-translates of this subset of the dominant coweights.

*Remark* 2.4.2. We could alternatively define $\mathcal{T} = W(P^\vee_{\text{dom}})$ to obtain a larger cone, in which case we obtain what could be thought of as an *extended* double affine Weyl semigroup. For our calculations, we fix the definition $\mathcal{T} = W(X_{\text{dom}})$ as above, but our results in Chapter 4 also apply in the larger 'extended' Tits cone $\mathcal{T} = W(P^\vee_{\text{dom}})$. In the literature, both of these are referred to as *the* Tits cone.

In the finite case, the Tits cone is simply the full set $X$, however in the affine case, the Tits cone is instead a strict subset. On the level of the Hecke algebra, this can be compared to how the Iwahori decomposition only applies to the sub-semigroup $G^+$.

**Definition 2.4.3.** *The* ***double affine Weyl semigroup*** *is* $W_{\mathcal{T}} = W \ltimes \mathcal{T}$*.*

Note that we obtain only a semigroup rather than a group, since if $\mu \in \mathcal{T}$ is away from the boundary of the Tits cone (see Remark 2.4.7 below), then $-\mu \notin \mathcal{T}$, and so $x \in W_{\mathcal{T}}$ is not guaranteed to have an inverse.

*Remark* 2.4.4. In some cases, we will reduce further by using the realisation $W \cong W_{\text{fin}} \ltimes \mathbb{Z}\Phi_{\text{fin}}^\vee$, where $W_{\text{fin}}$ and $\Phi_{\text{fin}}$ are the corresponding finite Weyl group and set of roots respectively. To avoid ambiguity when working explicitly in a double affine setting, we use $\varepsilon$ for the double affine component, and $\tau$ for the single affine component. That is, we write $w = w_o\tau^\lambda \in W$ for some $w_o \in W_{\text{fin}}$, $\lambda \in \mathbb{Z}\Phi^\vee$, and $x = w\varepsilon^\mu = w_o\tau^\lambda\varepsilon^\mu \in W_{\mathcal{T}}$ for some $\mu \in \mathcal{T}$.

*Remark* 2.4.5. Throughout the literature, including Muthiah and Puskás' paper [MP26] as well as Muthiah and Orr's paper [MO19], the convention used differs slightly. They write $x = w\pi^\mu \in W_{\mathcal{T}}$, where effectively $\pi^\mu = \varepsilon^{-\mu}$, meaning that these authors are working within the anti-dominant Tits cone $-\mathcal{T}$ from our perspective, and vice-versa (see [MP26, Equation 2.9]). We fix our convention to match that of Schremmer's [Sch24], and so some definitions and results in this thesis will differ slightly from those in the relevant work from which they are derived, as they have been converted to our notation.

We have the relation $w\varepsilon^\mu = \varepsilon^{w\mu}w$ in $W_{\mathcal{T}}$, which we can use to simplify products of elements of $W_{\mathcal{T}}$ when performing calculations. We have an explicit description of the Tits cone as follows.

**Proposition 2.4.6** ([Kac90])**.** *The Tits cone for affine type $G$ is:*

$$\mathcal{T} = \{\mu_o + m\delta + l\Lambda_0 \mid \mu_o \in \mathbb{Z}\Phi_{\text{fin}}^\vee, m \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 1}\} \cup \mathbb{Z}\delta. \tag{2.6}$$

*Remark* 2.4.7. Let $\mu \in \mathcal{T}$, and write $\mu = \mu_o + m\delta + l\Lambda_0$ as above. We call $l$ the *level* of $\mu$. We refer to elements in $W_{\mathcal{T}}$ of level 0 (which must take the form $w\varepsilon^{m\delta}$) as lying on the *boundary* of the Tits cone.

Despite $W_{\mathcal{T}}$ not being a Coxeter group for affine type $G$, we have the notion of a length function and a Bruhat order.

**Definition 2.4.8** ([MO19])**.** *Let $x = w\varepsilon^\mu \in W_{\mathcal{T}}$, Let $\mu^+$ be the unique dominant $W$-translate of $\mu$. Then the **length** of $x$ is:*

$$\ell(w\varepsilon^\mu) = 2\text{ht}(\mu^+) + |\{\tilde{\alpha} \in \text{Inv}(w) \mid \langle \mu, \tilde{\alpha}\rangle \geq 0\}| - |\{\tilde{\alpha} \in \text{Inv}(w) \mid \langle \mu, \tilde{\alpha}\rangle < 0\}|.$$

In particular, we see that $\ell(\varepsilon^\mu) = 2\text{ht}(\mu^+)$. Note that the sizes of the two sets in the formula must be finite, as $|\text{Inv}(w)| = \ell(w) < \infty$ for any $w \in W$, however $|\text{Inv}(w)|$ can take arbitrarily large values. In particular, the length function on $W_{\mathcal{T}}$ can take both positive and negative values of arbitrary size, and can also take rational values, however in our setting it will only take integer values. Also note that this definition can be applied to the single-affine case.

*Remark* 2.4.9. When computing lengths, determining the value of $\mathrm{ht}(\mu^+)$ involves a choice. In particular, we have that $\mathrm{ht}(\mu) = \langle \mu, \rho \rangle$, where $\rho$ is the sum of the fundamental weights. We thus have to choose how to define the pairing of the fundamental coweight $\Lambda_0$ with its dual. For the purposes of calculations, we fix this to be 0, but remark that the *differences* of lengths such as $\ell(xy) - \ell(x) - \ell(y)$ are independent of this choice.

**Example 2.4.10.** Let $G$ be the Kac-Moody group of type affine $\mathrm{SL}_2$, denoted $\widehat{\mathrm{SL}}_2$. The Weyl group $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 \rangle$ is the infinite dihedral group, and the Tits cone is

$$\mathcal{T} = \{k\alpha^\vee + m\delta + l\Lambda_0 \mid k \in \mathbb{Z}_{\neq 0}, m \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 1}\} \cup \mathbb{Z}\delta. \tag{2.7}$$

The double affine Weyl semigroup of type $\mathrm{SL}_2$ (or equivalently the Kac-Moody affine Weyl semigroup of type $\widehat{\mathrm{SL}}_2$) is then $W_{\mathcal{T}} = W \ltimes \mathcal{T}$. Consider $x = s_0 s_1 \varepsilon^{-\alpha^\vee + \Lambda_0}$ and $y = \varepsilon^{4\delta + \Lambda_0} \in W_{\mathcal{T}}$, following an example discussed in [MP26, Section 6.1], which we will return to in future chapters. We compute $xy$ and $yx$ as follows, using $s_0 s_1 = \tau^{\alpha^\vee}$:

$$\begin{aligned} xy &= s_0 s_1 \varepsilon^{-\alpha^\vee + \Lambda_0} \varepsilon^{4\delta + \Lambda_0} = s_0 s_1 \varepsilon^{-\alpha^\vee + 4\delta + 2\Lambda_0}, \\ yx &= \varepsilon^{4\delta + \Lambda_0} s_0 s_1 \varepsilon^{-\alpha^\vee + \Lambda_0} \\ &= s_0 s_1 \varepsilon^{s_1 s_0 (4\delta + \Lambda_0)} \varepsilon^{-\alpha^\vee + \Lambda_0} \\ &= s_0 s_1 \varepsilon^{\tau^{-\alpha^\vee}(4\delta + \Lambda_0) - \alpha^\vee + \Lambda_0} \\ &= s_0 s_1 \varepsilon^{-\alpha^\vee + 3\delta + \Lambda_0 - \alpha^\vee + \Lambda_0} = s_0 s_1 \varepsilon^{-2\alpha^\vee + 3\delta + 2\Lambda_0}. \end{aligned}$$

We also compute the length of $x$. Note that the unique dominant $W$-translate of $\mu_x = -\alpha^\vee + \Lambda_0$ is $\mu_x^+ = s_1 \tau^{\alpha^\vee}(-\alpha^\vee + \Lambda_0) = \delta + \Lambda_0$, since $\langle \delta + \Lambda_0, \alpha \rangle = 0$ and $\langle \delta + \Lambda_0, -\alpha + \delta \rangle = 1$, and hence $\langle \mu_x^+, \tilde{\alpha} \rangle \geq 0$ for all simple $\tilde{\alpha}$. We also have that $\mathrm{Inv}(w_x) = \mathrm{Inv}(s_0 s_1) = \{\alpha, \alpha + \delta\}$, with $\langle \mu_x, \alpha \rangle = -2$ and $\langle \mu_x, \alpha + \delta \rangle = -1$. Hence:

$$\begin{aligned} \ell(x) &= 2\mathrm{ht}(\delta + \Lambda_0) + 0 - 2 \\ &= 2\langle \delta + \Lambda_0, \rho \rangle - 2 \\ &= 2\langle \tilde{\alpha}_0 + \tilde{\alpha}_1 + \Lambda_0, \Lambda_0 + \Lambda_1 \rangle - 2 \\ &= 2(1 + 1 + 0) - 2 \ = 2. \end{aligned}$$

We now turn our attention to defining the Bruhat order on $W_{\mathcal{T}}$. Recall that the double affine roots of an affine root system $\Phi$ are $\Phi_{\mathrm{aff}} = \{\hat{\alpha} = \tilde{\alpha} + m\pi \mid \tilde{\alpha} \in \Phi, m \in \mathbb{Z}\}$. We have an action of $W_{\mathcal{T}}$ on $\Phi_{\mathrm{aff}}$ by generalising Equation (2.1):

$$w\varepsilon^\mu(\tilde{\alpha} + m\pi) = w(\tilde{\alpha}) + (m - \langle \mu, \tilde{\alpha} \rangle)\pi. \tag{2.8}$$

**Definition 2.4.11.** *The* ***double affine reflection*** $s_{\hat{\alpha}}$ *associated to a positive double affine root* $\hat{\alpha} = \tilde{\alpha} + m\pi$ *is:*

$$s_{\hat{\alpha}} = s_{\tilde{\alpha} + m\pi} = s_{\tilde{\alpha}} \varepsilon^{m\tilde{\alpha}^\vee}.$$

Note that these reflections do not generate $W_{\mathcal{T}}$, and in particular we have no sense of a *simple* reflection. However, we can use reflections to compare elements in the Bruhat order. We write $\hat{\alpha} > 0$ (respectively $\hat{\alpha} < 0$) to mean $\hat{\alpha} \in \Phi^+_{\mathrm{aff}}$ (respectively $\hat{\alpha} \in \Phi^-_{\mathrm{aff}}$).

**Definition 2.4.12.** *The **Bruhat order** for $W_{\mathcal{T}}$ is the partial order generated by the following relations for $x \in W_{\mathcal{T}}$ and $\hat{\alpha} \in \Phi_{\mathrm{aff}}$:*

$$xs_{\hat{\alpha}} > x \text{ if } x(\hat{\alpha}) > 0,$$
$$xs_{\hat{\alpha}} < x \text{ if } x(\hat{\alpha}) < 0.$$

As before, for $x, y \in W_{\mathcal{T}}$, we say $y$ *covers* $x$ if $x < y$ and if there is no $z \in W_{\mathcal{T}}$ such that $x < z < y$. The following result, first conjectured by Muthiah and Orr [MO19] and proved by Welch in type ADE [Wel19], links the length function and Bruhat order directly.

**Proposition 2.4.13** ([Phi24])**.** *The length function for $W_{\mathcal{T}}$ is a $\mathbb{Z}$-grading of the Bruhat order. In particular, if $x, y \in W_{\mathcal{T}}$ such that $x \leq y$, then $y$ covers $x$ if and only if $\ell(y) = \ell(x) + 1$.*

## § 2.5 Quantum Bruhat Graphs

Our goal is to define a double affine Demazure product for $W_{\mathcal{T}}$ and link it directly to the length function in a similar way to the Bruhat order. In order to do so, we must first define the quantum Bruhat graph, and link it to the (single) affine Demazure product [Sch24].

### § 2.5.1 Definition and Properties

The quantum Bruhat graph was initially defined by Brenti, Fomin and Postnikov [BFP98, Sec. 6] for a finite Weyl group, and generalised to affine Weyl groups by Mare and Mihalcea [MM18, Sec. 8.2]. For our purposes, we define it for a general Weyl group $W$ of either finite or affine type.

**Definition 2.5.1.** *The **quantum Bruhat graph** for $W$, denoted $\mathrm{QBG}(W)$, is the directed, weighted graph with vertex set $W$, with weights in $\mathbb{Z}\Phi^\vee$, and an edge $w \Rightarrow ws_\alpha$ for $\alpha \in \Phi^+$ if one of the following conditions is satisfied:*

(i) $\ell(ws_\alpha) = \ell(w) + 1$.

(ii) $\ell(ws_\alpha) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho \rangle$.

*Edges of type* (i) *are called **Bruhat edges** and have weight* $0$*. Edges of type* (ii) *are called **quantum edges** and have weight* $\alpha^\vee$*.*

Note that Bruhat edges correspond to covers in the (strong) Bruhat order. If $W$ is of finite or affine type, we refer to its quantum Bruhat graph as either finite or affine. We refer to Bruhat edges as "upward" and quantum edges as "downward", due to the length difference between the two relevant vertices. A root $\alpha$ for which there exists $w \in W$ such that $w \Rightarrow ws_\alpha$ is a quantum edge is called a *quantum root*, which have been studied in detail and classified by Hébert and Philippe [HP24]. We also define the following:

**Definition 2.5.2.** *Let $u, v \in W$.*

(i) *A **path** $p$ in* QBG$(W)$ *from $u$ to $v$ is a sequence of consecutive edges $(u, us_{\beta_1}, us_{\beta_1}s_{\beta_2}, ..., v)$.*

(ii) *The **length** of a path $p$ is the number of edges it contains.*

(iii) *The **weight** of a path $p$ is the sum of the weights along its edges.*

**Proposition 2.5.3.** *For $w_1, w_2 \in W$, there exists a path $w_1 \Rightarrow w_2$.*

*Proof.* Let $w \neq e \in W$ and consider a simple reflection $s$ such that $ws < w$. Then $ws \Rightarrow w$ is a Bruhat edge, and $w \Rightarrow ws$ is a quantum edge. Iterating this, we find paths $w \Rightarrow e$ and $e \Rightarrow w$. Concatenating these paths for $w_1$ and $w_2$ respectively yields a path $w_1 \Rightarrow w_2$. □

Given this result, we can make the following definition.

**Definition 2.5.4.** *The **distance function** $d : W \times W \to \mathbb{Z}_{\geq 0}$, denoted $d(u \Rightarrow v)$, is the length of a shortest path from $u$ to $v$ in* QBG$(W)$.

Note that $w_1 \leq w_2$ in the Bruhat order if and only if there exists a weight 0 path between them. In the case that $W$ is finite, we also have the following:

**Proposition 2.5.5.** [Pos05, Lemma 1] *Assume $W$ is finite, and let $w_1, w_2 \in W$. Then all shortest paths from $w_1$ to $w_2$ have the same weight, denoted* $\mathrm{wt}(w_1 \Rightarrow w_2)$.

Postnikov's proof of Proposition 2.5.5 relies on [BFP98, Section 6], where it is shown that any two shortest paths $w_1 \Rightarrow w_2$ can be obtained from one another by applying weight-preserving switches of pairs of edges. In particular, the following proposition, known as the *Diamond lemma*, is crucial to the proof.

**Proposition 2.5.6.** *Let $W$ be a finite type Weyl group. Let $u, v \in W$ such that $u \Rightarrow us_{\alpha_1} \Rightarrow us_{\alpha_1}s_{\alpha_2} = v$ is a length 2 path in* QBG$(W)$, *with $\alpha_1, \alpha_2 \in \Phi^+, \alpha_1 \neq \alpha_2$. Then there exists a unique length 2 path $u \Rightarrow us_{\beta_1} \Rightarrow us_{\beta_1}s_{\beta_2} = v$ for $\beta_1, \beta_2 \in \Phi^+$ and $\alpha_i \neq \beta_i \in \Phi^+$.*

In other words, given two elements of QBG($W$) which are a distance 2 away, there exist exactly 2 paths between them. Postnikov then verifies that such paths have the same total weight, and that any two shortest paths can be obtained by performing suitable weight-preserving swaps of edges using the Diamond lemma. This does not hold in the affine case, as can be seen in Figure 2.6, where there exists exactly one path from $u = s_0 s_1$ to $v = e$ in type $\widehat{\mathrm{SL}}_2$. We conjecture the following.

**Conjecture 2.5.7.** *The Diamond lemma fails in an affine quantum Bruhat graph exactly when the weight of a length 2 path is non-zero and purely imaginary, in which case there exist no other length 2 paths with the same start and end points.*

At present, there is no well-defined weight function for general affine quantum Bruhat graphs. Note that Postnikov's proof also relied on a particular choice of reflection order on $W$, which does not necessarily exist in affine type, as shown by Welch through the use of a counterexample in type $\widehat{\mathrm{SL}}_2$.

**Example 2.5.8.** The quantum Bruhat graph of types $\mathrm{SL}_3$ and $\widehat{\mathrm{SL}}_2$ are shown in Figure 2.5 and Figure 2.6 respectively.

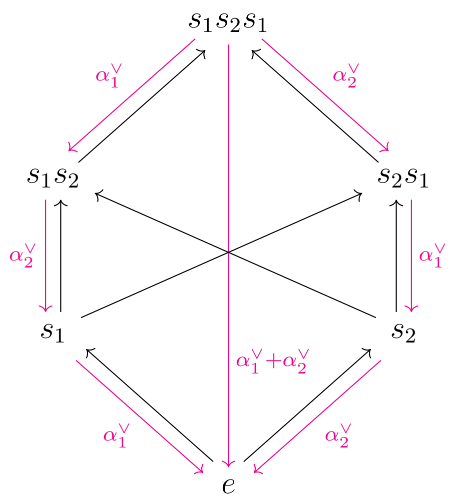


Figure 2.5: QBG($W$) for $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = e\rangle$, the Weyl group of type $\mathrm{SL}_3$, with root system $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Bruhat edges are in black, which each have weight 0 (unlabelled), and quantum edges are in magenta, labelled with their weight.

The quantum Bruhat graph carries many other properties that have uses within representation theory in general. For example, we have the notion of *quantum length* as defined in [LNS+15, Defn. 6.14], used to develop diamond lemmas for the *parabolic* quantum Bruhat graph, which we define here in the affine case for completeness.

**Definition 2.5.9.** *Let $w \in W$, of either finite or affine type. The* ***quantum length*** *of $w$ is $\ell_q(w) := d(w \Rightarrow e)$.*

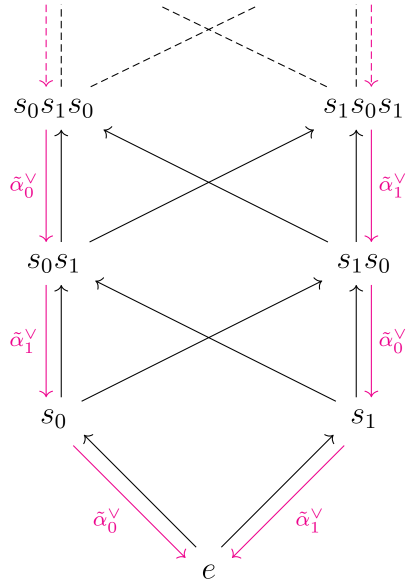


Figure 2.6: QBG($W$) for $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = e\rangle$, the Weyl group of type $\widehat{\mathrm{SL}}_2$, with root system $\Phi = \pm\{\alpha + n\delta \mid n \in \mathbb{Z}\}$, as calculated by [Wel19].

**Example 2.5.10.** In Figure 2.5, we see that in type $\mathrm{SL}_3$, $\ell_q(w) = \ell(w)$ for all $w \in W$ except $w = s_1s_2s_1$, in which case $\ell_q(s_1s_2s_2) = 1$.

Our hope is that the quantum length of an element could be used to perform induction with as part of a proof of the existence of a well-defined weight function for general type affine quantum Bruhat graphs.

### § 2.5.2 Double Affine Demazure Products

Schremmer's formula for the Demazure product relies on the notion of *length positivity* as well as the quantum Bruhat graph. For the following, we use the notation for the double-affine case, but note that the following is also defined when $G$ is of finite type. Write $\Phi^+(\cdot)$ to denote the indicator function for a positive root, that is:

$$\Phi^+(\tilde{\alpha}) = \begin{cases} 1 & \text{if } \tilde{\alpha} > 0, \\ 0 & \text{if } \tilde{\alpha} < 0. \end{cases}$$

**Definition 2.5.11.** *Let $x = w\varepsilon^\mu \in W_{\mathcal{T}}$.*

(i) *The* ***length functional*** *for $x$ for a given $\tilde{\alpha} \in \Phi$ is:*

$$\ell(x, \tilde{\alpha}) = \langle \mu, \tilde{\alpha}\rangle + \Phi^+(\tilde{\alpha}) - \Phi^+(w\tilde{\alpha}). \tag{2.9}$$

(ii) *$v \in W$ is* ***length positive*** *for $x$ if $\ell(x, v\tilde{\alpha}) \geq 0$ for every $\tilde{\alpha} \in \Phi^+$. We write* $\mathrm{LP}(x)$ *for the set of length positive elements for $x$.*

Given $x = w\varepsilon^{\mu} \in W_{\mathcal{T}}$, fix $u \in W$ such that $u\mu = \mu^+$, where $\mu^+$ is dominant. Then $u^{-1} \in \mathrm{LP}(x)$, since $u$ is determined uniquely by the condition that for every $\tilde{\alpha} \in \Phi^+$, $\langle u\mu, \tilde{\alpha}\rangle \geq \Phi^+(-u^{-1}\tilde{\alpha})$ [Sch23, Ex. 2.8]. In particular, since $\mu^+$ always exists, $\mathrm{LP}(x)$ is always non-empty.

**Example 2.5.12.** Fix $G$ of type $\widehat{\mathrm{SL}}_2$ and let $x = s_0 s_1 \varepsilon^{-\alpha^\vee + \Lambda_0} = \tau^{\alpha^\vee}\varepsilon^{-\alpha^\vee+\Lambda_0} \in W_{\mathcal{T}}$ as before. Also let $\tilde{\alpha} = \mathrm{sgn}(n)(\alpha + n\delta)$, and $u = s_1$. Then:

$$\begin{aligned}\ell(x, s_1\tilde{\alpha}) &= \langle -\alpha^\vee + \Lambda_0, s_1\tilde{\alpha}\rangle + \Phi^+(s_1\tilde{\alpha}) - \Phi^+(s_0\tilde{\alpha})\\ &= \mathrm{sgn}(n)\langle -\alpha^\vee + \Lambda_0, -\alpha + n\delta\rangle + \Phi^+(s_1\tilde{\alpha}) - \Phi^+(s_0\tilde{\alpha})\\ &= \mathrm{sgn}(n)(2+n) + \Phi^+(s_1\tilde{\alpha}) - \Phi^+(s_0\tilde{\alpha}).\end{aligned}$$

If $n \neq -1, -2$, then $\mathrm{sgn}(n)(2+n) \geq 1$ and hence $\ell(x, s_1\tilde{\alpha}) \geq 0$. We check the remaining cases explicitly:

$$\begin{aligned}n = -1: &\quad \ell(x, s_1(-\alpha+\delta)) = -1 + \Phi^+(\alpha+\delta) - \Phi^+(\alpha-\delta) = -1 + 1 - 0 = 0.\\ n = -2: &\quad \ell(x, s_1(-\alpha+2\delta)) = 0 + \Phi^+(\alpha+2\delta) - \Phi^+(\alpha) = 0 + 1 - 1 = 0.\end{aligned}$$

Hence $\ell(x, s_1\tilde{\alpha}) \geq 0$ for all $\tilde{\alpha} \in \Phi^+$, and so $s_1 \in \mathrm{LP}(x)$. Conversely, let $u = s_0$ and $\tilde{\alpha} = -\alpha + \delta$. Then:

$$\begin{aligned}\ell(x, s_0\tilde{\alpha}) &= \langle -\alpha^\vee + \Lambda_0, s_0\tilde{\alpha}\rangle + \Phi^+(s_0\tilde{\alpha}) - \Phi^+(s_0 s_1 s_0\tilde{\alpha})\\ &\leq \langle -\alpha^\vee + \Lambda_0, \alpha - \delta\rangle + 1\\ &= -2 - 1 + 1\\ &= -2 < 0.\end{aligned}$$

Hence there exists $\tilde{\alpha} \in \Phi^+$ with $\ell(x, s_0\tilde{\alpha}) < 0$ and so $s_0 \notin \mathrm{LP}(x)$.

In order to construct a double affine Demazure product, we rely on the following subset of pairs of length positive elements.

**Definition 2.5.13.** *Let $x, y \in W_{\mathcal{T}}$, with $y = w_y\varepsilon^{\mu_y}$ for some $w_y \in W, \mu_y \in \mathcal{T}$. The **distance minimising set** for the pair $(x, y)$ is:*

$$\begin{aligned}M_{x,y} := \{(u, v) \in \mathrm{LP}(x) \times \mathrm{LP}(y) \mid d(u \Rightarrow w_y v) \leq d(u' \Rightarrow w_y v')\\ \forall u' \in \mathrm{LP}(x), v' \in \mathrm{LP}(y)\}.\end{aligned}$$

We see that $M_{x,y}$ is always non-empty, as a shortest path between any two elements of $W$ always exists. We now define the double affine Demazure product, adapted from [Sch24, Thm 5.11].

**Definition 2.5.14.** *Let $x = w_x\varepsilon^{\mu_x}, y = w_y\varepsilon^{\mu_y} \in W_{\mathcal{T}}$, and let $(u, v) \in M_{x,y}$. Fix a shortest path $p : u \Rightarrow w_y v$ in $\mathrm{QBG}(W)$. We define the **generalised Demazure***

***product*** $*^p_{u,v}$ *on* $W_{\mathcal{T}}$ *as*

$$x *^p_{u,v} y := w_x uv^{-1}\varepsilon^{vu^{-1}\mu_x+\mu_y-v\mathrm{wt}(p)}. \tag{2.10}$$

If $W$ is of affine type, we refer to this as the *double affine Demazure product* on $W_{\mathcal{T}}$. Recall that in the finite case, $\mathrm{wt}(p)$ is independent of the choice of $p$, and so we write $*_{u,v} := *^p_{u,v}$. We have the following theorem due to Schremmer.

**Theorem 2.5.15.** [Sch24, Thm. 5.11, Prop. 5.12(b)] *Assume $W$ is of finite type. Let $x, y \in W_{\mathcal{T}} = \widehat{W}$, and $(u, v) \in M_{x,y}$. Then:*

(i) *$x *_{u,v} y = x * y$, the Demazure product of $x$ and $y$.*

(ii) *$uv^{-1}$ and $v\mathrm{wt}(u \Rightarrow w_y v)$ are independent of the choice of $(u, v) \in M_{x,y}$.*

This gives a method of calculating the Demazure product of two (extended) affine Weyl elements without directly using the Bruhat order, and instead considering the quantum Bruhat graph, which is finite when $W$ is of finite type. Our goal is to generalise this to the double affine case, using it as a definition of a Demazure product in $W_{\mathcal{T}}$, by showing that this product is independent of both the path $p$ and the choice of $(u, v) \in M_{x,y}$. In this thesis, we succeed in the case of $\widehat{\mathrm{SL}}_2$, furthermore proving that this product is associative, and provide some further Demazure-like qualities in the general case assuming well-definedness.

*Remark* 2.5.16. Schremmer proves Theorem 2.5.15(i) by considering the Bruhat order on $\widehat{W}$ directly. Uniqueness of the Demazure product then immediately proves Theorem 2.5.15(ii) as a corollary. *A priori*, there is no direct proof of part (ii) that does not depend on the Demazure product.

## Chapter 3

# Type Affine $\mathrm{SL}_2$

Our goal is to show that Schremmer's formula (2.10) defines a product on $W_{\mathcal{T}}$, resembling the Demazure product, which is independent of the choice of $(u, v) \in M_{x,y}$ and the path between them. We present a proof for type $\widehat{\mathrm{SL}}_2$, which we fix for the rest of this chapter unless stated otherwise.

We have $\Phi_{\text{fin}} = \{\pm\alpha\}$ and $W_{\text{fin}} = \langle s_1 \mid s_1^2 = e\rangle \cong S_2$. The set of roots is $\Phi = \{\pm\alpha + n\delta \mid n \in \mathbb{Z}\}$, with Weyl group $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = e\rangle \cong \widetilde{S}_2$, the infinite Dihedral group. We write $w\varepsilon^\mu \in W_{\mathcal{T}}$, for some $w \in W$ and $\mu \in \mathcal{T}$. Using the presentation $W \cong W_{\text{fin}} \ltimes \mathbb{Z}\Phi_{\text{fin}}^\vee$ and following Remark 2.4.4, we write $W = \langle w_o\tau^\lambda \mid w_o \in W_{\text{fin}}, \lambda \in \mathbb{Z}\alpha^\vee\rangle$ for the Weyl group of type $\widehat{\mathrm{SL}}_2$. We convert between presentations using the relation $s_0 = s_1\tau^{-\alpha^\vee}$, or equivalently $\tau^{\alpha^\vee} = s_0 s_1$.

## § 3.1 Classifying Length Positive Elements

Let $x, y \in W_{\mathcal{T}}$. In order to prove that the product $x *_{u,v}^p y$ as defined in Equation (2.10) is independent of the choice of $(u, v) \in M_{x,y}$, we need to be able to describe the length positive sets $\mathrm{LP}(x), \mathrm{LP}(y)$ from which $M_{x,y}$ is defined. To do this, we fix an arbitrary $x \in W_{\mathcal{T}}$ and expand on the conditions required for $v \in W$ to be length positive for $x$. We then determine the possible minimal values of the length functional and study these cases separately.

### § 3.1.1 Minimum Values of The Length Functional

Let $x = w\varepsilon^\mu = w_o\tau^{r\alpha^\vee}\varepsilon^\mu$ where $\mu = k\alpha^\vee + m\delta + l\Lambda_0$. Let $v = v_o\tau^{t\alpha^\vee} \in W$ for some $v_o \in W_{\text{fin}}, t \in \mathbb{Z}$. We seek the conditions on $v$ that force $v \in \mathrm{LP}(x)$. Fix $\tilde{\alpha} = \mathrm{sgn}(n)(\alpha + n\delta) \in \Phi^+$ as above for positive roots. Let $\mathrm{sgn}(v_o)$ denote the parity of $v_o$, such that $\mathrm{sgn}(e) = 1$ and $\mathrm{sgn}(s_1) = -1$. Expanding the action of $W$ on $\Phi$, we obtain the following:

$$
\begin{aligned}
v\tilde{\alpha} &= \operatorname{sgn}(n)(v_o\alpha + (n-2t)\delta), && (3.1)\\
wv\tilde{\alpha} &= \operatorname{sgn}(n)(w_o v_o\alpha + (n-2t-2r\operatorname{sgn}(v_o))\delta), && (3.2)\\
\langle \mu, v\tilde{\alpha}\rangle &= \operatorname{sgn}(n)(2k\operatorname{sgn}(v_o) + (n-2t)l). && (3.3)
\end{aligned}
$$

Each of these appear as terms in the length functional that we wish to consider:

$$\ell(x, v\tilde{\alpha}) = \langle \mu, v\tilde{\alpha}\rangle + \Phi^+(v\tilde{\alpha}) - \Phi^+(wv\tilde{\alpha}) \geq 0, \quad \forall \tilde{\alpha} \in \Phi^+. \tag{3.4}$$

*Remark* 3.1.1. The indicator component of Equation (3.4), $\Phi^+(v\tilde{\alpha}) - \Phi^+(w\tilde{\alpha})$, has absolute value at most 1. Hence, we must have $\langle \mu, v\tilde{\alpha}\rangle \geq -1$ for all $\tilde{\alpha} \in \Phi^+$. In particular, if we have $\langle \mu, v\tilde{\alpha}\rangle \geq 1$ for every positive root $\tilde{\alpha}$, then we immediately find that $v \in \mathrm{LP}(x)$.

The above remark suggests that we should split the calculation into cases concerning the minimum value attained by $\langle \mu, v\tilde{\alpha}\rangle$ as $\tilde{\alpha}$ varies over $\Phi^+$, denoted $\min\{\langle \mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}}$. We deal with the boundary case separately, as $\mu$ behaves differently. Our (disjoint) cases to consider are:

$$\text{(a)}\ \ l > 0, \begin{cases} \text{(i)} & \min\{\langle \mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} \geq 1. \\ \text{(ii)} & \min\{\langle \mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} = 0. \\ \text{(iii)} & \min\{\langle \mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} = -1. \end{cases}$$

(b) $l = 0$ (boundary case).

In the first three cases, we find conditions for $v$ such that $v \in \mathrm{LP}(x)$ and such that the corresponding minimum bound is attained. By Remark 3.1.1, one of these bounds must be attained, and so we will find all possible conditions which give $v \in \mathrm{LP}(x)$. To achieve this, we find equivalent conditions in terms of $t$ and $v_o$ which define $v$. It will be convenient to instead first find conditions on $\operatorname{sgn}(v_o)k - tl$, where $k, l, t, v_o$ are defined as before.

**Case (ai).** We require $\langle \mu, v\tilde{\alpha}\rangle \geq 1$ for any $\tilde{\alpha} = \operatorname{sgn}(n)(\alpha + n\delta) \in \Phi^+$. Each root is given by the corresponding integer $n$, and so we can equivalently enforce this condition for every $n \in \mathbb{Z}$. From Equation 3.3, this condition becomes:

$$\operatorname{sgn}(n)(2k\operatorname{sgn}(v_o) + (n-2t)l) \geq 1, \quad \forall n \in \mathbb{Z}. \tag{3.5}$$

Note that if this inequality holds for $n = 0$, then it holds for all $n \geq 0$, since $l \geq 0$. Similarly, if it holds for $n = -1$, then it must hold for all $n \leq -1$. Hence, we can reduce this to just two inequalities:

$$2k\,\mathrm{sgn}(v_o) - 2tl \geq 1, \tag{3.6}$$

$$-2k\,\mathrm{sgn}(v_o) + (1+2t)l \geq 1. \tag{3.7}$$

$$\iff \frac{1}{2} \;\leq\; k\,\mathrm{sgn}(v_o) - tl \;\leq\; \frac{l-1}{2}. \tag{3.8}$$

Hence, $v = v_o\tau^{t\alpha^\vee} \in \mathrm{LP}(x)$ and $\min\{\langle\mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} \geq 1$ if and only if $v_o$ and $t$ are chosen such that $\mathrm{sgn}(v_o)k - tl \in \{1, 2, ..., \lfloor\frac{l-1}{2}\rfloor\}$.

**Case (aii).** We now require $\langle\mu, v\tilde{\alpha}\rangle \geq 0$ for any $\tilde{\alpha} = \mathrm{sgn}(n)(\alpha+n\delta) \in \Phi^+$, attained for at least one root. As in the previous case, the inequalities for all $n$ can be reduced to just checking the cases $n = 0, -1$, and so we get the following:

$$0 \;\leq\; k\,\mathrm{sgn}(v_o) - tl \;\leq\; \frac{l}{2}. \tag{3.9}$$

Note the $n = 0$ and $n = -1$ cases correspond to the left and right hand inequalities respectively. One of these must be attained to ensure that $\min\{\langle\mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} = 0$, and so we get that $k \in \{\mathrm{sgn}(v_o)tl, \mathrm{sgn}(v_o)(\frac{l}{2} + tl)\}$ for $l$ even, and $k = \mathrm{sgn}(v_o)tl$ for $l$ odd. However, we need to check that the length positivity condition is also satisfied:

$$\begin{aligned}\ell(x, v\tilde{\alpha}) &= \langle\mu, v\tilde{\alpha}\rangle + \Phi^+(v\tilde{\alpha}) - \Phi^+(wv\tilde{\alpha}) \\ &= \Phi^+(v\tilde{\alpha}) - \Phi^+(wv\tilde{\alpha}) \;\geq\; 0\end{aligned}$$

$$\iff \Phi^+(v\tilde{\alpha}) = 1 \;\text{ or }\; \Phi^+(wv\tilde{\alpha}) = 0. \tag{3.10}$$

Expanding our form for $v\tilde{\alpha}$, we find that $\Phi^+(v\tilde{\alpha}) = 1$ if and only if the root $\mathrm{sgn}(n)(v_o\alpha + (n-2t)\delta) > 0$, where $n \in \{0, -1\}$ by the above. If $n = 0$, then in order to ensure that $v_o\alpha - 2t\delta > 0$, we must have that $t \leq \frac{1}{2}(\Phi^+(v_o\alpha) - 1)$, or equivalently $t \leq \Phi^+(v_o\alpha) - 1$. If $n = -1$, we similarly obtain $t \geq \frac{1}{2}(\Phi^+(v_o\alpha) - 1)$, equivalent to $t \geq 0$.

Applying the same method to the requirement $\Phi^+(wv\tilde{\alpha}) = 0$, where $w = w_o\tau^{r\alpha^\vee}$, for $n = 0$, $t$ must satisfy $t \geq \Phi^+(w_ov_o\alpha) - r\,\mathrm{sgn}(v_o)$, and for $n = -1$, $t$ must satisfy $t \leq -1 - r\,\mathrm{sgn}(v_o)$.

Summarising this case, we find that $v \in \mathrm{LP}(x)$ and $\min\{\langle\mu, v\tilde{\alpha}\rangle\}_{\tilde{\alpha}} = 0$ if and only if one of the following holds:

$$\mathrm{sgn}(v_o)k - tl = 0, \text{ and either } t \leq \Phi^+(v_o\alpha) - 1 \text{ or } t \geq \Phi^+(w_ov_o\alpha) - r\,\mathrm{sgn}(v_o), \tag{3.11}$$

$$\mathrm{sgn}(v_o)k - tl = \frac{l}{2} \text{ (with } l \text{ even), and either } t \geq 0 \text{ or } t \leq -1 - r\,\mathrm{sgn}(v_o). \tag{3.12}$$

**Case (aiii).** Continuing on as before, we require $\langle \mu, v\tilde{\alpha} \rangle \geq -1$, attained for at least one root, which leads to the following double-sided inequality.

$$-\frac{1}{2} \ \leq \ \mathrm{sgn}(v_o)k - tl \ \leq \ \frac{l+1}{2}. \tag{3.13}$$

This must be attained on one side, and so we must have that $\mathrm{sgn}(v_o)k - tl = \frac{l+1}{2}$, with $l$ odd, corresponding to $n = -1$. For $n = 0$, length positivity is ensured, since $\mathrm{sgn}(v_o)k - tl = \frac{l+1}{2} \geq \frac{1}{2}$ and so $\langle \mu, v\tilde{\alpha} \rangle \geq 1$ in this case. Checking the length positivity condition for $n = -1$, we must enforce:

$$\begin{aligned} &\Phi^+(v\tilde{\alpha}) - \Phi^+(wv\tilde{\alpha}) \geq 1 \\ \iff\ &\Phi^+(v\tilde{\alpha}) = 1 \text{ and } \Phi^+(wv\tilde{\alpha}) = 0. \end{aligned}$$

Using calculations done in Case (aii), this occurs for $n = -1$ when $0 \leq t \leq -1 - r\,\mathrm{sgn}(v_o)$. Hence, $v \in \mathrm{LP}(x)$ and $\min\{\langle \mu, v\tilde{\alpha} \rangle\}_{\tilde{\alpha}} = -1$ if and only if $v_o$ and $t$ are chosen such that $\mathrm{sgn}(v_o)k - tl = \frac{l+1}{2}$ and $0 \leq t \leq -1 - r\,\mathrm{sgn}(v_o)$. We summarise the previous cases in the following proposition.

**Proposition 3.1.2.** *Let $x = w_o\tau^{r\alpha^\vee}\varepsilon^{k\alpha^\vee + m\delta + l\Lambda_0} \in W_{\mathcal{T}}$ with $l > 0$, and let $v = v_o\tau^{t\alpha^\vee} \in W$. Write $j = k\,\mathrm{sgn}(v_o) - tl$. Then $v \in \mathrm{LP}(x)$ if and only if one of the following cases is satisfied:*

(i) $j \in \{1, 2, ..., \lfloor \frac{l-1}{2} \rfloor\}$.

(ii) $j = 0$, *with either* $t \leq \Phi^+(v_o\alpha) - 1$ *or* $t \geq \Phi^+(w_ov_o\alpha) - r\,\mathrm{sgn}(v_o)$, *or*
$j = \frac{l}{2}$, *with either* $t \geq 0$ *or* $t \leq -1 - r\,\mathrm{sgn}(v_o)$.

(iii) $j = \frac{l+1}{2}$ *and* $0 \leq t \leq -1 - r\,\mathrm{sgn}(v_o)$.

**Case (b).** Now let $l = 0$, and so $\mu \in \mathbb{Z}\delta$, giving us that $k = 0$ and hence $\langle \mu, v\tilde{\alpha} \rangle = 0$. The length positivity condition then becomes the same as in case (aii), namely that either $\Phi^+(v\tilde{\alpha}) = 1$ or $\Phi^+(wv\tilde{\alpha}) = 0$, however now we must have that at least one of these holds for every $n \in \mathbb{Z}$. Instead of looking at inequalities as above, we can classify the length positive set in this case using inversion sets, defined for $u \in W$ as $\mathrm{Inv}(u) = \{\tilde{\alpha} \in \Phi^+ \mid u\tilde{\alpha} < 0\}$. We use these to prove a simple description of $\mathrm{LP}(x)$.

**Proposition 3.1.3.** *Let $w\varepsilon^{m\delta} \in W_{\mathcal{T}}$ be a boundary element of the Tits cone in type $\widehat{\mathrm{SL}}_2$, where $w \in W$ and $m \in \mathbb{Z}$. If $w \neq e$, write $w = w's_i$, with $w' < w$ and $i \in \{0, 1\}$. Then*

$$\mathrm{LP}(w\varepsilon^{m\delta}) = \begin{cases} \{v \in W \mid s_iv > v\}, & \textit{if } w \neq e. \\ W, & \textit{otherwise.} \end{cases} \tag{3.14}$$

*Proof.* Write $x = w\varepsilon^{m\delta}$. Note that $v \notin \mathrm{LP}(x)$ if and only if $\Phi^+(v\tilde{\alpha}) = 0$ and $\Phi^+(wv\tilde{\alpha}) = 1$ for some $\tilde{\alpha} \in \Phi^+$. This occurs if and only if $v\tilde{\alpha} < 0$ and $w(-v\tilde{\alpha}) < 0$, equivalent to $\tilde{\alpha} \in \mathrm{Inv}(v)$ and $-v\tilde{\alpha} \in \mathrm{Inv}(w)$, or $-v\tilde{\alpha} \in -v\mathrm{Inv}(v) \cap \mathrm{Inv}(w)$. Hence $v \notin \mathrm{LP}(x) \iff \exists \tilde{\alpha} \in \mathrm{Inv}(w) \cap -v\mathrm{Inv}(v) \iff \mathrm{Inv}(w) \cap -v\mathrm{Inv}(v) \neq \emptyset$. Negating this statement, and noting that from [Hum90, Sec. 1.6] we have $-v\mathrm{Inv}(v) = \mathrm{Inv}(v^{-1})$, we find the description $\mathrm{LP}(x) = \{v \in W \mid \mathrm{Inv}(w) \cap \mathrm{Inv}(v^{-1}) = \emptyset\}$. We have that $\mathrm{Inv}(w) \cap \mathrm{Inv}(v^{-1}) = \emptyset \iff \ell(wv) = \ell(v) + \ell(w)$ from [Dye19, Lemma 4.1], i.e. $v, w$ are length additive, and so we have

$$\mathrm{LP}(w\varepsilon^{m\delta}) = \{v \in W \mid \ell(wv) = \ell(w) + \ell(v)\}. \tag{3.15}$$

Note that this equation holds for any general $W_{\mathcal{T}}$, and not necessarily just the case of $\widehat{\mathrm{SL}}_2$. First note that if $w = e$, then this condition holds for all $v \in W$, and so $\mathrm{LP}(x) = W$.

If we now fix $W_{\mathcal{T}}$ to be of type $\widehat{\mathrm{SL}}_2$ with $w \neq e$, then we have an easy description of length additivity. In particular, as there are no braid relations in $W$, we have that $v$ and $w$ are length additive if and only if, in reduced expressions for $v$ and $w$, $v = s_j v'$ and $w = w' s_i$ with $v > v', w > v'$, and $j \neq i$. This occurs if and only if $s_i v > v$, proving the claim. $\square$

*Remark* 3.1.4. We can describe $\mathrm{LP}(w\varepsilon^{m\delta})$ as consisting of the half of $W$ whose first simple reflection in a reduced expression is not the last simple reflection in a reduced expression for $w$. In particular, it is clear that $|\mathrm{LP}(x)| = \infty$.

**Corollary 3.1.5.** *Let $x = w\epsilon^{m\delta} \in W_{\mathcal{T}}$. Then $\mathrm{LP}(x)$ is connected in $\mathrm{QBG}(W)$.*

This can be seen graphically in Figure 2.6, where $\mathrm{LP}(x)$ consists of an entire left or right side, which are connected by vertical edges.

### § 3.1.2 Algorithm for computing $\mathrm{LP}(x)$

We can combine the results from above to give a short algorithm for computing $\mathrm{LP}(x)$. Firstly, if $l = 0$, we use Proposition 3.1.3, giving us $\mathrm{LP}(x)$ immediately. However, we are more interested in the non-boundary cases. Here we state and prove an algorithm using the above results for computing $\mathrm{LP}(x)$.

**Proposition 3.1.6.** *Let $x = w_o \tau^{r\alpha^\vee} \epsilon^{k\alpha^\vee + m\delta + l\Lambda_0} \in W_{\mathcal{T}}$, with $l > 0$. Then $\mathrm{LP}(x)$ can be computed using the following algorithm.*

1. *Compute $t \in \mathbb{Z}$ such that $-\frac{l}{2} < k - tl \leq \frac{l}{2}$, and write $j = k - tl$.*

2. *If $1 \leq \mathrm{sgn}(v_o)j \leq \frac{l-1}{2}$ for some choice of $v_o \in W_{\mathrm{fin}}$, then $\tau^{t\alpha^\vee} v_o \in \mathrm{LP}(x)$.*

3. *If $j = 0$, then $\tau^{t\alpha^\vee} \in \mathrm{LP}(x)$ if either $t \leq 0$ or $t \geq \Phi^+(w_o\alpha) - r$, and $\tau^{t\alpha^\vee} s_1 \in \mathrm{LP}(x)$ if either $t \geq 1$ or $t \leq \Phi^+(w_o\alpha) - r - 1$.*

4. *If $j = \frac{l}{2}$, then $\tau^{t\alpha^\vee} \in \mathrm{LP}(x)$ if either $t \geq 0$ or $t \leq -1 - r$, and $\tau^{t\alpha^\vee} s_0 \in \mathrm{LP}(x)$ if either $t \leq -1$ or $t \geq -r$.*

5. *If $j = -\operatorname{sgn}(v_o)\frac{l-1}{2}$ for some choice of $v_o \in W_{\text{fin}}$, then $\tau^{t\alpha^\vee} s_0 \in \mathrm{LP}(x)$ if $v_o = s_1$ and $-r \leq t \leq -1$, and $\tau^{t\alpha^\vee} s_1 s_0 \in \mathrm{LP}(x)$ if $v_o = e$ and $1 \leq t \leq -r$.*

*After applying step 5, we have found all elements of* $\mathrm{LP}(x)$.

*Proof.* We first show that each step of the algorithm holds true.

1. Such a $t$ exists, and is unique by the Euclidean algorithm.

2. If $j > 0$, then this immediately holds by equation 3.8, with $v_o = e$. If $j < 0$, then $-j = -k - (-t)l$, and so we apply equation 3.8 with $v_o = s_1$, using $-t$ instead of $t$.

3. Follows from 3.11, noting $\operatorname{sgn}(v_o)j = 0$ regardless of the choice of $v_o$, and if we choose $v_o = s_1$ then we need to replace $t$ with $-t$ using the same logic as in the proof of 2.

4. Follows from 3.12, noting that if $j = \frac{l}{2}$ then $-j = -\frac{l}{2}$, and so $-k - (-t-1)l = \frac{l}{2}$ also satisfies the required conditions with $v_o = e$ and using $-(t+1)$ instead of $t$.

5. Follows from Case (aiii) above, applying the same logic as in step 4 to shift $t$ to meet the required conditions.

As $t$ is unique, we can get at most one length positive element from each case (for a particular $v_o$). Hence, as we have covered all possible values of $j$ in the given range, we must have found all elements of $\mathrm{LP}(x)$. ☐

**Example 3.1.7.** Let $x = s_1\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee - \delta + \Lambda_0}$. We use Proposition 3.1.6 to compute $\mathrm{LP}(x)$, and then verify that one element we find is length positive directly from the definition.

1. We have $w_o = s_1, r = -1, k = 1, m = -1, l = 1$. We thus want to compute $t \in \mathbb{Z}$ such that $-\frac{1}{2} < j = 1 - t \leq \frac{1}{2}$, from which we find $t = 1$ and $j = 0 = \frac{l-1}{2}$.

2. We have $1 > \frac{l-1}{2} = 0$, and hence no length positive elements arise from this step.

3. We have $j = 0$, so this step applies. We have $1 = t \geq \Phi^+(w_o\alpha) - r = 1$ and so $\tau^{t\alpha^\vee} = s_0 s_1 \in \mathrm{LP}(x)$, and $t \geq 1$ so $\tau^{t\alpha^\vee} s_1 = s_0 \in \mathrm{LP}(x)$.

4. $j \neq \frac{l}{2}$ so we obtain no elements from this step.

5. $j = -\operatorname{sgn}(v_o)\frac{l-1}{2}$ for either of choice of $v_o$. We do not have $-r \leq t \leq -1$, however we do have $1 \leq t \leq -r$, and hence $\tau^{t\alpha^\vee} s_1 s_0 = e \in \mathrm{LP}(x)$.

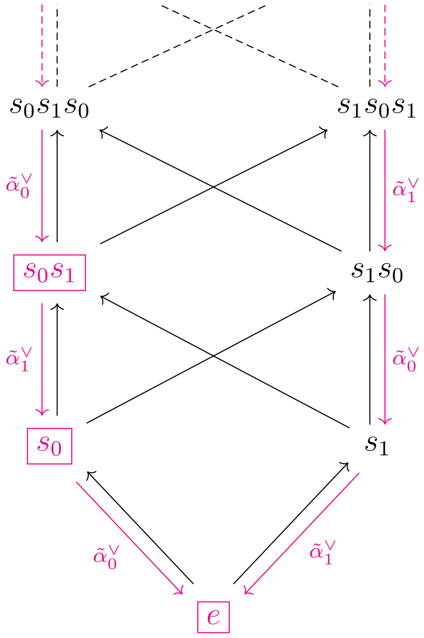


Figure 3.1: For $W$ of type $\widehat{\mathrm{SL}}_2$, the length positive set $\mathrm{LP}(s_1\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee-\delta+\Lambda_0})$ depicted within $\mathrm{QBG}(W)$.

Hence $\mathrm{LP}(x) = \{e, s_0, s_0s_1\}$, as pictured in Figure 3.1.

Given $x = w\varepsilon^{\mu_o+l\Lambda_0} \in W_{\mathcal{T}}$ for some $w \in W, \mu_o \in P^\vee$, we call $l$ the level of $x$, denoted $\mathrm{lev}(x)$.

**Corollary 3.1.8.** *Let $x \in W_{\mathcal{T}}$. Then $1 \leq |\mathrm{LP}(x)| \leq 2$ if $\mathrm{lev}(x) > 1$, and $1 \leq |\mathrm{LP}(x)| \leq 3$ if $\mathrm{lev}(x) = 1$.*

*Proof.* Fix $j = k - tl$ as in Proposition 3.1.6, and first assume that $l > 1$. Then if $j \notin \{0, \frac{l}{2}, \pm\frac{l-1}{2}\}$, Step 2 gives us exactly one element of $\mathrm{LP}(x)$.

If $j = 0$, then we certainly have a length positive element if $t \leq 0$. If instead $t \geq 0$, assume that $t \leq -(\Phi^+(w_o\alpha) - r)$. Then $\Phi^+(w_o\alpha) - r \leq 0$, and so $t \geq \Phi^+(w_o\alpha) - r$. Hence at least one condition is satisfied. There are also clearly at most two possibilities for length positive elements.

If $j = \frac{l}{2}$, then similarly to before we have a length positive element if $t \geq 0$. If instead $t \leq -1$, and $t \geq -r$, then we must have $r \leq 1$ and so $t \leq -1 + r$ is guaranteed. Hence, at least one condition is satisfied, and again there are at most two possibilities.

If $j = \pm\frac{l-1}{2}$, then we are guaranteed a length positive element from Case 2, and can only include at most one more element from this final step, giving at most two cases.

If instead $l = 1$, then we find no length positive elements from Step 2 of Prop. 3.1.6, however we have $j = 0 = \frac{l-1}{2}$. Step 3 returns either 1 or 2 elements depending on the values of $t, r$, and Step 5 can return at most 1 more element, giving $1 \leq |\mathrm{LP}(x)| \leq 3$. $\square$

**Corollary 3.1.9.** $\mathrm{LP}(x)$ *is connected in* $\mathrm{QBG}(W)$ *by edges of the form* $w \Rightarrow ws_i$.

*Proof.* Firstly, assume $\mathrm{lev}(x) > 0$. Then if $|\mathrm{LP}(x)| = 1$, the statement is true trivially.

If $|\mathrm{LP}(x)| = 2$, then following the algorithm, the only possible pairs of elements we find are sets of the form $\{\tau^{t\alpha^\vee}, \tau^{t\alpha^\vee}s_1\}, \{\tau^{t\alpha^\vee}, \tau^{t\alpha^\vee}s_0\}$, and $\{\tau^{t\alpha^\vee}s_1, \tau^{t\alpha^\vee}s_1s_0\}$, each of which are of the form $\{w, ws_i\}$ for $w \in W,\ i \in \{0,1\}$, which are connected in $\mathrm{QBG}(W)$ by a single edge.

If $|\mathrm{LP}(x)| = 3$, then we must have $l = 1$ and $j = 0 = \pm\frac{l-1}{2}$, forcing $\{\tau^{t\alpha^\vee}, \tau^{t\alpha^\vee}s_1\} \subset \mathrm{LP}(x)$ and either $\tau^{t\alpha^\vee}s_0 \in \mathrm{LP}(x)$ or $\tau^{t\alpha^\vee}s_1s_0 \in \mathrm{LP}(x)$. Hence, the triple of elements we find are connected in $\mathrm{QBG}(W)$ either as $\tau^{t\alpha^\vee}s_0 \Rightarrow \tau^{t\alpha^\vee} \Rightarrow \tau^{t\alpha^\vee}s_1$ or as $\tau^{t\alpha^\vee} \Rightarrow \tau^{t\alpha^\vee}s_1 \Rightarrow \tau^{t\alpha^\vee}s_1s_0$ as required.

If instead $\mathrm{lev}(x) = 0$, then either $\mathrm{LP}(x) = W$ in which case the result is immediate, or $\mathrm{LP}(x) = \{e, s_j, s_js_i, s_js_is_j, ...\}$ for some $i \neq j \in \{0,1\}$ by Proposition 3.1.3, from which it is clear that we have the edges $e \Rightarrow s_j \Rightarrow s_js_i \Rightarrow ...$ as required. □

**Corollary 3.1.10.** *If* $\mathrm{LP}(x) \neq W$*, then there exists some* $i \in \{0,1\}$ *such that* $s_iu > u$ *for every* $u \in \mathrm{LP}(x)$*.*

*Proof.* First assume $\mathrm{lev}(x) > 0$. From Proposition 3.1.3 and Corollary 3.1.9, this is immediate in all cases except $|\mathrm{LP}(x)| = 3$. The only situation left to rule out is that $\mathrm{LP}(x) \neq \{s_1, e, s_0\}$. In this case, we must have $\mathrm{lev}(x) = 1$ by Corollary 3.1.8. Hence, using the notation of Proposition 3.1.6, $j = 0$ and thus $\mathrm{LP}(x) \subset \{\tau^{t\alpha^\vee}, \tau^{t\alpha^\vee}s_1, \tau^{t\alpha^\vee}s_0, \tau^{t\alpha^\vee}s_1s_0\}$. For $e \in \mathrm{LP}(x)$, we therefore require $t = 0$ or $t = 1$. If $t = 1$, then $s_1 \notin \mathrm{LP}(x)$. If $t = 0$, then step 5 of the classification algorithm guarantees that $s_0 \notin \mathrm{LP}(x)$.

If instead $\mathrm{lev}(x) = 0$, then as in the proof of the previous corollary, we have $\mathrm{LP}(x) = \{e, s_j, s_js_i, s_js_is_j, ...\}$ for some $i \neq j \in \{0,1\}$, in which case $s_iu > u$ for every $u \in \mathrm{LP}(x)$ as required. □

**Corollary 3.1.11.** *Let* $x = w\varepsilon^\mu \in W_{\mathcal{T}}$ *with* $|\mathrm{LP}(x)| = 3$*. Then* $w\mathrm{LP}(x) = \{s_1, e, s_0\}$*.*

*Proof.* As before, $|\mathrm{LP}(x)| = 3$ implies that $\mathrm{lev}(x) = l = 1$, and so we can write $x = w_o\tau^{r\alpha^\vee}\varepsilon^{k\alpha^\vee + m\delta + \Lambda_0}$, with $w = w_o\tau^{r\alpha^\vee}$. Following Proposition 3.1.6, we have $t = k$ and $j = 0$. In order to have three length positive elements, we must have two from step 3 of the algorithm and one from step 1, as outlined in the proof of Corollary 3.1.8.

From step 3, we find $\tau^{k\alpha^\vee}, \tau^{k\alpha^\vee}s_1 \in \mathrm{LP}(x)$, with the condition that either $k \leq \min\{0, \Phi^+(w_o\alpha) - r - 1\}$ or $k \geq \max\{1, \Phi^+(w_o\alpha) - r\}$, one of which must be satisfied. From step 5, we have either $\tau^{k\alpha^\vee}s_0 \in \mathrm{LP}(x)$ if $-r \leq k \leq -1$, or $\tau^{k\alpha^\vee}s_1s_0 \in \mathrm{LP}(x)$ if $1 \leq k \leq -r$, and again one of these must be satisfied.

Assume first that $-r \leq k \leq -1$. Then $k < 0$ and so we must have $k \leq \min\{0, \Phi^+(w_o\alpha) - r - 1\}$, and so $k \leq \Phi^+(w_o\alpha) - r - 1$. However, we have assumed $k \geq -r$. These can only both be satisfied if $k = -r$ and $\Phi^+(w_o\alpha) = 1$, in which case $w_o = e$. Hence,

$$w\mathrm{LP}(x) = \tau^{-k\alpha^\vee}\{\tau^{k\alpha^\vee}, \tau^{k\alpha^\vee}s_1, \tau^{k\alpha^\vee}s_0\} = \{e, s_1, s_0\}. \tag{3.16}$$

In the second case, we have $1 \leq k \leq -r$, and so we must also have the inequality $k \geq \max\{1, \Phi^+(w_o\alpha) - r\} \geq \Phi^+(w_o\alpha) - r$, which forces $k = -r$ and $w_o = s_1$. We then find that:

$$w\mathrm{LP}(x) = s_1\tau^{-k\alpha^\vee}\{\tau^{k\alpha^\vee}, \tau^{k\alpha^\vee}s_1, \tau^{k\alpha^\vee}s_1s_0\} = \{s_1, e, s_0\}. \tag{3.17}$$

Hence, in both cases, we have $w\mathrm{LP}(x) = \{s_1, e, s_0\}$ as required. $\square$

In fact, we can classify those elements that have $|\mathrm{LP}(x)| = 3$. For $k \in \mathbb{Z}$, write $[k] = 1$ if $k \geq 0$, and $[k] = 0$ if $k < 0$. That is, $[k] = \frac{1}{2}(\mathrm{sgn}(k) + 1)$.

**Corollary 3.1.12.** $|\mathrm{LP}(x)| = 3 \iff x = s_1^{[k]}\tau^{-k\alpha^\vee}\varepsilon^{k\alpha^\vee + m\delta + \Lambda_0}$ *with* $k \neq 0$.

*Proof.* In the usual notation, from the proof of Corollary 3.1.11, we have that $|\mathrm{LP}(x)| = 3$ only if $l = 1$, and either $k = -r < 0, w_o = e$ or $k = -r > 0, w_o = s_1$. In either case, we have $r = -k$ and $w_o = s_1^{[k]}$. As shown, if an element $x$ satisfies these conditions, then algorithm from Proposition 3.1.6 returns 3 elements, and hence the statement is an if and only if, as required. $\square$

In particular, elements with $|\mathrm{LP}(x)| = 3$ are relatively rare, determined (up to an imaginary factor of $m\delta$) by one non-zero integer variable.

**Example 3.1.13.** We saw in Example 3.1.7 that, for $x = s_1\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee - \delta + \Lambda_0}$, we had $\mathrm{LP}(x) = \{e, s_0, s_0s_1\}$, and in particular $|\mathrm{LP}(x)| = 3$. As $w = s_1\tau^{-\alpha^\vee} = s_0$ we have $w\mathrm{LP}(x) = s_0\mathrm{LP}(x) = \{s_0, e, s_1\}$ as expected. We also see that $x = s_1^{[1]}\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee + m\delta + \Lambda_0}$ for $m = -1$, and hence takes the expected form from Corollary 3.1.12.

## § 3.2 The Double Affine Demazure Product for $\widehat{\mathrm{SL}}_2$

The main goal of this section is to show that the product $x *_{u,v}^{p} y$ is independent of the choice of $u, v$ and $p$ for $x, y \in W_{\mathcal{T}}$, $G = \widehat{\mathrm{SL}}_2$.

### § 3.2.1 Quantum Bruhat Graph of Type $\widehat{\mathrm{SL}}_2$

Firstly, we investigate the quantum Bruhat graph for $W$, $\mathrm{QBG}(W)$. This was studied in [MM18], giving the following description of the edges.

**Proposition 3.2.1.** *Let* $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = e\rangle$ *be the Weyl group of type* $\widehat{SL}_2$. *Then the upwards edges are of the form* $v \Rightarrow vs_{\tilde{\alpha}}$ *for* $\tilde{\alpha} \in \Phi^+$, *where* $\ell(vs_{\tilde{\alpha}}) = \ell(v) + 1$, *and the downwards edges are of the form* $v \Rightarrow vs_{\tilde{\alpha}}$ *for* $\tilde{\alpha} \in \Delta = \{\alpha, -\alpha + \delta\}$, *where* $\ell(vs_{\tilde{\alpha}}) = \ell(v) - 1$.

This graph was depicted earlier in Figure 2.6, with upwards edges depicted in black with zero weight, and downwards edges in magenta labelled with their weight.

*Remark* 3.2.2. Note that downwards edges are purely vertical (except at $e$), as they only appear associated to reflections with simple affine roots. Following this graphical description, we refer to an element $w \in W$ as being *left-sided* if $s_1 w > w$, and *right-sided* if $s_0 w > w$. It then follows from these descriptions that the identity $e$ is simultaneously left and right-sided. We call an edge $u \Rightarrow v$ *vertical* if $u, v$ are same-sided, and call it *diagonal* $u, v$ are opposite-sided. Note again that paths to and from $e$ are simultaneously vertical and diagonal.

In the language of the above remark, we can reword Corollary 3.1.10 as follows.

**Proposition 3.2.3.** *Let $x \in W_{\mathcal{T}}$ with $x \neq \varepsilon^{m\delta}$. Then $\mathrm{LP}(x)$ is either entirely left-sided or entirely right-sided.*

### § 3.2.2 Shortest Paths in $\mathrm{QBG}(W)$

We first aim to show that $*^p_{u,v} = *_{u,v}$, i.e. that the generalised Demazure product for $\widehat{\mathrm{SL}}_2$ is independent of the path $p$ chosen. We instead prove a stronger statement, showing that the weights of any two shortest paths must be equal, from which path-independence follows.

**Theorem 3.2.4.** *Let $u, v \in W$ of type $\widehat{\mathrm{SL}}_2$ and let $p_1, p_2 : u \Rightarrow v$ be any two shortest paths between them. Then $\mathrm{wt}(p_1) = \mathrm{wt}(p_2)$.*

*Proof.* First, note that upwards edges correspond to a length increase of one, and downwards edges correspond to a length decrease of one. Hence, for a path $p : u \Rightarrow v$, we have that $\ell(p) = |\ell(v) - \ell(u)|$ if and only if $p$ consists of solely upwards or downwards edges, in which case it must be shortest. Note also that if we can find such a path, then all shortest paths must be of the same form.

If $u \leq v$, then by definition there must be a path consisting of only upwards edges (arising form the corresponding Hasse diagram for the Bruhat order of $W$), and hence all shortest paths have this property. The weight of any such path is zero, and so all paths have the same weight.

If $u \geq v$, and $u, v$ are same-sided, then there exists a unique shortest path consisting of downwards vertical edges.

If none of the above cases hold, $u, v$ are not same-sided and $u \not\leq v$. Then it is clear from the diagram that a shortest path consists of vertical downwards edges and one upwards diagonal edge. Fix one such path $p$, and consider the path $p'$ consisting of the same edges, but with the diagonal taken one edge earlier. Then the weight of the subpath being removed is $\tilde{\alpha}_i^\vee + 0$ for some $i \in \{0, 1\}$, and the weight of the subpath being added is $0 + \tilde{\alpha}_i^\vee$ for the same $i$, as shown in the graph. Hence, $\mathrm{wt}(p') = \mathrm{wt}(p) - (\tilde{\alpha}_i^\vee + 0) + (\tilde{\alpha}_i^\vee + 0) = \mathrm{wt}(p)$. A similar argument applies if we consider the path taken with one edge later, and iterating this shows that all shortest paths must have the same weight. □

We henceforth write $\mathrm{wt}(u \Rightarrow v)$ for the weight of any shortest path $u \Rightarrow v$.

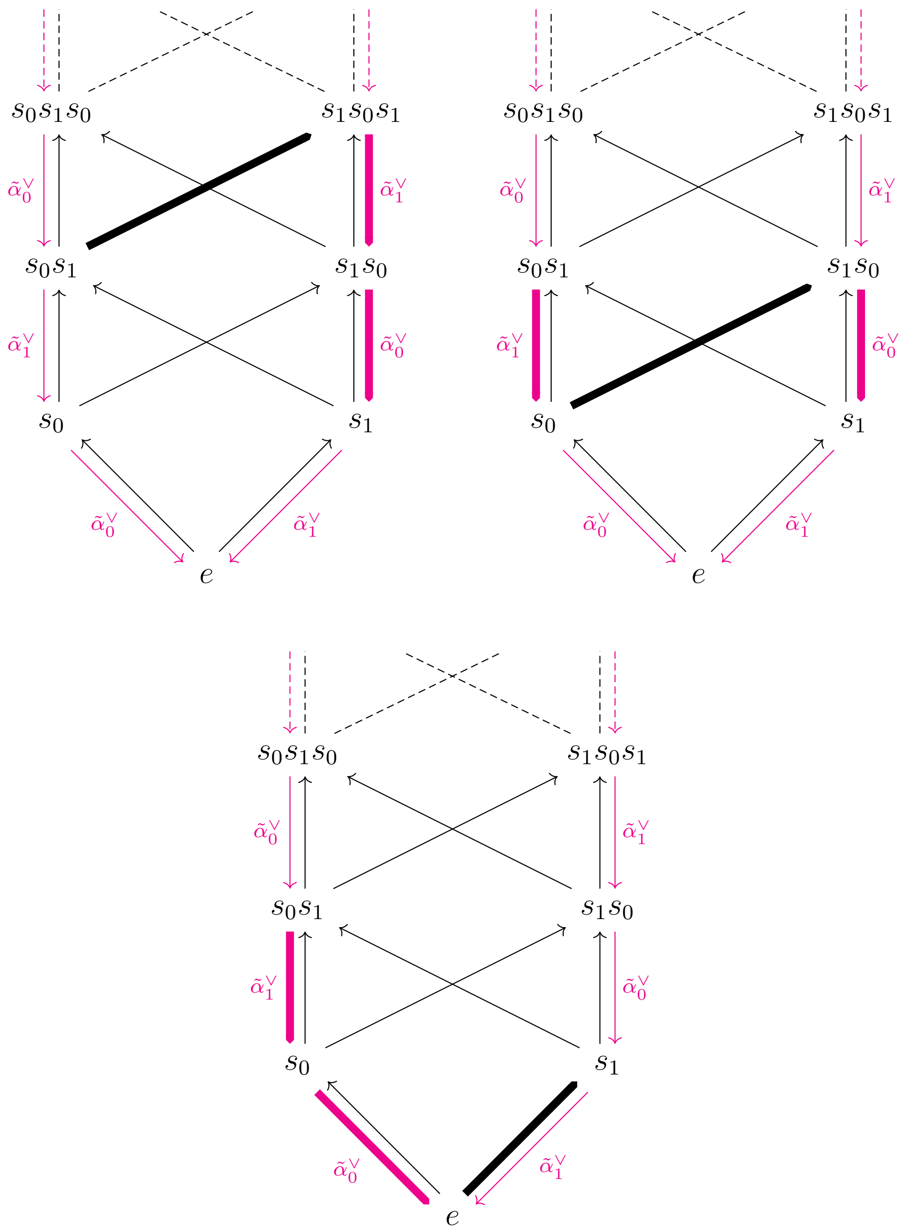


Figure 3.2: The three shortest paths from $s_0s_1$ to $s_1$ in $\mathrm{QBG}(W)$ for $W$ of type $\tilde{A}_1$, with paths labelled by bold arrows.

**Example 3.2.5.** Consider the shortest paths in $\mathrm{QBG}(W)$ from $s_0s_1$ to $s_1$, as depicted in Figure 3.2. In each case, the weight of the path is $\tilde{\alpha}_0^\vee + \tilde{\alpha}_1^\vee = \delta$, and hence the weight from $s_0s_1$ to $s_1$ is $\mathrm{wt}(s_0s_1 \Rightarrow s_1) = \delta$. Note that there exists a unique shortest path of length one from $s_1$ to $s_0 s_1$ of weight 0, from which it can be seen that the weight function is not symmetric.

From Theorem 3.2.4, we immediately obtain the following corollary.

**Corollary 3.2.6.** *For type $\widehat{\mathrm{SL}}_2$, the generalised Demazure product $*^p_{u,v} = *_{u,v}$ is independent of $p$.*

The following fact about paths in $\mathrm{QBG}(W)$ will also be useful when showing that the generalised Demazure product is independent of the choice of $(u, v) \in M_{x,y}$.

**Theorem 3.2.7.** *Let $x = w_x \varepsilon^{\mu_x} \in W_{\mathcal{T}}$, and let $u \in W$. Then $\ell(u \Rightarrow w_x v)$ is minimised by a unique $v \in \mathrm{LP}(x)$, as is $\ell(v \Rightarrow u)$.*

*Proof.* First assume that $\mathrm{lev}(x) > 0$, and consider $\ell(u \Rightarrow w_x v)$. As before, if $|\mathrm{LP}(x)| = 1$ then the statement is trivial, so assume that $|\mathrm{LP}(x)| = 2$. From Corollary 3.1.9, we can write $v, vs_i \in \mathrm{LP}(x)$ for some $i \in \{0, 1\}$.

From the proof of Theorem 3.2.4, we have that $\ell(u \Rightarrow w_x v) = |\ell(u) - \ell(w_x v)|$ if either $u \leq w_x v$ or if $u \geq w_x v$ and $u, w_x v$ are same-sided, otherwise $\ell(u \Rightarrow w_x v) = |\ell(u) - \ell(w_x v)| + 2$. It is then clear that $\ell(u \Rightarrow w_x v) \neq \ell(u \Rightarrow w_x v s_i)$, as these values must have different parities, since $\ell(w_x v s_i) = \ell(w_x v) \pm 1$, and so one of $v, vs_i$ will uniquely minimise the distance among that pair.

Finally assume $|\mathrm{LP}(x)| = 3$. Then by Corollary 3.1.11, $w_x \mathrm{LP}(x) = \{s_1, e, s_0\}$. If $u \in w_x \mathrm{LP}(x)$, the statement is trivial, so assume otherwise. If $s_1 u > s_1$, we have that $\ell(u \Rightarrow s_0) = |\ell(u) - 1|$ and $\ell(u \Rightarrow s_1) = |\ell(u) - 1| + 2$ by the above, which also gives $\ell(u \Rightarrow e) = |\ell(u) - 1| + 1$, and hence $v = w_x^{-1} s_0$ is the unique minimiser. If instead $s_1 u < s_1$, a similar argument shows that $v = w_x^{-1} s_1$ is the unique distance minimiser, and so in all cases we have a unique $v \in \mathrm{LP}(x)$ minimising $\ell(u \Rightarrow w_x v)$.

For $\ell(v \Rightarrow u)$, an analogous argument holds if $|\mathrm{LP}(x)| \leq 2$. If instead $|\mathrm{LP}(x)| = 3$, then from Corollaries 3.1.9 and 3.1.10, we may write $\mathrm{LP}(x) = \{v', v's_i, v's_i s_j\} \neq \{s_1, e, s_0\}$ for some $v' \in W$ and some $i \neq j \in \{0, 1\}$, such that $v' < v's_i < v's_i s_j$. If $u \in \mathrm{LP}(x)$, the statement is trivial, so assume otherwise.

First, consider $\mathrm{LP}(x)$ such that $v' \neq e$. We can use the same length formulae and argument as above to deduce the following. If $u \leq v'$, then $v = v'$ will uniquely minimise $\ell(v \Rightarrow u)$. Similarly, if $u \geq v's_i s_j$, then $v = v's_i s_j$ will instead uniquely minimise the distance. Otherwise, $u$ must have the same length as an element of $\mathrm{LP}(x)$. If $\ell(u) = \ell(v')$ or $\ell(u) = \ell(v's_i)$, then $v'$ will be the unique distance minimiser with distance 2 or 1 respectively. If $\ell(u) = \ell(v's_i s_j)$, then $v's_i$ is the unique distance minimiser.

Now, let $\mathrm{LP}(x) = \{e, s_i, s_i s_j\}$ for some $i \neq j$, and again assume that $u \notin \mathrm{LP}(x)$. Then $\ell(v \Rightarrow u)$ is uniquely minimised by $s_i s_j$ if $u > s_i s_j$. Otherwise, $\ell(v \Rightarrow s_j s_i)$ is uniquely minimised by $v = s_i$ and $\ell(v \Rightarrow s_j)$ is uniquely minimised by $v = e$, and hence we have a unique distance minimiser in all cases with $\mathrm{lev}(x) > 0$.

For $\mathrm{lev}(x) = 0$, the case $x = \varepsilon^{\mu_x}$ is trivial by Proposition 3.1.3. In all other cases, the proof uses the same computations, relying on the fact that $w_x\mathrm{LP}(x)$ remains either entirely left or right-sided. □

### § 3.2.3 Well-Definedness

We now state the main theorem of this section.

**Theorem 3.2.8.** *For type $\widehat{\mathrm{SL}}_2$, the generalised Demazure product $x *^p_{u,v} y = x * y$ is independent of the choice of $(u, v) \in M_{x,y}$ and $p : u \Rightarrow w_y v$.*

Before proving this, we first show its equivalence to a more explicit condition.

**Proposition 3.2.9.** *Let $x, y \in W_{\mathcal{T}}$, $(u, v) \in M_{x,y}$, and let $w_y \in W$ be the Weyl component of $y$. Then if $uv^{-1}$ and $v\mathrm{wt}(u \Rightarrow w_y v)$ are independent of the choice of $u, v$, Theorem 3.2.8 holds.*

*Proof.* Note that by Theorem 3.2.4, $*^p_{u,v}$ is certainly independent of $p$. Define the elements $\varphi_1 = uv^{-1} \in W$, $\varphi_2 = v\mathrm{wt}(u \Rightarrow w_y v) \in \mathbb{Z}\Phi^\vee$. Then the generalised Demazure product becomes

$$\begin{aligned} x *^p_{u,v} y = x *_{u,v} y &= w_x uv^{-1} \varepsilon^{vu^{-1}\mu_x + \mu_y - v\mathrm{wt}(u \Rightarrow w_y v)} \\ &= w_x \varphi_1 \varepsilon^{\varphi_1^{-1}\mu_x + \mu_y - \varphi_2}. \end{aligned}$$

Hence if $\varphi_1, \varphi_2$ are independent of the choice of $u, v$, the product is fully determined. □

*Proof of Theorem 3.2.8.* We first prove this holds for $x, y \in W_{\mathcal{T}}$ with $\mathrm{lev}(x), \mathrm{lev}(y) \neq 0$.

By Corollary 3.1.8, we have that $1 \leq |\mathrm{LP}(x)|, |\mathrm{LP}(y)| \leq 3$, and by Corollary 3.1.9, $\mathrm{LP}(x), \mathrm{LP}(y)$ are each connected by vertical edges in $\mathrm{QBG}(W)$. Note also that $w_y\mathrm{LP}(y)$ will also be connected by vertical edges, as multiplication on the left preserves this condition.

If either $|\mathrm{LP}(x)| = 1$ or $|\mathrm{LP}(y)| = 1$, then by Theorem 3.2.7, there is a unique distance minimising pair $M_{x,y} = \{(u, v)\}$, and so there are no choices to be made, in which case the theorem is true immediately.

Next we consider $|\mathrm{LP}(x)| = |\mathrm{LP}(y)| > 1$, with either $|\mathrm{LP}(x)| = 2$ or $|\mathrm{LP}(y)| = 2$. By Theorem 3.2.7, we must have that $|M_{x,y}| \leq 2$. Assume that $\mathrm{LP}(x) \cap w_y\mathrm{LP}(y) = \emptyset$, otherwise a distance minimising pair $(u, v)$ must have $u = w_y v$, in which case $uv^{-1} = w_y$ and $v\mathrm{wt}(u \Rightarrow w_y v) = 0$ are both forced.

If $\mathrm{LP}(x)$ and $w_y\mathrm{LP}(y)$ are same-sided in $\mathrm{QBG}(W)$, then it is clear from that graph that there is a unique distance minimising pair. If instead $\mathrm{LP}(x)$ and $w_y\mathrm{LP}(y)$ are opposite-sided, then there is again a unique distance minimising pair unless 2 elements of $\mathrm{LP}(x)$ are each connected to an element of $w_y\mathrm{LP}(y)$ by a diagonal edge, as pictured

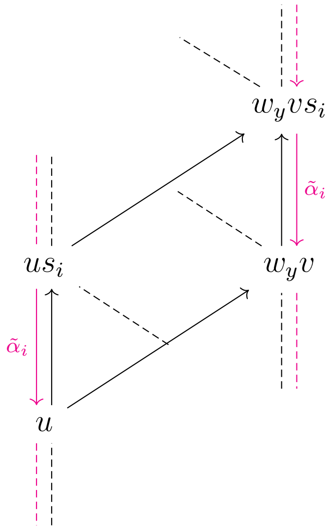


Figure 3.3: A portion of QBG$(W)$ when $\ell(w_y v) = \ell(us_i)$ and $w_y v, us_i$ are opposite-sided.

in Figure 3.3. In this case, $|M_{x,y}| = \{(u,v),(us_i,vs_i)\}$, where $\{u, us_i\} \subseteq \mathrm{LP}(x)$ and $\{v, vs_i\} \subseteq \mathrm{LP}(y)$, with $u < us_i$ and $v < vs_i$.

For each pair $(u_1, v_1) \in M_{x,y}$, we calculate $\varphi_1 = u_1 v_1^{-1}$ and $\varphi_2 = v_1 \mathrm{wt}(u_1 \Rightarrow w_y v_1)$:

$$\begin{aligned} \varphi_1: \quad & (us_i)(w_y v s_i)^{-1} && = us_i s_i v^{-1} w_y^{-1} = uv^{-1}w_y^{-1}. \\ & (u)(w_y v)^{-1} && = uv^{-1}w_y^{-1}. \end{aligned}$$

$$\begin{aligned} \varphi_2: \quad & vs_i \mathrm{wt}(us_i \Rightarrow w_y v s_i) && = vs_i(0) && = 0. \\ & v\mathrm{wt}(u \Rightarrow w_y v) && = v(0) && = 0. \end{aligned}$$

Hence $\varphi_1$ and $\varphi_2$ are independent of the choice of pair $(u_1, v_1) \in M_{x,y}$ as required.

The final non-boundary case to consider is when both $|\mathrm{LP}(x)| = 1$ and $|\mathrm{LP}(y)| = 3$. If $|M_{x,y}| = 1$, then we are done. If $|M_{x,y}| = 2$, then we must have a situation as in Figure 3.3, in which case the previous calculations apply and we are again done. The final option is $|M_{x,y}| = 3$, in which case $M_{x,y} = \{(u,v),(us_i,vs_i),(us_is_j,vs_is_j)\}$, for $i \neq j$, where $u, v$ are opposite-sided and $\ell(v) = \ell(u) + 1$. We have proven above that $\varphi_1, \varphi_2$ are constant for pairs of the form $(u,v),(us_i,vs_i)$ as previously described, and so must also be constant for $(us_i,vs_i),(us_is_j,vs_is_j)$. Hence, these values are constant for every pair in $M_{x,y}$, as required.

Now we must consider the case when at least one of $x, y$ are on the boundary of the Tits cone. Note that if either of $x$ or $y$ is of the form $\varepsilon^{m\delta}$, then its corresponding length positive set is the entirety of $W$, in which case we are always able to choose $u = w_y v$ and the result comes immediately, so assume that neither $x$ nor $y$ is of this form.

If $\mathrm{lev}(x) = 0$ but $\mathrm{lev}(y) > 0$, recall that $\mathrm{LP}(x)$ is either a complete left or right side of $\mathrm{QBG}(W)$ by Proposition 3.1.3, and $w_y\mathrm{LP}(y)$ is connected vertically. Hence, these sets are either same-sided or opposite-sided. In the first case, we must have $w_y\mathrm{LP}(y) \subset \mathrm{LP}(x)$, and so $\mathrm{LP}(x) \cap w_y\mathrm{LP}(y) \neq \emptyset$, forcing the choices of $\varphi_1$ and $\varphi_2$. If instead they are opposite-sided, then it is easy to see that a distance minimising path must have length 1, and we recover the same situation as in Figure 3.3, with either 2 or 3 distance minimising pairs. An analogous argument applies if $\mathrm{lev}(x) > 0$ and $\mathrm{lev}(y) = 0$.

Finally, if $\mathrm{lev}(x) = \mathrm{lev}(y) = 0$, then $w_y\mathrm{LP}(y)$ is single-sided, consisting of all elements above $w_y$. If $\mathrm{LP}(x)$ is same-sided, then we have an overlap and we are done, and if $\mathrm{LP}(x)$ is opposite-sided, we see that all distance minimising pairs are again of the same form as in Fig. 3.3, in which case $\varphi_1$ and $\varphi_2$ are constant. □

### § 3.2.4 Relation to the Single Affine Demazure Product

This section focuses on the statement and proof of the following theorem.

**Theorem 3.2.10.** *Let $x, y \in W_{\mathcal{T}}$ of type $\widehat{\mathrm{SL}}_2$ such that $x, y \in W$. Then the double affine Demazure product $x * y$ is equal to the Coxeter-theoretic single affine Demazure product.*

*Proof.* Throughout, we use the notation $x = w_x, y = w_y$ and $\mu_x = \mu_y = 0$. First assume $x = e$. Then $\mathrm{LP}(x) = W$ by Proposition 3.1.3. Hence we can choose $u = w_y v$ for some $v \in \mathrm{LP}(x)$, giving $\mathrm{wt}(u \Rightarrow w_y v) = 0$, and we find:

$$e * y = w_x u v^{-1} \varepsilon^{vu^{-1}\mu_x + \mu_y - v\mathrm{wt}(p)} = w_x w_y \varepsilon^0 = y,$$

which lines up with the single-affine product. Similarly, let $y = e$ such that $\mathrm{LP}(y) = W$. Then we can choose $v = w_y^{-1} u$ for some $u \in \mathrm{LP}(x)$ to get $x * e = x$ as expected.

Now assume that $x, y \neq e$, and so $\mathrm{LP}(x), \mathrm{LP}(y) \neq W$. By Proposition 3.1.3, we have that $\mathrm{LP}(x)$ and $\mathrm{LP}(y)$ each consist of a single side of $\mathrm{QBG}(W)$. From the description of $\mathrm{LP}(y) = \{v \in W \mid \ell(w_y v) = \ell(w_y) + \ell(v)\}$ from Equation (3.15), we have that $w_y\mathrm{LP}(y)$ consists of all elements of a single side that are greater than or equal to $w_y$ in the Bruhat order. Note that this side may be different from that of $\mathrm{LP}(y)$.

If $\mathrm{LP}(x)$ and $w_y\mathrm{LP}(y)$ are same-sided, then $\mathrm{LP}(x) \cap w_y\mathrm{LP}(y) \neq \emptyset$ and so we can choose $u \in \mathrm{LP}(x), v \in \mathrm{LP}(y)$ such that $u = w_y v$, and hence:

$$x * y = w_x w_y \varepsilon^0 = xy.$$

There must exist a unique $i \in \{0, 1\}$ such that $s_i \in \mathrm{LP}(x)$. This implies $\ell(w_x s_i) = \ell(w_x) + 1$, and in particular $x = ...s_j$ for some $j \neq i \in \{0, 1\}$. We must also have that $w_y \in w_y\mathrm{LP}(y)$ since $e \in \mathrm{LP}(y)$. Since $w_y\mathrm{LP}(y)$ and $\mathrm{LP}(x)$ are same-sided by assumption, we have that $w_y = s_i...$ . Hence, their single affine Demazure product is:

$$x * y = (...s_j)(s_i...) = xy,$$

as we have no braid relations in $W$, and so in this case the two separate Demazure products are aligned.

Finally, assume that $\mathrm{LP}(x)$ and $w_y\mathrm{LP}(y)$ are opposite-sided. Then we must have a shortest path $u \Rightarrow w_y v$ such that $w_y v = s_i u$ for some $i \in \{0,1\}$, which is a path of weight 0. Hence:

$$x * y = w_x u v^{-1} \varepsilon^0 = w_x(s_i w_y v)v^{-1} = x s_i y.$$

As before, fix $i$ such that $s_i \in \mathrm{LP}(x)$ and so $x = ...s_i s_j$ for $j \neq i$. Then $w_y \in \mathrm{LP}(y)$, but now $\mathrm{LP}(x)$ and $w_y\mathrm{LP}(y)$ are opposite-sided, so $w_y = s_j s_i....$ Therefore, the single affine Demazure product is

$$x * y = (...s_i s_j) * (s_j s_i...) = (...s_i) * (s_j * s_j) * (s_i...) = (...s_i) * s_j * (s_i...) = x s_i y,$$

and so in all cases the Demazure products agree. $\square$

## § 3.3 Associativity

Our goal in this section is to prove that the Demazure product for the double affine Weyl semigroup of type $\widehat{\mathrm{SL}}_2$ is associative, as would be expected of a Demazure product.

### § 3.3.1 Determining $\mathrm{LP}(x * y)$

In order to show associativity, we will first need to prove an extension of Theorem 5.12(b) in [Sch24] to the double-affine case. We write $M^y_{x,y} = \{v \in \mathrm{LP}(y) \mid (u,v) \in M_{x,y}$ for some $u \in \mathrm{LP}(x)\}$, the $\mathrm{LP}(y)$ component of the set of distance-minimising pairs. Firstly, we state the following proposition, an extension of a lemma by Schremmer ([Sch23, Lemma 2.14]).

**Proposition 3.3.1.** *Let $\tilde{\beta}$ be a simple affine root, and let $x \in W_{\mathcal{T}}$ with $u \in \mathrm{LP}(x)$. Then $us_{\tilde{\beta}} \in \mathrm{LP}(x) \iff \ell(x, u\tilde{\beta}) = 0$.*

*Proof.* Mimicking Schremmer's proof for the affine case, observe that the condition $u, us_{\tilde{\beta}} \in \mathrm{LP}(x)$ implies $\ell(x, u\tilde{\beta}), \ell(x, -u\tilde{\beta}) \geq 0 \implies \ell(x, u\tilde{\beta}) = 0$. Furthermore, if $\ell(x, u\tilde{\beta}) = 0$, then $\ell(x, u\tilde{\alpha}) \geq 0$ for any $\tilde{\alpha} \in \Phi^+ \cup \{-\tilde{\beta}\} = s_{\tilde{\beta}}(\Phi^+ \cup \{-\tilde{\beta}\})$, and so $\ell(x, us_{\tilde{\beta}}\tilde{\alpha}) \geq 0$ for all $\tilde{\alpha} \in \Phi^+$ as required. $\square$

**Theorem 3.3.2.** *For $x, y \in W_{\mathcal{T}}$, $\mathrm{LP}(x * y) = M^y_{x,y}$.*

*Proof.* Let $(u, v) \in M_{x,y}$, and write $x = w_x\varepsilon^{\mu_x}, y = w_y\varepsilon^{\mu_y}$ as before. Let $r \in W$. Then:

$$\begin{aligned}
\ell(x * y, r\tilde{\alpha}) &= \langle vu^{-1}\mu_x + \mu_y - v\mathrm{wt}(u \Rightarrow w_y v), r\tilde{\alpha}\rangle + \Phi^+(r\tilde{\alpha}) - \Phi^+(w_x uv^{-1}r\tilde{\alpha}) \\
&= \Big(\langle \mu_y, r\tilde{\alpha}\rangle + \Phi^+(r\tilde{\alpha}) - \Phi^+(w_y r\tilde{\alpha})\Big) \\
&\qquad + \Big(\langle \mu_x, uv^{-1}r\tilde{\alpha}\rangle + \Phi^+(uv^{-1}r\tilde{\alpha}) - \Phi^+(w_x uv^{-1}r\tilde{\alpha})\Big) \\
&\qquad - \langle \mathrm{wt}(u \Rightarrow w_y v), v^{-1}r\tilde{\alpha}\rangle + \Phi^+(w_y r\tilde{\alpha}) - \Phi^+(uv^{-1}r\tilde{\alpha}) \\
&= \ell(y, r\tilde{\alpha}) + \ell(x, uv^{-1}r\tilde{\alpha}) - \langle \mathrm{wt}(u \Rightarrow w_y v), v^{-1}r\tilde{\alpha}\rangle \\
&\qquad + \Phi^+(w_y r\tilde{\alpha}) - \Phi^+(uv^{-1}r\tilde{\alpha}).
\end{aligned}$$

Assume $r \neq v$. Our goal is to show that $r$ cannot be length positive for $x * y$ unless it is part of another distance-minimising pair. Write $r = vr_o$ for some $r_o \in W\backslash\{e\}$, so

$$\ell(x * y, r\tilde{\alpha}) = \ell(y, vr_o\tilde{\alpha}) + \ell(x, ur_o\tilde{\alpha}) - \langle \mathrm{wt}(u \Rightarrow w_y v), r_o\tilde{\alpha}\rangle + \Phi^+(w_y vr_o\tilde{\alpha}) - \Phi^+(ur_o\tilde{\alpha}). \tag{3.18}$$

Let $\tilde{\alpha} \in \mathrm{Inv}(r_o)$ and write $\tilde{\beta} = -r_o\tilde{\alpha} \in \mathrm{Inv}(r_o^{-1})$. Using the properties that $\ell(z, -\tilde{\alpha}) = -\ell(z, \tilde{\alpha})$ and $\Phi^+(-\tilde{\alpha}) = 1 - \Phi^+(\tilde{\alpha})$ ([Sch23, Lemma 2.6]), we find:

$$\ell(x * y, r\tilde{\alpha}) = -\ell(y, v\tilde{\beta}) - \ell(x, u\tilde{\beta}) + \langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle - \Phi^+(w_y v\tilde{\beta}) + \Phi^+(u\tilde{\beta}). \tag{3.19}$$

It is clear from the graph of $\mathrm{QBG}(W)$ that the weights of edges that contribute to $\mathrm{wt}(u \Rightarrow w_y v)$ alternate between $\tilde{\alpha}_0 = -\alpha + \delta$ and $\tilde{\alpha}_1 = \alpha$. Hence, writing $\mathrm{wt}(u \Rightarrow w_y v) = c_1\alpha + c_2\delta$, we must have $c_1 \in \{0, \pm 1\}$ and $c_2 \in \mathbb{Z}_{\geq 0}$. Therefore we must have that $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle \in \{-2, 0, 2\}$.

Note that $\ell(y, v\tilde{\beta}), \ell(x, u\tilde{\beta}) \geq 0$ as $u \in \mathrm{LP}(x), v \in \mathrm{LP}(y)$. Hence, if $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = -2$ for some $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$, then $\ell(x * y, r\tilde{\alpha}) \leq 0 + 0 - 2 - 0 + 1 = -1$, and so $r \notin \mathrm{LP}(x * y)$.

Assume that $r \in \mathrm{LP}(x * y)$. We first show that if $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 2$ for some $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$, then $|\mathrm{Inv}(r_o^{-1})| = 1$.

If $|\mathrm{Inv}(r_o^{-1})| > 1$, write $r_o^{-1} = ...s_j s_i$, for some $i, j \in \{0, 1\}, i \neq j$. Then by the classification of inversion sets ([Hum90, Sec. 5.6]), we must have that $\{\tilde{\alpha}_i, s_i(\tilde{\alpha}_j)\} \subset \mathrm{Inv}(r_o^{-1})$. It is easy to check that the classical parts of these two roots will differ by a sign. Then if $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 2$ for some $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$, it must also be non-zero for any $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$. Hence, the pairing of the weight with $\tilde{\alpha}_i$ and $s_i(\tilde{\alpha}_j)$ must both be non-zero, and also differ by a sign. We can therefore choose $\tilde{\beta} \in \{\tilde{\alpha}_i, s_i(\tilde{\alpha}_j)\}$ such that $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = -2$, implying that $r \notin \mathrm{LP}(x * y)$ by the above, giving a contradiction.

Therefore, we must have that either $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 2$ with $r_o = s_{\tilde{\beta}}$ and $\tilde{\beta}$ simple, or $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 0$ for all $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$.

In the first case, note that $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 2$ implies that we must have an odd number of downwards edges in a shortest path $u \Rightarrow w_y v$, and $\tilde{\beta}$ will be the simple root that occurs most often as a weight in this path. From the diagram of $\mathrm{QBG}(W)$, we then see that we must have $u\tilde{\beta} < 0$ and $w_y v\tilde{\beta} > 0$, and so:

$$\ell(x * y, r\tilde{\alpha}) = -\ell(y, v\tilde{\beta}) - \ell(x, u\tilde{\beta}) + 2 - 1 + 0 = -\ell(y, v\tilde{\beta}) - \ell(x, u\tilde{\beta}) + 1.$$

For length positivity to hold, we must therefore have at least one of $\ell(x, u\tilde{\beta})$ and $\ell(y, v\tilde{\beta})$ is zero, and so either $us_{\tilde{\beta}} \in \mathrm{LP}(x)$ or $vs_{\tilde{\beta}} \in \mathrm{LP}(y)$, breaking distance-minimality, giving a contradiction.

In the second case, we must have an even number of downwards edges (possibly zero) in a shortest path $u \Rightarrow w_y v$. If we do have downwards edges, then it can be seen from the diagram that $u, w_y v$ must end in the same simple reflection, from which it can be checked that $\Phi^+(u\tilde{\beta}) = \Phi^+(w_y v\tilde{\beta})$. For length positivity, we must therefore have that $\ell(x, u\tilde{\beta}) = \ell(y, v\tilde{\beta}) = 0$. By the classification of inversion sets, they must always contain a simple root, and so choosing $\tilde{\beta}$ to be simple, we find $us_{\tilde{\beta}} \in \mathrm{LP}(x), vs_{\tilde{\beta}} \in \mathrm{LP}(y)$, again breaking distance-minimality.

Hence, we must have no downwards edges in a shortest path, i.e. $u \leq w_y v$. Again, choose $\tilde{\beta}$ to be simple. Then since $\langle \mathrm{wt}(u \Rightarrow w_y v), \tilde{\beta}\rangle = 0$, we must have either $us_{\tilde{\beta}} \in \mathrm{LP}(x)$ or $vs_{\tilde{\beta}} \in \mathrm{LP}(y)$ to ensure length positivity. If $us_{\tilde{\beta}} \in \mathrm{LP}(x)$, then we must have $us_{\tilde{\beta}} < u$ for distance-minimality, unless $u, w_y v$ differ by a single diagonal edge. This would then give that $\Phi^+(u\tilde{\beta}) = 0$, and so we also need $vs_{\tilde{\beta}} \in \mathrm{LP}(y)$. Distance-minimality then also implies that $\Phi^+(w_y v\tilde{\beta}) = 1$, forcing $r \notin \mathrm{LP}(x * y)$. The same occurs if we first assume that $vs_{\tilde{\beta}} \in \mathrm{LP}(y)$.

Therefore, the only possible situation that can occur is that $u, w_y v$ differ by a single diagonal edge, giving $\Phi^+(u\tilde{\beta}) = \Phi^+(w_y v\tilde{\beta})$, and forcing $\ell(x, u\tilde{\beta}) = \ell(y, v\tilde{\beta}) = \ell(x * y, v\tilde{\beta}) = 0$. This also tells us that $\mathrm{Inv}(u) \subset \mathrm{Inv}(w_y v)$. We now check that in this case, $v \in \mathrm{LP}(x * y)$.

$$\begin{aligned}\ell(x * y, v\tilde{\alpha}) &= \ell(y, v\tilde{\alpha}) + \ell(x, u\tilde{\alpha}) + \Phi^+(w_y v\tilde{\alpha}) - \Phi^+(u\tilde{\alpha}) \\ &< 0 \iff \ell(x, u\tilde{\alpha}) = \ell(y, v\tilde{\alpha}) = 0,\ \Phi^+(w_y v\tilde{\alpha}) = 0,\ \Phi^+(u\tilde{\alpha}) = 1.\end{aligned}$$

This occurs when $\tilde{\alpha} \in \mathrm{Inv}(w_y v)/\mathrm{Inv}(u) = \{u^{-1}\tilde{\alpha}_i\}$ for some $i \in \{0, 1\}$ satisfying $w_y v = s_i u$, and when $us_{\tilde{\alpha}} \in \mathrm{LP}(x), vs_{\tilde{\alpha}} \in \mathrm{LP}(y)$. This implies that $\ell(x * y, v\tilde{\alpha}) < 0$ only when $\tilde{\alpha} = \tilde{\beta}$, since $us_{\tilde{\beta}} \in \mathrm{LP}(x)$. However, $\ell(x * y, v\tilde{\beta}) = 0$ by the above, a contradiction. Hence $\ell(x * y, v\tilde{\alpha}) \geq 0$ for every positive $\tilde{\alpha}$ and so $v \in \mathrm{LP}(x * y)$.

Since $\ell(x, u\tilde{\beta}) = \ell(y, v\tilde{\beta}) = \ell(x * y, v\tilde{\beta}) = 0$, by Prop. 3.3.1, we have that $\tilde{\beta}$ is simple $\implies us_{\tilde{\beta}} \in \mathrm{LP}(x) \iff vs_{\tilde{\beta}} \in \mathrm{LP}(y) \iff vs_{\tilde{\beta}} \in \mathrm{LP}(x * y)$.

Note that if $|M_{x,y}| = 1$, then we have shown $r \neq v$ is never length positive for $x * y$. However, since length positive sets are non-empty, we must have that $v \in \mathrm{LP}(x * y)$

and so $\mathrm{LP}(x * y) = M^y_{x,y}$.

If instead $|M_{x,y}| > 1$, then we have shown that if $(u, v) \in M_{x,y}$, then $v \in \mathrm{LP}(x * y)$, and if $vr_0 \in \mathrm{LP}(x*y)$, then there exists a simple $\tilde{\beta} \in \mathrm{Inv}(r_o^{-1})$ such that $vs_{\tilde{\beta}} \in \mathrm{LP}(x*y)$ and $vs_{\tilde{\beta}} \in M^y_{x,y}$. By setting $v' = vs_{\tilde{\beta}}$, we can iterate along such elements, using the connectivity of length positive sets to find that $vr_o \in \mathrm{LP}(x*y)$ if and only if $vr_o \in M^y_{x,y}$ for all possible $r_o$, and so $\mathrm{LP}(x * y) = M^y_{x,y}$ as required. □

### § 3.3.2 Conditions for Associativity

We fix the notation $M_{x_1,x_2} = M_{1,2}$, and we fix notation such that the Demazure product of two elements $x_1, x_2 \in W_{\mathcal{T}}$ with $(u_{1,2}, v_{1,2}) \in M_{1,2}$ is given as follows:

$$x_1 * x_2 = (w_1\varepsilon^{\mu_1}) * (w_2\varepsilon^{\mu_2}) = w_1u_{1,2}v_{1,2}^{-1}\varepsilon^{v_{1,2}u_{1,2}^{-1}\mu_1+\mu_2-v_{1,2}\mathrm{wt}(u_{1,2}\Rightarrow w_2v_{1,2})}.$$

We also use the notation $x_i * x_j = x_{ij} = w_{ij}\varepsilon^{\mu_{ij}}$ throughout, extending this to other objects such as $(u_{ij,k}, v_{ij,k}) \in M_{ij,k} \subseteq \mathrm{LP}(x_{ij})\times\mathrm{LP}(x_k)$. The triple products $(x_1*x_2)*x_3$ and $x_1 * (x_2 * x_3)$ are given in this notation below.

$$\begin{aligned}(x_1 * x_2) * x_3 \\ &= w_1u_{1,2}v_{1,2}^{-1}u_{12,3}v_{12,3}^{-1}\varepsilon^{v_{12,3}u_{12,3}^{-1}(v_{1,2}u_{1,2}^{-1}\mu_1+\mu_2-v_{1,2}\mathrm{wt}(u_{1,2}\Rightarrow w_2v_{1,2}))+\mu_3-v_{12,3}\mathrm{wt}(u_{12,3}\Rightarrow w_3v_{12,3})},\end{aligned} \tag{3.20}$$

$$\begin{aligned}x_1 * (x_2 * x_3) \\ &= w_1u_{1,23}v_{1,23}^{-1}\varepsilon^{v_{1,23}u_{1,23}^{-1}\mu_1+v_{2,3}u_{2,3}^{-1}\mu_2+\mu_3-v_{2,3}\mathrm{wt}(u_{2,3}\Rightarrow w_3v_{2,3})-v_{1,23}\mathrm{wt}(u_{1,23}\Rightarrow w_2u_{2,3}v_{2,3}^{-1}v_{1,23})}.\end{aligned} \tag{3.21}$$

In the rest of this section, we aim to show that, for some suitable choices of $(u_{i,j}, v_{i,j}) \in M_{i,j}$, these formulae are equal, and are therefore always equal by well-definedness.

We know that each element $x \in W_{\mathcal{T}}$ can be written uniquely as $x = w\varepsilon^\mu$ for some $w \in W, \mu \in \mathcal{T}$. Hence, for the left and right triple products to be equal, we must have that the Weyl components $w_{12,3}, w_{1,23}$ and the coweight lattice components $\mu_{12,3}, \mu_{1,23}$ are each equal.

### § 3.3.3 Weyl Component

We can simplify the condition for the Weyl components to be equal as follows

$$\begin{aligned}w_1u_{1,2}v_{1,2}^{-1}u_{12,3}v_{12,3}^{-1} = w_1u_{1,23}v_{1,23}^{-1} \iff & u_{1,2}v_{1,2}^{-1}u_{12,3}v_{12,3}^{-1} = u_{1,23}v_{1,23}^{-1} \\ \iff & g := u_{1,23}^{-1}u_{1,2}v_{1,2}^{-1}u_{12,3}v_{12,3}^{-1}v_{1,23} = e.\end{aligned}$$

Our goal is then to show that $g = e$. We first simplify $g$ before splitting into cases.

**Proposition 3.3.3.** *There is some choice of distance minimising pairs such that $v_{1,2} = u_{12,3}$ and $v_{2,3} = v_{1,23}$.*

*Proof.* Note that $u_{12,3} \in \mathrm{LP}(x_{12})$, and so there is some pair $(-, u_{12,3}) \in M_{1,2}$. However, we may choose any pair $(u_{1,2}, v_{1,2}) \in M_{1,2}$ for the calculation of the product $x_1 * x_2$, and so we are free to fix $v_{1,2}$ by fixing the pairs $(u_{1,2}, v_{1,2}) = (-, u_{12,3})$, giving us the choice $v_{1,2} = u_{12,3}$.

Similarly, $v_{1,23} \in \mathrm{LP}(x_{23})$, and so we can fix the pairs $(u_{2,3}, v_{2,3}) = (-, v_{1,23})$, giving us a choice of $v_{2,3} = v_{1,23}$ as required. □

With these choices, we obtain the following simplified expression for $g$.

$$g = u_{1,23}^{-1} u_{1,2} v_{1,2}^{-1} u_{12,3} v_{12,3}^{-1} v_{1,23} = u_{1,23}^{-1} u_{1,2} u_{12,3}^{-1} u_{12,3} v_{12,3}^{-1} v_{1,23} = u_{1,23}^{-1} u_{1,2} v_{12,3}^{-1} v_{1,23}.$$

Next, we investigate when the element $u_{1,23}^{-1} u_{1,2} = e$. To do this, we use the following statement.

**Proposition 3.3.4.** *If $u_{12,3} = u_{2,3}$, then both $v_{12,3} = v_{1,23}$ and $u_{1,23} = u_{1,2}$.*

*Proof.* Theorem 3.2.7 immediately gives us the first part of the claim by noting that $u_{12,3}, u_{2,3} \in \mathrm{LP}(x_2)$. For the second part, note that if $w_2 v_{1,2} = w_{23} v_{1,23}$, then $u_{1,2} = u_{1,23}$, by Theorem 3.2.7, and so we consider the following calculation:

$$(w_2 v_{1,2})^{-1}(w_{23} v_{1,23}) = v_{1,2}^{-1} w_2^{-1} w_2 u_{2,3} v_{2,3}^{-1} v_{1,23} = v_{1,2}^{-1} u_{2,3} = u_{12,3}^{-1} u_{2,3}. \tag{3.22}$$

Hence, if $u_{12,3} = u_{2,3}$, then $(w_2 v_{1,2})^{-1}(w_{23} v_{1,23}) = u_{12,3}^{-1} u_{2,3} = e$, giving us that $w_2 v_{1,2} = w_{23} v_{1,23}$ and so $u_{1,2} = u_{1,23}$, proving the rest of the claim. □

As an immediate consequence, we get that if $u_{12,3} = u_{2,3}$, then $g = e$ and we are done, and so henceforth we will assume that we cannot have $u_{12,3} = u_{2,3}$, so in particular $|\mathrm{LP}(x_2)| > 1$. Note that we can rewrite the condition that $g = e$ as:

$$\begin{aligned}
u_{1,23}^{-1} u_{1,2} v_{12,3}^{-1} v_{1,23} = e &\iff v_{12,3}^{-1} v_{2,3} = u_{1,2}^{-1} u_{1,23} \\
&\iff (w_3 v_{12,3})^{-1}(w_3 v_{2,3}) = u_{1,2}^{-1} u_{1,23} \\
&\iff (w_2 w_3 v_{12,3})^{-1}(w_2 w_3 v_{2,3}) = u_{1,2}^{-1} u_{1,23},
\end{aligned}$$

which relates to the following paths:

$$\begin{aligned}
u_{1,2} \Rightarrow w_2 v_{1,2} \quad &= w_2 u_{12,3} \Rightarrow w_2 w_3 v_{12,3}, \\
u_{1,23} \Rightarrow w_{23} v_{1,23} &= w_2 u_{2,3} \quad \Rightarrow w_2 w_3 v_{2,3}.
\end{aligned}$$

We see that the condition that $g = e$ corresponds to the 'difference' in the start and ends of these paths being the same. Note that everything we have stated so far applies to any double affine Weyl semigroup with a well-defined Demazure product, however for the rest of this proof we will work explicitly with the case of $\widehat{\mathrm{SL}}_2$.

For now, we fix that $\mathrm{lev}(x_1), \mathrm{lev}(x_2) > 1$, so that $|\mathrm{LP}(x_i)| \leq 2$. By Theorem 3.2.7, $\mathrm{LP}(x_2)$ is connected vertically in $\mathrm{QBG}(W)$. We may therefore write $u_{12,3} = u_{2,3}s_i$ for some $i \in \{0, 1\}$, which gives us $w_{23}v_{1,23} = w_2v_{1,2}s_i$. We split the remaining work into two cases; either $u_{1,23} \neq u_{1,2}$, or $u_{1,23} = u_{1,2}$.

In the first instance, note that $\mathrm{LP}(x_2) = \{v_{1,2}, v_{1,2}s_i\}$ and $\mathrm{LP}(x_1) = \{u_{1,2}, u_{1,23}\}$. Hence, for distance minimality to hold, we must have a situation as in Fig. 3.3, otherwise we would require $u_{1,2} = u_{1,23}$. Therefore $|M_{1,2}| = 2$, and from the diagram we find that $u_{1,2} = u_{1,23}s_i$. Furthermore, by assumption we cannot choose $u_{12,3} = u_{2,3}$ despite having two options for $u_{12,3}$, and so again we must have an alignment as in Fig. 3.3. Therefore $v_{12,3} = v_{2,3}s_i$, and so $g = u_{1,23}^{-1}u_{1,2}v_{12,3}^{-1} = s_is_i = e$.

In the second instance, we have $u_{1,23} = u_{1,2}$. If we also had $|M_{2,3}| = 2$, then $\mathrm{LP}(x_3) = \{v_{2,3}, v_{2,3}s_i\}$, and so we would have the freedom to choose $v_{1,23} = v_{2,3}$. For distance minimality we would therefore need $w_{23}v_{1,23} = w_2v_{1,2}$, a contradiction. Hence $|M_{2,3}| = 1$ and so $v_{2,3} = v_{12,3}$ is forced, giving $g = e$ as required. We have therefore proven the following:

**Proposition 3.3.5.** *The generalised Demazure product associated to affine $\widehat{SL}_2$ is associative in the Weyl component for level greater than one.*

### § 3.3.4 Lattice Component

In this section, we continue to assume that $\mathrm{lev}(x), \mathrm{lev}(y) > 1$. We can write down the conditions in the coweight lattice part for associativity directly from the left and right triple products that were calculated in Equations (3.20) and (3.21):

$$\begin{aligned} &v_{12,3}u_{12,3}^{-1}(v_{1,2}u_{1,2}^{-1}\mu_1 + \mu_2 - v_{1,2}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2})) + \mu_3 \\ &\qquad\qquad\qquad\qquad\qquad\qquad - v_{12,3}\mathrm{wt}(u_{12,3} \Rightarrow w_3v_{12,3}) \\ &= v_{1,23}u_{1,23}^{-1}\mu_1 + v_{2,3}u_{2,3}^{-1}\mu_2 + \mu_3 - v_{2,3}\mathrm{wt}(u_{2,3} \Rightarrow w_3v_{2,3}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3}v_{2,3}^{-1}v_{1,23}). \end{aligned}$$

We can immediately reduce this expression; the $\mu_1$ components cancel out, as the corresponding equations are simply associativity in the Weyl part, and the $\mu_3$ components cancel trivially. So, as long as we make the same choices as we did previously, namely that $v_{1,2} = u_{12,3}$ and $v_{2,3} = v_{1,23}$, we obtain the following condition:

$$\begin{aligned} &v_{12,3}u_{12,3}^{-1}\mu_2 - v_{12,3}u_{12,3}^{-1}v_{1,2}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{12,3}\mathrm{wt}(u_{12,3} \Rightarrow w_3v_{12,3}) \\ &= v_{2,3}u_{2,3}^{-1}\mu_2 - v_{2,3}\mathrm{wt}(u_{2,3} \Rightarrow w_3v_{2,3}) - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3}v_{2,3}^{-1}v_{1,23}). \end{aligned}$$

We can then apply our choices and group together different elements to further simplify this expression as much as we can before investigating individual components, giving

$$\begin{aligned} (v_{12,3}u_{12,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2) - (v_{12,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3})) \\ - (v_{12,3}\mathrm{wt}(u_{12,3} \Rightarrow w_3v_{12,3}) - v_{1,23}\mathrm{wt}(u_{2,3} \Rightarrow w_3v_{2,3})) \ = \ 0. \end{aligned}$$

Looking at this, we define the following three objects $\eta_1, \eta_2, \eta_3 \in P^\vee$ such that our conditions reduce to showing that $\eta_1 - \eta_2 - \eta_3 = 0$.

$$\begin{aligned} \eta_1 &:= v_{12,3}u_{12,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2, \\ \eta_2 &:= v_{12,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3}), \\ \eta_3 &:= v_{12,3}\mathrm{wt}(u_{12,3} \Rightarrow w_3v_{12,3}) - v_{1,23}\mathrm{wt}(u_{2,3} \Rightarrow w_3v_{2,3}). \end{aligned}$$

We now begin to investigate what values the elements $\eta_1, \eta_2, \eta_3$ can take, and what conditions need to be met for these values to be obtained.

**Proposition 3.3.6.** *If the following three conditions are met, then $\eta_2 = -v_{2,3}(\tilde{\alpha}_i^\vee)$, otherwise $\eta_2 = 0$.*

(i) $u_{2,3} = v_{1,2}s_i$.

(ii) $v_{2,3} = v_{12,3}$.

(iii) $\ell(u_{1,2}) \geq \ell(w_2v_{1,2}) > \ell(w_2v_{1,2}s_i)$.

*Proof.* First, assume condition (i) cannot hold. Since $|\mathrm{LP}(x)| \leq 2$, we must have that $u_{2,3} = v_{1,2}$, giving us the following.

$$w_{23}v_{1,23} = w_2u_{2,3}v_{2,3}^{-1}v_{1,23} = w_2u_{2,3} = w_2v_{1,2}$$

Where we have used the choice that $v_{2,3} = v_{1,23}$. However, uniqueness of distance minimising elements then forces $u_{1,2} = u_{1,23}$. We also have $u_{2,3} = v_{1,2} = u_{12,3}$ from our previous choice, and so $v_{2,3} = v_{12,3}$ by uniqueness in the other direction. We then have:

$$\begin{aligned} \eta_2 &= v_{12,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3}) \\ &= v_{2,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{2,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) \ = \ 0. \end{aligned}$$

Now we assume that condition (i) holds, and that $u_{2,3} = v_{1,2}s_i$ for some $i \in \{0, 1\}$. It then immediately follows that $|\mathrm{LP}(x_2)| = 2$, with $w_{23}v_{1,23} = v_2 v_{1,2} s_i$ following an earlier calculation, and $u_{2,3} = u_{12,3}s_i$ from our choices. We can then split this into 4 different cases depending on the positioning of $u_{1,2}, u_{1,23}, w_2v_{1,2}, w_{23}v_{1,23}$ within $\mathrm{QBG}(W)$, up to the involution $s_0 \leftrightarrow s_1$ which corresponds to mirroring the diagrams along the vertical centre. This is depicted in Figure 3.3.4 below.

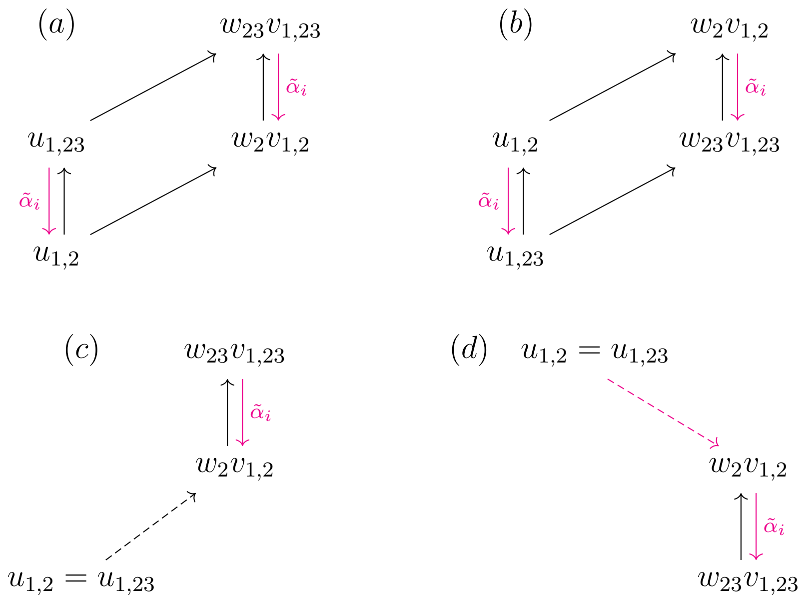


Figure 3.4: Portions of $\mathrm{QBG}(W)$ to be considered in associativity.

This exhausts all possible cases, as any other arrangement will violate either uniqueness of distance-minimising elements or the minimality itself.

We use the graphs in Figure 3.3.4 to split our work into cases. In Cases (a), (b), and (c), the weights that $\eta_2$ depends on are both 0 and so $\eta_2 = 0$. Hence, the only case left to check is (d), which occurs precisely when we have $\ell(u_{1,2}) \geq \ell(w_2v_{1,2}) > \ell(w_2v_{1,2}s_i)$, i.e. when condition (iii) holds.

We can now further split this into 2 more cases depending on whether or not condition (ii) holds. First, assume that it does not hold, and so $v_{2,3} = v_{12,3}s_i$. Then we must have that $|M_{2,3}| = 2$, as we have two different distance minimising paths from $\mathrm{LP}(x_2)$ to $w_3\mathrm{LP}(x_3)$, namely $u_{2,3} \Rightarrow w_3v_{2,3}$ and $u_{12,3} \Rightarrow w_3v_{12,3}$. However, this means that the element $v_{1,23}$ is unrestricted, which contradicts diagram (d), since the element $w_{23}v_{12,3}s_i = w_{23}v_{2,3}$ would be closer to $u_{1,23}$ than $v_{1,23}$. Hence, condition (ii) must hold and so $v_{2,3} = v_{12,3}$. We are then left to calculate $\eta_2$ directly. First, note that $\mathrm{wt}(u_{1,23} \Rightarrow w_{23}v_{1,23}) = \mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) + \tilde{\alpha}_i^\vee$, as indicated from Diagram (d). Hence:

$$\begin{aligned}\eta_2 &= v_{12,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{1,23}\mathrm{wt}(u_{1,23} \Rightarrow w_2u_{2,3}) \\ &= v_{2,3}\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) - v_{2,3}(\mathrm{wt}(u_{1,2} \Rightarrow w_2v_{1,2}) + \tilde{\alpha}_i^\vee) = -v_{2,3}(\tilde{\alpha}_i^\vee).\end{aligned}$$

□

Now that we have proven Proposition 3.3.6, we next want to investigate $\eta_3$. The following proposition gives the following values of $\eta_3$. The proof is analogous to the case of $\eta_2$.

**Proposition 3.3.7.** *If the following three conditions are met, then $\eta_3 = v_{2,3}(\tilde{\alpha}_i^\vee)$, otherwise $\eta_3 = 0$:*

(i) $u_{2,3} = v_{1,2}s_i$.

(ii) $v_{2,3} = v_{12,3}$.

(iii) $\ell(u_{2,3}s_i) > \ell(u_{2,3}) \geq \ell(w_3 v_{2,3})$.

We now have complete descriptions of the behaviour of $\eta_2, \eta_3$, so now we want to consider the value of $\eta_1$ under the same conditions in order to show that $\eta_1 = \eta_2 + \eta_3$. Firstly, we list the different cases that can appear, along with the value of $\eta_2 + \eta_3$ in each case.

(i) $u_{2,3} = u_{12,3}, \implies \eta_2 + \eta_3 = 0.$

(ii) $u_{2,3} = u_{12,3}s_i,\ v_{2,3} = v_{12,3}s_i, \implies \eta_2 + \eta_3 = 0.$

(iii) $u_{2,3} = u_{12,3}s_i,\ v_{2,3} = v_{12,3}$, and:

  (a) $\ell(u_{1,2}) < \ell(w_2 v_{1,2}) < \ell(w_2 v_{1,2} s_i),\ \ell(u_{2,3}s_i) < \ell(u_{2,3}) < \ell(w_3 v_{2,3}), \implies \eta_2 + \eta_3 = 0.$

  (b) $\ell(u_{1,2}) \geq \ell(w_2 v_{1,2}) > \ell(w_2 v_{1,2} s_i),\ \ell(u_{2,3}s_i) > \ell(u_{2,3}) \geq \ell(w_3 v_{2,3}), \implies \eta_2 + \eta_3 = 0.$

  (c) $\ell(u_{1,2}) \geq \ell(w_2 v_{1,2}) > \ell(w_2 v_{1,2} s_i),\ \ell(u_{2,3}s_i) < \ell(u_{2,3}) < \ell(w_3 v_{2,3}), \implies \eta_2 + \eta_3 = -v_{2,3}(\tilde{\alpha}_i^\vee).$

  (d) $\ell(u_{1,2}) < \ell(w_2 v_{1,2}) < \ell(w_2 v_{1,2} s_i),\ \ell(u_{2,3}s_i) > \ell(u_{2,3}) \geq \ell(w_3 v_{2,3}), \implies \eta_2 + \eta_3 = v_{2,3}(\tilde{\alpha}_i^\vee).$

Note that this exhausts all cases, as we must have either $\ell(u_{1,2}) < \ell(w_2 v_{1,2}) < \ell(w_2 v_{1,2} s_i)$ or $\ell(u_{1,2}) \geq \ell(w_2 v_{1,2}) > \ell(w_2 v_{1,2} s_i)$, by uniqueness of distance minimising elements in $\mathrm{QBG}(W)$. Finally, we are left to calculate the value of $\eta_1$ in each case, and show that we must have $\eta_1 = \eta_2 + \eta_3$.

**Case (i)** We have that $u_{2,3} = u_{12,3}$ and so $v_{2,3} = v_{12,3}$ by uniqueness, which immediately gives:

$$\eta_1 = v_{12,3}u_{12,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2 = v_{2,3}u_{2,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2 = 0.$$

**Case (ii)** We have $u_{2,3} = u_{12,3}s_i$ and $v_{2,3} = v_{12,3}s_i$, and again we immediately find:

$$\eta_1 = v_{12,3}u_{12,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2 = v_{2,3}s_is_iu_{2,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2 = 0.$$

**Case (iii)** Before considering the four subsequent cases separately, let us first simplify $\eta_1$.

$$\begin{aligned}\eta_1 = v_{12,3}u_{12,3}^{-1}\mu_2 - v_{2,3}u_{2,3}^{-1}\mu_2 &= v_{2,3}(s_iu_{2,3}^{-1}\mu_2 - u_{2,3}^{-1}\mu_2)\\ &= v_{2,3}(u_{2,3}^{-1}\mu_2 - \langle u_{2,3}^{-1}\mu_2, \tilde{\alpha}_i\rangle\tilde{\alpha}_i^\vee - u_{2,3}^{-1}\mu_2)\\ &= -\langle\mu_2, u_{2,3}\tilde{\alpha}_i\rangle v_{2,3}(\tilde{\alpha}_i^\vee). \qquad (3.23)\end{aligned}$$

And so we seek to compute the value of $\langle\mu_2, u_{2,3}\tilde{\alpha}_i\rangle$ in each sub-case. We do this using the classification of length positive elements to explicitly find the possible values of $\mu_2$ and $u_{2,3}$, showing that the bilinear product takes the values 0, 1, or -1 in the relevant cases labelled above.

In all sub-cases, we have $|\mathrm{LP}(x_2)| = 2$. Write $\mu_2 = k\alpha^\vee + m\delta + l\Lambda_0$, and fix $t \in \mathbb{Z}$ such that $-\frac{l}{2} < j = k - tl \leq \frac{l}{2}$. From the proof of Corollary 3.1.8, we must have that $j \in \{0, \frac{l}{2}, \pm\frac{l-1}{2}\}$. By Proposition 3.1.6, the possible values of pairs $(j, u)$ for $u \in \mathrm{LP}(x_2) = \{u_{2,3}, v_{1,2}\}$ are:

$$\begin{aligned}&(0, \tau^{t\alpha^\vee}), \quad (\frac{l}{2}, \tau^{t\alpha^\vee}), \quad (\frac{l-1}{2}, \tau^{t\alpha^\vee}), \quad (-\frac{l-1}{2}, \tau^{(t-1)\alpha^\vee}),\\ &(0, \tau^{t\alpha^\vee}s_1), (\frac{l}{2}, \tau^{t\alpha^\vee}s_0), (\frac{l-1}{2}, \tau^{t\alpha^\vee}s_0), (-\frac{l-1}{2}, \tau^{(t-1)\alpha^\vee}s_0).\end{aligned}$$

Write $u_{2,3} = v_o\tau^{t'\alpha^\vee}$. Taking $i = 0$ and $i = 1$ respectively, we find:

$$\begin{aligned}\langle\mu_2, u_{2,3}\tilde{\alpha}_1\rangle &= \langle k\alpha^\vee + m\delta + l\Lambda_0, v_o\tau^{t'\alpha^\vee}(\alpha)\rangle\\ &= \langle k\alpha^\vee + m\delta + l\Lambda_0, v_o\alpha + 2t'\delta\rangle \ = 2(k\,\mathrm{sgn}(v_o) - lt').\end{aligned}$$

$$\begin{aligned}\langle\mu_2, u_{2,3}\tilde{\alpha}_0\rangle &= \langle k\alpha^\vee + m\delta + l\Lambda_0, v_o\tau^{t'\alpha^\vee}(-\alpha + \delta)\rangle\\ &= \langle k\alpha^\vee + m\delta + l\Lambda_0, -v_o\alpha + (2t'+1)\delta\rangle \ = -2(k\,\mathrm{sgn}(v_o) - lt') + l.\end{aligned}$$

Before investigating the sub-cases, we calculate these products for each possible pair $(j, u)$, and then investigate which pairs are valid for which sub-cases. It is clear from the relevant pairs which value of $i$ is required, as $j$ takes the same value for a fixed $x \in W_{\mathcal{T}}$, and hence is the same for $u_{2,3}, v_{1,2}$, where $u_{2,3} = v_{1,2}s_i$. We label the pairs in the form $(j, u_o)$, where $u_o$ is the finite Weyl component of $u$.

| $(j, u_o)$ | $(0, e)$ | $(0, s_1)$ | $(\frac{l}{2}, e)$ | $(\frac{l}{2}, s_1)$ | $(\frac{l-1}{2}, e)$ | $(\frac{l-1}{2}, s_1)$ | $(-\frac{l-1}{2}, e)$ | $(-\frac{l-1}{2}, s_1)$ |
|---|---|---|---|---|---|---|---|---|
| $\langle\mu_2, u_{2,3}\tilde{\alpha}_i\rangle$ | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 |

For the following, we write $w_2 = w_o\tau^{r\alpha^\vee}$, where $w_o \in W_{\mathrm{fin}}$ and $r \in \mathbb{Z}$.

**Case (a)** $v_{1,2} < u_{2,3}$ and $w_2v_{1,2} < w_2u_{2,3}$. If $j = \frac{l-1}{2}$, it then follows that $u_{2,3}, v_{1,2} \in \{\tau^{t\alpha^\vee}, s_1\tau^{-(t+1)\alpha^\vee}\}$. Note that from Proposition 3.1.6, the latter of these being length positive requires $-r \le t \le -1$, forcing $r + t \ge 0$. If $u_{2,3} = s_1\tau^{-(t+1)\alpha^\vee}$, then for $u_{2,3} > v_{1,2}$ we must have $t \ge 0$, a contradiction. If instead $u_{2,3} = \tau^{t\alpha^\vee}$, then $w_2u_{2,3} = w_o\tau^{r\alpha^\vee}\tau^{t\alpha^\vee} = w_o\tau^{(r+t)\alpha^\vee} = w_o(s_0s_1)^{r+t}$. However, $w_2v_{1,2} = w_2u_{2,3}s_0 = w_o(s_0s_1)^{r+t}s_0 > w_2u_{2,3}$ as $r + t \ge 0$, contradicting the case. If $j = -\frac{l-1}{2}$, an analogous argument leads to the same contradictions. Hence $j = 0$ or $j = \frac{l}{2}$, and in either case $\langle \mu_2, u_{2,3}\tilde{\alpha}_i\rangle = 0$ as required.

**Case (b)** $v_{1,2} > u_{2,3}$ and $w_2v_{1,2} > w_2u_{2,3}$. We mirror the argument in case (a): If $j = \frac{l-1}{2}$, then as before we require $t \le -1$ and $r + t \ge 0$. If $u_{2,3} = \tau^{t\alpha^\vee}$, then for $u_{2,3} < v_{1,2}$ we require $t \ge 0$, a contradiction. If instead $u_{2,3} = s_1\tau^{-(t+1)\alpha^\vee}$, then $w_2u_{2,3} = w_os_1\tau^{-1-r-t} = w_os_0(s_1s_0)^{r+t}$, and $w_2v_{1,2} = w_os_0(s_1s_0)^{r+t}s_0 < w_2u_{2,3}$ since $r + t \ge 0$, another contradiction. Repeating the argument with $j = -\frac{l-1}{2}$ gives the same result, and so as before we must have $j = 0$ or $j = \frac{l}{2}$, and so $\langle \mu_2, u_{2,3}\tilde{\alpha}_i\rangle = 0$.

**Case (c)** $v_{1,2} < u_{2,3}$ and $w_2v_{1,2} > w_2u_{2,3}$. If $j = 0$ then $u_{2,3}, v_{1,2}$ take the values $\tau^{t\alpha^\vee}, s_1\tau^{-t\alpha^\vee}$. If $u_{2,3} = \tau^{t\alpha^\vee}$, then we must have $t \ge 1$ to ensure $v_{1,2} < u_{2,3}$. Hence by Proposition 3.1.6, we must have $t \ge \Phi^+(w_o\alpha) - r$. However, $w_2v_{1,2} > w_2u_{2,3} \implies w_os_1\tau^{-(t+r)\alpha^\vee} > w_o\tau^{(t+r)\alpha^\vee} \implies t + r \le \Phi^+(w_o\alpha) - 1$, a contradiction. Similarly, if $u_{2,3} = s_1\tau^{-t\alpha^\vee}$, we require $t \le 0$ for $v_{1,2} < u_{2,3}$ and $t \le \Phi^+(w_o\alpha) - r - 1$ for length positivity. However, $w_2v_{1,2} > w_2u_{2,3} \implies t + r \ge \Phi^+(w_o\alpha)$, a contradiction, and so $j \ne 0$. If $j = \frac{l}{2}$, an analogous argument leads to the same contradictions, and so $j = \pm\frac{l-1}{2}$.

If $j = \frac{l-1}{2}$ and $u_{2,3} = \tau^{t\alpha^\vee}s_0$, then we require $t \ge 0$ to ensure $u_{2,3} > v_{1,2}$, and we require $-r \le t \le -1$ for length positivity, a contradiction. If $j = -\frac{l-1}{2}$ and $u_{2,3} = \tau^{(t-1)\alpha^\vee}$, then we require $t \le 0$ to ensure $u_{2,3} > v_{1,2}$, and $1 \le t \le -r$ for length positivity, another contradiction.

In all remaining possibilities, $\langle \mu_2, u_{2,3}\tilde{\alpha}_i\rangle = 1$, and so $\eta_1 = -v_{2,3}(\tilde{\alpha}_i^\vee)$ by Equation (3.23), as required.

**Case (d)** $v_{1,2} > u_{2,3}$ and $w_2v_{1,2} < w_2u_{2,3}$. We mirror the argument in case (c):

If $j = 0$, then $u_{2,3}, v_{1,2}$ take the values $\tau^{t\alpha^\vee}, s_1\tau^{-t\alpha^\vee}$. If $u_{2,3} = \tau^{t\alpha^\vee}$, then we need $t \le 0$ to ensure $v_{1,2} > u_{2,3}$, and therefore we need $t \le \Phi^+(w_o\alpha) - r - 1$ for length positivity. For $w_2v_{1,2} < w_2u_{2,3}$, we need $t + r \ge \Phi^+(w_o\alpha)$, a contradiction. If instead $u_{2,3} = s_1\tau^{-t\alpha^\vee}$, then we require $t \ge 1$ and $t \ge \Phi^+(w_o\alpha) - r$, but for $w_2v_{1,2} < w_2u_{2,3}$, we need $t + r \le \Phi^+(w_o\alpha) - 1$, a contradiction. The same occurs if $j = \frac{l}{2}$, and so $j = \pm\frac{l-1}{2}$.

If $j = \frac{l-1}{2}$ and $u_{2,3} = \tau^{t\alpha^\vee}$, then we require $t \leq -1$ and $1 \leq t \leq -r$, a contradiction. If $j = -\frac{l-1}{2}$ and $u_{2,3} = \tau^{(t-1)\alpha^\vee} s_0$, then we require $t \geq 1$ and $-r \leq t \leq -1$, another contradiction.

Similarly to the previous case, in all remaining possibilities we have $\langle \mu_2, u_{2,3}\tilde{\alpha}_i \rangle = -1$, and so $\eta_1 = v_{2,3}(\tilde{\alpha}_i^\vee)$ as required. This completes all possible cases, and so $\eta_1 = \eta_2 + \eta_3$. Hence, we have shown that the lattice component of the generalised Demazure product is associative, and so we have proven the following.

**Theorem 3.3.8.** *The generalised Demazure product for $W_\mathcal{T}$ of type $\widehat{\mathrm{SL}}_2$ is associative for elements with level greater than one.*

*Remark* 3.3.9. In the case of $\mathrm{lev}(x) = 1$, the above arguments fail when $|\mathrm{LP}(x)| = 3$, for which more cases appear from the distance minimality conditions. In this edge case, we have failed to find any non-associative examples, but the calculations become increasingly arduous. However, an analogous argument does hold for those level one elements whose length positive set is at most two elements. With this in mind, we make the following conjecture.

**Conjecture 3.3.10.** *The generalised Demazure product for $W_\mathcal{T}$ of type $\widehat{\mathrm{SL}}_2$ is associative.*

**Example 3.3.11.** Consider the elements:

$$x_1 = s_1\varepsilon^{-4\delta}, \ x_2 = \tau^{\alpha^\vee}\varepsilon^{-\alpha^\vee+\Lambda_0}, \ x_3 = s_1\tau^{-\alpha^\vee}\varepsilon^{-2\alpha^\vee+3\Lambda_0}.$$

We have $\mathrm{LP}(x_1) = \{e, s_0, s_0s_1, s_0s_1s_0, ...\}$ by Equation 3.15. For $x_2, x_3$, we use Proposition 3.1.6 to find $\mathrm{LP}(x_2) = \{s_1, s_1s_0, s_1s_0s_1\}, \mathrm{LP}(x_3) = \{s_1s_0\}$.

Note $e = w_2s_1s_0 \in w_2\mathrm{LP}(x_2)$, and hence we can choose $(e, s_1s_0) \in M_{1,2}$ with corresponding weight $\mathrm{wt}(e \Rightarrow w_2s_1s_0) = 0$ to give:

$$x_1 * x_2 = s_1e(s_1s_0)^{-1}\varepsilon^{s_1s_0e^{-1}(-4\delta)+(-\alpha^\vee+\Lambda_0)-0} = s_1\tau^{\alpha^\vee}\varepsilon^{-\alpha^\vee-4\delta+\Lambda_0}.$$

We then have that $s_1s_0 \in \mathrm{LP}(x_2)$ minimises the distance to $w_3\mathrm{LP}(x_3) = \{s_0s_1s_0\}$, and so $(s_1s_0, s_1s_0) \in M_{2,3}$ with $\mathrm{wt}(s_1s_0 \Rightarrow w_3s_1s_0) = 0$. Hence:

$$x_2 * x_3 = \tau^{\alpha^\vee}(s_1s_0)(s_1s_0)^{-1}\varepsilon^{s_1s_0(s_1s_0)^{-1}(-\alpha^\vee+\Lambda_0)+(-2\alpha^\vee+3\Lambda_0)-0} = \tau^{\alpha^\vee}\varepsilon^{-3\alpha^\vee+4\Lambda_0}.$$

Using either Proposition 3.1.6 or Theorem 3.3.2, we find that $\mathrm{LP}(x_1 * x_2) = \{s_1, s_1s_0\}$ and $\mathrm{LP}(x_2 * x_3) = \{s_1s_0\}$, giving $(s_1s_0, s_1s_0) \in M_{12,3}$ and $(e, s_1s_0) \in M_{1,23}$, both with weight 0 shortest-length paths, and so we find:

$$\begin{aligned}
(x_1 * x_2) * x_3 &= s_1\tau^{\alpha^\vee}(s_1s_0)(s_1s_0)^{-1}\varepsilon^{s_1s_0(s_1s_0)^{-1}(-\alpha^\vee-4\delta+\Lambda_0)+(-2\alpha^\vee+3\Lambda_0)-0} \\
&= s_1\tau^{\alpha^\vee}\varepsilon^{-3\alpha^\vee-4\delta+4\Lambda_0}, \\
x_1 * (x_2 * x_3) &= s_1(e)(s_1s_0)^{-1}\varepsilon^{s_1s_0e^{-1}(-4\delta)+(-3\alpha^\vee+4\Lambda_0)-0} \\
&= s_1\tau^{\alpha^\vee}\varepsilon^{-3\alpha^\vee-4\delta+4\Lambda_0},
\end{aligned}$$

and so $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$.

# Chapter 4

# General Type

For this chapter, we consider a general Kac-Moody group of affine type as in Chapter 2, and we work with the full Tits cone $\mathcal{T} = W(P^\vee_{\text{dom}})$ instead of the subcone $W(X_{\text{dom}})$ (see Remark 2.4.2 for details), as our calculations now apply in full generality. As such, $\mu \in \mathcal{T}$ takes the form $\mu = \mu_o + m\delta + l\Lambda_0$ for some $\mu_o \in P^\vee_{\text{fin}}, m \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0}$.

*Remark* 4.0.1. Unless otherwise stated, the results in this chapter rely on the assumption that we are working in a Kac-Moody type for which the generalised Demazure product (2.10) for $W_{\mathcal{T}}$ is well-defined.

## § 4.1 Length Additivity

### § 4.1.1 Theorem Statement and Propositions.

The goal of this section is to investigate and ultimately prove the following theorem.

**Theorem 4.1.1.** *Let $x, y \in W_{\mathcal{T}}$, with $\operatorname{lev}(y) > 0$, and assume that $*^p_{u,v} = *$ is well-defined. Then $\ell(x * y) = \ell(x) + \ell(y) \iff \ell(xy) = \ell(x) + \ell(y)$. In this case, $x * y = xy$.*

To prove this, we collect some results below, which mostly apply in full generality, that is we allow $\operatorname{lev}(y) = 0$. We state explicitly in the result if the assumption $\operatorname{lev}(y) > 0$ applies. Firstly, we aim to prove the following extension of work by Schremmer [Sch24, Thm. 5.11] to the double affine case.

**Theorem 4.1.2.** *Assume that $*^p_{u,v} = *$ is well-defined, and let $(u, v) \in M_{x,y}$. Then:*

$$\ell(x * y) = \ell(x) + \ell(y) - d(u \Rightarrow w_y v).$$

To prove this, we need two extend two more propositions by Schremmer to the double-affine case. The first of these can be proved almost immediately by mimicking Schremmer's proof in the single-affine case.

It is important to note that, by assuming that $*$ is well-defined, we are also assuming that the weight function on $\mathrm{QBG}(W)$ is well-defined, and so the weight of two shortest paths is the same. This has yet to be proved for general type, although calculations suggest that this is reasonable to assume.

**Proposition 4.1.3.** *Let $w_1, w_2 \in W$, and assume that the weight function for $\mathrm{QBG}(W)$ is well-defined. Then:*

$$\langle \mathrm{wt}(w_1 \Rightarrow w_2), 2\rho \rangle = d(w_1 \Rightarrow w_2) + \ell(w_1) - \ell(w_2).$$

*Proof.* First, note that if $w_2 = w_1$, all factors are 0 and the claim holds. Now assume that $w_2 = w_1 s_{\tilde{\alpha}}$ such that $w_1 \Rightarrow w_2$ is an edge in $\mathrm{QBG}(W)$, leading to one of two cases.

(i) $w_1 \Rightarrow w_1 s_{\tilde{\alpha}}$ is a Bruhat edge: Then $\ell(w_2) = \ell(w_1) + 1$ and $\mathrm{wt}(w_1 \Rightarrow w_2) = 0$.

(ii) $w_1 \Rightarrow w_1 s_{\tilde{\alpha}}$ is a quantum edge: Then $\ell(w_2) = \ell(w_1) + 1 - \langle \tilde{\alpha}^\vee, 2\rho \rangle$ and $\mathrm{wt}(w_1 \Rightarrow w_2) = \tilde{\alpha}^\vee$.

In either case, we have that $\ell(w_1 s_{\tilde{\alpha}}) = \ell(w_1) + 1 - \langle \mathrm{wt}(w_1 \Rightarrow w_1 s_{\tilde{\alpha}}), 2\rho \rangle$. Now consider a general-length path, where $w_2 = w_1 p$, and $p = s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_n}$ corresponds to some minimal-distance path in $\mathrm{QBG}(W)$. We can iterate the above process to find the following.

$$\begin{aligned}
\ell(w_2) &= \ell(w_1 s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_n}) \\
&= \ell(w_1 s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_{n-1}}) + 1 - \langle \mathrm{wt}(w_1 s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_{n-1}} \Rightarrow w_1 s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_n}), 2\rho \rangle \\
&= \cdots \\
&= \ell(w_1) + n - \langle \mathrm{wt}(w_1 \Rightarrow w_1 s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_n}), 2\rho \rangle \\
&= \ell(w_1) + d(w_1 \Rightarrow w_2) - \langle \mathrm{wt}(w_1 \Rightarrow w_2), 2\rho \rangle.
\end{aligned}$$

Rearranging then gives the desired result. □

The second ingredient we need to prove Theorem 4.1.2 is the following.

**Proposition 4.1.4.** *Let $x = w_x \varepsilon^{\mu_x} \in W_{\mathcal{T}}$, and $u \in \mathrm{LP}(x)$. Then:*

$$\ell(x) = \langle u^{-1}\mu_x, 2\rho \rangle - \ell(u) + \ell(w_x u).$$

*Proof.* Let $w = s_{i_n} \cdots s_{i_1} \varepsilon^{\mu_x}$, such that $\ell(w) = n$ and $s_i = s_{\tilde{\beta}_i}$ for some simple affine root $\tilde{\beta}_i$. By the definition of the length function, we have the following.

$$\ell(x) = \ell(\varepsilon^{\mu_x}) + |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha} \rangle \geq 0\}| - |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha} \rangle < 0\}|.$$

From [MO19, Prop. 3.10], we have that:

$$\ell(\varepsilon^{\mu_x}) = 2\langle \mu_x, \rho \rangle - 2 \sum_{\tilde{\beta} \in \Phi^+, \langle \mu_x, \tilde{\beta} \rangle < 0} \langle \mu_x, \tilde{\beta} \rangle.$$

Let $u \in \mathrm{LP}(x)$. Since the dominant translate of $\mu_x$, denoted $\mu_x^+$, is unique, $u^{-1}\mu_x$ and $\mu_x$ must have the same dominant translate. Hence, from the definition $\ell(\varepsilon^{\mu_x}) = 2\langle \mu_x^+, \rho\rangle$, we must have that $\ell(\varepsilon^{u^{-1}\mu_x}) = \ell(\varepsilon^{\mu_x})$. Therefore:

$$\ell(\varepsilon^{\mu_x}) = 2\langle u^{-1}\mu_x, \rho\rangle - 2 \sum_{\tilde{\beta}\in\Phi^+, \langle \mu_x, u\tilde{\beta}\rangle < 0} \langle \mu_x, u\tilde{\beta}\rangle.$$

Note also that, by length positive and Remark 3.1.1, if $\langle \mu_x, u\tilde{\beta}\rangle < 0$, then it must equal -1. Hence we can rewrite the sum to obtain the formula:

$$\ell(\varepsilon^{\mu_x}) = 2\langle u^{-1}\mu_x, \rho\rangle + 2|\{\tilde{\alpha} \in \Phi^+ : \langle \mu_x, u\tilde{\alpha}\rangle = -1\}|.$$

Note that $\langle \mu_x, u\tilde{\alpha}\rangle = -1 \implies u\tilde{\alpha} \in \mathrm{Inv}(w)$, and $\langle \mu_x, u\tilde{\alpha}\rangle \geq -1$ by length positivity. Hence:

$$\begin{aligned}
&|\{\tilde{\alpha} \in \Phi^+ : \langle \mu_x, u\tilde{\alpha}\rangle = -1\}| \\
&\qquad = |\{\tilde{\alpha} \in \Phi^+ \cap u^{-1}\mathrm{Inv}(w) : \langle \mu_x, u\tilde{\alpha}\rangle < 0\}| \\
&\qquad = |\{\tilde{\alpha} \in u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&\qquad = |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| - |\{\tilde{\alpha} \in -u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}|
\end{aligned}$$

$$\begin{aligned}
\implies \ell(x) &= 2\langle u^{-1}\mu_x, \rho\rangle \\
&\qquad + 2|\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&\qquad - 2|\{\tilde{\alpha} \in -u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&\qquad + |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle \geq 0\}| - |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&= 2\langle u^{-1}\mu_x, \rho\rangle \\
&\qquad + |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle \geq 0\}| + |\{\tilde{\alpha} \in \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&\qquad - 2|\{\tilde{\alpha} \in -u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&= 2\langle u^{-1}\mu_x, \rho\rangle + |\mathrm{Inv}(w)| - 2|\{\tilde{\alpha} \in -u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}|.
\end{aligned}$$

We have that $|\mathrm{Inv}(w)| = \ell(w)$. Note also the following:

$$\begin{aligned}
\tilde{\alpha} \in -u\Phi^+ &\iff \tilde{\alpha} = -u\tilde{\beta} \text{ for some } \tilde{\beta} \in \Phi^+ \\
&\iff u^{-1}\tilde{\alpha} = -\tilde{\beta} \text{ for some } \tilde{\beta} \in \Phi^+ \\
&\iff u^{-1}\tilde{\alpha} < 0.
\end{aligned}$$

However, $\tilde{\alpha} \in \mathrm{Inv}(w) \implies \tilde{\alpha} \in \Phi^+$. Hence, $\tilde{\alpha} \in \mathrm{Inv}(u^{-1})$, and so $-u^{-1}\tilde{\alpha} > 0$. Applying length positivity to this element, we get the following.

$$\ell(x, u(-u^{-1}\tilde{\alpha})) = \ell(x, -\tilde{\alpha}) = -\langle \mu_x, \tilde{\alpha}\rangle + 0 - 1 \ \geq 0 \implies \langle \mu_x, \tilde{\alpha}\rangle \leq -1.$$

Hence the condition that $\langle \mu_x, \tilde{\alpha}\rangle < 0$ in the above set is redundant. Using [Wel19, Prop. 2.4.6], we therefore conclude the following.

$$\begin{aligned}
\ell(x) &= 2\langle u^{-1}\mu_x, \rho\rangle + |\mathrm{Inv}(w)| - 2|\{\tilde{\alpha} \in -u\Phi^+ \cap \mathrm{Inv}(w) : \langle \mu_x, \tilde{\alpha}\rangle < 0\}| \\
&= 2\langle u^{-1}\mu_x, \rho\rangle + \ell(w) - 2|\mathrm{Inv}(u^{-1}) \cap \mathrm{Inv}(w)| \\
&= 2\langle u^{-1}\mu_x, \rho\rangle + \ell(w) + (\ell(wu) - \ell(w) - \ell(u)) \\
&= 2\langle u^{-1}\mu_x, \rho\rangle - \ell(u) + \ell(wu).
\end{aligned}$$

As required. □

*Proof of Theorem 4.1.2.* Let $(u, v) \in M_{x,y}$, noting that this implies $v \in \mathrm{LP}(x * y)$. Using the definition of the Demazure product (which we assume to be well-defined), and using the formulae found in the above claims, we perform the following calculation, mimicking Schremmer's proof.

$$\begin{aligned}
\ell(x * y) &= \langle v^{-1}(vu^{-1}\mu_x + \mu_y - v\mathrm{wt}(u \Rightarrow w_y v)), 2\rho\rangle - \ell(v) + \ell(w_{x*y}v) \\
&= \langle u^{-1}\mu_x, 2\rho\rangle + \langle v^{-1}\mu_y, 2\rho\rangle - \langle \mathrm{wt}(u \Rightarrow w_y v), 2\rho\rangle - \ell(v) + \ell(w_x u) \\
&= \ell(x) + \ell(u) - \ell(w_x u) + \ell(y) + \ell(v) - \ell(w_y v) \\
&\qquad\qquad - d(u \Rightarrow w_y v) - \ell(u) + \ell(w_y v) - \ell(v) + \ell(w_x u) \\
&= \ell(x) + \ell(y) - d(u \Rightarrow w_y v).
\end{aligned}$$

□

As a corollary, we have the following important fact for discussing length additivity.

**Corollary 4.1.5.** *Let $x, y \in W_{\mathcal{T}}$, and assume that the product $*$ is well-defined. Then:*

$$\ell(x * y) = \ell(x) + \ell(y) \iff |\mathrm{LP}(y) \cap w_y^{-1}\mathrm{LP}(x)| > 0.$$

We also having the following result due to Muthiah and Puskás. First, let our double affine roots be $\Phi_{\mathrm{daf}} = \{\widehat{\beta} = \tilde{\beta} + m\pi \mid \tilde{\beta} \in \Phi, m \in \mathbb{Z}\}$, with positive roots $\Phi^+_{\mathrm{daf}} = \{\mathrm{sgn}(m)(\tilde{\beta} + m\pi) \mid \tilde{\beta} \in \Phi^+, m \in \mathbb{Z}\}$. For $x \in W_{\mathcal{T}}$, we then define its inversion set to be $\mathrm{Inv}(x) := \{\hat{\beta} \in \Phi^+_{\mathrm{daf}} \mid x(\hat{\beta}) < 0\}$, where $W_{\mathcal{T}}$ acts on $\Phi_{\mathrm{daf}}$ as an affinisation of equation 2.1:

$$w\varepsilon^{\mu}(\tilde{\beta} + m\pi) = w\tilde{\beta} + (m - \langle \tilde{\beta}, \mu\rangle)\pi. \tag{4.1}$$

Note that these inversion sets are typically infinite, but their intersections are finite, as seen in the following.

**Proposition 4.1.6** ([MP26, Thm. 5.4])**.** *Let $x, y \in W_{\mathcal{T}}$. Then:*

$$\ell(xy) = \ell(x) + \ell(y) - 2|\{\mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1})\}|.$$

From this, it follows that the product is length additive when the corresponding intersection of inversion sets is empty.

### § 4.1.2 Proof of Thm. 4.1.1

In order to prove Thm. 4.1.1, we need the following proposition, adapted from [Sch23, Prop. 2.9], [Sch23, Lemma 2.11(b)], and [Sch23, Lemma 2.3] to the double affine case. For a fixed $\tilde{\alpha} \in \Phi$, we write $\mathrm{Inv}_{\tilde{\alpha}}(x)$ for the subset of $\mathrm{Inv}(x)$ consisting of double affine roots of the form $\tilde{\alpha} + m\pi$, for some $m \in \mathbb{Z}$. Furthermore, we write $I_x(v) = \{\tilde{\alpha} \in \Phi^+ \mid \ell(x, v\tilde{\alpha}) < 0\} \cup \{\tilde{\alpha} \in \Phi^- \mid \ell(x, v\tilde{\alpha}) > 0\}$ for $v \in W$.

**Proposition 4.1.7.** *Let $x = w\varepsilon^{\mu} \in W_{\mathcal{T}}, v \in W$, and fix $\tilde{\alpha} \in \Phi$. Then*

(i) $\mathrm{Inv}_{\tilde{\alpha}}(x) = \{\tilde{\alpha} + m\pi \mid 0 \leq m - \Phi^+(-\tilde{\alpha}) < \ell(x, \tilde{\alpha})\}$.

(ii) $|\mathrm{Inv}_{\tilde{\alpha}}(x)| = \max\{0, \ell(x, \tilde{\alpha})\}$.

(iii) $\ell(x^{-1}, \tilde{\alpha}) = -\ell(x, w^{-1}\tilde{\alpha})$.

(iv) $|I_x(vs_{\tilde{\alpha}})| < |I_x(v)|$ *if* $\tilde{\alpha} \in I_x(v)$ *and if* $|I_x(v)| < \infty$.

Schremmer's proofs of these statements in the single affine case generalise immediately to the double affine case.

We also require the following statement.

**Proposition 4.1.8.** *Let $x \in W_{\mathcal{T}}$ with* $\mathrm{lev}(x) > 0$. *Then* $|\mathrm{LP}(x)| < \infty$.

*Proof.* Let $v \in \mathrm{LP}(x)$, such that $\ell(x, v\tilde{\alpha}) \geq 0$ for all $\tilde{\alpha} \in \Phi^+$. Consider this inequality for $\tilde{\alpha} = \alpha$ and $-\alpha + \delta$ respectively, where $\alpha \in \Phi_{\text{fin}}$. This gives us the following pair of inequalities:

$$\langle \mu, v\alpha \rangle + \Phi^+(v\alpha) - \Phi^+(wv\alpha) \geq 0,$$

$$\langle \mu, v(-\alpha + \delta) \rangle + \Phi^+(v(-\alpha + \delta)) - \Phi^+(wv(-\alpha + \delta)) \geq 0.$$

Letting $\mu = \mu_o + m\delta + l\Lambda_0$ with $l > 0$, and writing $v = v_o\tau^{\lambda}$ for some fixed $v_o \in W_{\text{fin}}$, these inequalities become

$$\langle \mu_o, v_o\alpha \rangle - 1 \leq l\langle \alpha, \lambda \rangle \leq \langle \mu_o, v_o\alpha \rangle + 1.$$

Let $\Delta_{\text{fin}} = \{\alpha_i\}_i$ be the set of simple roots for $W_{\text{fin}}$, and write $\lambda = \sum_i c_i\alpha_i$ for some $c_i \in \mathbb{Z}$. Then, by setting $\alpha = \alpha_i$ in the above inequality, it is clear that there are only finitely many values $c_i$ can take. As $\Delta_{\text{fin}}$ is itself a finite set, this gives only a finite number of possibilities for the value of $\lambda$. Finally, note that there were also only a finite number of choices for $v_o$, and hence $\mathrm{LP}(x)$ must be a finite set. $\square$

These propositions give us the following, from which Theorem 4.1.1 follows immediately.

**Proposition 4.1.9.** *Let $x, y \in W_{\mathcal{T}}$ with $y = w_y \varepsilon^{\mu_y}$ and $\mathrm{lev}(y) > 0$. Then:*

$$\mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1}) \neq \emptyset \iff \mathrm{LP}(y) \cap w_y^{-1}\mathrm{LP}(x) = \emptyset.$$

*Proof.* Assume first that $\mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1})$ is non-empty, so that there exists some $\widehat{\beta} = \tilde{\beta} + m\pi \in \mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1})$. From Proposition 4.1.7(ii), we must have that $\ell(x, \tilde{\beta}) > 0$, and $\ell(y^{-1}, \tilde{\beta}) > 0$, or equivalently $\ell(y, w_y^{-1}\tilde{\beta}) < 0$ by Proposition 4.1.7(iii), and so $\ell(x, \tilde{\beta})$ and $\ell(y, w_y^{-1}\tilde{\beta})$ must have opposite signs.

Also note that $v \in \mathrm{LP}(y) \cap w_y^{-1}\mathrm{LP}(x)$ if and only if $\ell(x, w_y v\tilde{\alpha}) \geq 0$ and $\ell(y, v\tilde{\alpha}) \geq 0$ for every $\tilde{\alpha} \in \Phi^+$. Let $\tilde{\alpha} = \pm v^{-1} w_y^{-1}\tilde{\beta}$, with the sign chosen such that $\tilde{\alpha} \in \Phi^+$. Then we must have that $\pm\ell(x, \tilde{\beta}) \geq 0$ and $\pm\ell(y, w_y^{-1}\tilde{\beta}) \geq 0$, and so $\ell(x, \tilde{\beta})$ and $\ell(y, w_y^{-1}\tilde{\beta})$ have the same sign, or one of them is zero, contradicting an earlier statement. Hence, no such $v$ can exist, and $\mathrm{LP}(y) \cap w_y^{-1}\mathrm{LP}(x)$ must be empty.

Now instead assume that $\mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1})$ is empty. Then if $\widehat{\beta} \in \mathrm{Inv}_{\tilde{\beta}}(x)$, we must have that $\widehat{\beta} \notin \mathrm{Inv}_{\tilde{\beta}}(y^{-1})$, and vice versa. Hence, using Proposition 4.1.7(i) and (iii), and setting $m = \Phi^+(-\tilde{\beta})$, we find the following implications.

$$\ell(x, \tilde{\beta}) > 0 \implies \ell(y, w_y^{-1}\tilde{\beta}) \geq 0, \qquad \ell(y, w_y^{-1}\tilde{\beta}) < 0 \implies \ell(x, \tilde{\beta}) \leq 0, \quad \forall \tilde{\beta} \in \Phi.$$

Letting $v \in \mathrm{LP}(y)$ and writing $\tilde{\beta} = -w_y v\tilde{\alpha}$ gives:

$$\ell(x, w_y v\tilde{\alpha}) < 0 \implies \ell(y, v\tilde{\alpha}) \leq 0, \qquad \ell(y, v\tilde{\alpha}) > 0 \implies \ell(x, w_y v\tilde{\alpha}) \geq 0, \quad \forall \tilde{\alpha} \in \Phi.$$

Now, consider only $\tilde{\alpha} \in \Phi^+$, which forces $\ell(y, v\tilde{\alpha}) \geq 0$. If $\ell(x, w_y v\tilde{\alpha}) \geq 0$ for all $\tilde{\alpha} \in \Phi^+$, then $w_y v \in \mathrm{LP}(x)$ and we are done. Otherwise, if $\ell(x, w_y v\tilde{\alpha}) < 0$ for some $\tilde{\alpha} \in \Phi^+$, we must have $\ell(y, v\tilde{\alpha}) = 0$, in which case $vs_{\tilde{\alpha}} \in \mathrm{LP}(y)$ by Proposition 3.3.1. Since $\mathrm{LP}(y)$ is finite by Proposition 4.1.8, there can only be finitely many positive roots $\tilde{\alpha}$ with $\ell(x, w_y v\tilde{\alpha}) < 0$, and so we have that $|I_x(w_y vs_{\tilde{\alpha}})| < |I_x(w_y v)| < \infty$ by Proposition 4.1.7(iv). If $|I_x(w_y vs_{\tilde{\alpha}})| = 0$, then $w_y vs_{\tilde{\alpha}} \in \mathrm{LP}(x)$ and we are done. Otherwise, we iterate this procedure to find an element $u = vs_{\tilde{\alpha}_1} \ldots s_{\tilde{\alpha}_n}$ which lies in $\mathrm{LP}(y)$ and $w_y^{-1}\mathrm{LP}(x)$ as required. $\square$

*Proof of Theorem 4.1.1.* We have that $\ell(x*y) = \ell(x) + \ell(y) \iff \mathrm{LP}(y) \cap w_y^{-1}\mathrm{LP}(x) \neq \emptyset$ by Corollary 4.1.5, which occurs if and only if $\mathrm{Inv}(x) \cap \mathrm{Inv}(y^{-1}) = \emptyset$ by Proposition 4.1.9, which in turn occurs if and only if $\ell(xy) = \ell(x) + \ell(y)$ by Proposition 4.1.6, as required. $\square$

### § 4.1.3 Examples Relating to the Hecke Algebra

Recall from Remark 2.4.5 that our conventions differ from those in Muthiah and Puskás paper [MP26], and we have converted their examples into our notation for the following.

**Length Additive Pair.** We compute the Demazure product in type $\widehat{\mathrm{SL}}_2$ of the elements $x = \varepsilon^{\delta+\Lambda_0}s_0s_1, y = \varepsilon^{4\delta+\Lambda_0}$, aligning with Section 6.1 of [MP26], with changes made to account for differences in convention. They compute $T_xT_y = T_{xy}$ in the Hecke algebra for this chosen $x, y$. Hence, our goal is to show that $x * y = xy$.

Using the action of $W$ on $\mathcal{T}$ as defined in Equation (2.5), we find:

$$x = s_0s_1\varepsilon^{s_1s_0(\delta+\Lambda_0)} = \tau^{\alpha^\vee}\varepsilon^{-\alpha^\vee+\Lambda_0}.$$

We now compute $\mathrm{LP}(x)$ and $\mathrm{LP}(y)$ using Proposition 3.1.6. Following the previous notation, for $\mathrm{LP}(x)$ we have $l = 1, k = -1$ which gives $t = -1$ and $j = 0 = \pm\frac{l-1}{2}$. Therefore $t \leq 0$, $t \leq \Phi^+(w_o\alpha) - r - 1 = -1$, and $-r \leq t \leq -1$, so we have $\mathrm{LP}(x) = \{\tau^{-\alpha^\vee}, \tau^{-\alpha^\vee}s_1, \tau^{-\alpha^\vee}s_0\} = \{s_1s_0, s_1s_0s_1, s_1\}$ from steps 3 and 5 respectively.

For $y = \varepsilon^{4\delta+\Lambda_0}$, we have $k = 0, r = 0, l = 1, j = 0 = \pm\frac{l-1}{2}$, and so $t = 0$. We therefore have $t \leq 0$ and $t \leq \Phi^+(w_o\alpha) - r - 1 = 0$, so from step 3 we have $\mathrm{LP}(y) = \{\tau^{0\alpha^\vee}, \tau^{0\alpha^\vee}s_1\} = \{e, s_1\}$.

Hence $w_y\mathrm{LP}(y) = \{e, s_1\}$, and so we have a non-empty intersection of sets such that $\mathrm{LP}(x) \cap w_y\mathrm{LP}(y) = \{s_1\}$. Hence, by Theorems 4.1.1 and 4.1.2, we should expect $x * y = xy$, as computed by Muthiah and Puskás. We have $M_{x,y} = \{(s_1, s_1)\}$ with $d(s_1 \Rightarrow w_ys_1) = 0$, and we compute the Demazure product using the notation of Definitio 2.5.14:

$$\begin{aligned} x * y \;=\; w_xuv^{-1}\varepsilon^{vu^{-1}\mu_x+\mu_y-v\mathrm{wt}(p)} &= s_0s_1s_1s_1^{-1}\varepsilon^{s_1s_1^{-1}(-\alpha^\vee+\Lambda_0)+(4\delta+\Lambda_0)-s_1(0)} \\ &= s_0s_1\varepsilon^{-\alpha^\vee+4\delta+2\Lambda_0} \\ &= s_0s_1\varepsilon^{-\alpha^\vee+\Lambda_0}\varepsilon^{4\delta+\Lambda_0} \;=\; xy. \end{aligned}$$

**Non-Length Additive Pair**. Now let $x = \varepsilon^{\Lambda_0}s_0s_1s_0, y = \varepsilon^{\Lambda_0}s_0$ as per [MP26, Sec. 6.2]. In their notation, with $\pi = \varepsilon^{-1}$ they compute the following:

$$T_xT_y = qT_{xy} + (q-1)T_{\pi^{2\Lambda_0-\delta-\tilde{\alpha}_0}s_0},$$

and so $T_xT_y \equiv -T_{\pi^{2\Lambda_0-\delta-\tilde{\alpha}_0^\vee}s_0} \mod q$. Hence, following Conjecture 1.1.1 and converting to our notation, we wish to show that $x * y = s_0\varepsilon^{2\Lambda_0-\delta-\tilde{\alpha_0}^\vee}$.

First we rewrite these elements as before, giving $x = s_1\tau^{-2\alpha^\vee}\varepsilon^{2\alpha^\vee-4\delta+\Lambda_0}$ and $y = s_1\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee-\delta+\Lambda_0}$. For $x$, we have $w_o = s_1, r = -2, k = 1, l = 1$ giving $t = 2$ and

$j = 0 = \pm\frac{l-1}{2}$. By Proposition 3.1.6, we find that $\mathrm{LP}(x) = \{\tau^{2\alpha^\vee}, \tau^{2\alpha^\vee}s_1, \tau^{2\alpha^\vee}s_1s_0\} = \{s_0s_1s_0s_1, s_0s_1s_0, s_0s_1\}$. For $y$, we have $w_o = s_1, r = -1, k = 1, l = 1$ giving $t = 1$ and $j = 0 = \pm\frac{l-1}{2}$, and hence $\mathrm{LP}(y) = \{\tau^{\alpha^\vee}, \tau^{\alpha^\vee}s_1, \tau^{\alpha^\vee}s_1s_0\} = \{s_0s_1, s_0, e\}$, and so $w_y\mathrm{LP}(y) = \{s_1, e, s_0\}$.

We now have $\mathrm{LP}(x) \cap w_y\mathrm{LP}(y) = \emptyset$, and so a shortest path will have length at least one. It is easy to see that $s_0s_1 \Rightarrow s_0$ is a path of length exactly one, and therefore minimal. Hence $(s_0s_1, w_y^{-1}s_0) = (s_0s_1, e) \in M_{x,y}$, with $\mathrm{wt}(s_0s_1 \Rightarrow s_0) = \tilde{\alpha}_1^\vee = \alpha^\vee$. Using Definition 2.5.14 with this choice of distance minimising pair, we find that:

$$\begin{aligned} x * y = w_x uv^{-1}\varepsilon^{vu^{-1}\mu_x+\mu_y-v\mathrm{wt}(p)} &= s_1\tau^{-2\alpha^\vee}\tau^{\alpha^\vee}e\varepsilon^{\tau^{-\alpha^\vee}(2\alpha^\vee-4\delta+\Lambda_0)+\alpha^\vee-\delta+\Lambda_0-e(\alpha^\vee)} \\ &= s_1\tau^{-\alpha^\vee}\varepsilon^{\alpha^\vee-2\delta+2\Lambda_0} \\ &= s_0\varepsilon^{-\tilde{\alpha}_0{}^\vee-\delta+2\Lambda_0}. \end{aligned}$$

This aligns with the Muthiah and Puskás' result, taking the $q = 0$ term in their Hecke product. It can also be verified that $\ell(x * y) = \ell(x) + \ell(y) - 1$, an example of Thm. 4.1.2.

## § 4.2 Double Affine Demazure Products of Single Affine Elements

In this section, we give some work towards a generalisation of Theorem 3.2.10 to arbitrary type. Throughout, we fix $x = w \in W \subset W_{\mathcal{T}}$, but refer to both $x$ and $w$ depending on the context, to match previous notation.

The following propositions help to give us an idea of what our length positive sets look like for single affine elements of $W_{\mathcal{T}}$. Throughout, we take $W_{\mathcal{T}}$ to be explicitly double affine, so that the underlying Weyl group $W$ is affine.

**Proposition 4.2.1.** *Let $W_{\mathcal{T}}$ be a double affine Weyl semigroup, and let $x \in W_{\mathcal{T}}$ such that $x \in W$. Then $|\mathrm{LP}(x)| = \infty$.*

*Proof.* Recall from Equation (3.15) that for any such $x = w$ in general type, we have $\mathrm{LP}(w) = \{v \in W \mid \ell(wv) = \ell(w) + \ell(v)\}$. Fix a $v \in \mathrm{LP}(w)$. Note that $e \in \mathrm{LP}(w)$ for any $w \in W$. We want to show that there exists a simple reflection $s_i$ such that $vs_i > v$ and $wvs_i > wv$. Assume for a contradiction that this does not hold, so for any $s_i \in S$ we have that if $wvs_i > wv$ then $vs_i < v$.

From standard Coxeter theory (see e.g. [Hum90]), we have that $vs_i < v$ if and only if $v\tilde{\alpha}_i < 0$, and $wvs_i > wv$ if and only if $wv\tilde{\alpha}_i > 0$, in which case $v\tilde{\alpha}_i \in -\mathrm{Inv}(w)$. Hence, for every $i \in I$, we have that if $\tilde{\alpha}_i \in -v^{-1}\mathrm{Inv}(w)$, then $\tilde{\alpha}_i \in \mathrm{Inv}(v)$. However, following the logic of the proof of Prop. 3.1.3, we find that $-v^{-1}\mathrm{Inv}(w) \cap \mathrm{Inv}(v) = \emptyset$

in order for $v$ to be length positive. Hence no such $\tilde{\alpha}_i$ can exist, and so $-v^{-1}\text{Inv}(w)$ contains no simple roots. In particular, there is no $\tilde{\alpha}_i$ such that $wvs_i > wv$. This implies that $wv$ is a longest element, which does not exist for affine type, which we have assumed as $|W| = \infty$, and so we have found a contradiction.

Therefore we can choose a simple reflection $s_i \in S$ as required. In this case, we find:

$$\ell(wvs_i) = \ell(wv) + 1 = \ell(w) + \ell(v) + 1 = \ell(w) + \ell(vs_i).$$

Hence $vs_i \in \text{LP}(x)$. We can then iterate this process starting with $e \in \text{LP}(w)$ to obtain infinitely many elements as required. □

The previous proposition shows how we can construct longer elements of $\text{LP}(x)$ beginning from shorter ones. The next proposition instead discusses how to recover shorter elements from longer ones in order to prove connectivity.

**Proposition 4.2.2.** *For $x \in W \subset W_{\mathcal{T}}$ as above, $\text{LP}(x)$ is connected.*

*Proof.* Write $x = w$, and let $u \in \text{LP}(x)$, so that $\ell(wu) = \ell(w) + \ell(u)$. Let $u = s_{i_1}...s_{i_n}$ be a reduced decomposition for $u$. Write $u' = s_{i-1}...s_{i_{n-1}}$. We have $\ell(wu) = \ell(wu') \pm 1$. First assume that $\ell(wu) = \ell(wu') - 1$. Then:

$$\begin{aligned} \ell(wu) = \ell(wu') - 1 &\implies \ell(w) + \ell(u) = \ell(wu') - 1 \\ &\implies \ell(w) + \ell(u') + 1 = \ell(wu') - 1 \\ &\implies \ell(wu') = \ell(w) + \ell(u') + 2. \end{aligned}$$

However, $\ell(wu') \leq \ell(w) + \ell(u')$, a contradiction. Hence $\ell(wu) = \ell(wu') + 1$, and so:

$$\ell(wu') = \ell(wu) - 1 = \ell(w) + \ell(u) - 1 = \ell(w) + \ell(u').$$

Therefore $u' \in \text{LP}(x)$. Note also that $u \Rightarrow u' = us_{i_n}$ is a quantum edge in $\text{QBG}(W)$, as $\ell(u's_{i_n}) = \ell(u') + 1 - 2 = \ell(u') + 1 - 2\text{ht}(\tilde{\alpha}_{i_n})$. Similarly, $u' \Rightarrow u$ is a Bruhat edge as $\ell(u) = \ell(u's_{i_n}) = \ell(u') + 1$. Hence we have a double edge $u \Leftrightarrow u'$ in $\text{QBG}(W)$. We can now iterate along such elements to obtain a sequence of double edges:

$$s_{i_1}...s_{i_n} \Leftrightarrow s_{i_1}...s_{i_{n-1}} \Leftrightarrow ... \Leftrightarrow s_{i_1} \Leftrightarrow e.$$

Hence each element of $\text{LP}(x)$ is connected both to and from the identity. Given $u, v \in \text{LP}(x)$, we can thus concatenate such paths to obtain a path $u \Rightarrow e \Rightarrow v$, and hence $\text{LP}(x)$ is strongly connected within $\text{QBG}(W)$ as required. □

**Proposition 4.2.3.** *For $x = w \in W \subset W_{\mathcal{T}}$ as above, we have that $w \leq wv$ for all $v \in \text{LP}(x)$. In particular, $w$ is the smallest element of the set $w\text{LP}(x)$.*

*Proof.* From our description of $\text{LP}(x)$, we have:

$$w\text{LP}(x) = \{wv \mid \ell(wv) = \ell(w) + \ell(v)\}.$$

Note that $w \in w\mathrm{LP}(x)$, and $w \leq wv$ for all $v \in \mathrm{LP}(x)$ as $\ell(wv) = \ell(w) + \ell(v)$, from which the proposition follows. $\square$

In order to verify a generalisation of Theorem 3.2.10 to arbitrary type, we would need a well-defined weight function on the affine quantum Bruhat graph, which we do not have. However, we can say the following in the case of length additive elements.

**Proposition 4.2.4.** *Let $x, y \in W_{\mathcal{T}}$ such that $x, y \in W$. The single affine Demazure product gives $x * y = xy$ if and only if the double affine Demazure product is well-defined and gives $x * y = xy$.*

*Proof.* If the single affine Demazure product gives $x * y = xy$, then $\ell(xy) = \ell(x) + \ell(y)$. In particular, $y \in \mathrm{LP}(x)$. As previously mentioned, $y \in y\mathrm{LP}(y)$, and hence any distance minimising path must have length 0. In particular, any choice of $(u, v) \in M_{x,y}$ gives $uv^{-1} = w_y = y$ and $v\mathrm{wt}(u \Rightarrow w_y v) = 0$. Therefore:

$$x *^p_{u,v} y = xuv^{-1}\varepsilon^0 = xy.$$

Hence the double affine Demazure product is well-defined, and equal to the single affine product.

Now assume that the double affine Demazure product gives $x *^p_{u,v} y = xy$. Then $xuv^{-1} = xy$ and $\mathrm{wt}(u \Rightarrow yv) = 0$, which are both simultaneously true if and only if $uv^{-1} = y$, i.e. $u = yv$. Therefore there exists $(u, v) \in M_{x,y}$ such that $yv \in \mathrm{LP}(x)$, that is $\ell(xyv) = \ell(x) + \ell(yv)$. We also have that $\ell(yv) = \ell(y) + \ell(v)$, and so by similar arguments to before, we find that:

$$\ell(xy) = \ell(xyv) - \ell(v) = \ell(x) + \ell(yv) - \ell(v) = \ell(x) + \ell(y).$$

In which case the single affine Demazure product gives $x * y = xy$ as required. $\square$

## § 4.3 Semi-Infinite Bruhat Order

### § 4.3.1 Definition and Properties

The semi-infinite Bruhat order is a different partial order on Weyl groups of affine type [Pet97], which has many uses including within the context of crystal bases (see e.g. [Ish22]). In [Lus80], a link was developed between the semi-infinite Bruhat order and the (finite) quantum Bruhat graph, discused in more detail in [INS16]. In this section, we discuss recent results regarding the semi-infinite Bruhat order and its connection to the quantum Bruhat graph, with an aim to generalise these results to a double affine setting.

**Definition 4.3.1.** *Let $w = w_o\tau^\lambda \in W$. The **semi-infinite length function** is defined as:*

$$\ell^{\frac{\infty}{2}}(w) := \ell(w_o) + 2\langle \rho_{\text{fin}}, \lambda\rangle = \ell(w_o) + 2\text{ht}(\lambda).$$

Note that if $w = w_o$, we recover the standard length function $\ell^{\frac{\infty}{2}}(w_o) = \ell(w_o)$, however in general the semi-infinite length function does not satisfy all properties of the standard one. For example, we have:

$$\ell^{\frac{\infty}{2}}(w^{-1}) = \ell^{\frac{\infty}{2}}(w_o^{-1}\tau^{-w_o\lambda}) = \ell(w_o^{-1}) + 2\text{ht}(-w_o\lambda) = \ell(w_o) - 2\text{ht}(w_o\lambda),$$

and so we see that generally $\ell^{\frac{\infty}{2}}(w^{-1}) \neq \ell^{\frac{\infty}{2}}(w)$.

The following proposition gives us some insight into this function.

**Proposition 4.3.2.** *If $\lambda$ is dominant, then $\ell^{\frac{\infty}{2}}(w_o\tau^\lambda) = \ell(w_o\tau^\lambda)$.*

*Proof.* Recall the single-affine equivalent of the double affine length function (Defn. 2.4.8):

$$\ell(w_o\tau^\lambda) = 2\text{ht}(\lambda^+) + |\{\alpha \in \text{Inv}(w_o) \mid \langle \lambda, \alpha\rangle \geq 0\}| - |\{\alpha \in \text{Inv}(w_o) \mid \langle \lambda, \alpha\rangle < 0\}| \quad (4.2)$$

Now take $\lambda$ to be dominant, such that $\lambda^+ = \lambda$. Then $\langle \lambda, \alpha\rangle \geq 0$ for all $\alpha \in \Phi^+_{\text{fin}}$, and hence:

$$\ell(w_o\tau^\lambda) = 2\text{ht}(\lambda) + |\{\alpha \in \text{Inv}(w_o)\}| = 2\text{ht}(\lambda) + \ell(w_o) = \ell^{\frac{\infty}{2}}(w_o\tau^\lambda),$$

and so if the $\mathbb{Z}\Phi^\vee_{\text{fin}}$-component of is dominant, the regular length function and the semi-infinite length function coincide. □

We can therefore view the difference between the length function and the semi-infinite length function partly as a measure of how much $\lambda$ fails to be dominant. Using this, we define a new partial order on $W$.

**Definition 4.3.3.** *The **semi-infinite Bruhat order** on $W$ is the partial order graded by $\ell^{\frac{\infty}{2}}(\cdot)$, generated by the following relations for $w \in W, \tilde{\alpha} \in \Phi^+$:*

$$w \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}}w \iff \ell^{\frac{\infty}{2}}(s_{\tilde{\alpha}}w) = \ell^{\frac{\infty}{2}}(w) + 1.$$

Note that we can define an equivalent right-sided semi-infinite Bruhat order, but in most contexts within the literature our left-sided definition is more appropriate.

*Remark* 4.3.4. One feature of note of the semi-infinite Bruhat order is its lack of relation to the translation part of the corresponding elements. Let $w = w_o\tau^\lambda \in W, \tilde{\alpha} = \alpha + n\delta \in$

$\Phi^+$, and consider the following calculation:

$$\begin{aligned} w \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}} w &\iff w_o \tau^\lambda \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}} w_o \tau^\lambda \\ &\iff w_o \tau^\lambda \leq^{\frac{\infty}{2}} s_\alpha \tau^{n\alpha^\vee} w_o \tau^\lambda = s_\alpha w_o \tau^{n w_o^{-1}\alpha^\vee + \lambda} \\ &\iff \ell^{\frac{\infty}{2}}(s_\alpha w_o \tau^{n w_o^{-1}\alpha^\vee + \lambda}) = \ell^{\frac{\infty}{2}}(w_o \tau^\lambda) + 1 \\ &\iff \ell(s_\alpha w_o) + 2\mathrm{ht}(n w_o^{-1}\alpha^\vee + \lambda) = \ell(w_o) + 2\mathrm{ht}(\lambda) + 1 \\ &\iff \ell(s_\alpha w_o) + 2\mathrm{ht}(n w_o^{-1}\alpha^\vee) = \ell(w_o) + 1. \end{aligned}$$

Hence we see that the condition $w \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}} w$ is independent of the value of $\lambda$, as expected given the result in the previous theorem. In fact, we have the following statement.

**Proposition 4.3.5** ([NW17, Prop. 2.2.5])**.** *Let $w = w_o \tau^\lambda \in W, \tilde{\beta} = \beta + n\delta \in \Phi^+$. Then:*

$$\ell^{\frac{\infty}{2}}(w) < \ell^{\frac{\infty}{2}}(s_{\tilde{\beta}} w) \iff w_o^{-1}\beta > 0.$$

There also exists a link to the Bruhat order more directly. We say a coweight $\mu$ is *regular* if $\langle \mu, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$.

**Proposition 4.3.6** ([NW17, Prop. 2.2.2])**.** *Let $w, w' \in W$. Then $w \leq^{\frac{\infty}{2}} w' \iff w\tau^\mu \leq w'\tau^\mu$ for some sufficiently regular dominant coweight $\mu$.*

Hence we can view the semi-infinite Bruhat order as being equivalent to the regular Bruhat order once translated into a more dominant part of the coweight lattice. By sufficiently regular, we mean that there exists some sufficiently large $C$ such that $|\langle \mu, \alpha \rangle| \geq C$ for all $\alpha \in \Phi$ in order for the statement to hold.

### § 4.3.2 Linking QBG($W$) with the Semi-Infinite Bruhat Order

Recently, a direct link between the semi-infinite Bruhat order and the quantum Bruhat graph has been developed.

**Theorem 4.3.7** ([INS16, Prop. A.1.2])**.** *Let $w = w_o \tau^\lambda \in W$.*

(i) *If $w \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}} w$ is a cover for some $\tilde{\alpha} \in \Phi$, then $\tilde{\alpha} = w_o\beta + n\delta$ for some $\beta \in \Phi^+_{\mathrm{fin}}$ with $n \in \{0, 1\}$, and $w_o \Rightarrow w_o s_\beta$ is an edge in QBG($W_{\mathrm{fin}}$) of Bruhat type if $n = 0$, or of quantum type if $n = 1$.*

(ii) *If $w_o \Rightarrow w_o s_\beta$ is an edge in QBG($W_{\mathrm{fin}}$), then $w \leq^{\frac{\infty}{2}} s_{\tilde{\alpha}} w$ is a cover, where $\tilde{\alpha} = w_o\beta$ if the edge is Bruhat, or $\tilde{\alpha} = w_o\beta + \delta$ if the edge is quantum.*

Hence we have a correspondence between edges in QBG($W_{\mathrm{fin}}$) and covering relations in the semi-infinite Bruhat order. Note that this is certainly not a one-to-one correspondence, in particular QBG($W_{\mathrm{fin}}$) has finitely many edges whereas the semi-infinite Bruhat order on $W$ has infinitely many covering relations.

The proof of Theorem 4.3.7 uses another structure known as the level-zero weight lattice, defined in [LNS+15]. We write $P_{\text{fin}}$ for the finite weight lattice and $P$ for the affine weight lattice.

**Definition 4.3.8.** *Let $Z = \{\mu \in P^\vee \mid \langle \mu, \tilde{\alpha}_i \rangle = 0\ \forall\ \tilde{\alpha}_i \in \Delta\}$. The **level-zero weight lattice** is $P^0 = \{\omega \in P \mid \langle \mu, \omega \rangle = 0\ \forall \mu \in Z\}$.*

We first verify that this definition of level-zero lines up with our previous definition of the level of an affine (co)weight.

**Proposition 4.3.9.** *Let $\omega = \omega_o + m\delta + l\Lambda_0 \in P$, for some $\omega_o \in P_{\text{fin}}$. Then $\omega \in P^0$ if and only if* $\text{lev}(\omega) = l = 0$.

*Proof.* Write $\mu = \mu_o + n\delta + j\Lambda_0 \in Z$ for some $\mu_o \in P^\vee_{\text{fin}}$. By definition, we have that $\langle \mu, \alpha_i \rangle = \langle \mu, \delta \rangle = 0$ for all $i \in I$. Note that both the (finite) root lattice and the (finite) weight lattice have the same rank, as each set spans the underlying vector space, and hence we can write every weight as an $\mathbb{R}$-linear combination of the simple roots. Using this fact, we write $\omega_o = \sum_i c_i \alpha_i$ for some $c_i \in \mathbb{R}$, and calculate:

$$\begin{aligned} 0 = \langle \mu, \omega \rangle = \langle \mu, \omega_o + m\delta + l\Lambda_0 \rangle &= \sum_i c_i \langle \mu, \alpha_i \rangle + m\langle \mu, \delta \rangle + \langle \mu, l\Lambda_0 \rangle \\ &= 0 + 0 + \langle \mu_o + n\delta + j\Lambda_o, l\Lambda_0 \rangle \\ &= nl. \end{aligned}$$

Note that $n\delta \in Z$ for all $n \in \mathbb{Z}$ as $\langle n\delta, \tilde{\alpha}_i \rangle = 0$ for all $i \in I$, and so we can choose $\mu$ such that $n$ is non-zero. Hence if $\omega \in P^0$, then $nl = 0$ for some non-zero $n$, which forces $l = 0$. Similarly, if instead we first assume that $l = 0$, then we have that $\langle \mu, \omega \rangle = 0$ and so $\omega \in P^0$ as required. □

**Corollary 4.3.10.** *$P^0 = P_{\text{fin}} \oplus \mathbb{Z}\delta = \{\omega_o + m\delta \mid \omega_o \in P_{\text{fin}}, m \in \mathbb{Z}\}$.*

Note that if $\omega \in P^0$, then $w(\omega) \in P^0$ for all $w \in W$. This allows us to make the following definition.

**Definition 4.3.11.** *Fix a dominant finite weight $\omega_o \in P_{\text{fin}}$. The **level-zero weight poset** for $\omega_o$ is $P^0(\omega_o) := W(\omega_o)$, the $W$-translates of $\omega_o$, with the poset structure defined by the transitive closure of the following relation for $\lambda \in P^0(\omega_o), \tilde{\alpha} \in \Phi^+$:*

$$\lambda \prec s_{\tilde{\alpha}}\lambda \iff \langle \tilde{\alpha}^\vee, \lambda \rangle > 0.$$

The proof of Theorem 4.3.7 follows from the following two propositions, using the level-zero weight poset as a stepping stone by connecting it to the quantum Bruhat graph and the semi-infinite Bruhat order respectively.

**Proposition 4.3.12** ([LNS+15, Thm. 6.5])**.** *Let $\mu = w(\omega_o) = w_o\tau^\lambda(\omega_o) \in P^0(\omega_o)$.*

(i) *If $\mu \prec s_{\tilde{\alpha}}\mu$ is a cover for some $\tilde{\alpha} \in \Phi$, then $\tilde{\alpha} = w_o\beta + n\delta$ for some $\beta \in \Phi^+_{\text{fin}}$ with $n \in \{0, 1\}$, and $w_o \Rightarrow w_o s_\beta$ is an edge in $\text{QBG}(W_{\text{fin}})$ of Bruhat type if $n = 0$, or of quantum type if $n = 1$.*

(ii) *If $w_o \Rightarrow w_o s_\alpha$ is an edge in $\text{QBG}(W_{\text{fin}})$, then $\mu \prec s_{\tilde{\alpha}}\mu$ is a cover, where $\tilde{\alpha} = w_o\beta$ if the edge is Bruhat, or $\tilde{\alpha} = w_o\beta + \delta$ if the edge is quantum.*

**Proposition 4.3.13** ([INS16, Prop. 4.2.1])**.** *Let $w \in W$ and $\tilde{\beta} \in \Phi^+$, and fix a dominant finite weight $\omega_o$. Then $w \leq^{\frac{\infty}{2}} s_{\tilde{\beta}}w$ is a cover if and only if $w\omega_o \prec s_{\tilde{\beta}}w\omega_o$ is a cover.*

### § 4.3.3 The Double Affine Semi-Infinite Bruhat Order

Our aim is to generalise the definitions of the previous subsection to the double affine case, with the long term goal being to develop a deeper understanding of the affine quantum Bruhat graph. In what follows, we propose affinisations of the previous concepts and give some preliminary results in order to support their introduction.

**Definition 4.3.14.** *Let $x = w\varepsilon^\mu \in W_{\mathcal{T}}$.*

(i) *The **double affine semi-infinite length function** is:*

$$\ell^{\frac{\infty}{2}}(x) := \ell(w) + \langle 2\rho, \mu\rangle$$

(ii) *The **semi-infinite Bruhat order** on $W_{\mathcal{T}}$ is the partial order graded by $\ell^{\frac{\infty}{2}}(\cdot)$, and generated by the following relations for $x \in W_{\mathcal{T}}, \hat{\alpha} \in \Phi^+_{\text{aff}}$:*

$$x \leq^{\frac{\infty}{2}} s_{\hat{\alpha}}x \iff \ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) = \ell^{\frac{\infty}{2}}(x) + 1.$$

Note that the semi-infinite Bruhat order is itself already single affine, but we use the terminology 'double affine' regardless for clarity. Many of the properties of this order generalise to the double affine case.

**Theorem 4.3.15.** *Let $x = w\varepsilon^\mu \in W_{\mathcal{T}}$ and $\hat{\alpha} = \tilde{\alpha} + m\pi \in \Phi^+_{\text{aff}}$.*

(i) $\ell^{\frac{\infty}{2}}(w) = \ell(w)$.

(ii) *If $\mu$ is dominant then $\ell^{\frac{\infty}{2}}(w\varepsilon^\mu) = \ell(w\varepsilon^\mu)$.*

(iii) $x \leq^{\frac{\infty}{2}} s_{\hat{\alpha}}x \iff \ell(s_{\tilde{\alpha}}w) - \ell(w) = 1 - 2\langle\rho, mw^{-1}\tilde{\alpha}^\vee\rangle$.

(iv) $\ell^{\frac{\infty}{2}}(x) < \ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) \iff w^{-1}\tilde{\alpha} > 0$.

*Proof.* (i) follows from the definition of the order. The proof of Proposition 4.3.2 generalises immediately to give (ii), and similarly the calculation in Remark 4.3.4 generalises to give (iii). For (iv), we mimic the proof of Proposition 4.3.5 found in [NW17, Prop. 2.2.5]. We find:

$$s_{\hat{\alpha}}x = s_{\tilde{\alpha}}\varepsilon^{m\tilde{\alpha}^\vee}w\varepsilon^{\mu} = s_{\tilde{\alpha}}w\varepsilon^{mw^{-1}\tilde{\alpha}^\vee+\mu} = ws_{w^{-1}\tilde{\alpha}}\varepsilon^{mw^{-1}\tilde{\alpha}^\vee+\mu},$$

and hence:

$$\begin{aligned}\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) &= \ell(ws_{w^{-1}\tilde{\alpha}}) + \langle 2\rho, mw^{-1}\tilde{\alpha}^\vee + \mu\rangle - \ell(w) + \langle 2\rho, \mu\rangle \\ &= \ell(ws_{w^{-1}\tilde{\alpha}}) - \ell(w) + 2m\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle.\end{aligned}$$

First assume that $w^{-1}\tilde{\alpha} < 0$. We have that $\ell(s_{w^{-1}\tilde{\alpha}}) \leq 2\mathrm{ht}(w^{-1}\tilde{\alpha}) - 1$ by [HP24, Lemma 2.5], and hence, since $w^{-1}\tilde{\alpha} < 0$, we get that:

$$\ell(s_{w^{-1}\tilde{\alpha}}) \leq -2\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle - 1.$$

Then we calculate:

$$\begin{aligned}\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) &= \ell(ws_{w^{-1}\tilde{\alpha}}) - \ell(w) + 2m\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle \\ &\leq \ell(w) + \ell(s_{w^{-1}\tilde{\alpha}}) - \ell(w) + 2m\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle \\ &\leq -2\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle - 1 + 2m\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle \\ &= 2(m-1)\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle - 1.\end{aligned}$$

Note that $\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle < 0$ by assumption, and hence $\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) > 0$ if and only if $m = 0$, in which case we have $\ell^{\frac{\infty}{2}}(x) = \ell(w)$ and $\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) = \ell(ws_{w^{-1}\tilde{\alpha}})$ by part (i). However, we also have $\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) = \ell(ws_{w^{-1}\tilde{\alpha}}) - \ell(w)$ from the above calculation, from which we must have $\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) = 0$, a contradiction. Therefore $\ell^{\frac{\infty}{2}}(x) > \ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x)$ as required.

Now instead assume $w^{-1}\tilde{\alpha} > 0$. If $\tilde{\alpha} > 0$, then $ws_{w^{-1}\tilde{\alpha}} = s_{\tilde{\alpha}}w > w$ and so:

$$\begin{aligned}\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) &= \ell(ws_{w^{-1}\tilde{\alpha}}) - \ell(w) + 2m\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle \\ &\geq \ell(ws_{w^{-1}\tilde{\alpha}}) - \ell(w) + m(\ell(s_{w^{-1}\tilde{\alpha}}) + 1) \\ &> 0,\end{aligned}$$

as required. If $\tilde{\alpha} < 0$, then we apply a similar calculation to before to find:

$$\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) \geq 2(m-1)\langle \rho, w^{-1}\tilde{\alpha}^\vee\rangle + 1 > 0,$$

and so in both cases we have $\ell^{\frac{\infty}{2}}(s_{\hat{\alpha}}x) - \ell^{\frac{\infty}{2}}(x) > 0$. □

In order to link the double affine semi-infinite Bruhat order to the affine quantum Bruhat graph, we want to introduce an affinisation of the level-zero weight poset. Initially this seems difficult, as the definition of this poset already relies on the affine weight lattice, which we do not have a natural generalisation for. However, the presentation we have found in Corollary 4.3.10 does allow for a more natural affinisation.

**Definition 4.3.16.** *We define the **double affine level-zero weight lattice** to be* $P^0_{\mathrm{aff}} := P \oplus \mathbb{Z}\pi = \{\omega + m\pi \mid \omega \in P_{\mathrm{aff}}, m \in \mathbb{Z}\}$.

In order to define a double affine level-zero weight poset, we first need to define an action of $W_{\mathcal{T}}$ on $P^0_{\mathrm{aff}}$. We do this by extending the action of $W_{\mathcal{T}}$ on $\Phi_{\mathrm{aff}}$ as defined in Equation (2.8), using the action of $W$ on $P$ from Equaiton (2.5). Namely, for $x = w\varepsilon^\mu \in W_{\mathcal{T}}$ with $w \in W$ and $\mu \in \mathcal{T}$, we write:

$$x(\lambda + m\pi) = w\varepsilon^\mu(\lambda + m\pi) := w(\lambda) + (m - \langle \mu, \lambda \rangle)\pi, \tag{4.3}$$

where $\lambda \in P, m \in \mathbb{Z}$.

We also need to define a double affine version of the pairing $\langle \tilde{\alpha}^\vee, \lambda \rangle$ in order to give the partial ordering on the set, which seemingly has no obvious affinisation. However, writing $\tilde{\alpha} = \alpha + n\delta$ and $\lambda = w\omega_o = w_o\tau^\lambda\omega_o$, we have:

$$\begin{aligned} \langle \tilde{\alpha}^\vee, \mu \rangle = \langle \tilde{\alpha}^\vee, w(\omega_o) \rangle &= \langle \alpha^\vee + n\delta, w_o\tau^\lambda(\omega_o) \rangle \\ &= \langle \alpha^\vee + n\delta, w_o(\omega_o) - \langle \lambda, \omega_o \rangle \delta \rangle = \langle \alpha^\vee, w_o(\omega_o) \rangle, \end{aligned}$$

which can be affinised much more naturally. Hence, we define the following:

**Definition 4.3.17.** *Fix a dominant (affine) weight* $\omega \in P$. *The **double affine level-zero weight poset** for* $\omega$ *is* $P^0_{\mathrm{aff}}(\omega) := W_{\mathcal{T}}(\omega)$, *the* $W_{\mathcal{T}}$*-translates of* $\omega$, *with the poset structure defined by the transitive closure of the following relation for* $\lambda = w\varepsilon^\mu(\omega) \in P^0_{\mathrm{aff}}(\omega), \hat{\alpha} = \mathrm{sgn}(m)(\tilde{\alpha} + m\delta) \in \Phi^+_{\mathrm{aff}}$, $\tilde{\alpha} \in \Phi^+$:

$$\lambda \prec s_{\hat{\alpha}}\lambda \iff \langle \tilde{\alpha}^\vee, w(\omega) \rangle > 0.$$

Unfortunately, neither of the proofs of Proposition 4.3.12 nor Proposition 4.3.13 generalise smoothly to our double affine setting, however the concepts used within the respective proofs seem to remain well-defined once affinised. Our hope is that a link between the double affine semi-infinite Bruhat order and the affine quantum Bruhat graph can be developed using this new double affine level-zero weight poset, or otherwise, from which a greater understanding of both concepts could be gathered.

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