Representation Theory of Quantised Function Algebras at Roots of Unity

by

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A thesis submitted to
the Faculty of Science
at the University of Glasgow
for the degree of
Doctor of Philosophy

October 8, 1998

To

the Clan Gordon

" To cook a hare - you must first catch it;"

The Devils, Dostoyevsky.

Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Chapters 1 and 2 cover basic definitions, notations and mostly known results.

Chapters 3,4,5 and 6 are the original work of the author. Some of the results therein will be published in [40], [41] and [42].

Acknowledgements

Firstly, I would like to express my gratitude to my supervisor Professor K. A. Brown for all his guidance, assistance and encouragement. He has always supported my work and has given me a free hand to follow my interests. Secondly, I want to thank everyone in Room 403, without whom things would not have been nearly so interesting. I also thank the Engineering and Physical Sciences Research Council for funding me with a three year award. My final and special thanks go to my parents for all their love, support and patience.

Summary

The subject of this thesis is quantum group theory. More precisely this thesis is concerned with the representation theory of quantised function algebras of semisimple algebraic groups at roots of unity and of quantised enveloping algebras of Borel subalgebras of semisimple Lie algebras at roots of unity.

Chapter 1 is an introduction to the theory of quantum groups. We describe generic quantum groups and their specialisations to roots of unity over $\mathbb{C}$, following [23] and [25]. We discuss various ring theoretic features enjoyed by these algebras which allow us define a family of finite dimensional $\mathbb{C}$-algebras parametrised by an algebraic group. We end this chapter by discussing Drinfel'd duality and recalling a number of the features of the representation theory of these factors, proved in [24] and [27].

In Chapter 2 we consider some cohomological aspects of augmented algebras and Hopf algebras. We introduce two spectral sequences for augmented algebras: firstly, following [7], an analogue of the Lyndon-Hochschild-Serre spectral sequence of group theory and Lie algebras; secondly a spectral sequence which relates, for a class of filtered rings, the cohomology of the associated graded ring with the cohomology of the original ring. We discuss various aspects of these spectral sequences including their relations with inflation and restriction of cohomology. We then pursue links between the cohomology of a Hopf algebra and its representation theory. Following the well-trodden path of [34], [35], [31] and [74] we acquaint ourselves with the complexity of a finite dimensional algebra, support varieties for Hopf algebras and how these relate to the representation type of a finite dimensional algebra. One necessary condition on the success of this theory is that the Hopf algebras we consider must have finitely generated cohomology rings.

Chapter 3 is dedicated to proving that quantised function algebras at roots of unity have finitely generated cohomology, allowing the results of Chapter 2 to be applied. In fact, under a restriction on the order of the root of unity, we explicitly describe the structure of

the cohomology ring - it is simply a polynomial ring concentrated in even degrees. There is already an analogue of this result in the literature for quantised enveloping algebras (of Borel subalgebras) at roots of unity, [39]. We also discuss the finite generation of cohomology for much less restrictive conditions on the order of the root of unity. In these situations, however, we are unable to prove explicit structure theorems. Much of this chapter will be published in [40].

Following Chapter 3, we compute the complexities of the algebras occuring in the family of factors of quantum groups. We first do our calculations for factors of the quantised enveloping algebra of Borel subalgebras. This has a simple description in terms of the length function on the Weyl group associated to the Borel subalgebra (or more precisely its associated semisimple Lie algebra). To prove this we have to invoke a number of results from Chapters 1 and 2. Having done this we deduce the result for factors of quantised function algebras by using Drinfel'd duality. Again this has a description in terms of the length function on the Weyl group. The results in this chapter comprise the core of [42].

In Chapter 5 we discuss the ramifications of the results of Chapter 4. In particular we completely describe the factors (for both quantised function algebras and quantised enveloping algebras of Borel subalgebras) which have finite representation type. It turns out that this question is more easily answered in the function algebra case - for quantised enveloping algebras we get a result which is perhaps indicative of something more general. We also show that there are no tame factors. To prove this we have to discuss the Azumaya locus of the quantised function algebras. It turns out that almost all the algebras we need lie on this locus and, thanks to a result of [13] and a description of the centre of the quantised function algebra in [30], we can completely describe the structure of these algebras. Having done this it's easy to see that these are wild whenever they don't have finite representation type.

Finally, in Chapter 6, we discuss the most degenerate case of the above factors, namely the analogues of the restricted enveloping algebra of a restricted Lie algebra. Since these factors are basic algebras they can be described as path algebras with relations. We set forth this description in the widest setting possible, generalising the results of [21] for the case of quantised enveloping algebras. The work in this chapter constitutes [41].

Introduction

This thesis seeks to illustrate the combined strength of cohomology, Hopf algebras, ring theory and geometry through the example of the representation theory of quantised function algebra at roots of unity. It is the interplay of these areas which allows us to prove strong and often beautiful results. The Hopf algebras in this thesis are not cocommutative (in fact they are not even quasitriangular) so much of the known cohomology theory is lost; the rings we study are either finite dimensional algebras or affine PI algebras; the geometry appearing is (mostly) the double Bruhat decomposition of a semisimple algebraic group, defined in (1.2). The theorems we prove describe cohomology rings, support varieties, complexity and the representation type of finite dimensional algebras naturally associated to quantised function algebras at roots of unity and Morita theorems and presentations by quivers and relations for some of these algebras too.

In the beginning were Drinfel'd and Jimbo. The background from which Drinfel'd and Jimbo introduced quantum groups was the quantum Yang-Baxter equation from statistical mechanics and the quantum inverse scattering method of integrable systems, [50] and [28]. From their birth in physics quantum groups have found applications in knot theory, invariants of 3-manifolds, category theory, Hopf algebras, ring theory and Lie theory. It is the last three subjects which interest us.

Quantum groups in Lie theory begin with generic quantised enveloping algebras, $U_q(\mathfrak{g})$, and generic quantised function algebras, $\mathcal{O}_q[G]$. These Hopf algebras are deformations over $\mathbb{C}$ of the enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ and of the ring of regular functions of a simply-connected, semisimple algebraic group G respectively. The (finite dimensional) representation theory of $U_q(\mathfrak{g})$ turns out to be very similar to the representation theory of the enveloping algebra of $\mathfrak{g}$; the category of finite dimensional representations of $U_q(\mathfrak{g})$ does however have an interesting monoidal structure, [19, Chapter 10]. The representation theory of $\mathcal{O}_q[G]$ is more interesting. The prime spectrum of $\mathcal{O}_q[G]$ has been described, first by Hodges and Levasseur for $\mathcal{O}_q[SL(n)]$ in [44] and then for general $\mathcal{O}_q[G]$ in [51]: it is stratified by locally closed pieces parametrised by $W \times W$ (here W is the Weyl group of G), the closure ordering being given by the (reverse) Bruhat order on $W \times W$. The spectrum of each locally closed piece of this stratification is homeomorphic to the prime spectrum of a Laurent polynomial ring, whose maximal ideals correspond bijectively to the primitive ideals of $\mathcal{O}_q[G]$ contained in the relevant locally closed strata.

The representation theory of quantum groups at roots of unity is, at first sight, radically different. Lusztig introduced restricted quantised enveloping algebras at roots of unity, $\widehat{U}_\epsilon(\mathfrak{g})$, in an attempt to prove his conjecture which describes the characters of simple modules of simply-connected, semisimple algebraic groups in positive characteristic, [58]. Here $\epsilon \in \mathbb{C}$ is a primitive ℓ^{th} root of unity (with a mild restriction on the order ℓ depending on $\mathfrak{g}$). The algebra $\widehat{U}_\epsilon(\mathfrak{g})$ is a characteristic zero analogue of the hyperalgebra of G in positive characteristic. Lusztig hoped his quantised enveloping algebra would provide a link between the relatively well understood representation theory of affine Kac-Moody Lie algebras in characteristic zero and the representation theory of algebraic groups in characteristic $p > 0$. This has been realised in a series of papers [53], [54], [55] and [3] and has culminated in a proof of this conjecture for sufficiently large p depending on G. Unfortunately it is unknown how large p should be.

Perhaps inspired by Lusztig's approach De Concini and Kac introduced the non-restricted quantised enveloping algebra at roots of unity, $U_\epsilon(\mathfrak{g})$, [23]. In this case $U_\epsilon(\mathfrak{g})$ is a characteristic zero deformation of the enveloping algebra of $\mathfrak{g}$ in positive characteristic, $U(\mathfrak{g})$. The quantised and classical algebras share many properties: they are Noetherian Hopf domains; they are finitely generated over central sub-Hopf algebras; their representation theory is largely governed by the orbits of the coadjoint action of G (on a suitably defined space). In contrast with the restricted enveloping algebra at a root of unity, the representation theory of unrestricted quantised enveloping algebras at roots of unity limps behind the positive characteristic case. This is illustrated perfectly by the Kac-Weisfeiler theorem which relates the dimension of simple $U(\mathfrak{g})$-modules (respectively simple $U_\epsilon(\mathfrak{g})$-modules) to the dimension of the above coadjoint orbits: this is now a theorem for $U(\mathfrak{g})$, [68]; for $U_\epsilon(\mathfrak{g})$ it has been proven for type A_n only, using reduction mod p and the $U(\mathfrak{g})$ theorem, [14].

Quantised function algebras at roots of unity, $\mathcal{O}_\epsilon[G]$, are much better understood, [25]. These are Hopf algebra deformations over $\mathbb{C}$ of $\mathcal{O}[G]$, the ring of regular functions on G, but unlike $\mathcal{O}_q[G]$ the representation theory of $\mathcal{O}_\epsilon[G]$ behaves like modular representation theory. Indeed $\mathcal{O}_\epsilon[G]$ is finitely generated over a central sub-Hopf algebra isomorphic to $\mathcal{O}[G]$. In particular all simple $\mathcal{O}_\epsilon[G]$-modules are finite dimensional over $\mathbb{C}$ and so, by Schur's lemma, $Z := Z(\mathcal{O}_\epsilon[G])$, the centre of $\mathcal{O}_\epsilon[G]$, acts by scalar multiplication on simple $\mathcal{O}_\epsilon[G]$-modules. This is illustrated by the following picture

$$\mathcal{I}rr\mathcal{O}_\epsilon[G] \xrightarrow{\chi} \text{Maxspec }(Z) \xrightarrow{\pi} G, \tag{1}$$

where $\mathcal{I}rr\mathcal{O}_\epsilon[G]$ consists of the isomorphism classes of simple $\mathcal{O}_\epsilon[G]$-modules. Here χ sends a simple module to its central character (defined by the scalar multiplication above) whilst π is the morphism associated to the inclusion of commutative algebras $\mathcal{O}[G] \longrightarrow Z$. Both χ and π are finite surjective maps. Moreover, for any $g \in G$, the elements of $(\pi \circ \chi)^{-1}(g)$ are precisely the simple modules of the finite dimensional algebra

$$\mathcal{O}_\epsilon[G](g) := \frac{\mathcal{O}_\epsilon[G]}{\mathfrak{m}_g\mathcal{O}_\epsilon[G]}, \tag{2}$$

where $\mathfrak{m}_g$ is the maximal ideal of $\mathcal{O}[G]$ associated to $g \in G$. It is in this picture that we see the combination of ring theory, Hopf algebras and geometry. For example ring theory and geometry combine to make G a Poisson manifold. As a result G is foliated by symplectic leaves and, using the Hopf structure of $\mathcal{O}[G]$ (which of course we can consider as being induced from $\mathcal{O}_\epsilon[G]$), if g and g' lie in the same symplectic leaf then $\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g')$, [26]. Following calculations of Hodges and Levasseur, [44], this was strengthened to $\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g')$ if g and g' lie in the same double Bruhat cell of G, [25]. (Here we see a similarity with the representation theory of $\mathcal{O}_q[G]$ - this is no coincidence.) To this point a similar analysis works for $U_\epsilon(\mathfrak{g})$, leading to a decomposition of a central sub-Hopf algebra of $U_\epsilon(\mathfrak{g})$ into symplectic leaves, the coadjoint orbits mentioned in the previous paragraph, along which representation theory is constant, [26], mirroring the representation theory of $U(\mathfrak{g})$, [49]. Henceforth, however, our results mostly do not enjoy this generality.

In [27], a paper fundamental to us, the interaction between the PI theory of $\mathcal{O}_\epsilon[G]$ and the geometry of the double Bruhat cells of G is explored. De Concini and Procesi prove that, for any $g \in G$, any simple module of $\mathcal{O}_\epsilon[G](g)$ has dimension $\ell^{\frac{1}{2}\dim S_g}$ where S_g is the symplectic leaf containing g. This is a strong analogue of the Kac-Weisfeiler theorem. An inductive construction of the simple modules is also described.

It is a basic philosophy that the finite dimensional algebra $\mathcal{O}_\epsilon[G](1)$ is the most complicated and important algebra in the family defined by (2). There are several straightforward reasons for this: the trivial module, $\mathbb{C}$, is a module for this algebra; $1 \in G$ is in the closure of every double Bruhat cell and so in the most degenerate position geometrically. However from our point of view the algebra $\mathcal{O}_\epsilon[G](1)$ is crucial since it inherits a Hopf structure from $\mathcal{O}_\epsilon[G]$ and, using comultiplication, many problems concerning $\mathcal{O}_\epsilon[G](g)$ can be transferred to problems in $\mathcal{O}_\epsilon[G](1)$. We demonstrate this with the following example. The point $1 \in G$ is itself a symplectic leaf of G so the result of De Concini and Procesi implies that all the irreducible representations of $\mathcal{O}_\epsilon[G](1)$ are one dimensional. Since $\mathcal{O}_\epsilon[G](1)$ is a Hopf algebra we can tensor representations together, showing in particular that the simple $\mathcal{O}_\epsilon[G](1)$-modules form a group, denoted $Pic(\mathcal{O}_\epsilon[G](1))$ (and isomorphic to $\mathbb{Z}_\ell^r$, [25]). For any $g \in G$ it is immediate that $Pic(\mathcal{O}_\epsilon[G](1))$ acts on the simple modules of $\mathcal{O}_\epsilon[G](g)$ by tensoring and a simple argument shows that this is a transitive action. Consequently much of the representation theory of $\mathcal{O}_\epsilon[G](g)$ can be deduced from studying just one simple module. Before getting to the heart of the thesis let's discuss one more aspect of $\mathcal{O}_\epsilon[G](1)$. As we've observed the simple $\mathcal{O}_\epsilon[G](1)$-modules are all one dimensional and so $\mathcal{O}_\epsilon[G](1)$ is isomorphic to the path algebra of a quiver with relations. From the standpoint of the representation theory of finite dimensional algebras it is important to describe this quiver and its relations. We do this in Theorem 6.2, concluding that the quiver is (essentially) a Cayley graph of $\mathbb{Z}_\ell^r$, reflecting the importance of $Pic(\mathcal{O}_\epsilon[G](1))$.

Given that we have, in principle, a description of the simple $\mathcal{O}_\epsilon[G](g)$-modules for any $g \in G$ it is natural to try to describe the indecomposable modules of $\mathcal{O}_\epsilon[G](g)$. (This, however, will not even describe all the finite dimensional indecomposable modules of $\mathcal{O}_\epsilon[G]$.) This problem has to be refined first - we have to decide whether it is possible to describe the indecomposable representations of $\mathcal{O}_\epsilon[G](g)$, that is it is necessary to find the representation type of $\mathcal{O}_\epsilon[G](g)$. We'll be looking for $g \in G$ such that $\mathcal{O}_\epsilon[G](g)$ has finite or tame representation type since if $\mathcal{O}_\epsilon[G](g)$ has wild representation type it will be impossible to describe all the indecomposable $\mathcal{O}_\epsilon[G](g)$-modules. The philosophy of the previous paragraph is perfect for this problem. Everything starts with the observation of Crawley-Boevey, [22], formalised by Rickard, [74], which gives a sufficient condition for a finite dimensional algebra to have wild representation type in terms of the maximal rate of growth of (iterations of) Auslander-Reiten translates of the indecomposable modules of this algebra, or, equivalently for self-injective algebras, in terms of the maximal rate of growth of minimal projective resolutions of the simple modules of the algebra. This second rate of growth is called the complexity of the algebra and is a homological condition, easily described by the rate of growth of certain Ext groups of the algebra. In the case of the algebras $\mathcal{O}_\epsilon[G](g)$ we can then use the well-known isomorphisms, for finite dimensional $\mathcal{O}_\epsilon[G](g)$-modules M and N,

$$\mathrm{Ext}^i_{\mathcal{O}_\epsilon[G](g)}(N, M) \cong \mathrm{Ext}^i_{\mathcal{O}_\epsilon[G](1)}(\mathbb{C}, M \otimes N^*),$$

to transfer the problem to the algebra $\mathcal{O}_\epsilon[G](1)$. Since we need only consider simple M and N this approach reduces to the following:

1. Knowing that the algebras $\mathcal{O}_\epsilon[G](g)$ are self-injective;

2. Having a good description of the simple $\mathcal{O}_\epsilon[G](g)$-modules;

3. Being able to calculate cohomology in $\mathcal{O}_\epsilon[G](1)$.

Generalities on Hopf algebras show Number 1 holds whilst we have already seen that [27] has dealt with Number 2. Only Number 3 remains and this takes up a large part of the thesis.

Cohomology is generally difficult to calculate explicitly; that we can do so for $\mathcal{O}_\epsilon[G](1)$ goes back to the opening paragraph of the introduction. Basically the combination of cohomology and ring theory allows us to replace the algebra $\mathcal{O}_\epsilon[G](1)$ by a simpler associated graded algebra, constructed with respect to some filtration depending on $g \in G$; the geometry and ring theory discussed in [27] give us the starting point for an induction argument which describes the cohomology of the simpler associated graded algebra; all the while the Hopf structure on $\mathcal{O}_\epsilon[G](1)$ lets us keep track of the various products in cohomology. (This scheme is slightly simplified: we actually do these calculations for different quantised function algebras, $O_\epsilon[B^\pm](1)$, where $B^\pm$ are opposite Borel subgroups of G with respect to some maximal torus of G. Using Drinfel'd duality the algebras $O_\epsilon[B^\pm](1)$ can be described as factors of quantised enveloping algebras of Borel subalgebras of $\mathfrak{g}$, with which it is easier to work. We deduce the result for $\mathcal{O}_\epsilon[G](1)$ by patching together $O_\epsilon[B^+](1)$ and $O_\epsilon[B^-](1)$ through a quantisation of the dominant morphism $B^+ \times B^- \longrightarrow G$ given by multiplication.) As a result we are able to explicitly describe the cohomology ring $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G](1)}(\mathbb{C}, \mathbb{C})$ in Theorem 3.7 and also the complexity of all the algebras $\mathcal{O}_\epsilon[G](g)$ in Theorem 4.15. The complexity is simply described in terms of the Weyl group of G.

As a result of the complexity calculations we find families of algebras of wild representation type and also families of finite representation type, Theorem 5.1 and Proposition 5.3. Our calculations leave only the case when $\mathcal{O}_\epsilon[G](g)$ has complexity two to describe.

Cohomology alone is not sufficient to determine the representation type of these - it only rules out the possibility of finite representation type. To describe these algebras we return to our picture (1),

$$\mathcal{I}rr\mathcal{O}_\epsilon[G] \xrightarrow{\chi} \text{Maxspec }(Z) \xrightarrow{\pi} G,$$

where $Z = Z(\mathcal{O}_\epsilon[G])$. Let $p \in$ Maxspec (Z). A theorem of Brown and Goodearl, [13], says that $\chi^{-1}(p)$ is a singleton if and only if $\chi^{-1}(p)$ is a simple module of maximal dimension if and only if p is a smooth point of Maxspec (Z). We call such points Azumaya. If p is Azumaya we show that $\mathcal{O}_\epsilon[G](\pi(p))$ is Morita equivalent to $Z_{\pi(p)} := Z/\mathfrak{m}_{\pi(p)}Z$ where $\mathfrak{m}_{\pi(p)}$ is the maximal ideal of $\mathcal{O}[G]$ corresponding to $\pi(p) \in G$. In fact $\mathcal{O}_\epsilon[G](\pi(p))$ is isomorphic to a matrix ring over $Z_{\pi(p)}$. We can observe from the dimension formulas of De Concini and Procesi that nearly all the algebras $\mathcal{O}_\epsilon[G](g)$ of complexity two have simple modules of maximal dimension so we need only calculate Z_g for such $g \in G$. A theorem of Enriquez, [30], describes Z, the centre of $\mathcal{O}_\epsilon[G]$, precisely. We use this theorem to find some singular points of Maxspec (Z) which, by the above theorem, cannot be central characters of maximal dimension simple modules. Having discounted these nasty points we are able to calculate Z_g exactly for $g \in G$ such that $\chi^{-1}(g)$ consists of Azumaya points. A simple consequence of this, together with another calculation which wipes up the missing $\mathcal{O}_\epsilon[G](g)$ of complexity two, shows that the algebras $\mathcal{O}_\epsilon[G](g)$ are never of tame representation type, Theorem 5.17. Moreover the above calculation also describes all the algebras of finite representation type: they are direct sums of matrix rings over truncated polynomial algebras in one variable, Corollary 5.11.

Throughout the thesis we also consider the above problems for the algebras $O_\epsilon[B^+]$ and $O_\epsilon[B^-]$, where $B^\pm$ are opposite Borel subgroups of G with respect to some maximal torus of G. With the exception of Morita theorems for algebras whose simple modules have maximal dimension we prove analogues of all the results above: in particular we calculate complexities, show there are no tame factors and explicitly describe all the factors having finite representation type. Indeed we often prove a result for these algebras first and the deduce the $\mathcal{O}_\epsilon[G]$ version afterwards.

The thesis is arranged as follows. In Chapter 1 we introduce quantum groups. The cohomology background required for the later chapters is found in Chapter 2, whilst the calculation of cohomology rings and complexity is found in Chapter 3 and Chapter 4 respectively. Chapter 5 deals with representation type, complexity two factors and the factors whose simple modules have maximal dimension. In Chapter 6 we describe presentations of $\mathcal{O}_\epsilon[G](1)$ and $O_\epsilon[B^{\pm}](1)$ as path algebras of quivers with relations. At the end of each chapter we include bibliographic remarks.

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Chapter 1

Quantum groups

This chapter introduces the protagonists of the piece: quantised function algebras of semisimple algebraic groups and quantised enveloping algebras of Borel subalgebras of semisimple Lie algebras. To present these we need a lot of notation from Lie theory (which it's good to fix in any case) and also the idea of generic quantum groups. Our algebras arise from specialising certain integral forms of generic quantum groups, a procedure which often allows us to translate results from the generic case to the root of unity situation. We show how the above quantised function algebras and enveloping algebras are closely related through Drinfel'd duality, a fact which we'll often exploit in later chapters. A substantial amount of work on the representation theory of these quantum groups has been conducted already, notably in [24], [25] and [27]. We present some highlights from these papers which will be of use later.

1.1 Notation

Let G be a semisimple, simply-connected, connected algebraic group defined over $\mathbb{C}$. Fix a maximal torus of G, say T, and let B^+ be a Borel subgroup of G containing T. Define B^- to be the Borel subgroup of G opposite B^+. Let $r = \mathrm{rank}(G)$.

Differentiating the multiplication map $G \times G \longrightarrow G$ gives us a semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, a Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T)$ and Borel subalgebras, $\mathfrak{b}^+$ and $\mathfrak{b}^-$, opposite with respect to $\mathfrak{h}$. Let (a_{ij}) be the Cartan matrix of $\mathfrak{g}$.

Associated to $T \subseteq B^+ \subseteq G$ we have a root system Δ and choice of positive roots Δ^+. Let Q be the root lattice associated to this root system and P the weight lattice. There is a non-degenerate bilinear form

$$(\ ,\ ) : P \times Q \longrightarrow \mathbb{Z}.$$

Under this form the simple roots, $\Pi = \{\alpha_i : 1 \le i \le r\} \subseteq Q$, are dual to the fundamental weights, $\{\varpi_i : 1 \le i \le r\}$. When dealing with (not necessarily simple) roots we will use the notation β. Always we let ρ be the Weyl weight of P, that is $\rho = \sum_i \varpi_i$.

Let W be the Weyl group of Δ, which we can realise as $N_G(T)/T$. For $\alpha_i \in \Pi$ write the corresponding simple reflection in W as s_i. Given $w \in W$ let $\ell(w)$ be the length of w and let w_0 be the (unique) longest word of W. We have $\ell(w_0) = N = |\Delta^+|$.

Associated to any reduced expression of w_0 we have a total ordering of Δ^+. Such an ordering will be denoted by $\beta_1 < \beta_2 < \cdots < \beta_N$ where $N = |\Delta^+|$. To be explicit suppose $w_0 = s_{i_1} \dots s_{i_N}$ is a reduced expression. Then we define the (positive) roots β_j by $\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})$.

Let $s(w)$ be the rank of the linear map $(w - e)$ on $\mathbb{R} \otimes_{\mathbb{Z}} P$. For $w \in W$ define

$$P^w = \{\lambda \in P : w\lambda = \lambda\}.$$

By definition we have $\dim(\mathbb{R} \otimes_{\mathbb{Z}} P^w) = r - s(w)$. The integer $s(w)$ is equal to smallest number, n, such that w can be written as a product of n reflections in arbitrary roots, [52, Lemma A.1.18].

We'll denote the Coxeter number of W by h. In other words h is the order in W of the element $s_1 s_2 \dots s_r$ (for any enumeration of the simple roots).

For each α_i define the integer

$$d_i = \frac{(\alpha_i, \alpha_i)}{2} \in \{1, 2, 3\}. \tag{1.1}$$

More generally if $\beta = w\alpha_i$ for some $w \in W$ we let $d_\beta = d_i$. There are cell decompositions of both B^- and G induced from the Bruhat decomposition of G:

$$B^- = \coprod_{w \in W} B^+ w B^+ \cap B^- \quad \text{and} \quad G = \coprod_{(w_1, w_2) \in W \times W} B^+ w_1 B^+ \cap B^- w_2 B^-. \tag{1.2}$$

In both cases we have abused notation: we should pick representatives of $w \in W = N_G(T)/T$ in $N_G(T)$ in the above decompositions. However, since $T \subseteq B$, the decompositions are independent of this choice. For $w_1, w_2 \in W$ we will write

$$X_{w_1, w_2} = B^+ w_1 B^+ \cap B^- w_2 B^-.$$

We have for all $w_1, w_2 \in W$

$$\dim(X_{w_1, w_2}) = \ell(w_1) + \ell(w_2) + r.$$

1.2 Generic quantum groups

We will assume q is an indeterminate over $\mathbb{C}$. For $m \in \mathbb{Z}$, $d, j \in \mathbb{N}$ define

$$[m]_d = \frac{q^{dm} - q^{-dm}}{q^d - q^{-d}}; \quad [m]_d! = [1]_d[2]_d \dots [m]_d;$$

$$\begin{bmatrix} m \\ j \end{bmatrix}_d = \frac{[m]_d[m-1]_d \dots [m-j+1]_d}{[j]_d!}; \quad \begin{bmatrix} m \\ 0 \end{bmatrix}_d = 1.$$

For $\beta \in \Delta$ let $q_\beta = q^{d_\beta}$ where d_β is as in the previous section.

Definition 1. Fix a lattice P', $Q \subseteq P' \subseteq P$. The *quantised enveloping algebra of* $\mathfrak{g}$ *associated to* P', denoted $U_q(\mathfrak{g}, P')$ is the $\mathbb{C}(q)$-algebra with generators E_i, F_i $(i = 1, \dots, r)$ and K_α $(\alpha \in P')$ subject to the following relations

$$K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_0 = 1, \tag{1.3}$$

$$K_\alpha E_i K_\alpha^{-1} = q^{(\alpha,\alpha_i)} E_i, \tag{1.4}$$

$$K_\alpha F_i K_\alpha^{-1} = q^{-(\alpha,\alpha_i)} F_i, \tag{1.4'}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}, \tag{1.5}$$

and the quantum Serre relations, for $i \neq j$,

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0.$$

It's clear from the above relations that $U_q(\mathfrak{g}, P')$ is a Q-graded algebra with $\deg(E_i) = \alpha_i, \deg(F_i) = -\alpha_i$ and $\deg(K_\alpha) = 0$. So we can speak of the weight of any element of $U_q(\mathfrak{g}, P')$ and it follows from (1.3), (1.4) and (1.4′) that $x \in U_q(\mathfrak{g}, P')$ has weight $\lambda \in Q$ if and only if $K_\alpha x K_\alpha^{-1} = q^{(\alpha,\lambda)} x$ for all $\alpha \in P'$.

The algebra $U_q(\mathfrak{g}, P')$ is a Hopf algebra with comultiplication Δ, antipode S and counit η defined by

$$\begin{aligned} \Delta E_i &= E_i \otimes 1 + K_{\alpha_i} \otimes E_i, & SE_i &= -K_{-\alpha_i} E_i, & \eta E_i &= 0, \\ \Delta F_i &= F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, & SF_i &= -F_i K_{\alpha_i}, & \eta F_i &= 0, \\ \Delta K_\alpha &= K_\alpha \otimes K_\alpha, & SK_\alpha &= K_{-\alpha}, & \eta K_\alpha &= 1. \end{aligned}$$

Definition 2. We say $U_q(\mathfrak{g}, P)$ is of *simply-connected type* and $U_q(\mathfrak{g}, Q)$ is of *adjoint type.*

Remark 1.1. Always in what follows if the lattice P' is not specified for a quantised enveloping algebra take the simply-connected type, that is we take $P' = P$.

Define conjugation on $\mathbb{C}(q)$ as the $\mathbb{C}$-algebra map sending q to q^{-1}. Then there is a $\mathbb{C}(q)$-conjugate linear anti-automorphism of $U_q(\mathfrak{g}, P')$ defined by

$$\omega(E_i) = F_i,\ \omega(F_i) = E_i,\ \omega(K_\alpha) = K_{-\alpha},\ \omega(q) = q^{-1}.$$

Definition 3. Let U_q^+ (respectively U_q^-) be the subalgebra of $U_q(\mathfrak{g}, P')$ generated by $\{E_i : i = 1, \dots, r\}$ (respectively $\{F_i : i = 1, \dots r\}$) and $U_q^0(P')$ be the sub-Hopf algebra generated by $\{K_\lambda : \lambda \in P'\}$.

It is clear that $\omega(U_q^{\pm}) = U_q^{\mp}$ whilst $\omega(U_q^0(P')) = U_q^0(P')$. We have a triangular decomposition, as vector spaces,

$$U_q^- \otimes U_q^0(P') \otimes U_q^+ \cong U_q(\mathfrak{g}, P'). \tag{1.6}$$

Definition 4. Let $U_q^{\geq 0}(P')$ (respectively $U_q^{\leq 0}(P')$) be the sub-Hopf algebra of $U_q(\mathfrak{g}, P')$ generated by $\{E_i, K_\alpha : \alpha \in P'\}$ (respectively $\{F_i, K_\alpha : \alpha \in P'\}$).

As vector spaces, $U_q^{\geq 0}(P') = U_q^+ \otimes U_q^0(P')$ (respectively $U_q^{\leq 0} = U_q^0(P') \otimes U_q^-$). This algebra can be thought of as a quantisation of the enveloping algebra of $\mathfrak{b}^+$ (respectively of $\mathfrak{b}^-$).

For each reduced expression of w_0, the longest word of the Weyl group, there exists a "PBW-type" basis associated to $U_q(\mathfrak{g}, P')$, details of which can be found in [59] and [26, Section 9]. To be a little more specific, taking a reduced expression $w_0 = s_{i_1} \dots s_{i_N}$, we get our total ordering on Δ^+ given by $\beta_1 < \beta_2 < \cdots < \beta_N$. Then associated to any $\beta \in \Delta^+$ we can construct E_β in U_q^+, depending on this reduced expression, [23, Sections 1.6 and 1.7]. The elements E_{α_i}, however, are equal to E_i for all $i = 1, \dots, r$.

Theorem 1.2. *[59] The monomials $\{E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N} : k_j \in \mathbb{N}, 1 \leq j \leq N\}$ are a $\mathbb{C}(q)$-basis of U_q^+.*

Letting $F_\beta = \omega(E_\beta)$ we similarly obtain a monomial basis for U_q^-. If we choose a basis of the lattice P', then we get an obvious basis of $U_q^0(P')$, which, by the triangular decomposition (1.6), combines with the bases of U_q^+ and U_q^- to yield our "PBW-type" basis of $U_q(\mathfrak{g}, P')$.

Remark 1.3. It is clear that an appropriate subset of this basis affords a basis of any of the subalgebras above.

Lemma 1.4. *[3, Proposition C5] Let $\beta = w\alpha \in \Delta^+$ for some $\alpha \in \Pi$. Then there are weights γ_v with $0 < \gamma_v < \beta$ and $w^{-1}\gamma_v < 0$ and $w^{-1}(\beta - \gamma_v) > 0$ for all j such that*

$$\Delta(E_\beta) = E_\beta \otimes 1 + K_\beta \otimes E_\beta + \sum_j x_v \otimes y_v,$$

with $x_v \in U_q(\mathfrak{g}, P')$ of weight γ_v and $y_v \in U_q(\mathfrak{g}, P')$ of weight $\beta - \gamma_v$.

Using ω and the identity $\Delta \circ \omega = (\omega \otimes \omega) \circ \tau \circ \Delta$, where τ is the twist sending $x \otimes y$ to $y \otimes x$, there is an analogous statement for the elements F_β.

To define the quantised function algebra of G we need a well-known result.

Theorem 1.5. *[48, 5.1-5.3 and Theorem 5.10] Let $U = U_q(\mathfrak{g}, Q)$. Then the set*

$$\bigcap \{Ann(V) : V \text{ finite dimensional irreducible } U\text{-module}\}$$

is zero and all finite dimensional irreducible representations are parametrised by $\mathbb{Z}_2^r \times P^+$.

Let $\mathfrak{C}$ denote the class of finite dimensional representations of U on which the K_{α_i}'s act as powers of q (this corresponds to taking the irreducible representations parametrised by $\{1\} \times P^+$). Given $V \in \mathfrak{C}$ and $v \in V$, $\phi \in V^*$ we define a functional

$$c^V_{\phi,v} : U = U_q(\mathfrak{g}, Q) \longrightarrow \mathbb{C}(q),$$

called a *matrix coefficient*, by the rule

$$c^V_{\phi,v}(x) = \phi(x.v)$$

for all $x \in U$. Since the objects of $\mathfrak{C}$ are finite dimensional, the kernel of $c^V_{\phi,v}$ contains an ideal of finite codimension so these matrix coefficients belong to the Hopf dual of $U_q(\mathfrak{g}, Q)$. We can multiply any two matrix coefficients together as follows,

$$c^V_{\phi,v} c^W_{\theta,w}(x) = \sum_x c^V_{\phi,v}(x_{(1)}) c^W_{\theta,w}(x_{(2)}).$$

Definition 5. The $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{g}, Q)^o$, the Hopf dual of $U_q(\mathfrak{g}, Q)$, generated by the matrix coefficients $c^V_{\phi,v}$ for $V \in \mathfrak{C}$ is called the *quantised function algebra of G*. We denote it by $\mathcal{O}_q[G]$.

Since $\mathfrak{C}$ is closed under taking tensor products and duals, and contains the trivial module, $\mathcal{O}_q[G]$ is spanned, as a $\mathbb{C}(q)$-vector space, by the matrix coefficients. Moreover the Hopf algebra structure of $U_q(\mathfrak{g}, Q)$ makes $\mathcal{O}_q[G]$ into a Hopf algebra.

As all finite dimensional U-modules are completely reducible, [48, Theorem 5.17], it is enough to consider only the matrix coefficients of irreducible representations in the above definition, since these will generate $\mathcal{O}_q[G]$. For V an irreducible representation in $\mathfrak{C}$ parametrised by $\lambda \in P^+$, we'll denote the matrix coefficient $c^V_{\phi,v}$ by $c^\lambda_{\phi,v}$ where $v \in V$ and $\phi \in V^*$.

Remark 1.6. The principal problem with this definition is lack of a complete set of relations satisfied by the matrix coefficients. In the case $G = SL(n)$ there is, however, a presentation by generators and relations, [44]. The algebra $\mathcal{O}_q[SL(n)]$ is generated by elements X_{rs} for $1 \le r, s \le n$ with relations

$$\begin{aligned}
X_{rs}X_{ts} &= qX_{ts}X_{rt} && \text{for } r < t \\
X_{rs}X_{rt} &= qX_{rt}X_{rs} && \text{for } s < t \\
X_{rs}X_{tu} &= X_{tu}X_{rs} && \text{for } r < t \text{ and } s > u \\
X_{rs}X_{tu} - X_{tu}X_{rs} &= (q - q^{-1})X_{ru}X_{ts} && \text{for } r < t \text{ and } s < u
\end{aligned}$$

$$\mathrm{Det}_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} X_{\sigma(1),1} \dots X_{\sigma(n),n} = 1.$$

Here Det_q is the quantum determinant. With this realisation many of the matrix coefficients arising from irreducible representations associated with fundamental weights can be interpreted as quantum minors.

We can make one useful observation at this point. Let η be the augmentation of $\mathcal{O}_q[G]$, defined by $\eta(c^\lambda_{\phi,v}) = \phi(1.v)$, and let $\mathcal{O}_q[G]_+$ denote the augmentation ideal. Then we see

$$c^\lambda_{\phi,v} \in \mathcal{O}_q[G]_+ \iff \phi(v) = 0. \tag{1.7}$$

Recall ρ is the Weyl weight of P. Let $\phi \in V(\rho)^* \cong V(\rho)$, $v \in V(\rho)$ be respectively highest and lowest weights, dual to one another, where $V(\rho)$ is the irreducible U-module with parameter ρ. If we define

$$z^\rho = c^\rho_{\phi,v}, \tag{1.8}$$

then we see that, by (1.7),

$$z^\rho \notin \ker(\eta). \tag{1.9}$$

1.3 Quantum groups at roots of unity

To avoid such potential catastrophes as dividing by zero in the construction of quantum groups at roots of unity we impose the following restriction on the order of roots of unity we consider.

Hypothesis 1. *Let $\epsilon \in \mathbb{C}$ be an ℓ^{th} root of unity where ℓ is an odd positive integer, prime to the numbers d_i defined in equation (1.1).*

In practice this will mean that ℓ is odd and prime to 3 if G has a component of type G_2. Whenever we talk about quantum groups at a root of unity ϵ we will assume that ϵ satisfies Hypothesis 1 and perhaps we'll remind ourselves of this from time to time.

Quantised enveloping algebras at roots of unity. Following [23, Section 1] one can construct an integral form of $U_q(\mathfrak{g}, P')$ which allows specialisation to ϵ. This integral form, $U_{\mathcal{A}}(\mathfrak{g}, P')$, is simply the $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ subalgebra of $U_q(\mathfrak{g}, P')$ generated by the elements E_i, F_i, K_α and $\frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}}$ for $\alpha \in P'$. For ϵ an ℓ^{th} primitive root of unity, where ℓ satisfies Hypothesis 1, the specialisation $U_{\mathcal{A}}(\mathfrak{g}, P') \otimes_{\mathcal{A}} \mathbb{C}$ obtained by sending q to ϵ is simply the algebra with generators and relations identical to those of $U_q(\mathfrak{g}, P')$, with q replaced throughout by ϵ.

Definition 6. The specialised algebra above is called the *(non-restricted) quantised enveloping algebra of* $\mathfrak{g}$ *at a root of unity,* ϵ, and is denoted by $U_\epsilon(\mathfrak{g}, P')$.

Remark 1.7. The algebra $U_\epsilon(\mathfrak{g}, Q)$ is certainly not the same as the quantised enveloping algebra of $\mathfrak{g}$ at a root of unity ϵ introduced in [59] and [60]. This difference is analogous to the difference in non-zero characteristic between the enveloping algebra of $\mathfrak{g}$ and its hyperalgebra (when $\mathfrak{g}$ is suitably defined). More details on this can be found in [19, Chapter 9].

By abuse of notation we will write the generators of $U_\epsilon(\mathfrak{g}, P')$ as E_i, F_i and K_α and will understand that we are discussing $U_\epsilon(\mathfrak{g}, P')$ unless otherwise stated.

The algebra $U_\epsilon(\mathfrak{g}, P')$ inherits the structure of a $\mathbb{C}$-Hopf algebra from $U_q(\mathfrak{g}, P')$ with formulae for comultiplication simply obtained by replacing q's with ϵ's. In particular Lemma 1.4 remains valid for $U_\epsilon(\mathfrak{g}, P')$. One may also construct a "PBW-type" basis for $U_\epsilon(\mathfrak{g}, P')$ using exactly the same methods: such a basis still depends on a choice of reduced expression for $w_0 \in W$.

Definition 7. Let U_ϵ^+ (resp. U_ϵ^-) be the subalgebra of $U_\epsilon(\mathfrak{g}, P')$ generated by $\{E_i : i = 1, \ldots, r\}$ (resp. $\{F_i : i = 1, \ldots r\}$) and $U_\epsilon^0(P')$ be the sub-Hopf algebra generated by $\{K_\lambda : \lambda \in P'\}$. Similarly $U_\epsilon^{\geq 0}(P')$ (resp. $U_\epsilon^{\leq 0}(P')$) is defined to be the sub-Hopf algebra of $U_\epsilon(\mathfrak{g}, P')$ generated by $\{E_i, K_\alpha : \alpha \in P'\}$ (resp.$\{F_i, K_\alpha : \alpha \in P'\}$).

We will be concerned with the representation theory of the algebras $U_\epsilon^{\geq 0}$ and $U_\epsilon^{\leq 0}$. (Recall this means take the simply-connected versions, that is take $P' = P$.) However there is an algebra isomorphism

$$\varphi : U_\epsilon^{\geq 0} \longrightarrow U_\epsilon^{\leq 0}$$

given by $\varphi(E_i) = F_i$ and $\varphi(K_\alpha) = K_{-\alpha}$. Hence we'll often state results for $U_\epsilon^{\geq 0}$ only.

Pick a reduced expression for w_0,

$$w_0 = s_{i_1} \ldots s_{i_N}, \tag{1.10}$$

and, for $1 \leq j \leq N$, let E_{β_j} be the corresponding element in the "PBW-type" basis. Let $Z_0^{\geq 0}$ be the subalgebra of $U_\epsilon^{\geq 0}$ generated by the elements $E_{\beta_j}^\ell$ for $1 \leq j \leq N$ and K_α^ℓ for $\alpha \in P$.

Proposition 1.8. *The algebra $Z_0^{\geq 0}$ is a central sub-Hopf algebra of $U_\epsilon^{\geq 0}$ and, as a Hopf algebra, is isomorphic to $\mathcal{O}[B^-]$, the ring of regular functions on B^-. Moreover it is independent of the choice of reduced expression for w_0. As a $Z_0^{\geq 0}$-module, $U_\epsilon^{\geq 0}$ is free of rank ℓ^{N+r}.*

Proof. The final statement of the proposition is clear from the "PBW-type" basis of $U_\epsilon^{\geq 0}$. It's also clear from the relations in $U_\epsilon^{\geq 0}$ that the elements K_α^ℓ and E_i^ℓ are central. The remainder of the claims are proved in [23, Proposition 3.3] and [26, Section 14].

It follows from Proposition 1.8 that the algebra U_ϵ^+ is free of rank ℓ^N over the central subalgebra generated by $E_{\beta_j}^\ell$.

Keep the choice of reduced expression for w_0 fixed as in (1.10). Define the degree (not to be confused with the Q-grading on $U_\epsilon^{\geq 0}$) of the monomial

$$M_{k,\alpha} = E_{\beta_1}^{k_1} \ldots E_{\beta_N}^{k_N} K_\alpha \in U_\epsilon^{\geq 0} \tag{1.11}$$

to be

$$d(M_{k,\alpha}) = (k_1, \ldots, k_N) \in \mathbb{N}^N.$$

Let $\mathbb{N}^N$ be ordered lexicographically with the convention that $u_1 < u_2 < \ldots < u_N$ where $u_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{iN})$. The following lemma is crucial.

Lemma 1.9. *[57] For $i < j$ we have*

$$E_{\beta_i}E_{\beta_j} = \epsilon^{(\beta_j,\beta_i)}E_{\beta_j}E_{\beta_i} + \sum_{k\in\mathbb{N}^N} c_k M_{k,0},$$

where $c_k \in \mathbb{C}$ and $c_k \neq 0$ only when $k = (k_1, \dots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$.

In particular Proposition 1.9 shows that the degree induces a filtration on $U_\epsilon^{\geq 0}$. The following proposition is immediate from the relations in $U_\epsilon^{\geq 0}$ and the "PBW-type" basis.

Proposition 1.10. *[23, Proposition 1.7(d)] Let $U_\epsilon^{\geq 0}$ be filtered as above. Then the associated graded ring of $U_\epsilon^{\geq 0}$, written $GrU_\epsilon^{\geq 0}$, is a localised quantised polynomial algebra. Indeed $GrU_\epsilon^{\geq 0}$ has generators $\bar{E}_{\beta_i}$ for $1 \leq i \leq N$ and $\bar{K}_\alpha$ for $\alpha \in P$ satisfying the relations*

$$\begin{aligned} \bar{K}_\alpha \bar{K}_{\alpha'} &= \bar{K}_{\alpha+\alpha'} && \alpha, \alpha' \in P \\ \bar{K}_\alpha \bar{E}_{\beta_i} &= \epsilon^{(\alpha,\beta_i)} \bar{E}_{\beta_i} \bar{K}_\alpha && \alpha \in P, 1 \leq i \leq N \\ \bar{E}_{\beta_i} \bar{E}_{\beta_j} &= \epsilon^{(\beta_i,\beta_j)} \bar{E}_{\beta_j} \bar{E}_{\beta_i} && 1 \leq i < j \leq N. \end{aligned}$$

Let us discuss a couple of variants of this filtration. Suppose that we have chosen a reduced expression for w_0 as in (1.10) and let $w = s_{i_1} \dots s_{i_t}$ for some t. Let U_w be the subalgebra of $U_\epsilon^{\geq 0}$ generated by the elements $E_{\beta_1}, \dots, E_{\beta_t}$ and K_α for $\alpha \in P$. A priori this algebra depends on the choice of reduced expression of w (coming from the reduced expression of w_0). Indeed any reduced expression for $w \in W$ can be extended to a reduced expression for w_0, so, in principal, we have versions of U_w for each reduced expression of w. However this turns out not to be the case.

Lemma 1.11. *[26, Proposition 9.3(a)] The algebra U_w is independent of the choice of reduced expression of w.*

It follows from Lemma 1.9 that the monomials $E_{\beta_1}^{k_1} \dots E_{\beta_t}^{k_t} K_\alpha$ form a basis of U_w. Therefore U_w inherits a filtration from $U_\epsilon^{\geq 0}$ such that $\mathrm{Gr}U_w$ is isomorphic to the subalgebra of $\mathrm{Gr}U_\epsilon^{\geq 0}$ generated by $\bar{E}_{\beta_1}, \dots, \bar{E}_{\beta_t}$ and $\bar{K}_\alpha$ for $\alpha \in P$.

Let $w \in W$ be such that $\ell(w) = t$ and assume our reduced expression of w_0 extends w. It will also be useful to coarsen the filtration on $U_\epsilon^{\geq 0}$. To this end for $M_{k,\alpha}$ as in (1.11) define the t-degree to be

$$d_t(M_{k,\alpha}) = (k_{t+1}, \dots, k_N) \in \mathbb{N}^{N-t}.$$

The proof of the following lemma is again a consequence of the "PBW-type" basis, the relations in $U_\epsilon^{\geq 0}$ and Lemma 1.9.

Lemma 1.12. *Fix $w \in W$ with $\ell(w) = t$ and a reduced expression of w_0 extending w as above. Let $U_\epsilon^{\geq 0}$ have the above "t-degree" filtration. Call its associated graded ring $Gr^{(t)}U_\epsilon^{\geq 0}$. Then*

(i) The algebra U_w is a subalgebra of $Gr^{(t)}U_\epsilon^{\geq 0}$.

(ii) Let I_t be the two-sided ideal of U_w generated by $E_{\beta_1}, \dots, E_{\beta_t}$. Then $I_t Gr^{(t)}U_\epsilon^{\geq 0} = Gr^{(t)}U_\epsilon^{\geq 0} I_t$ is a two-sided ideal of $Gr^{(t)}U_\epsilon^{\geq 0}$.

(iii) The factor algebra $Gr^{(t)}U_\epsilon^{\geq 0}/(I_t Gr^{(t)}U_\epsilon^{\geq 0})$ is isomorphic to the subalgebra of the localised quantised polynomial ring, $GrU_\epsilon^{\geq 0}$, generated by $\bar{E}_{\beta_{t+1}}, \dots, \bar{E}_{\beta_N}$ and $\bar{K}_\alpha$ for $\alpha \in P$.

Remark 1.13. There are similar results for the algebra U_ϵ^+ which will be required later. We leave the formulation of these to the reader.

Quantised function algebras at roots of unity. There also exists an integral form of $\mathcal{O}_q[G]$ which is defined by looking at certain $\mathcal{A} = \mathbb{C}[q, q^{-1}]$-integral matrix coefficients, [25, Sections 3 and 4]. Call this form $\mathcal{A}_q[G]$.

Definition 8. For ϵ an ℓ^{th} primitive root of unity, where ℓ satisfies Hypothesis 1, the specialisation $\mathcal{A}_q[G] \otimes_A \mathbb{C}$ obtained by sending q to ϵ is called the *quantised function algebra of G at a root of unity,* ϵ, and is denoted by $\mathcal{O}_\epsilon[G]$.

As with $U_\epsilon^{\geq 0}$ the algebra $\mathcal{O}_\epsilon[G]$ has a large central sub-Hopf algebra.

Proposition 1.14. *[25, Proposition 6.4 and Theorem 7.2] The algebra $\mathcal{O}_\epsilon[G]$ has a central sub-Hopf algebra which, as a Hopf algebra, is isomorphic to $\mathcal{O}[G]$. As an $\mathcal{O}[G]$-module, $\mathcal{O}_\epsilon[G]$ is projective of rank $\ell^{dim(G)}$.*

Let's spend some time describing the whole centre of $\mathcal{O}_\epsilon[G]$. The simple $U_q(\mathfrak{g})$-modules parametrised by $\lambda \in P^+$ have an integral form allowing us to obtain matrix coefficients $c_{\phi,v}^\lambda$ in $\mathcal{O}_\epsilon[G]$ by specialising, [58, Section 4]. For ease of notation we'll suppress the superscript λ from these matrix coefficients: we'll always assume that these arise from (specialisations of) irreducible $U_q(\mathfrak{g})$-modules. There is a commutation relation in $\mathcal{O}_q[G]$ for such elements, discovered in [57]. Assume that v, w have weights μ_1, μ_2 and that ϕ, ψ have weights ν_1, ν_2. Then

$$c_{\phi,v}c_{\psi,w} = q^{-(\mu_1,\mu_2)+(\nu_1,\nu_2)}c_{\psi,w}c_{\phi,v} + \sum_i c_{\psi_i,w_i}c_{\phi_i,v_i}, \tag{1.12}$$

where

$$\psi_i \otimes \phi_i = p_i(q)(M_i(E) \otimes M_i(F))\psi \otimes \phi, \; w_i \otimes v_i = p_i'(q)(M_i'(E) \otimes M_i'(F))w \otimes v, \tag{1.13}$$

where the p_i, p_i' are in $\mathbb{C}(q)$ and the M_i, M_i' are monomials, not both constant. Let v_i be the highest weight in the irreducible module corresponding to $\varpi_i \in P^+$ and let ϕ_i be the highest weight in the dual module (everything is defined up to scalars). Similarly let v_i' be the lowest weight in the irreducible module corresponding to $-w_0\varpi_i$ and let ϕ_i' be the lowest weight in the dual module. It is immediate from (1.12) and (1.13) that

$$c_{\psi,w}c_{\phi_i,v_i} = q^{-(\mu,\varpi_i)+(\nu,-w_0\varpi_i)}c_{\phi_i,v_i}c_{\psi,w}$$

and

$$c_{\phi_i',v_i'}c_{\psi,w} = q^{-(\mu,-\varpi_i)+(\nu,w_0\varpi_i)}c_{\psi,w}c_{\phi_i',v_i'},$$

for ψ having weight ν and w having weight μ. (We should really be careful here and introduce roots of q; when we specialise to ϵ, however, we'll have no problems.)

Definition 9. For $1 \leq i \leq r$, let b_i (respectively c_i) be the specialisations of c_{ϕ_i,v_i} (respectively $c_{\phi_i',v_i'}$) to $\mathcal{O}_\epsilon[G]$.

The above equations show that the elements $b_i^k c_i^{\ell-k}$ are central for $0 \leq k \leq \ell$. The following proposition describes the centre of $\mathcal{O}_\epsilon[G]$.

Proposition 1.15. *[30][25, Appendix]*

(i) Let Z_q be the central subalgebra of $\mathcal{O}_\epsilon[G]$ generated by the elements $b_i^k c_i^{\ell-k}$ for $1 \leq i \leq r$ and $0 \leq k \leq \ell$. Then Z_q is isomorphic to the algebra $\mathbb{C}[\alpha_i(k) : 1 \leq i \leq r, 0 \leq k \leq \ell]/I$ where I is generated by the relations, for $1 \leq i \leq r$,

$$\begin{cases} \alpha_i(k)\alpha_i(k') - \alpha_i(0)\alpha_i(k+k') & \textit{if } k+k' \leq \ell, \\ \alpha_i(k)\alpha_i(k') - \alpha_i(\ell)\alpha_i(k+k'-\ell) & \textit{if } k+k' > \ell. \end{cases}$$

This isomorphism sends $b_i^k c_i^{\ell-k}$ to $\alpha_i(k)$.

(ii) Let B_i and C_i be the elements of $\mathcal{O}[G]$ defined exactly as b_i and c_i (using the Peter-Weyl theorem). Sending $X_i \longmapsto B_i$ and $Y_i \longmapsto C_i$ induces an algebra map

$$\mathbb{C}[X_1, \dots, X_r, Y_1, \dots, Y_r] \longrightarrow \mathcal{O}[G]$$

whilst $X_i \longmapsto \alpha_i(0)$ and $Y_i \longmapsto \alpha_i(\ell)$ induces an algebra map

$$\mathbb{C}[X_1, \dots, X_r, Y_1, \dots, Y_r] \longrightarrow Z_q.$$

There is an algebra isomorphism

$$Z(\mathcal{O}_\epsilon[G]) \cong \mathcal{O}[G] \otimes_{\mathbb{C}[X_1,\dots,X_r,Y_1,\dots,Y_r]} Z_q.$$

(iii) Let $Spec Z_q = C_\ell^r$. As a variety, $Spec Z(\mathcal{O}_\epsilon[G])$ is the fibre product $G \times_{\mathbb{A}^{2r}} C_\ell^r$.

Remark 1.16. It's clear that $Z_q \cong \bigotimes_{i=1}^r Z_q^i$, where $Z_q^i = \mathrm{alg}\{\alpha_i(k) : 0 \le k \le \ell\}$. Also, as an $\mathcal{O}[G] \cap Z_q$-module, Z_q is finitely generated and free: indeed each Z_q^i is free over $\mathcal{O}[G] \cap Z_q^i$ with basis $\alpha_i(k)$ for $0 \le k \le \ell - 1$.

1.4 Drinfel'd Duality

A key feature of quantum groups which is missing from the classical theory is Drinfel'd duality.

Definition 10. The *quantised function algebra of* B^-, written $\mathcal{O}_q[B^-]$, is obtained from $\mathcal{O}_q[G]$ by restricting the matrix coefficients to $U_q^{\le 0}(Q)$.

Since $U_q^{\le 0}(Q)$ is a Hopf subalgebra of $U_q(\mathfrak{g}, Q)$, $\mathcal{O}_q[B^-]$ is a Hopf algebra and the restriction map

$$\pi_- : \mathcal{O}_q[G] \longrightarrow \mathcal{O}_q[B^-] \tag{1.14}$$

is an epimorphism of Hopf algebras. We now introduce Drinfel'd duality.

Proposition 1.17. *[48, 6.13(1), 6.13(4) and Proposition 8.29] Choose a reduced expression of w_0 and hence a "PBW-type" basis for $U_q^{\ge 0}$ and $U_q^{\le 0}(Q)$. There is a perfect Hopf algebra pairing between $U_q^{\le 0}(Q)$ and $(U_q^{\ge 0})_{op}$ given by*

$$\left(\prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha, \prod_{i=N}^{1} E_{\beta_i}^{j_i} K_\lambda \right) = q^{(\alpha,\lambda)} \prod_{i=N}^{1} \delta_{h_i,j_i} (h_i)_{q_{\beta_i}^{-2}}! (q_{\beta_i}^{-1} - q_{\beta_i})^{-h_i},$$

where $(h)_q = \frac{q^h - 1}{q-1}$.

Here $(U_q^{\ge 0})_{\mathrm{op}}$ means that we take the same multiplication but the opposite comultiplication on $U_q^{\ge 0}$. The definition of a Hopf pairing can be found in [52, 3.2.1]. In particular we have, for $x, x' \in U_q^{\le 0}(Q)$ and $y, y' \in U_q^{\ge 0}$,

$$(x, yy') = \sum (x_{(1)}, y)(x_{(2)}, y') \quad \text{and} \quad (xx', y) = \sum (x, y_{(2)})(x', y_{(1)}).$$

This pairing results in a Hopf algebra map from $(U_q^{\ge 0})_{\mathrm{op}}$ to $\mathcal{O}_q[B^-]$. The following proposition is exciting news.

Proposition 1.18. *[25, Lemma 3.4, Proposition 4.2] There is an isomorphism of Hopf algebras $(U_q^{\ge 0})_{op} \cong \mathcal{O}_q[B^-]$.*

Remark 1.19. The algebra $\mathcal{O}_q[B^+]$ is constructed similarly and analogues of Proposition 1.17 and Proposition 1.18 hold for it.

Define γ to be the following composition of algebra maps

$$\mathcal{O}_q[G] \xrightarrow{\Delta} \mathcal{O}_q[G] \otimes \mathcal{O}_q[G] \xrightarrow{\pi_+ \otimes \pi_-} \mathcal{O}_q[B^+] \otimes \mathcal{O}_q[B^-] \xrightarrow{\delta} U_q^{\leq 0} \otimes U_q^{\geq 0}, \quad (1.15)$$

where Δ is comultiplication, π_+ and π_- are restrictions from (1.14) and δ is the tensor of the isomorphisms induced by Drinfel'd duality as in Proposition 1.18.

Theorem 1.20. *[25, Theorem 4.6]*
(i) The map $\gamma : \mathcal{O}_q[G] \longrightarrow U_q^{\leq 0} \otimes U_q^{\geq 0}$ is an injective algebra homomorphism.
(ii) The matrix coefficient z^ρ, defined in (1.8), is normal in $\mathcal{O}_q[G]$ and its image under γ is invertible in $U_q^{\leq 0} \otimes U_q^{\geq 0}$. Thus γ extends to an injection (also denoted by γ)

$$\gamma : \mathcal{O}_q[G][(z^\rho)^{-1}] \longrightarrow U_q^{\leq 0} \otimes U_q^{\geq 0}. \quad (1.16)$$

The image of $\mathcal{O}_q[G][(z^\rho)^{-1}]$ under γ is generated by the elements $1 \otimes E_\beta$, $F_\beta \otimes 1$ and $K_{-\lambda} \otimes K_\lambda$ ($\beta \in \Delta^+$, $\lambda \in P$).

In [25, Sections 2,3,4] the above constructions are carried out for the integral form $\mathcal{A}_q[G]$ introduced in Section 1.3. Therefore we can specialise the above to a root of unity, ϵ, and obtain similar results (explaining why $\mathcal{O}[B^-]$ lies in the centre of $U_\epsilon^{\geq 0}$). In the following portmanteau theorem we collect the results for the root of unity case which will be important for us.

Theorem 1.21. *[25, Sections 4 and 6][27, Proposition 4.4]*
(i) There is an injective algebra map,

$$\gamma_\epsilon : \mathcal{O}_\epsilon[G] \longrightarrow U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}.$$

(ii) The element z^ρ is normal in $\mathcal{O}_\epsilon[G]$ and its image under γ_ϵ is invertible in $U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$ and so γ_ϵ extends to a map (also denoted γ_ϵ)

$$\gamma_\epsilon : \mathcal{O}_\epsilon[G][(z^\rho)^{-1}] \longrightarrow U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}.$$

The image of $\mathcal{O}_\epsilon[G][(z^\rho)^{-1}]$ under this embedding is generated by the elements $1 \otimes E_\beta$, $F_\beta \otimes 1$ and $K_{-\lambda} \otimes K_\lambda$ for $\beta \in \Delta^+, \lambda \in P$.
(iii) The divisor corresponding to $(z^\rho)^\ell \in O[G]$ is the complement of the big cell, $B^+B^- \subseteq G$ and γ_ϵ restricted to $O[G][((z^\rho)^\ell)^{-1}]$ is the algebra map associated to the dominant map $m : B^+ \times B^- \longrightarrow G$ given by multiplication.

1.5 Little Frobenius algebras and big Hopf algebras.

Throughout this section k is an algebraically closed field of arbitrary characteristic. Recall that a finite dimensional k-algebra T is a Frobenius algebra if there is a T-module isomorphism between T and its dual, $\mathrm{Hom}_k(T,k)$. It is a straightforward consequence of the definition that a Frobenius algebra is self-injective, a useful fact in homological algebra. It's well-known that all finite dimensional Hopf algebras are Frobenius, [64, Theorem 2.1.3]; here we generalise this result to Hopf algebras which are finitely generated modules over central sub-Hopf algebras, using the idea of a Frobenius extension. We use this to derive a family of finite dimensional Frobenius algebras parametrised by the closed points of the group scheme associated to the central sub-Hopf algebra. Hopf algebras satisfying such a condition are, as we have seen in Propositions 1.8 and 1.14, central to the study of quantum groups at roots of unity.

Definition 11. Given $S \subseteq T$ rings we say T is a Frobenius extension of S if T is projective as a left S-module and there is an (T,S)-bimodule isomorphism

$$T \cong \mathrm{Hom}_S(T,S),$$

where, for $f \in \mathrm{Hom}_S(T,S)$,we have $(tfs)(t') = f(t't)s$ for $s \in S$ and $t, t' \in T$, see [9].

Remark 1.22. This generalises the notion of a Frobenius algebra. Indeed T is a Frobenius algebra if and only if T is a Frobenius extension of k.

Suppose we have a Frobenius extension $S \subseteq T$ with $\theta : T \xrightarrow{\sim} \mathrm{Hom}_S(T,S)$ the associated (T,S)-bimodule isomorphism. Then there is an induced non-degenerate bilinear form

$$<,>: T \times T \longrightarrow S$$

defined by $<t,t'>= \theta(t')(t)$ for $t,t' \in T$. It's clear that this bilinear form is S-bilinear and T-associative, that is it has the property that $<tt',t''>=<t,t't''>$ for all $t,t',t'' \in T$.

We'll consider Hopf algebras satisfying the following hypothesis:

Hypothesis 2. *H is a Hopf k-algebra which is a finitely generated module over a central affine sub-Hopf algebra H_0.*

Let $\chi : H_0 \longrightarrow k$ be an algebraic character of H_0 and let $\mathfrak{m}_\chi$ be the element of $\mathrm{Maxspec}(H_0)$ associated to χ, that is the kernel of χ. We define a finite dimensional algebra

$$H_\chi = \frac{H}{\mathfrak{m}_\chi H}. \tag{1.17}$$

By abuse of notation let $\eta : H_0 \longrightarrow k$ be the augmentation of H_0 too. Then H_η is a Hopf algebra which we will denote by $\mathbf{H}$.

Remark 1.23. These finite dimensional algebras arise naturally in representation theory. It's not hard to see that, since H is finitely generated over a central subalgebra, any simple H-module, say M, is finite dimensional. Schur's lemma then says that H_0 acts on M via an algebraic character, χ, and so M is a simple H_χ-module. On the other hand any simple H_χ-module is necessarily simple as an H-module, so it's clear that, in order to understand the representation theory of H, we must understand the representation theory of each H_χ.

Lemma 1.24. *The algebra H is a Frobenius extension of H_0.*

Proof. It is straightforward to see that $H_0 \subset H$ is a $\mathbf{H}$-Galois extension, see [56, 1.4] for the definition and [79, Section 1, (4)] for more details. Now [56, Theorem 1.7(5)] tells us that H is a Frobenius extension of H_0.

Remark 1.25. Note that this lemma says that, in particular, H is projective as an H_0-module.

Theorem 1.26. *The algebras H_χ are Frobenius for all characters χ of H_0.*

Proof. By [9, Theorem 1.1] there exists a dual projective pair $\{a_i\}$, $\{b_i\}$ with respect to the non-degenerate bilinear form described above. In other words for all $h \in H$ we have

$$h = \sum_i b_i < a_i, h > = \sum_i < h, b_i > a_i. \tag{1.18}$$

For a given H_0-character χ define the bilinear form

$$(\, , \,) : H_\chi \times H_\chi \longrightarrow k$$

by

$$(h + \mathfrak{m}_\chi H, h' + \mathfrak{m}_\chi H) = < h, h' > + \mathfrak{m}_\chi.$$

It is easy to check that $(\, , \,)$ is well-defined and H_χ-associative. The non-degeneracy of this form is guaranteed by (1.18).

The Hopf structure on H has other effects on the representation theory of the rings H_χ. Indeed if M is a finite dimensional H_χ-module and N is a finite dimensional $H_{\chi'}$-module it is straightforward to check that the tensor product $M \otimes N$ is naturally a finite dimensional $H_{\chi \cdot \chi'}$-module via the comultiplication in H. Moreover, using the antipode of H, there is a natural $H_{\chi^{-1}}$-module structure on M^*, the dual of M. These observations often allow us to rephrase questions about H_χ in terms of another factor, say $H_{\chi'}$, an algebra about which we may know more. We record one very useful lemma in this vein.

Lemma 1.27. *Let L and M be H_χ-modules and let N be an H-module. Then there is an isomorphism*

$$\mathrm{Hom}_H(N, L \otimes M^*) \cong \mathrm{Hom}_{H_\chi}(N \otimes M, L).$$

Proof. This is a straightforwad calculation using the usual adjunction isomorphism between homomorphisms and tensor products.

1.6 Representations of quantum groups.

By Propositions 1.8 and 1.14 we see that the algebras $U_\epsilon^{\geq 0}$ and $\mathcal{O}_\epsilon[G]$ satisfy hypothesis 2 of Section 1.5. Following Remark 1.23 it is important to study the finite dimensional quotients of these algebras obtained by factoring out maximal ideals of the appropriate central sub-Hopf algebra.

Definition 12. Given $b \in B^-$ and $g \in G$ let $\mathfrak{m}_b \triangleleft O[B^-]$ and $\mathfrak{m}_g \triangleleft O[G]$ denote the associated maximal ideals. We define the algebra $U_\epsilon^{\geq 0}(b)$ as

$$U_\epsilon^{\geq 0}(b) \equiv \frac{U_\epsilon^{\geq 0}}{\mathfrak{m}_b U_\epsilon^{\geq 0}},$$

and the algebra $\mathcal{O}_\epsilon[G](g)$ as

$$\mathcal{O}_\epsilon[G](g) \equiv \frac{\mathcal{O}_\epsilon[G]}{\mathfrak{m}_g \mathcal{O}_\epsilon[G]}.$$

By Theorem 1.26 and Propositions 1.8 and 1.14 these are Frobenius algebras and have dimension ℓ^{N+r} and ℓ^{2N+r} respectively.

Recall from (1.2) the Bruhat decompositions of B^- and G,

$$B^- = \coprod_{w \in W} X_{w,e}, \quad G = \coprod_{(w_1,w_2) \in W \times W} X_{w_1,w_2},$$

where $X_{w_1,w_2} = B^+w_1B^+ \cap B^-w_2B^-$. The following theorem relating these decompositions to representation theory is a consequence of the Poisson geometry on B^- and G which essentially arises from the existence of (non-commutative) integral versions of $\mathcal{O}[B^-]$ and $\mathcal{O}[G]$ which lift the rings of regular functions to subalgebras of quantum groups, [26, Sections 11.7 and 11.8].

Theorem 1.28. *i) [24, Theorem 4.4] Suppose $b, b' \in X_{w,e}$ for some $w \in W$. Then there is an isomorphism of algebras*

$$U_\epsilon^{\geq 0}(b) \cong U_\epsilon^{\geq 0}(b').$$

ii) [25, Section 9] Suppose $g, g' \in X_{w_1,w_2}$ for some $w_1, w_2 \in W$. Then there is an isomorphism of algebras

$$\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g').$$

In fact each cell is a T-orbit of symplectic leaves which have dimension $\ell(w) + s(w)$ and $\ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1)$ in $X_{w,e}$ and X_{w_1,w_2} respectively, [44, Appendix].

We concentrate on $U_\epsilon^{\geq 0}$ first. Consider $\bar{U}_\epsilon^{\geq 0} = U_\epsilon^{\geq 0}(1)$. In this case we are factoring out from $U_\epsilon^{\geq 0}$ the relations

$$E_{\beta_j}^\ell = 0 \text{ for } 1 \leq j \leq N \text{ and } K_\alpha^\ell = 1 \text{ for } \alpha \in P.$$

It's easy to see the simple modules of $\bar{U}_\epsilon^{\geq 0}$ are one dimensional and in bijective correspondence with the elements of $Q_\ell \equiv Q/\ell Q$. Indeed, given $\mu \in Q$, let $\mathbb{C}_\mu$ denote the following one dimensional $\bar{U}_\epsilon^{\geq 0}$-module:

$$\begin{aligned} E_{\beta_j}.1 &= 0 && \text{for } 1 \leq j \leq N, \\ K_\lambda.1 &= \epsilon^{(\mu,\lambda)} && \text{for } \lambda \in P. \end{aligned}$$

It is immediate that, as $\bar{U}_\epsilon^{\geq 0}$-modules, we have

$$\mathbb{C}_\mu \otimes \mathbb{C}_{\mu'} \cong \mathbb{C}_{\mu+\mu'}.$$

This just says that the Picard group of $\bar{U}_\epsilon^{\geq 0}$, that is the set of isomorphism classes of one dimensional representations under the tensor product, is isomorphic to the abelian group Q_ℓ. We'll write this as $\mathrm{Pic}(\bar{U}_\epsilon^{\geq 0})$.

Fix $b \in B^-$ and let $S(b)$ be the set of isomorphism classes of simple $U_\epsilon^{\geq 0}(b)$-modules. There is a left (resp. right) action of $\mathrm{Pic}(\bar{U}_\epsilon^{\geq 0})$ on $S(b)$ obtained by sending $\mathbb{C}_\mu$ to the map which takes $S \in \mathcal{S}(b)$ to $\mathbb{C}_\mu \otimes S$ (resp. $S \otimes \mathbb{C}_\mu$). Both these actions are transitive. Indeed let $S, S' \in \mathcal{S}$. As $S' \otimes S^*$ is a finite dimensional $U_\epsilon^{\geq 0}$-module there exists $\mu \in Q_\ell$ such that $\mathbb{C}_\mu$ is contained in $\mathrm{Soc}(S' \otimes S^*)$. Thus, by Lemma 1.27, we have

$$0 \neq \mathrm{Hom}_{U_\epsilon^{\geq 0}}\left(\mathbb{C}_\mu, S' \otimes S^*\right) \cong \mathrm{Hom}_{U_\epsilon^{\geq 0}(b)}\left(\mathbb{C}_\mu \otimes S, S'\right).$$

Therefore, noting that $\mathbb{C}_\mu \otimes S$ is irreducible (apply $\mathbb{C}_{-\mu}$), we obtain $\mathbb{C}_\mu \otimes S \cong S'$ as required. To get the transitivity of the right action we just need to observe that taking duals provides a bijection between $\mathcal{S}(b)$ and $\mathcal{S}(b^{-1})$ which reverses the order of tensor products. We'll describe the stabiliser of these actions in Theorem 1.29.

Fix an element $w \in W$ and assume $\ell(w) = t$. Recall the definition of U_w from Section 1.3. Let A_w be the subalgebra of U_w generated by the elements $E_{\beta_1}, \dots, E_{\beta_t}$ only. As we've already remarked both these algebras are independent of the choice of reduced expression for w, see Lemma 1.11.

Definition 13. Given $b \in B^-$, define $A_w(b)$ as

$$A_w(b) \equiv \frac{A_w}{(\mathfrak{m}_b \cap A_w)A_w}, \tag{1.19}$$

and $U_w(b)$ as

$$U_w(b) \equiv \frac{U_w}{(\mathfrak{m}_b \cap U_w)U_w}. \tag{1.20}$$

These algebras have dimension ℓ^t and ℓ^{t+r} respectively.

We say a positive integer is good if it is prime to the bad primes for the root system of G. In other words if

$$\tilde{\alpha} = \sum_{i=1}^{r} a_i \alpha_i$$

is the longest root in Δ^+ then the integer is good if it is prime to all a_i. Only the integers 2, 3 and 5 can occur as bad primes.

Hypothesis 3. *The integer ℓ is odd and good.*

This subsumes Hypothesis 1 but is more restrictive for types E and F.

Theorem 1.29. *[27, Theorems 4.4 and 4.5] Suppose that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$.*
i) Restriction provides a bijection between the simple $U_\epsilon^{\geq 0}(b)$-modules and the simple $U_w(b)$-modules.

ii) Let L be a simple $U_\epsilon^{\geq 0}(b)$-module. A complete list of simple $U_\epsilon^{\geq 0}(b)$-modules is given by $\mathbb{C}_\mu \otimes L$ for $\mu \in Q_\ell$. Moreover L is isomorphic to $\mathbb{C}_\mu \otimes L$ if and only if $(\lambda, \mu) \in \ell\mathbb{Z}$ for all $\lambda \in P^w$. In particular there are exactly $\ell^{r-s(w)}$ simple $U_\epsilon^{\geq 0}(b)$-modules.
iii) The simple $U_\epsilon^{\geq 0}(b)$-modules have dimension $\ell^{\frac{1}{2}(\ell(w)+s(w))}$.

The remark following Theorem 1.28 shows that Part (iii) can be considered as a "Kac-Weisfeiler" theorem for $U_\epsilon^{\geq 0}$, in the sense of [68]. A simple counting argument, using Theorem 1.29 (ii) and (iii), gives the following corollary.

Corollary 1.30. *Suppose that ℓ satisfies Hypothesis 3 and let $b \in X_{w,e}$. Then the algebra $U_w(b)$ is semisimple: it has precisely $\ell^{r-s(w)}$ simple modules, each of dimension $\ell^{\frac{1}{2}(\ell(w)+s(w))}$.*

Remark 1.31. There are analogues of all the above results for the (analogously defined) factors $U_\epsilon^{\leq 0}(b)$ for $b \in B^+ = \coprod_{w \in W} X_{e,w}$.

Now let's tackle $\mathcal{O}_\epsilon[G]$. Theorem 1.21 offers us an opportunity to relate the factors of $\mathcal{O}_\epsilon[G]$, $U_\epsilon^{\geq 0}$ and $U_\epsilon^{\leq 0}$. Indeed as both $X_{w_1,e}$ and X_{e,w_2} are invariant under multiplication by T we can find unipotent elements $b_- \in X_{w_1,e}$ and $b_+ \in X_{e,w_2}$. So we can choose $g \equiv b_+ b_- \in X_{w_1,w_2} \cap B^+ B^-$.

Proposition 1.32. *Let $g \in X_{w_1,w_2}$ be chosen as above. Then*
(i) The map γ_ϵ from Theorem 1.21 induces an injection

$$\gamma_g : \mathcal{O}_\epsilon[G](g) \longrightarrow U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-).$$

with image a skew group ring extension of $A_{w_0}^-(b_+) \otimes A_{w_0}(b_-)$ by $(\mathbb{Z}/\ell\mathbb{Z})^r \cong U_\epsilon^0$.
(ii) In case $g = 1 \in X_{e,e}$ the map

$$\gamma_1 : \mathcal{O}_\epsilon[G] \longrightarrow U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0} \tag{1.21}$$

is an embedding of augmented algebras.

Proof. By Theorem 1.21(iii) γ_ϵ lifts the map $B^+ \times B^- \longrightarrow G$ so it's clear from the choice of g, b_+ and b_- that we have an injective algebra map

$$\gamma_g : \mathcal{O}_\epsilon[G](g)[(z^\rho)^{-1}] \longrightarrow U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-),$$

which, by Theorem 1.21(ii), has the desired image. Since regular elements in artinian rings are invertible, localising at z^ρ has no effect, proving (i).

To prove (ii) it is sufficient to prove that γ_ϵ is an embedding of augmented algebras. The augmentation ideal of $\mathcal{O}_\epsilon[G][(z^\rho)^{-1}]$, say M, is induced from the augmentation on $\mathcal{O}_\epsilon[G]$ (under which z^ρ is sent to a non-zero element of the field $\mathbb{C}$ as observed in (1.9)). The augmentation ideal of $U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$, say M', is generated by the augmentation ideals of $U_\epsilon^{\leq 0}$ and $U_\epsilon^{\geq 0}$. So it is enough to show $\gamma_\epsilon(M) \subseteq M'$. But this follows immediately from (1.15) since each individual map in the composition preserves the relevant augmentation.

Remark 1.33. Note that the algebras $A_{w_0}^{\pm}(1)$ are local. Thus, using the map γ_1 discussed above, one sees that the simple $\mathcal{O}_\epsilon[G]$-modules are all one dimensional and in one-to-one correspondence with the elements of Q_ℓ. We denote the representation associated with $\mu \in Q_\ell$ by $\mathbb{C}_\mu$. Moreover, by [25, Theorem 10.8], for good ℓ we have $\mathrm{Pic}(\mathcal{O}_\epsilon[G]) \cong Q_\ell$, and we can repeat the arguments preceding Definition 13 to see that $\mathrm{Pic}(\mathcal{O}_\epsilon[G])$ acts transitively on the simple representations of $\mathcal{O}_\epsilon[G](g)$ for any $g \in G$ by tensoring on the left.

Fix reduced expressions for $w_1, w_2 \in W$

$$w_1 = s_{i_1} \dots s_{i_t}, \quad w_2 = s_{i'_1} \dots s_{i'_{t'}}.$$

We define A_{w_1,w_2} to be the algebra generated by the elements $F_{\beta'_1} \otimes 1, \dots F_{\beta'_{t'}} \otimes 1$ and $1 \otimes E_{\beta_1}, \dots 1 \otimes E_{\beta_t}$ and U_{w_1,w_2} to be the algebra generated by A_{w_1,w_2} and the elements $K_\alpha \otimes K_{-\alpha}$ for $\alpha \in P$. We define finite dimensional algebras $A_{w_1,w_2}(b_+, b_-)$ and $U_{w_1,w_2}(b_+, b_-)$ for $b_+ \in U^+$ and $b_- \in U^-$ as in (1.19) and (1.20).

Theorem 1.34. *[27, Proposition 4.10 and Theorem 4.10] Suppose that ℓ satisfies Hypothesis 3. Let $g \in X_{w_1,w_2}$ such that $g = b_+ b_-$ for $b_+ \in U^+$ and $b_- \in U^-$.*
i) There is a bijection, induced by restriction, between the simple $\mathcal{O}_\epsilon[G](g)$-modules and the simple $U_{w_1,w_2}(b_+, b_-)$-modules.
ii) Let L be a simple $\mathcal{O}_\epsilon[G](g)$-module. A complete list of simple $\mathcal{O}_\epsilon[G](g)$-modules is given by $\mathbb{C}_\mu \otimes L$ for $\mu \in Q_\ell$. Moreover L and $\mathbb{C}_\mu \otimes L$ are isomorphic if and only if $(w_1(\lambda), \mu) \in \ell\mathbb{Z}$ for all $\lambda \in P^{w_2^{-1} w_1}$.
iii) The simple $\mathcal{O}_\epsilon[G](g)$-modules have dimension $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+s(w_2^{-1}w_1))}$.

It follows from Theorem 1.28 that this theorem completely describes the dimensions and number of simple $\mathcal{O}_\epsilon[G](g)$-modules for any $g \in G$.

1.7 Additional remarks

1. Most of Section 1.2 was cribbed from [48], where there is a much more detailed discussion of generic quantum groups.

2. With the exception of Lemma 1.12, Section 1.3 is well-known.

3. A more complete account of Drinfel'd duality can be found in [48, Chapters 6 and 8] or in [25, Sections 2 and 3] where further references can be found. The importance of considering the map γ_ϵ of Theorem 1.21 as a lift to quantum groups of a well-known algebraic geometry result (or indeed linear algebra result) is apparent from [25]. This point of view is pushed further in [27].

4. Section 1.5 is presumably folklore.

5. Theorem 1.28 is an example of a general, and as yet poorly understood, link between the representation theory of certain PI algebras and the geometry of their centres, [26, Section 11]. Theorems 1.29 and 1.34 are simply interpretations of the principal results in [27].

Chapter 2

Cohomology

The purpose of this chapter is to collect a number of cohomological results which will be vital in later chapters. A frequent difficulty in the study of quantum groups is their "rigidity". In the present context this problem forces us to work with augmented algebras instead of Hopf algebras. This gives us several technical problems, most notably when dealing with augmented algebra analogues of the classical Lyndon-Hochshild-Serre spectral sequence which is well-known in both group and Lie theory. In what is germane to us we will finesse these issues, but it is perhaps worthwhile noting that it seems it is exactly this difficulty which prevents the beginnings of an adequate theory of support varieties for quantised enveloping algebras at roots of unity.

Throughout this chapter k denotes an algebraically closed field of characteristic zero.

Before we start let's introduce an important notion.

Definition 14. Suppose B is an augmented k-algebra and $A \subset B$ is an augmented subalgebra: then A *is a normal subalgebra of* B if $A_+B = BA_+$, where A_+ denotes the augmentation ideal of A.

In case A is a normal subalgebra of B we can always form the algebra

$$B//A \equiv B \equiv \frac{B}{BA_+}.$$

Throughout this section we will adopt the following hypothesis.

Hypothesis 4. *The algebra A is a normal subalgebra of B. Moreover, as a (left) A-module, B is projective.*

Clearly, if such a B is finitely generated as an A-module then B will have finite k-dimension.

2.1 Warm-up

We kick off with a couple of simple lemmas. The following is of course common knowledge.

Lemma 2.1. *Let S be an augmented algebra. Then, as vector spaces,*

$$\mathrm{Ext}^1_S(k,k) \cong (S_+/S_+^2)^*. \tag{2.1}$$

Remark 2.2. For our purposes Lemma 2.1 is best seen from the description of $\mathrm{Ext}^*_S(k,k)$ as $H^*(S,k)$, Hochschild cohomology. In this case it is well-known that, see [81, Lemma 9.2.1] for instance,

$$H^1(S,k) \cong \mathrm{Der}_k(S_+,k)/\mathrm{PDer}_k(S_+,k)$$

where $\mathrm{Der}_k(S_+,k)$ is the space of k-derivations of S and $\mathrm{PDer}_k(S_+,k)$ the subspace of principal k-derivations. Since k is a trivial S-module $\mathrm{PDer}_k(S_+,k)=0$, so

$$H^1(S,k) \cong \mathrm{Der}_k(S_+,k).$$

In this case, since a functional on S_+ can be uniquely extended to a k-derivation of S, we see

$$\mathrm{Der}_k(S,k) \cong (S_+/S_+^2)^*.$$

From this description it is clear that the isomorphism of Lemma 2.1 is (contravariantly) functorial in S.

Let I be a homogeneous two-sided ideal in the tensor algebra $T(V)$ where V is a finite dimensional vector space. Define $S := T(V)/I$ and $R = I/(T^+(V)I + IT^+(V))$.

Lemma 2.3. *If I has no degree one elements then* $\mathrm{Ext}^2_S(k,k) \cong \mathrm{Hom}_k(R,k)$.

Proof. Let $J := IT^+(V)$ and $R' := I/J$. We have an exact S-sequence

$$0 \longrightarrow T^+(V)/I \longrightarrow S \longrightarrow k \longrightarrow 0$$

whose associated long exact sequence in cohomology gives an isomorphism $\mathrm{Ext}^2_S(k,k) \cong \mathrm{Ext}^1_S(T^+(V)/I,k)$. Consider another exact S-sequence

$$0 \longrightarrow R' \longrightarrow T^+(V)/J \longrightarrow T^+(V)/I \longrightarrow 0$$

which induces the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(T^+(V)/I,k) \longrightarrow \mathrm{Hom}_S(T^+(V)/J,k) \longrightarrow \mathrm{Hom}_S(R',k) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_S\left(T^+(V)/I, k\right) \longrightarrow \mathrm{Ext}^1_S\left(T^+(V)/J, k\right) \longrightarrow \dots$$

The map from $\mathrm{Hom}_S\left(T^+(V)/J, k\right) \longrightarrow \mathrm{Hom}_S\left(R', k\right)$ is induced by inclusion. Since $I \subseteq T^+(V)^2$ and $T^+(V).k = 0$ it follows this is the zero map. Moreover, if V has a basis $\{v_1, \dots v_k\}$ then $T^+(V)/J$ is a free S-module with generators $\{v_1, \dots v_k\}$. Therefore $\mathrm{Ext}^1_S\left(T^+(V)/J, k\right) = 0$.

The lemma now follows from the isomorphism $\mathrm{Hom}_S\left(R', k\right) \cong \mathrm{Hom}_k\left(R, k\right)$.

2.2 Multiplicative structures in cohomology.

Products in cohomology can be found in numerous guises; however there often appears to be no relation between these different versions. Here we present three types of product which, under suitable conditions, coincide. Each product has its own virtue and often a result which is opaque for one becomes transparent with another. An example of this is Proposition 2.7 which seems surprising when considered from the point of view of Hopf algebras. Throughout this section, T is an augmented k-algebra.

1. *The Yoneda product.* Let M be a left T-module and suppose $x \in \mathrm{Ext}^p_T\left(k, k\right)$ and $y \in \mathrm{Ext}^q_T\left(k, M\right)$. We represent x and y by the long exact sequences

$$\zeta : 0 \longrightarrow k \to A_{p-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow k \longrightarrow 0$$

and

$$\xi : 0 \longrightarrow M \longrightarrow B_{q-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow k \longrightarrow 0.$$

Then the Yoneda product of x and y, written (for the moment) $y * x$, is obtained by splicing together ζ and ξ as follows:

$$\begin{tikzcd}
\xi * \zeta : 0 \arrow[r] & M \arrow[r] & B_{q-1} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[dr] & & A_{p-1} \arrow[r] & \cdots \arrow[r] & A_0 \arrow[r] & k \arrow[r] & 0 \\
& & & & & k \arrow[ur] \arrow[dr] & & & & & \\
& & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

Taking $M = k$ in the construction gives $\mathrm{Ext}^*_T\left(k, k\right)$ the structure of a graded k-algebra.

2. *Hochschild cohomology.* For this we need T-bimodules, say M and N, equipped with a homomorphism $\psi : M \otimes_T N \longrightarrow M$.

Definition 15. The *(two-sided) bar resolution* of T, written $B(T)$, has p^{th} component $B^p(T) = T \otimes T^{\otimes p} \otimes T$ with the obvious T-bimodule structure. The differential is given by

$$\delta^p(t_0 \otimes t_1 \otimes \ldots \otimes t_{p+1}) = \sum_{i=0}^{p} (-1)^i t_0 \otimes \ldots \otimes t_i t_{i+1} \otimes \ldots \otimes t_{p+1}$$

Remark 2.4. When dealing with left modules instead of bimodules there is an analogous one-sided bar resolution, [7, Chapter I, Section 5].

Hochschild cohomology, written $H^*(T, M)$, is defined to be the homology of the complex $\mathrm{Hom}_{T-\mathrm{bimod}}(B(T), M)$. We identify $\mathrm{Hom}_{T-\mathrm{bimod}}(B^p(T), M)$ with $\mathrm{Hom}_k(T^{\otimes p}, M)$ by sending $F \in \mathrm{Hom}_{T-\mathrm{bimod}}(B^p(T), M)$ to the map $f \in \mathrm{Hom}_k(T^{\otimes p}, M)$ defined by

$$f(t_1 \otimes \ldots \otimes t_p) = F(1 \otimes t_1 \otimes \ldots \otimes t_p \otimes 1).$$

The inverse sends f to the map F defined by

$$F(t_0 \otimes t_1 \otimes \ldots \otimes t_p \otimes t_{p+1}) = t_0 f(t_1 \otimes \ldots \otimes t_p) t_{p+1}.$$

In this way we see that Hochschild cohomology is just the homology of the complex whose p^{th} component is $\mathrm{Hom}_k(T^{\otimes p}, M)$ and which has differentials

$$\begin{aligned}(d^p(f))(t_0 \otimes \ldots \otimes t_p) &= t_0 f(t_1 \otimes \ldots \otimes t_p) \\ &\quad + \sum_{i=0}^{p-1} (-1)^{i+1} f(t_0 \otimes \ldots \otimes t_i t_{i+1} \otimes \ldots \otimes t_p) \\ &\quad + (-1)^{p+1} f(t_0 \otimes \ldots \otimes t_{p-1}) t_p. \end{aligned} \tag{2.2}$$

for $f \in \mathrm{Hom}_k(T^{\otimes p}, M)$.

For $f \in \mathrm{Hom}_k(T^{\otimes p}, M)$ and $g \in \mathrm{Hom}_k(T^{\otimes q}, N)$ we define a cup product as follows

$$f \cup g(t_1 \otimes \ldots \otimes t_{p+q}) = (-1)^q \psi(f(t_1 \otimes \ldots \otimes t_p) \otimes g(t_{p+1} \otimes \ldots \otimes t_{p+q})).$$

It's easy to check the identity

$$d(f \cup g) = d(f) \cup g + (-1)^p f \cup d(g)$$

for $f \in \mathrm{Hom}_k(T^{\otimes p}, M)$. Thus the cup product, which depends on ψ, passes to cohomology,

$$H^p(T, M) \otimes H^q(T, N) \xrightarrow{\ \cup\ } H^{p+q}(T, M).$$

In particular we can take $M = N = k$ to find a graded k-algebra structure on $H^*(T, k)$, whilst if M is a T-bimodule with trivial right action we obtain a right $H^*(T, k)$-module structure on $H^*(T, M)$ via the multiplication map $M \otimes_T k \longrightarrow M$.

3. *Products for Hopf algebras.* It T is a Hopf algebra we can use comultiplication to define products. Indeed, let $x \in \mathrm{Ext}_T^p(k, k)$ and $y \in \mathrm{Ext}_T^q(k, M)$ be represented by the long exact sequences

$$\zeta : 0 \longrightarrow k \to A_{p-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow k \longrightarrow 0$$

and

$$\xi : 0 \longrightarrow M \longrightarrow B_{q-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow k \longrightarrow 0,$$

respectively. If we tensor these sequences together, using comultiplication to define a T-structure on the tensor product of T-modules, we obtain a long exact sequence representing an element of $\mathrm{Ext}_T^{p+q}(k, M)$. This element is the product of x and y. In this manner $\mathrm{Ext}_T^*(k,k)$ becomes an algebra and $\mathrm{Ext}_T^*(k,M)$ a right module for this algebra.

Theorem 2.5. *Let T be an augmented algebra and M a left T-module. Then there is a natural graded isomorphism $H^*(T,M) \cong \mathrm{Ext}_T^*(k,M)$ where, in the first argument, we consider M to have trivial right action. Under this identification the above product structures 1 and 2 agree. Moreover if T is a Hopf algebra then the product 3, defined via the comultiplication, agrees with both of the above.*

Proof. Given left T-modules M and N, the space $\mathrm{Hom}_k(N,M)$ becomes a T-bimodule through

$$(t.f.t')(n) = t(f(t'n)),$$

for $t, t' \in T$ and $f \in \mathrm{Hom}_k(N,M)$. By [81, Lemma 9.1.9] there is a natural isomorphism

$$H^p(T, \mathrm{Hom}_k(N,M)) \cong \mathrm{Ext}_T^p(N,M).$$

Taking $N = k$ gives the first statement. The remaining claims can be found in [38, 13.7] and [16, Theorem 6.6] or [10, Proposition 3.2.1].

Remark 2.6. In view of Theorem 2.5 we will never differentiate between different product constructions and, from now on, will write products by concatenation.

Definition 16. Suppose $S = \oplus S^p$ is a $\mathbb{Z}$-graded algebra. We say S is *graded commutative* if for $x \in S_p$ and $y \in S_q$ we have $xy = (-1)^{pq} yx$.

The following proposition is clear for cocommutative Hopf algebras, in view of Definition 3 of multiplication in $\mathrm{Ext}_T^*(k,k)$.

Proposition 2.7. *For any Hopf algebra T, $\mathrm{Ext}_T^*(k,k)$ is a graded commutative k-algebra.*

Proof. This is proved in [39, Corollary 5.6].

2.3 Lyndon-Hochschild-Serre spectral sequences.

An important tool in studying the cohomology of groups and of Lie algebras is the Lyndon-Hochschild-Serre spectral sequence. For example, in the case of groups this spectral sequence relates the cohomology of a group to the cohomology of a normal subgroup and the corresponding factor group. Therefore it is ideally suited for induction. In the case of groups and Lie algebras it is a multiplicative spectral sequence, a consequence of the Hopf algebra structure on group algebras and enveloping algebras. However, this need no longer be the case if we only consider augmented algebras. Below we present the basic properties which we will use later.

Recall that Hypothesis 4 is in force throughout this section. That is A is a normal subalgebra of an augmented algebra B, and, as an A-module, B is projective.

Lemma 2.8. *For any B-module M and for all $i \geq 0$ there is a natural B-module structure on $\operatorname{Ext}^i_A(k, M)$.*

Proof. We have an isomorphism

$$\operatorname{Hom}_A(k, -) \cong \operatorname{Hom}_B(B \otimes_A k, -). \tag{2.3}$$

This yields, since B is a projective A-module, $\operatorname{Ext}^i_A(k, M) \cong \operatorname{Ext}^i_B(B \otimes_A k, M)$ for all left B-modules M. But $B \otimes_A k = B$ and the left B-action on $\operatorname{Ext}^i_B(B, M)$ is induced by right multiplication of B on itself.

Remark 2.9. We'll also use a result related to Lemma 2.8. Suppose G is a finite abelian group (whose order is invertible in k of course) and let R be a k-algebra on which G acts by k-algebra automorphisms. Let $R * G$ be the associated skew group algebra. Let M be an $R * G$-module, finite dimensional over k. Then for any $\chi \in X(G)$, the character group of G, we can define ${}^{\chi}M$ to be the $R * G$-module which is isomorphic to M as a k-vector space and where $(ag) \cdot m = \chi^{-1}(g)(agm)$. Note too that M is completely reducible as a G-module, that is

$$M = \bigoplus_{\chi \in X(G)} M_\chi,$$

where $M_\chi = \{m \in M : gm = \chi(g)m \text{ for all } g \in G\}$.

Given $R * G$-modules M and N there is a natural G-action on $\operatorname{Hom}_R(M, N)$ given by $(g \cdot f)(m) = gf(g^{-1}m)$. Thus there is a decomposition

$$\operatorname{Hom}_R(M, N) \cong \bigoplus_{\chi \in X(G)} \operatorname{Hom}_R(M, N)_\chi.$$

It is easy to see there is a natural isomorphism

$$\operatorname{Hom}_{R*G}(M, {}^{\chi}N) \cong \operatorname{Hom}_R(M,N)_{\chi} \tag{2.4}$$

induced by restriction.

As in Lemma 2.8, for any $R*G$-modules M and N there is an isomorphism

$$\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{R*G}(R*G\otimes_R M, N), \tag{2.5}$$

which is natural in both entries. There is a G-action on $R*G\otimes_R M$ given by

$$g.(x\otimes_R m) = xg\otimes_R g^{-1}m.$$

With this action the isomorphism (2.5) becomes G-equivariant. Since $R*G$ is a projective R-module it follows that we can calculate the cohomology $\operatorname{Ext}^i_R(M,N)$ together with its G-action by considering an $R*G$-projective resolution of M, say P, and then calculating the homology of the induced complex $\operatorname{Hom}_{R*G}(R*G\otimes_R P, N)$. Combining (2.4) with (2.5) we see there is a natural isomorphism

$$\operatorname{Hom}_{R*G}(R*G\otimes_R M, N)_{\chi} \cong \operatorname{Hom}_{R*G}(M, {}^{\chi}N),$$

given by sending $\theta \in \operatorname{Hom}_{R*G}(R*G\otimes_R M, N)_{\chi}$ to the map θ which sends $m\in M$ to $\theta(1\otimes_R m)$. This proves the following lemma.

Lemma 2.10. *Keep the above notation. In particular G is a finite abelian group and R is a k-algebra on which G acts by k-algebra automorphisms. Then, for any finite dimensional $R*G$-modules M and N, there is a G-action on $\operatorname{Ext}^i_R(M,N)$ and there is a natural isomorphism*

$$\operatorname{Ext}^i_R(M,N)_{\chi} \cong \operatorname{Ext}^i_{R*G}(M, {}^{\chi}N).$$

In particular there is an isomorphism

$$\operatorname{Ext}^i_R(M,N) \cong \bigoplus_{\chi\in X(G)} \operatorname{Ext}^i_{R*G}(M, {}^{\chi}N).$$

Note also that in case $R*G$ is an augmented algebra such that R is a normal subalgebra the G-action on $\operatorname{Ext}^i_R(M,N)$ defined above agrees with the G-action from Lemma 2.8.

Lemma 2.11 ([17, XVI, 6.17]). *For any B-module M there is a spectral sequence,*

$$E_2^{p,q}(M) = \operatorname{Ext}^p_{B}\left(k, \operatorname{Ext}^q_A(k,M)\right) \Longrightarrow \operatorname{Ext}^{p+q}_B(k,M). \tag{2.6}$$

Proof. We prove this using Grothendieck spectral sequences. For the equivalence of this approach to [17, XVI, 6.17] see [7, Chapters VI, VII and VIII].

Define two functors F and G as follows:

$$\mathrm{F} = \mathrm{Hom}_B(k, -) :_B \mathfrak{M} \longrightarrow \mathfrak{Ab},$$

$$\mathrm{G} = \mathrm{Hom}_A(k, -) :_B \mathfrak{M} \longrightarrow_B \mathfrak{M}.$$

Clearly F is left exact. Let I be an injective B-module. Then $\mathrm{Hom}_B(B, I)$ is an injective B-module as, for all B-modules N,

$$\mathrm{Hom}_B\left(N, \mathrm{Hom}_B(B, I)\right) \cong \mathrm{Hom}_B(N, I)$$

by the adjoint isomorphism [77, 2.11]. By (2.3) GI is injective, so, in particular, it is right F-acyclic, see [77, p.302].

Therefore the relevant Grothendieck spectral sequence yields

$$E_2^{p,q}(M) = R^p\mathrm{F}(R^q\mathrm{G}(M)) \Longrightarrow R^n(\mathrm{FG}(M))$$

where R^n denotes the nth derived functor as in, for example, [77, 11.38]. Finally, since, for any ring R and any two-sided ideal $I \triangleleft R$, we have a natural transformation from the category of R-modules

$$\mathrm{Hom}_R(R/I, -) \cong \mathrm{Ann}_-(I)$$

we deduce that

$$\mathrm{Hom}_B\left(k, \mathrm{Hom}_A(k, -)\right) \cong \mathrm{Ann}_-(B_+) \cong \mathrm{Hom}_B(k, -)$$

implying the result.

Definition 17. Let M be a B-module and let $(P)_* \to k$ be a B-projective resolution of k. Since B is a projective A-module the forgetful functor from B-modules to A-modules makes this resolution an A-projective resolution of k. Thus the forgetful functor yields a map on cohomology, called *restriction*,

$$\mathrm{res} : \mathrm{Ext}_B^p(k, M) \longrightarrow \mathrm{Ext}_A^p(k, M).$$

Remark 2.12. It's clear that the definition of restriction does not require that A be a normal subalgebra of B, only that B is projective as an A-module.

It's possible to define restriction in the settings of Hochschild cohomology and long exact sequences representing Ext groups. In the first case let $\psi : A \longrightarrow B$ be inclusion. Then restriction is just induced by the pullback on cohomology

$$\psi^* : H^p(B, M) \longrightarrow H^p(A, M).$$

In the second case the forgetful functor gives restriction. In particular, it is clear from both cases that restriction preserves the product structures introduced in the previous section.

Lemma 2.13. *Let $e : \mathrm{Ext}^q_B (k, M) \longrightarrow \mathrm{Hom}_B \left(k, \mathrm{Ext}^q_A (k, M)\right)$ be the vertical edge map of 2.11 and let $\iota : \mathrm{Hom}_B \left(k, \mathrm{Ext}^q_A (k, M)\right) \longrightarrow \mathrm{Ext}^q_A (k, M)$ be the natural inclusion. Then the composition $\iota \circ e$ is given by restriction.*

Proof. The classical construction of the spectral sequence, due to Hochschild and Serre, is presented in [7, Chapter IV]. From this it is clear that the construction is functorial. That is if $A \subseteq B$ and $A' \subseteq B'$ are normal extensions and $\theta : B' \longrightarrow B$ is a map of augmented algebras such that $\theta(A') \subseteq A$ then there is a map of spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = \mathrm{Ext}^p_B \left(k, \mathrm{Ext}^q_A (k, M)\right) & \Rightarrow & \mathrm{Ext}^{p+q}_B (k, M) \\ \Big\downarrow \theta^* & & \Big\downarrow \theta^* \\ {}'E_2^{p,q} = \mathrm{Ext}^p_{B'} \left(k, \mathrm{Ext}^q_{A'} (k, M)\right) & \Rightarrow & \mathrm{Ext}^{p+q}_{B'} (k, M), \end{array}$$

where, on the left hand side, θ^* indicates the pullbacks of the induced maps $\theta : A' \longrightarrow A$ and $\theta : B' \longrightarrow B$. In particular, consider the normal extensions $A \subseteq B$ and $A \subseteq A$. Then this yields a map of spectral sequences as above. Since the lower spectral sequence is trivial we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^q_B (k, M) & \xrightarrow{e} & \mathrm{Hom}_B \left(k, \mathrm{Ext}^q_A (k, M)\right) \\ {\scriptstyle \mathrm{res}}\Big\downarrow & & \Big\downarrow{\scriptstyle \iota} \\ \mathrm{Ext}^q_A (k, M) & \xrightarrow{\mathrm{id}} & \mathrm{Ext}^q_A (k, M), \end{array}$$

proving the lemma.

Definition 18. Suppose N is a B-module, M is a B-module and let M^F denote the B-module $\mathrm{Hom}_A (k, M)$, which is naturally isomorphic to $\mathrm{Hom}_B (B, M)$. Let $(P)_* \to N$ be a B-projective resolution of N and $(Q)_* \to N$ a B-projective resolution of N. The comparison lemma gives a B-lifting of id_N to $(Q)_* \to (P)_*$, inducing a map

$$\mathrm{Ext}^p_B \left(N, M^F\right) \longrightarrow \mathrm{Ext}^p_B \left(N, M^F\right).$$

Composing this with the B-inclusion $M^F \hookrightarrow M$ yields *inflation*

$$\inf^p : \mathrm{Ext}^p_{\underline{B}}\left(N, M^F\right) \longrightarrow \mathrm{Ext}^p_B(N, M).$$

It is the lift to cohomology of the isomorphism $\mathrm{Hom}_{\underline{B}}\left(N, M^F\right) \cong \mathrm{Hom}_B(N, M)$, see [17, Chapter VI.4].

Lemma 2.14. *Keep the preceding notation. Then*

1. *If $N = k$ the horizontal edge map of the spectral sequence of Lemma 2.11*

$$E_2^{p,0} = \mathrm{Ext}^p_{\underline{B}}\left(k, M^F\right) \longrightarrow \mathrm{Ext}^p_B(k, M)$$

is inflation.

2. *If $M = N = k$ inflation is a graded algebra map.*

3. *Inflation preserves products. In other words if $M = M^F$ is a B-module then for $x \in \mathrm{Ext}^p_{\underline{B}}(k, k)$ and $m \in \mathrm{Ext}^q_{\underline{B}}(k, M)$ we have*

$$\inf^{p+q}(x.m) = \inf^p(x).\inf^q(m).$$

Proof. 1. This follows from [17, Chapter XVI.5, $(3)_4$].

2 and 3. By Theorem 2.5 we need only consider the module structure on Hochschild cohomology. Choose the relevant bar resolutions as projective resolutions. In this case $\bar{\pi}$, the morphism of resolutions in the definition of inflation, is induced from the canonical map $\pi : B \twoheadrightarrow \underline{B}$. One easily checks that the following diagram of complexes is commutative

$$\begin{array}{ccccc}
\mathrm{Hom}_k\left(B(\underline{B}), M\right) & \otimes & \mathrm{Hom}_k\left(B(\underline{B}), k\right) & \xrightarrow{\cup} & \mathrm{Hom}_k\left(B(\underline{B}), M\right) \\
\downarrow{\scriptstyle\bar{\pi}} & & \downarrow{\scriptstyle\bar{\pi}} & & \downarrow{\scriptstyle\bar{\pi}} \\
\mathrm{Hom}_k\left(B(B), M\right) & \otimes & \mathrm{Hom}_k\left(B(B), k\right) & \xrightarrow{\cup} & \mathrm{Hom}_k\left(B(B), M\right).
\end{array}$$

Passing to cohomology yields the third claim. The second claim is the special case $M = k$.

The following lemma will prove useful in the future, cf. [31, Proposition 7.3.2].

Lemma 2.15. *Suppose, in addition to Hypothesis 4, that there exists an algebra $C \subset B$ such that B is a projective C-module and the composition*

$$C \hookrightarrow B \twoheadrightarrow \underline{B}.$$

provides an isomorphism of augmented algebras between C and B. Then restriction

$$\mathrm{res} : \mathrm{Ext}^*_B(k,k) \longrightarrow \mathrm{Ext}^*_C(k,k)$$

is an algebra epimorphism, and inflation

$$\mathrm{inf} : \mathrm{Ext}^*_B(k,k) \longrightarrow \mathrm{Ext}^*_B(k,k)$$

is an algebra monomorphism.

Proof. We have already seen that both restriction and inflation are algebra maps. Let $\pi : B \longrightarrow B$ be the canonical map. Then π induces a chain map of bar resolutions

$$\bar{\pi} : B(B) \longrightarrow B(B)$$

which is clearly a B-map. So we have a commutative diagram of chain complexes,

$$\begin{array}{ccc}
\mathrm{Hom}_B(B(B),k) & \xrightarrow{\bar{\pi}} & \mathrm{Hom}_B(B(B),k) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_C(B(B),k) & \xrightarrow{\bar{\pi}} & \mathrm{Hom}_C(B(B),k).
\end{array}$$

Passing to cohomology yields a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^*_B(k,k) & \xrightarrow{\mathrm{inf}} & \mathrm{Ext}^*_B(k,k) \\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\
\mathrm{Ext}^*_C(k,k) & \longrightarrow & \mathrm{Ext}^*_C(k,k).
\end{array}$$

The left hand map is an isomorphism since it is just a transfer of structure from B to C, whilst the bottom map is an isomorphism, being a lift to cohomology of the identity.

For the remainder of this section we will also need the following hypothesis in addition to Hypothesis 4.

Hypothesis 5. *The algebra A is Noetherian and has finite global dimension n and is central in B.*

We first prove a well-known result.

Proposition 2.16. *Assume Hypothesis 5. There is an isomorphism of graded algebras*

$$\mathrm{Ext}^*_A(k,k) \cong \wedge^*(k^n).$$

Proof. First we localise at A_+, the augmentation ideal of A. Note that, since A_+ annihilates $\operatorname{Ext}^*_A(k,k)$ and is a maximal ideal of A, A_+ lies in the support of $\operatorname{Ext}^*_A(k,k)$. The natural map

$$\operatorname{Ext}^*_A(k,k) \longrightarrow (\operatorname{Ext}^*_A(k,k))_{A_+} \tag{2.7}$$

is thus an algebra isomorphism.

Since $(\operatorname{Ext}^*_A(k,k))_{A_+} \cong \operatorname{Ext}^*_{A_{A_+}}(k,k)$ we have reduced to the local case. We can now take our projective resolution of k to be the Koszul resolution from which the result follows easily, for example [81, Section 4.5].

Theorem 2.17. *Assume Hypotheses 4 and 5. The action of B on $\operatorname{Ext}^*_A(k,k)$ is trivial.*

Proof. Recall that B acts on $\operatorname{Ext}^*_A(k,k)$ by Lemma 2.8. We note the isomorphism (2.7) introduced above is B-equivariant since $B_{A_+} \cong B$. This means we can assume A is local and that $\operatorname{Ext}^*_A(k,k)$ can be calculated via the Koszul resolution:

$$\cdots \to A \otimes \wedge^1(k^n) \to A \otimes \wedge^0(k^n) \to k \to 0.$$

Since B is a projective A-module applying $B \otimes_A -$ yields a B-free resolution of $B \otimes_A k \cong B$. Since A is central in B we see by Lemma 2.8 and its proof (cf. proof of Lemma 3.4), the B-action is induced from the natural B-action on the complex

$$\cdots \longleftarrow \operatorname{Hom}_B\left(B \otimes \wedge^1(k^n), k\right) \longleftarrow \operatorname{Hom}_B\left(B \otimes \wedge^0(k^n), k\right) \longleftarrow 0.$$

That is, on $\operatorname{Hom}_B\left(B \otimes \wedge^i(k^n), k\right)$, B acts by right multiplication on B and trivially on $\wedge^i(k^n)$ and k. It is therefore trivial: for $b \in B$ and $f \in \operatorname{Hom}_B\left(B \otimes \wedge^i(k^n), k\right)$ we have

$$(b.f)(b' \otimes x) = f(b'b \otimes x) = b'b.(f(1 \otimes x)) = \eta(b'b)f(1 \otimes x) = \eta(b)f(b' \otimes x).$$

Taking $M = k$ in (2.6) we consider the differential $d_2 : E_2^{0,1} \longrightarrow E_2^{2,0}$ to obtain the following corollary to Theorem 2.17.

Corollary 2.18. *The above map has image central in $\operatorname{Ext}^*_B(k,k)$.*

Proof. The second page of the spectral sequence (2.6) has an induced multiplicative structure via the Yoneda product on $\operatorname{Ext}^*_A(k,k)$. Moreover, since $\operatorname{Ext}^*_A(k,k)$ is a trivial B-module we have an isomorphism of graded algebras

$$\operatorname{Ext}^*_B(k, \operatorname{Ext}^*_A(k,k)) \cong \operatorname{Ext}^*_B(k,k) \otimes \operatorname{Ext}^*_A(k,k)$$

induced from the isomorphism

$$\mathrm{Hom}_B(k, \mathrm{Ext}_A^*(k,k)) \cong \mathrm{Hom}_B(k,k) \otimes \mathrm{Ext}_A^*(k,k).$$

Therefore for any $x \in \mathrm{Ext}_B^p(k,k)$ and $y \in \mathrm{Ext}_A^1(k,k)$, we have

$$(x \otimes 1)(1 \otimes y) = (-1)^p (1 \otimes y)(x \otimes 1).$$

Since d_2 is a graded derivation, meaning that $d_2(fg) = d_2(f)g + (-1)^p f d_2(g)$ for $f, g \in \mathrm{Ext}_B^*(k,k) \otimes \mathrm{Ext}_A^*(k,k)$and f having cohomological degree p, and $d_2(E_2^{p,0}) = 0$, we have

$$(-1)^p x.d_2 y = (-1)^p d_2 y.x$$

as required.

2.4 A spectral sequence for filtered rings

It is a general fact that spectral sequences arise naturally from filtrations so it is only natural to expect the cohomology of a filtered ring to be related to the cohomology of its associated graded ring by a spectral sequence. As we show here such a spectral sequence behaves well with respect to products and restriction.

Suppose T is an augmented algebra with a finite multiplicative filtration $k = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = T$. Let $\mathrm{Gr}T$ denote the associated graded algebra. Given any T-module M we can filter M by $M_i = M$ for all $i \geq 0$. Then $\mathrm{Gr}M$ becomes a $\mathrm{Gr}T$-module such that any homogeneous element of $\mathrm{Gr}T$ with positive degree annihilates $\mathrm{Gr}M$. In particular, taking $M = k$, it follows that $\mathrm{Gr}T$ is an augmented algebra.

Lemma 2.19. *Let M be a T-module. Then there is a spectral sequence*

$$E_1^{p,q}(M) = \mathrm{Ext}_{GrT}^{p+q}(k,k)_{(p)} \otimes M \Longrightarrow \mathrm{Ext}_T^{p+q}(k,M).$$

Furthermore, this spectral sequence is compatible with the right $\mathrm{Ext}_T^*(k,k)$-module structure on $\mathrm{Ext}_T^*(k,M)$ given by the Yoneda product.

Proof. Let $H^*(T,-)$ denote Hochschild cohomology. We will first construct a spectral sequence

$$H^{p+q}((\mathrm{Gr}T),k)_{(p)} \otimes N \Longrightarrow H^{p+q}(T,N)$$

for any T-bimodule N and then specialize to the situation of the theorem.

The filtration on T induces a filtration on $T^{\otimes n}$ defined as follows,

$$(T^{\otimes n})_i := \sum_{i_1+\ldots i_n \leq i} T_{i_1} \otimes \cdots \otimes T_{i_n}.$$

Consider the complex $C(N)$ used to calculate Hochschild cohomology:

$$C^n(N) = \mathrm{Hom}_k \left(T^{\otimes n}, N\right),$$

with differentials, d, as in (2.2). We can filter this complex to get $C_0(N) \subseteq C_1(N) \subseteq \cdots$ where

$$C_i^n(N) = \mathrm{Hom}_k \left((T^{\otimes n})_i, N\right)$$

and

$$d_i^n = d|_{C_i^n(N)}.$$

We then have a homomorphism

$$\frac{C_i^n(N)}{C_{i-1}^n(N)} \cong \mathrm{Hom}_k \left(\mathrm{Gr}_i(T^{\otimes n}), N\right),$$

where $\mathrm{Gr}_i(T^{\otimes n})$ denotes the homogeneous component of $\mathrm{Gr}(T^{\otimes n})$ of degree i. This is an isomorphism of chain complexes where, on the right hand side, N is considered a trivial $\mathrm{Gr}T$-bimodule. Further, as chain complexes

$$\mathrm{Hom}_k \left(\mathrm{Gr}_i(T^{\otimes n}), N\right) \cong \mathrm{Hom}_k \left(\mathrm{Gr}_i(T^{\otimes n}), k\right) \otimes N.$$

Thus we have a spectral sequence

$$H^{p+q}(\mathrm{Gr}T, k)_{(p)} \otimes N \Longrightarrow H^{p+q}(T, N),$$

where the subscript (p) simply means that we must consider graded cohomology, as defined above.

The chain complex $C(k)$ is a differential graded algebra and the filtration on it is clearly multiplicative with respect to the cup product. Therefore by [81, 5.4.8] we have a multiplicative spectral sequence

$$H^{p+q}(\mathrm{Gr}T, k)_{(p)} \Longrightarrow H^{p+q}(T, k).$$

Moreover, since the filtration on T is finite, the spectral sequence converges, [81, 5.5.1]. The lemma now follows from Theorem 2.5.

Suppose S is a filtered algebra with $S_0 = k$ and $\psi : S \longrightarrow T$ is an augmented algebra map. If ψ is filtered, that is $\psi(S_i) \subseteq T_i$ then there is an induced algebra map $\mathrm{Gr}\psi$: $\mathrm{Gr}S \longrightarrow \mathrm{Gr}T$, which preserves the augmentation. This is the case if S is a subalgebra of T. Then S inherits a filtration from T, given by $S_i = S \cap T_i$, from which it follows that, with this filtration, $\mathrm{Gr}S$ is a subalgebra of $\mathrm{Gr}T$.

Lemma 2.20. *Suppose $S \xrightarrow{\psi} T$ is a filtered map of augmented algebras. Then there is an induced map of spectral sequences*

$$\begin{tikzcd} E_1^{p,q} = \mathrm{Ext}^{p+q}_{GrT}(k,k)_{(p)} \arrow[r] \arrow[d, "(Gr\psi)^*"] & \mathrm{Ext}^{p+q}_{T}(k,k) \arrow[d, "\psi^*"] \\ {}'E_1^{p,q} = \mathrm{Ext}^{p+q}_{GrS}(k,k)_{(p)} \arrow[r] & \mathrm{Ext}^{p+q}_{S}(k,k). \end{tikzcd}$$

Moreover, if ψ is an inclusion and, under $Gr\psi$, GrT is projective as an GrS-module then the vertical maps coincide with restriction.

Proof. These spectral sequences arise from filtrations on bar resolutions induced by the respective ring filtrations. Thus it is clear that ψ induces a filtered map on these resolutions, whence the map of spectral sequences.

If $\mathrm{Gr}T$ is a projective $\mathrm{Gr}S$-module then, by [62, Proposition 7.6.16], T is a projective S-module under ψ. In this situation the vertical maps agree with restriction by the remarks preceeding Lemma 2.13.

2.5 Complexity and representation type

Given a module M for some finite dimensional algebra T, the complexity of M is a measure of the rate of growth of the minimal projective resolution of M. If T happens to be a self-injective algebra and M is indecomposable then there is a relation between the syzygies of this resolution and the Auslander-Reiten translates of M, [5, Proposition 3.7]. This is the first step towards finding a direct link between the representation theory of T and its cohomology. Indeed we'll see that complexity bears upon the representation type of T, and is in fact a more subtle invariant of T than its representation type.

Definition 19. Let T be an arbitrary finite dimensional algebra and M a finitely generated T-module. Let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be the minimal T-projective resolution of M, [31, Section 2.4]. We define the *complexity of* M to be the smallest non-negative integer c such that there is a non-zero constant $b \in \mathbb{Q}$ with $\dim P_n \leq b.n^{c-1}$ for each $n \in \mathbb{N}$ (if c does not exist we set the complexity of M to infinity). We will write the complexity as $\mathrm{cx}_T(M)$. We define the *complexity of the algebra* T as

$$\mathrm{cx}(T) \equiv \max\{\mathrm{cx}_T(M) : M \text{ a finitely generated } T\text{-module}\}.$$

Remark 2.21. In fact to determine $\mathrm{cx}(T)$ it suffices to determine $\mathrm{cx}_T(S)$ for all simple T-modules S. Indeed consider the short exact sequence of T-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0, \tag{2.8}$$

and suppose that $(P')_*$ and $(P'')_*$ are the minimal T-projective resolutions of M' and M'' respectively. By the "Horseshoe Lemma", [10, Lemma 2.5.1], there exists a T-projective resolution, $(Q)_*$, of M lifting the short exact sequence (2.8), that is a compatible exact sequence of resolutions

$$0 \longrightarrow (P')_* \longrightarrow (Q)_* \longrightarrow (P'')_* \longrightarrow 0.$$

Let $(P)_*$ be the minimal T-projective resolution of M, so we have

$$\dim P_i \leq \dim Q_i = \dim P'_i + \dim P''_i.$$

It follows from this that $\mathrm{cx}_T(M) \leq \max\{\mathrm{cx}_T(M'), \mathrm{cx}_T(M'')\}$. In other words

$$\mathrm{cx}(T) = \max\{\mathrm{cx}_T(S) : S \text{ simple } T\text{-module}\}. \tag{2.9}$$

Let us recall the definition of representation type for a finite dimensional k-algebra.

Definition 20. [22, Section 6] Let T be a finite dimensional algebra. Then

i) T has *finite representation type* if there are a finite number of mutually non-isomorphic indecomposable T-modules;

ii) T has *tame representation type* if T does not have finite representation type and if, for each dimension $d > 0$, there are a finite number of $T - k[x]$-bimodules M_i which are free as right $k[x]$-modules such that every indecomposable T-module of dimension d is isomorphic to $M_i \otimes_{k[x]} N$ for some i and some simple $k[x]$-module T;

iii) T has *wild representation type* if there is a finitely generated $T - k\langle x, y\rangle$-bimodule M which is free as a right $k\langle x, y\rangle$-module and such that the functor $F(N) = M \otimes_{k\langle x,y\rangle} N$ from the category of finite dimensional $k\langle x, y\rangle$-modules to the category of finite dimensional T-modules preserves indecomposability and isomorphism classes.

Sometimes we will say T is finite, tame or wild to refer to one of the above classes.

Remarks 2.22. i) Any finite dimensional algebra falls into precisely one of the above classes, [22, Corollary C].

ii) The definitions above say roughly that a finite dimensional algebra has tame representation type if its indecomposable modules of a fixed dimension lie in a finite number of one parameter families, for example the group algebra of the Klein-four group over a field of characteristic two. Similarly a finite dimensional algebra is wild if there is no hope of classifying its indecomposable modules (since the representation theory of finite dimensional $k\langle x, y\rangle$-modules is undecidable)!

The role of complexity in representation theory is amply illustrated by the following theorem.

Theorem 2.23. *Let T be a self-injective algebra. Then*
a) T is semisimple if and only if $\mathrm{cx}(T) = 0$,
b) T has wild representation type if $\mathrm{cx}(T) \geq 3$.

Part a) is clear, since a finite dimensional self-injective algebra of finite global dimension is semisimple, [77, Exercise 9.1]. Part b) comprises [74].

In view of Theorem 2.23 we would like to describe the self-injective algebras having complexity one or two. This, however, appears to be a more delicate question. Indeed it is known that finite dimensional group algebras of complexity one can be either finite or tame whilst group algebras of complexity two can be tame or wild, [10, Theorem 4.4.4]. The case for reduced enveloping algebras of finite dimensional restricted Lie algebras is different: complexity one implies finite representation type, [69, Lemma 5.1].

As a partial result in this direction we can show that if T is a self-injective algebra such that $\mathrm{cx}(T) = 2$ then T has either tame or wild representation type.

Proposition 2.24. *Let T be a self-injective algebra having finite representation type. Then* $\mathrm{cx}(T) \leq 1$.

Proof. Let M be a finite dimensional T-module. For $t \in \mathbb{Z}$ let $\Omega^t(M)$ denote the t-th syzygy of M, [10, Section 1.5]. Since T is self-injective it follows from Schanuel's lemma, [10, Lemma 1.5.3], that

$$\begin{aligned} M &\cong \Omega\Omega^{-1}(M) \oplus (\text{projective}) \\ &\cong \Omega^{-1}\Omega(M) \oplus (\text{projective}). \end{aligned}$$

Therefore, if M has no projective summands, we see M is indecomposable if and only if $\Omega(M)$ is indecomposable.

Since T has finite representation type there is a bound, C, on the dimensions of the indecomposable T-modules. By (2.9), there exists a simple T-module S such that $\mathrm{cx}_T(S) = \mathrm{cx}(T)$. By the above Ω^t sends indecomposables to indecomposables or zero. If $\Omega^t(S) = 0$ then the minimal resolution of S is finite. This implies that $\mathrm{cx}(T) = \mathrm{cx}_T(S) = 0$. Therefore we may assume that $\Omega^t(S)$ is non-zero for all $t \in \mathbb{N}$. Associated to the minimal resolution of S,

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0$$

we have, for each $t \in \mathbb{N}$, a short exact sequence

$$0 \longrightarrow \Omega^{t+1}(S) \longrightarrow P_t \longrightarrow \Omega^t(S) \longrightarrow 0.$$

This implies

$$\dim(P_t) = \dim(\Omega^{t+1}(S)) + \dim(\Omega^t(S)) \leq 2C.$$

Since $P_n \neq 0$ for all $n \in \mathbb{N}$ it follows that $\mathrm{cx}(T) = \mathrm{cx}_T(S) = 1$ as required.

2.6 Support varieties

From our point of view support varieties complete the link between the representation type of an algebra and its cohomology. For an augmented algebra (satisfying a hypothesis), T, and a T-module M, support varieties are geometric objects carrying information about $\mathrm{Ext}^*_T(k, M)$ considered as a module over the cohomology ring $\mathrm{Ext}^*_T(k, k)$. We show here that the complexity of such algebras can be calculated using these support varieties, so, by the results of the previous section, this gives us a means of studying representation type. Following the ground work of Friedlander and Parshall in [35], this approach has been successfully followed in [69], where the complexity and the representation type of reduced enveloping algebras of reductive Lie algebras are studied. Here we introduce support varieties for the families of finite dimensional algebras arising in Section 1.5. This follows the approach in [34] and [35]. We just note that the arguments in these references generalise to the case of non cocommutative Hopf algebras (with just a little care).

Recall the following hypothesis from Section 1.5.

Hypothesis 2. *The Hopf algebra H is a finitely generated over an affine central sub-Hopf* *algebra* H_0.

Recall too the family of finite dimensional algebras in (1.17), denoted H_χ for $\chi : H_0 \longrightarrow k$, and the finite dimensional Hopf algebra $H = H_\eta$. We will require the following hypothesis on this algebra.

Hypothesis 6. *The algebra* $\mathrm{Ext}_H^{2*}(k,k)$ *is a finitely generated commutative algebra, and for* M*, a finitely generated* H*-module,* $\mathrm{Ext}_H^{*}(k,M)$ *is a finitely generated* $\mathrm{Ext}_H^{2*}(k,k)$*-module.*

Remark 2.25. It is possible that this hypothesis is no restriction at all. Indeed it has been shown in [36] that all finite dimensional cocommutative Hopf algebras satisfy this condition. Moreover, as far as we are aware, there is no counterexample to this for non cocommutative Hopf algebras. Indeed, it's a major part of this thesis to prove that this hypothesis is satisfied by quantised function algebras, whilst the case for quantised enveloping algebras has already been shown in [39]. Of course, by Proposition 2.7, these (even degree) cohomology rings are always commutative.

Throughout this section we'll assume that H satisfies Hypotheses 2 and 6. Consider the variety,

$$\mathfrak{V}_H = \mathrm{Maxspec}(\mathrm{Ext}_H^{2*}(k,k)).$$

Definition 21. Let M be a finitely generated H_χ-module. We define the *support variety of* M to be the subvariety of $\mathfrak{V}_H$ associated to the ideal

$$\mathfrak{I}_M \equiv \mathrm{Ann}_{\mathrm{Ext}_H^{2*}(k,k)}(\mathrm{Ext}_H^{*}(k, M \otimes M^*)).$$

We denote this by $\mathfrak{V}_H(M)$.

Remark 2.26. It is clear that $\mathfrak{V}_H(k) = \mathfrak{V}_H$. Also, by hypothesis 6, $\mathrm{Ext}_H^{*}(k, M \otimes M^*)$ is a finitely generated $\mathrm{Ext}_H^{2*}(k,k)$-module so the definition above coincides with the (usual commutative algebra notion of) support variety of $\mathrm{Ext}_H^{*}(k, M \otimes M^*)$, that is

$$\mathfrak{V}_H(M) \equiv \{\mathfrak{m} \in \mathrm{maxspec}(\mathrm{Ext}_H^{2*}(k,k)) : (\mathrm{Ext}_H^{*}(k, M \otimes M^*))_{\mathfrak{m}} \neq 0\},$$

see [61, pp.25-26].

Lemma 2.27. *Let* M *be a finitely generated* H_χ*-module. Then*

$$\mathfrak{V}_H(M) = \bigcup_{\substack{L \text{ irreducible} \\ H_\chi\text{-module}}} \mathrm{Supp}(\mathrm{Ext}_H^{*}(k, L \otimes M^*)). \tag{2.10}$$

Proof. Let $\{L_1, \dots, L_n\}$ be a complete set of representatives of the isomorphism classes of simple H_χ-modules. Then $M \otimes M^*$ has a filtration as an H-module with factors $L_i \otimes M^*$. Suppose we have a short exact sequence of H-modules

$$0 \to N' \to N \to N'' \to 0.$$

This induces a triangle in cohomology

$$\begin{tikzcd} \mathrm{Ext}^*_H(k, N') \arrow[rr] & & \mathrm{Ext}^*_H(k, N) \arrow[dl] \\ & \mathrm{Ext}^*_H(k, N'') \arrow[ul] & \end{tikzcd}$$

which in turn shows that

$$\mathrm{Supp}(\mathrm{Ext}^*_H(k, N)) \subseteq \mathrm{Supp}(\mathrm{Ext}^*_H(k, N')) \cup \mathrm{Supp}(\mathrm{Ext}^*_H(k, N'')).$$

This proves one inclusion of (2.10).

To see the other inclusion we note first that for H_χ-modules N and N' the isomorphism of Lemma 1.27

$$\mathrm{Hom}_H(k, N' \otimes N^*) \xrightarrow{\sim} \mathrm{Hom}_{H_\chi}(N, N')$$

induces a graded isomorphism on the level of cohomology

$$\mathrm{Ext}^*_H(k, N' \otimes N^*) \xrightarrow{\sim} \mathrm{Ext}^*_{H_\chi}(N, N'), \tag{2.11}$$

see [35, Proposition 5.1]. Under (2.11) the $\mathrm{Ext}^*_H(k,k)$-module structure on $\mathrm{Ext}^*_{H_\chi}(M, M)$ is given by taking an exact sequence representing an element of $\mathrm{Ext}^n_H(k,k)$ and tensoring it by M.

So let $\zeta \in \mathrm{Ext}^n_H(k,k)$ be represented by the H-exact sequence

$$0 \longrightarrow k \to A_{n-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow k \longrightarrow 0 \tag{2.12}$$

and $\rho \in \mathrm{Ext}^m_{H_\chi}(M, L_i)$ be represented by the H_χ-exact sequence

$$0 \longrightarrow L_i \longrightarrow B_{m-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow M \longrightarrow 0. \tag{2.13}$$

Then $\rho.\zeta$ is represented by H_χ-exact sequence

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & L_i \arrow[r] & B_{m-1} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[dr] \arrow[rr] & & A_{m-1} \otimes M \arrow[r] & \cdots \arrow[r] & A_0 \otimes M \arrow[r] & M \arrow[r] & 0 \\ & & & & & M \arrow[ur] \arrow[dr] & & & & & \\ & & & & 0 \arrow[ur] & & 0 & & & & \end{tikzcd}$$

On the other hand tensoring (2.12) by M and then applying (2.13) yields the same H_χ-exact sequence showing that the right $\mathrm{Ext}^*_H(k,k)$ action on $\mathrm{Ext}^*_H(k, L_i \otimes M^*)$ factors through

$$\theta : \mathrm{Ext}^*_H(k,k) \xrightarrow{\otimes M} \mathrm{Ext}^*_{H_\chi}(M,M).$$

Since $\mathfrak{I}_M = \ker\theta$ the other inclusion of (2.10) follows.

Theorem 2.28. *Let M be a finitely generated H_χ-module. Then* $\mathrm{cx}_{H_\chi}(M) = \dim \mathfrak{V}_H(M)$.

Proof. Let $\{L_1, \cdots, L_n\}$ be a complete set of representatives of the isomorphism classes of simple H_χ-modules and let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{2.14}$$

be the minimal H_χ-projective resolution of M. Each P_t is a direct sum $\sum_{i=1}^n s_{i,t}P(L_i)$ where $P(L_i)$ is the projective indecomposable module with head L_i. Moreover, by [10, Lemma 1.7.6], we have the equality

$$s_{i,t} = \dim(\mathrm{Hom}_{H_\chi}(P_t, L_i)).$$

Since (2.14) is a minimal resolution of M and each L_i is irreducible we see that for all $t \geq 0$ and $1 \leq i \leq n$

$$\mathrm{Ext}^t_{H_\chi}(M, L_i) = \mathrm{Hom}_{H_\chi}(P_t, L_i).$$

We deduce that

$$\dim P_t = \sum_{i=1}^n \dim(\mathrm{Ext}^t_{H_\chi}(M, L_i)) \dim P(L_i).$$

It follows that $\mathrm{cx}_{H_\chi}(M)$ equals the maximum of the rates of growth of $\mathrm{Ext}^*_{H_\chi}(M, L_i)$ for $1 \leq i \leq n$.

By Lemma 2.27 we have seen that $\mathfrak{V}_H(M)$ equals the union of the supports of the $\mathrm{Ext}^*_H(k,k)$-modules $\mathrm{Ext}^*_H(k, L_i \otimes M^*)$. By [61, p.31] we have

$$\dim \mathrm{Supp}(\mathrm{Ext}^*_H(k, L_i \otimes M^*)) = \mathrm{Kdim}\, \frac{\mathrm{Ext}^{2*}_H(k,k)}{\mathrm{Ann}(\mathrm{Ext}^*_H(k, L_i \otimes M^*))}.$$

It follows from Hypothesis 6 that $\mathrm{Ext}^*_H(k, L_i \otimes M^*)$ is a finitely generated, faithful module over $\mathrm{Ext}^{2*}_H(k,k)/\mathrm{Ann}(\mathrm{Ext}^*_H(k, L_i \otimes M^*))$ and so $\dim \mathrm{Supp}(\mathrm{Ext}^*_H(k, L_i \otimes M^*))$ equals the rate of growth of $\mathrm{Ext}^*_H(k, L_i \otimes M^*)$. By (2.11) we see that $\mathrm{Ext}^*_H(k, L_i \otimes M^*) \cong \mathrm{Ext}^*_{H_\chi}(M, L_i)$ and so the theorem follows.

Following (2.9) and Theorem 2.28 we can describe the complexity of the algebras in terms of support varieties.

Corollary 2.29. *Let $\{L_1, \cdots, L_n\}$ be a complete set of representatives of the isomorphism classes of simple H_χ-modules. Then*

$$\mathrm{cx}(H_\chi) = max_{i=1}^{n}\{\dim(\mathfrak{V}_H(L_i)\}.$$

Remark 2.30. It also follows from the proof of Lemma 2.28 that $\mathrm{cx}(H_\chi)$ equals the rate of growth of $\mathrm{Ext}^*_{H_\chi}(\oplus_i L_i, \oplus_i L_i)$ where $\{L_i\}_i$ is the set of simple H_χ-modules.

2.7 Additional remarks

1. As with a lot of cohomology the results of Section 2.1 and 2.2 are well-known, but references are difficult to find. Products in cohomology are discussed well in [16, Chapter 6], while a lot of the original work goes back to [37].

2. The Lyndon-Hochschild-Serre spectral sequence is discussed (ad nauseam) in [7]. We follow [39, Appendix] from Hypothesis 5 to the end of Section 2.3.

3. A construction similar to that of the spectral sequence in Lemma 2.19 is dicussed in [47, I: 9.12 and 9.13].

4. The definition of complexity goes back to the study of the cohomology of groups in [2]. The representation type trichotomy is due to Drozd, [29]. A readable account can be found in [22]. Presumably Proposition 2.24 is well-known.

5. Support varieties were first introduced in group cohomology, [15], where profound connections were discovered between the mod-p cohomology of a group G and the mod-p cohomology of its elementary abelian p-subgroups, [72] and [6]. Support varieties were introduced in restricted Lie algebra theory in [33] and [34] and afterwards in the non-restricted case in [35]. Our treatment follows [35] carefully. Some results on support varieties for quantised enveloping algebras at roots of unity can be found in [66] and [65].

Chapter 3

The cohomology of quantised function algebras.

In this chapter we prove the validity of Hypothesis 6 for the Hopf algebra $\mathcal{O}_\epsilon[G]$, under a slight restriction on ℓ. That is we prove that $\mathrm{Ext}^{2*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is a finitely generated commutative algebra and that if M is a finite dimensional $\mathcal{O}_\epsilon[G]$-module then $\mathrm{Ext}^{*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},M)$ is a finitely generated (right) $\mathrm{Ext}^{*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$-module. In fact we prove more, and it's crucial for us later that we do. We show, assuming that ℓ is sufficiently large, that $\mathrm{Ext}^{2*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is a polynomial ring in $|\Delta|$ variables. The approach to this follows [39] where the cohomology of the "restricted" quantised enveloping algebra of $\mathfrak{g}$, written $U_\epsilon(\mathfrak{g})$, is studied. For what it's worth, although the results are stated for $\mathbb{C}$, everything in this chapter is true for an arbitrary algebraically closed field of characteristic zero.

Throughout this chapter we will assume that we have fixed a reduced expression of w_0 and hence an ordering on Δ^+,

$$\beta_1 < \beta_2 < \ldots < \beta_N,$$

and elements $E_{\beta_j} \in U_\epsilon^+$ and $F_{\beta_j} \in U_\epsilon^-$ for all j between 1 and N.

3.1 The cohomology of $\mathcal{O}_\epsilon[G]$.

In this section we determine the ring $\mathrm{Ext}^{*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$. We follow closely the methods employed in [39] where the cohomology of $U_\epsilon(\mathfrak{g})$ is calculated. Recall by Proposition 1.14 $\mathcal{O}_\epsilon[G]$ is projective of finite rank over the central subalgebra $\mathcal{O}[G]$. Moreover $\mathcal{O}[G]$ is the coordinate ring of a reduced affine group scheme so has finite global dimension, see [80, 11.6]. Therefore Hypotheses 4 and 5 are satisfied so we can apply the results of Section 2.3 at will.

Definition 22. Let the *quantised exterior algebras* Λ_ϵ and Λ'_ϵ be the $\mathbb{C}$-algebras with generators $\{f_\beta : \beta \in \Delta^+\}$ and $\{e_\beta : \beta \in \Delta^+\}$ respectively and relations

$$f_{\beta_1} f_{\beta_2} + \epsilon^{-(\beta_1,\beta_2)} f_{\beta_2} f_{\beta_1} = 0, \quad e_{\beta_1} e_{\beta_2} + \epsilon^{(\beta_1,\beta_2)} e_{\beta_2} e_{\beta_1} = 0 \qquad \text{if } \beta_1 > \beta_2$$

$$f_\beta^2 = 0, \quad e_\beta^2 = 0 \qquad \text{for all } \beta \in \Delta^+$$

Both Λ_ϵ and Λ'_ϵ have $\mathbb{N}$-gradings where the above generators are given degree one. Further each has a natural U_ϵ^0 action given, on the generators, by

$$K_\lambda . f_\beta = \epsilon^{-(\beta,\lambda)} f_\beta, \quad K_\lambda . e_\beta = \epsilon^{(\beta,\lambda)} e_\beta$$

Consider the algebra $S = U_\epsilon^- \otimes U_\epsilon^+$. By Proposition 1.10 S has a $\mathbb{N}^{2N}$-filtration whose associated graded ring, GrS, is a quantised polynomial algebra with generators $\bar{E}_{\beta_j}$ and $\bar{F}_{\beta_j}$. Moreover S has a natural U_ϵ^0-action given by the action by weight on $U_\epsilon^\pm$ composed with the homomorphism

$$\theta : K_\lambda \longmapsto K_\lambda^{-1} \otimes K_\lambda \tag{3.1}$$

Note that in this case the element $F_{\beta_1} \otimes E_{\beta_2}$ has weight $\beta_1 + \beta_2$.

Recall, from Theorem 1.21(ii), that $\mathcal{O}_\epsilon[G][(z^\rho)^{-1}]$ is isomorphic to the algebra $T \subseteq U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$ generated by $F_i \otimes 1, 1 \otimes E_i$ $(i = 1, \dots r)$ and $K_{-\lambda} \otimes K_\lambda$ $(\lambda \in P)$. Clearly S is a normal subalgebra of T and $T//S \cong U^0$ via the inverse of θ given in (3.1). Noting the multiplicative filtration on S can be extended to T by including $K_{-\lambda} \otimes K_\lambda$ in the zeroth part of the filtration we see GrS is a normal subalgebra of GrT and Gr$T//$Gr$S \cong$ Gr$U_\epsilon^0 \cong U_\epsilon^0$ (by the above observation). Moreover GrT is a projective GrS-module. Because of this Lemma 2.8 applies to give a left U_ϵ^0-structure on $\mathrm{Ext}^*_{\mathrm{Gr}S}(\mathbb{C}, \mathbb{C})$.

Proposition 3.1. *As graded algebras* $\mathrm{Ext}^*_{\mathrm{Gr}S}(\mathbb{C}, \mathbb{C}) \cong \Lambda_\epsilon \otimes \Lambda'_\epsilon$. *Moreover this isomorphism is* U_ϵ^0*-equivariant.*

Proof. We show that GrS is a homogeneous Koszul algebra in the sense of [71, Section 2]. Once this has been done the first claim follows immediately from [71, Theorem 2.5].

We have a basis for GrS given by the set

$$\{\bar{F}_{\beta_1}^{r_1} \dots \bar{F}_{\beta_N}^{r_N} \bar{E}_{\beta_N}^{t_N} \dots \bar{E}_{\beta_1}^{t_1} : (r_1, \dots r_N, t_N, \dots t_1) \in \mathbb{N}^{2N}\}$$

By Proposition 1.10 the $\tilde{F}$'s and the $\tilde{E}$'s are both PBW bases in the sense of [71, 5.1]. Moreover the product of these satisfy the conditions given in [71, 5.1].

For the U_ϵ^0-equivariance we use the (normalised) one-sided bar resolution of GrS. This has the same differentials as the one-sided bar resolution but the n^{th}-degree piece is $\mathrm{Gr}S \otimes (\mathrm{Gr}S)_+^{\otimes n}$ instead of $\mathrm{Gr}S \otimes (\mathrm{Gr}S)^{\otimes n}$, where $(\mathrm{Gr}S)_+$ is the augmentation ideal of GrS. If we use this to calculate cohomology we see, by [71, Theorem 2.5], that $f_{\beta_i} \in \mathrm{Ext}^1_{\mathrm{Gr}S}(\mathbb{C},\mathbb{C})$ is represented by the homomorphism $\phi_{\beta_i} \in \mathrm{Hom}_{\mathrm{Gr}S}(\mathrm{Gr}S \otimes (\mathrm{Gr}S)_+, \mathbb{C})$ dual to $\tilde{F}_{\beta_i}$. That is, for $x \in \mathrm{Gr}S$,

$$\phi_{\beta_i}(x \otimes \tilde{F}^{r_1}_{\beta_1} \dots \tilde{F}^{r_N}_{\beta_N} \tilde{E}^{t_N}_{\beta_N} \dots \tilde{E}^{t_1}_{\beta_1}) = \begin{cases} x.1 & \text{if } r_j = \delta_{ij} \text{ and } t_j = 0 \text{ for all } 1 \leq j \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for e_{β_i}.

It follows from Lemma 2.8 that the action of K_λ on cohomology, for $\lambda \in P$, is induced from the maps θ^i_λ below which lift right multiplication on $GrT//GrS \cong U^0_\epsilon$ by K_λ,

$$\begin{array}{ccccccc} \dots \longrightarrow & \mathrm{Gr}T \otimes (\mathrm{Gr}S)_+ & \xrightarrow{1\otimes d_1} & \mathrm{Gr}T & \xrightarrow{d_0} & \mathrm{Gr}T \otimes_{\mathrm{Gr}S} \otimes \mathbb{C} \\ & \theta^1_\lambda \big\downarrow & & \theta^0_\lambda \big\downarrow & & \times K_\lambda \big\downarrow \\ \dots \longrightarrow & \mathrm{Gr}T \otimes (\mathrm{Gr}S)_+ & \xrightarrow{1\otimes d_1} & \mathrm{Gr}T & \xrightarrow{d_0} & \mathrm{Gr}T \otimes_{\mathrm{Gr}S} \otimes \mathbb{C} \end{array}$$

Here $(1 \otimes d_1)(t \otimes s) = ts$, whilst $d_0(t) = t.1$. It is easy to check that we can take θ^0_λ to be right multiplication by K_λ on GrT and that we can take

$$\theta^1_\lambda(t \otimes \tilde{F}^{r_1}_{\beta_1} \dots \tilde{F}^{r_N}_{\beta_N} \tilde{E}^{t_N}_{\beta_N} \dots \tilde{E}^{t_1}_{\beta_1}) = \epsilon^{-(\sum(r_j+t_j)\beta_j,\lambda)} t K_\lambda \otimes \tilde{F}^{r_1}_{\beta_1} \dots \tilde{F}^{r_N}_{\beta_N} \tilde{E}^{t_N}_{\beta_N} \dots \tilde{E}^{t_1}_{\beta_1}.$$

From this it follows that $K_\lambda.\phi_{\beta_i} = \epsilon^{-(\beta_i,\lambda)}\phi_{\beta_i}$, proving the U^0_ϵ-equivariance for the elements f_β and similarly e_β. The general case follows since U^0_ϵ acts algebraically, as in Lemma 3.4 below.

Remark 3.2. Note that all U^0_ϵ-actions discussed so far have involved powers of ϵ, an ℓth root of unity, so all the actions are in fact $\underline{U}^0_\epsilon$-actions, where, of course, $\underline{U}^0_\epsilon = U^0_\epsilon/\langle K^l_\lambda - 1 : \lambda \in P^+\rangle$.

Consider the list of elements of Δ^+:

$$\beta_1,\ \beta_2, \dots,\ \beta_N,\ \beta_N, \dots,\ \beta_1.$$

and let γ_j be the jth element in the above list for $j = 1, \dots, 2N$. Now define elements of

S as follows:

$$y_j = \begin{cases} F^{\ell}_{\gamma_j} \otimes 1 & \text{if } 1 \le j \le N, \\ 1 \otimes E^{\ell}_{\gamma_j} & \text{if } N+1 \le j \le 2N. \end{cases} \tag{3.2}$$

Let V_j be the space spanned by the elements $\{y_1, \dots, y_j\}$. Let $M_j = S(V_j)$, the symmetric algebra with generators the elements of V_j. Give each of these generators degree two. By Proposition 1.8 for each j the algebra M_j is central in T for each j. Since the elements $F^{\ell}_{\gamma_j}$ and $E^{\ell}_{\gamma_j}$ are homogeneous it is clear that $M_j \cong \mathrm{Gr} M_j \subseteq \mathrm{Gr} S$. Thus we may define B_j to be $\mathrm{Gr} S//M_j$ for $j = 0, \dots, 2N$, letting $B_0 = \mathrm{Gr} S$. Let A_{j+1} be the subalgebra of B_j generated by the image of y_{j+1}. Clearly A_{j+1} is a polynomial ring in one variable, B_j is a projective A_{j+1}-module and $B_j//A_{j+1} \cong B_{j+1}$ for $0 \le j < 2N$.

We prove a useful general lemma, following the proof of [39, Proposition 2.3.2].

Proposition 3.3. *Suppose $\ell > h$, where, as in Section 1.1, h denotes the Coxeter number of G. For $j = 0, 1, \dots, 2N$ there is a natural graded algebra isomorphism*

$$\mathrm{Ext}^*_{B_j}(\mathbb{C}, \mathbb{C}) \cong \Lambda_\epsilon \otimes \Lambda'_\epsilon \otimes S(V_j^*) \tag{3.3}$$

which is also a U^0_ϵ-map.

Proof. We use induction on j, the case $j = 0$ being given by Proposition 3.1. Thus suppose the proposition is true for all $i \le j$. By Lemma 2.11 we have a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{B_{j+1}}\left(\mathbb{C}, \mathrm{Ext}^q_{A_{j+1}}(\mathbb{C}, \mathbb{C})\right) \Longrightarrow \mathrm{Ext}^{p+q}_{B_j}(\mathbb{C}, \mathbb{C}). \tag{3.4}$$

By Lemma 2.14 the horizontal edge map of (3.4) is inflation and so provides a graded algebra homomorphism

$$\inf : \mathrm{Ext}^*_{B_{j+1}}(\mathbb{C}, \mathbb{C}) \longrightarrow \mathrm{Ext}^*_{B_j}(\mathbb{C}, \mathbb{C}).$$

We claim this is an epimorphism:

By our induction hypothesis $\mathrm{Ext}^*_{B_j}(\mathbb{C}, \mathbb{C})$ is generated in degree one and two so it is sufficient to prove surjectivity there. It follows from the proof of Lemma 2.14 that inflation is induced from $B_j \twoheadrightarrow B_{j+1}$. Since the relations induced by B_{j+1} are obtained from elements which are contained in the square of the augmentation ideal (for ℓ is certainly not 1) we have a vector space isomorphism $(B_j)_+/((B_j)_+)^2 \cong (B_{j+1})_+/((B_{j+1})_+)^2$, and thanks to Lemma 2.1 this takes care of degree one. From the proof of Lemma 2.3 it follows, using the naturality of long exact sequences, that $\inf^2$ agrees with the map $\mathrm{Hom}_{\mathbb{C}}(R_{j+1}, \mathbb{C}) \longrightarrow$

$\mathrm{Hom}_{\mathbb{C}}(R_j, \mathbb{C})$ induced by $R_j \longrightarrow R_{j+1}$, where we adopt the obvious notation for ideals of the appropriate tensor algebra. Hence we need only prove this is surjective (with the obvious notation). But this follows immediately since $R_j \longrightarrow R_{j+1}$ is an injection since each relation in B_j is still a non-trivial relation in B_{j+1} and so R_j is a $\mathbb{C}$-subspace of R_{j+1}. This proves the claim.

We have the following commutative diagram for all $p \geq 0$:

$$\begin{tikzcd}
E_2^{p-2,1} \arrow[r, "d_2"] & E_2^{p,0} \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & E_\infty^{p,0} \arrow[d, "i"] \\
 & \mathrm{Ext}^p_{B_{j+1}}(\mathbb{C},\mathbb{C}) \arrow[r, "\mathrm{inf}^p"] & \mathrm{Ext}^p_{B_j}(\mathbb{C},\mathbb{C})
\end{tikzcd}$$

where π is the canonical epimorphism and i is the obvious injection, according to the filtration of $\mathrm{Ext}^p_{B_j}(\mathbb{C},\mathbb{C})$ provided by the spectral sequence (3.4). Since inf^p is surjective by the previous paragraph, it follows that i is in fact an isomorphism and so

$$E_\infty^{p,0} \cong \mathrm{Ext}^p_{B_j}(\mathbb{C},\mathbb{C}). \tag{3.5}$$

Therefore

$$E_\infty^{p,q} = 0 \text{ for all } q > 0. \tag{3.6}$$

Since A_{j+1} is a central subalgebra of B_j isomorphic to a polynomial ring Lemma 2.17 implies the action of B_{j+1} on $\mathrm{Ext}^*_{A_{j+1}}(\mathbb{C},\mathbb{C})$ is trivial. Hence Corollary 2.18 implies

$$\mathrm{Ext}^1_{A_{j+1}}(\mathbb{C},\mathbb{C}) \xrightarrow{d_2} \mathrm{Ext}^2_{B_{j+1}}(\mathbb{C},\mathbb{C})$$

has image central in $\mathrm{Ext}^*_{B_{j+1}}(\mathbb{C},\mathbb{C})$. As A_{j+1} is a polynomial algebra in one variable $\mathrm{Ext}^1_{A_{j+1}}(\mathbb{C},\mathbb{C})$ is one dimensional. Let v denote the image of a non-zero element, y, under d_2. We claim that

$$ker(\mathrm{inf}) = \langle v \rangle, \tag{3.7}$$

where $\langle v \rangle$ denotes the two-sided ideal generated by v:

By definition $E_2^{p,0}/d_2(E_2^{p-2,1}) \cong E_3^{p,0}$. Since $\mathrm{Ext}^r_{A_{j+1}}(\mathbb{C},\mathbb{C}) = 0$ for all $r \geq 2$ it follows that $E_3^{p,0} \cong E_\infty^{p,0}$. By (3.5) this is isomorphic to $\mathrm{Ext}^p_{B_j}(\mathbb{C},\mathbb{C})$. This sequence of isomorphisms corresponds to travelling clockwise around the commutative diagram above. Hence $\ker(\mathrm{inf}^p) = d_2(E_2^{p-2,1})$. Note that $E_2^{p-2,1}$ is isomorphic to $\mathrm{Ext}^{p-2}_{B_{j+1}}(\mathbb{C},\mathbb{C}) \otimes \mathrm{Ext}^1_{A_{j+1}}(\mathbb{C},\mathbb{C})$. Since d_2 is a derivation and $d_2(E_2^{p-2,0}) = 0$, it follows that, for all $x \in \mathrm{Ext}^{p-2}_{B_{j+1}}(\mathbb{C},\mathbb{C})$,

$$d_2(x \otimes y) = d_2((x \otimes 1)(1 \otimes y)) = (-1)^{p-2} x d_2(y) = (-1)^{p-2} x v$$

proving (3.7).

Since $\mathrm{Ext}^r_{A_{j+1}}(\mathbb{C},\mathbb{C}) = 0$ for all $r \geq 2$ we have $E_3^{p,1} \cong E_\infty^{p,1}$ which is zero by (3.6). Thus $d_2|_{E_2^{p,1}}$ is injective since $d_2(E_2^{p-2,2}) = 0$. Therefore if

$$0 \neq x \otimes y \in \mathrm{Ext}^p_{B_{j+1}}(\mathbb{C},\mathbb{C}) \otimes \mathrm{Ext}^1_{A_{j+1}}(\mathbb{C},\mathbb{C}) \cong E_2^{p,1}$$

then $d_2(x \otimes y) = (-1)^p x.v \neq 0$. In other words v is not a zero divisor in $\mathrm{Ext}^*_{B_{j+1}}(\mathbb{C},\mathbb{C})$.

In order to prove the induction step it only remains to prove that we have a graded algebra splitting of the homomorphism $\inf : \mathrm{Ext}^*_{B_{j+1}}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ext}^*_{B_j}(\mathbb{C},\mathbb{C})$, since it is clear that it commutes with the U^0_ϵ-actions. To this end, by induction, we have only to lift the generators of the algebras $\Lambda_\epsilon \otimes \Lambda'_\epsilon$ and $S(V_j^*)$ to $\mathrm{Ext}^*_{B_{j+1}}(\mathbb{C},\mathbb{C})$ and check that the correct relations continue to hold. Consider the commutative diagram

$$\begin{array}{ccc} V^*_{j+1} & \xrightarrow{d_2} & \mathrm{Ext}^2_{B_{j+1}}(\mathbb{C},\mathbb{C}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \inf^2} \\ V_j^* & \xrightarrow{d_2} & \mathrm{Ext}^2_{B_j}(\mathbb{C},\mathbb{C}) \end{array}$$

where $i : V_j \hookrightarrow V_{j+1}$ is the natural injection. Any splitting of $i^* : V^*_{j+1} \twoheadrightarrow V_j^*$ provides a lifting of generators along $\inf^2$. The image is necessarily central since for any $r \geq 0$ Lemma 2.17 shows the natural action of B_{r+1} on $\mathrm{Ext}^*_{A_r}(\mathbb{C},\mathbb{C})$ is trivial and so Corollary 2.18 applies.

The generators of $\Lambda_\epsilon \otimes \Lambda'_\epsilon$ are homogeneous of degree one and we have seen in the second paragraph of the proof that $\inf^1$ is a U^0_ϵ-equivariant isomorphism. Hence applying $(\inf^1)^{-1}$ yields unique U^0_ϵ-equivariant lifts of these generators. Suppose, for all $\beta \in \Delta^+$, f_β and e_β are lifts of f_β and e_β respectively.

We prove first that $f_{\beta_1}.f_{\beta_2} + \epsilon^{-(\beta_1,\beta_2)} f_{\beta_1}.f_{\beta_2} = 0$ if $\beta_1 > \beta_2$. Since the image of the left-hand side under $\inf^1$ is zero we know that it must be a scalar multiple of v. The element v has weight 0 (as it is the image of an element which has trivial action by an argument identical to Lemma 2.17) whereas $f_{\beta_1} f_{\beta_2} + \epsilon^{-(\beta_1,\beta_2)} f_{\beta_1} f_{\beta_2}$ has weight $\beta_1 + \beta_2$. Since β_1 and β_2 are positive roots it follows from [45, Table 1, p.45] that either $-1 \leq (\beta_1, \beta_2^\vee) \leq 1$ or $-1 \leq (\beta_2, \beta_1^\vee) \leq 1$. In the first case $1 \leq (\beta_1 + \beta_2, \beta_2^\vee) \leq 3$ whilst in the second $1 \leq (\beta_1 + \beta_2, \beta_1^\vee) \leq 3$. Hence, for $\ell > h$, $\beta_1 + \beta_2 \notin \ell Q$, so the scalar is zero.

The same argument works for the pullback of ${f_\beta}^2 = 0$ since $2\beta \notin \ell Q$ as we are assuming ℓ is odd. Exactly the same arguments apply for e's replacing f's. Finally we note that the left-hand side of the equation $f_{\beta_1} e_{\beta_2} + e_{\beta_2} f_{\beta_1} = c'v$ has weight $\beta_1 + \beta_2$ so, by a similar analysis to the above, the scalar c' is zero in this case also. This completes the proof of the induction step, and hence of the proposition.

Let T_0 denote the (central) subalgebra of T generated by $F_\beta^\ell \otimes 1$, $1 \otimes E_\beta^\ell$ and $K_\lambda^{\pm\ell} \otimes K_\lambda^{\mp\ell}$ and $S_0 := S \cap T_0$. Let $T = T//T_0$ and $S = S//S_0$. Then, by Proposition 1.32, T is isomorphic, as an augmented algebra, to $\mathcal{O}_\epsilon[G]$. Recall that U_ϵ^0 is isomorphic to the subalgebra of T generated by the images (under the canonical projection) of $K_\lambda^{\pm 1} \otimes K_\lambda^{\mp 1}$. Therefore we have $U_\epsilon^0 = T//S$.

We observe that T_0 and S_0 are compatible with the filtration on T induced from Proposition 1.10, in the sense that

$$\mathrm{Gr}T \cong \mathrm{Gr}T//\mathrm{Gr}T_0 \cong \mathrm{Gr}T//T_0,$$

and

$$\mathrm{Gr}S \cong \mathrm{Gr}S//\mathrm{Gr}S_0 \cong \mathrm{Gr}S//S_0.$$

The second isomorphisms on each line follow from the fact that T_0 and S_0 are central in T and S respectively. In particular we note that $\mathrm{Gr}S \cong B_{2N}$ where B_{2N} is as defined before Proposition 3.3.

Let M be a finite dimensional T-module. We prepare for our main theorems with several lemmas concerning U_ϵ^0-actions on the cohomology of M.

Lemma 3.4. *Multiplication*

$$\mathrm{Ext}_S^p(\mathbb{C}, M) \otimes \mathrm{Ext}_S^q(\mathbb{C}, \mathbb{C}) \longrightarrow \mathrm{Ext}_S^{p+q}(\mathbb{C}, M)$$

is a U_ϵ^0-map, where the module structure on the left is induced from the Hopf algebra structure of U_ϵ^0.

Proof. Fix a weight $\lambda \in P$ and thus an element $K_\lambda \in U_\epsilon^0$. Let $m_\lambda : T \otimes_S \mathbb{C} \longrightarrow T \otimes_S \mathbb{C}$ denote right multiplication on T by K_λ. As S has a basis of weight vectors with respect to K_λ we can construct a T-map

$$\theta_{0,\lambda} : T \otimes_S S \longrightarrow T \otimes_S S$$

by $\theta_{0,\lambda}(t \otimes s) = \epsilon^{-(\lambda,\mu)} t K_\lambda \otimes s$ for $s \in S$ of weight μ. This construction ensures $\theta_{0,\lambda}$ is S-balanced. It is easy to check that, since all elements of S of non-zero weight are in the augmentation ideal of S, the following diagram commutes

$$\begin{array}{ccc} T\otimes_S S & \xrightarrow{1\otimes\epsilon} & T\otimes_S \mathbb{C} \\ {\scriptstyle \theta_{0,\lambda}}\downarrow & & \downarrow{\scriptstyle m_\lambda} \\ T\otimes_S S & \xrightarrow{1\otimes\epsilon} & T\otimes_S \mathbb{C} \end{array}$$

Similarly we can define an T-map

$$\theta_{p,\lambda} : T\otimes_S S\otimes S^{\otimes p} \longrightarrow T\otimes_S S\otimes S^{\otimes p}$$

by $\theta_{p,\lambda}(t\otimes s_0\otimes s_1\otimes\cdots\otimes s_p) = \epsilon^{-(\lambda,\sum\mu_i)}tK_\lambda\otimes s_0\otimes s_1\otimes\cdots\otimes s_p$ where $s_i\in S$ has weight μ_i. Adapting the bar resolution to left modules, it is clear that $\theta_{p,\lambda}$ provides a map of chain complexes, lifting m_λ

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & T\otimes_S S\otimes S^{\otimes p} & \longrightarrow & \cdots & \longrightarrow & T\otimes_S S & \xrightarrow{1\otimes\epsilon} & T\otimes_S\mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{p,\lambda}} & & & & \downarrow{\scriptstyle \theta_{0,\lambda}} & & \downarrow{\scriptstyle m_\lambda} & & \\ \cdots & \longrightarrow & T\otimes_S S\otimes S^{\otimes p} & \longrightarrow & \cdots & \longrightarrow & T\otimes_S S & \xrightarrow{1\otimes\epsilon} & T\otimes_S\mathbb{C} & \longrightarrow & 0. \end{array}$$

Let $f \in \mathrm{Hom}_T\left(T\otimes_S S\otimes S^{\otimes p}, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, M)\right)$ and $g \in \mathrm{Hom}_T\left(T\otimes_S S\otimes S^{\otimes q}, \mathbb{C}\right)$. Then the cup product translates to the following formula

$$(f\cup g)(t\otimes s_0\otimes\ldots\otimes s_{p+q}) = (-1)^q f(t\otimes s_0\otimes\ldots\otimes s_p)g(1\otimes 1\otimes s_{p+1}\otimes\ldots\otimes s_{p+q}).$$

From this we can easily see that

$$(K_\lambda.f)\cup(K_\lambda.g) = K_\lambda.(f\cup g).$$

Since $\Delta(K_\lambda) = K_\lambda\otimes K_\lambda$ the result follows.

Recall that $\mathrm{Gr}T//\mathrm{Gr}S \cong \mathrm{Gr}U^0_\epsilon \cong U^0_\epsilon$. Thus by Lemma 2.8 we have a U^0_ϵ-module structure on both $\mathrm{Ext}^*_S(\mathbb{C}, M)$ and $\mathrm{Ext}^*_{\mathrm{Gr}S}(\mathbb{C}, M)$, where M is considered a trivial GrS-module.

Lemma 3.5. *Let M be T-module, considered as a trivial GrS-module as above. The spectral sequence of Lemma 2.19,*

$$\mathrm{Ext}^{p+q}_{GrS}(\mathbb{C},\mathbb{C})_{(p)}\otimes M \Longrightarrow \mathrm{Ext}^{p+q}_S(\mathbb{C}, M)$$

preserves the U^0_ϵ-action.

Proof. From the description of the U_ϵ^0-action given in the proof of Lemma 3.4 it is clear that U_ϵ^0 preserves the filtration on the complex $C(\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, M))$. Thus it remains only to prove the identification of the chain complexes

$$C(\mathbb{C}) \otimes N \longrightarrow C(N)$$

given by

$$f \otimes n \longmapsto (s_1 \otimes \cdots \otimes s_p \mapsto f(s_1 \otimes \cdots \otimes s_p)n) \tag{3.8}$$

for a trivial S-module N preserves the U_ϵ^0-action. This becomes clear from the description of the U_ϵ^0-action in Lemma 3.4 when we translate (3.8) to the following:

$$\mathrm{Hom}_T\left(T \otimes_S S \otimes S^{\otimes p}, \mathbb{C}\right) \otimes N \longrightarrow \mathrm{Hom}_T\left(T \otimes_S S \otimes S^{\otimes p}, N\right)$$

$$g \otimes n \longmapsto h$$

where $h(t \otimes s_0 \otimes \ldots \otimes s_p) = t.(g(1 \otimes s_0 \otimes \ldots \otimes s_p)n)$.

Lemma 3.6. *Let M be a T-module. Then there is a natural isomorphism*

$$\mathrm{Ext}_S^*(\mathbb{C}, M)^{U_\epsilon^0} \cong \mathrm{Ext}_T^*(\mathbb{C}, M).$$

Moreover this is an isomorphism of $\mathrm{Ext}_T^*(\mathbb{C}, \mathbb{C})$-modules.

Proof. The first part is clear from Lemma 2.10. The second claim follows since this is effected by restriction.

For the following theorem we add to our hypotheses. Consider

$$\sum_{i=1}^{N} \beta_i = \sum_{j=1}^{r} a_j \alpha_j,$$

and let $d = \max_{j=1}^{r}\{a_j\}$.

Hypothesis 7. *The integer ℓ is greater than $2d$.*

This number turns out always to be greater than the Coxeter number. For example, when G is of type A_r this implies

$$\ell > \begin{cases} r(r/2+1) & \text{if } r \text{ is even,} \\ (r+1)^2/2 & \text{if } r \text{ is odd.} \end{cases} \tag{3.9}$$

Theorem 3.7. *Assume Hypotheses 1 and 7 hold. Then* $\mathrm{Ext}^{odd}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C}) = 0$ *and there is a natural graded algebra isomorphism*

$$\mathrm{Ext}^{2*}_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C}) \cong S(V_{2N}^*).$$

Proof. It follows from Proposition 1.32(ii) that $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C}) \cong \mathrm{Ext}^*_T(\mathbb{C},\mathbb{C})$ as graded algebras, since cohomological products depend only on the structure of $\mathcal{O}_\epsilon[G]$ as an augmented algebra by Theorem 2.5. By Lemma 3.6 we have an algebra isomorphism

$$\mathrm{Ext}^*_T(\mathbb{C},\mathbb{C}) \cong \mathrm{Ext}^*_S(\mathbb{C},\mathbb{C})^{U^0_\epsilon}. \tag{3.10}$$

Using the isomorphism $\mathrm{Gr}S \cong B_{2N}$, the spectral sequence of Lemma 2.19 gives

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{B_{2N}}(\mathbb{C},\mathbb{C})_{(p)} \Longrightarrow \mathrm{Ext}^{p+q}_S(\mathbb{C},\mathbb{C})$$

which commutes with the natural U^0_ϵ-actions by Lemma 3.5. Hence, by (3.3) and (3.10), there is a spectral sequence

$$\mathrm{Ext}^*_{B_{2N}}(\mathbb{C},\mathbb{C})^{U^0_\epsilon}_{(p)} \cong (\Lambda_\epsilon \otimes \Lambda'_\epsilon \otimes S(V^*_{2N}))^{U^0_\epsilon}_{(p)} \Longrightarrow \mathrm{Ext}^*_T(\mathbb{C},\mathbb{C}) \tag{3.11}$$

where $S(V^*_{2N})$ has trivial U^0_ϵ-action and we have (ab)used the usual notation for (p).

Following the definition of the U^0_ϵ-action on Λ_ϵ and Λ'_ϵ the only weights appearing in $\Lambda_\epsilon \otimes \Lambda'_\epsilon$ are of the form

$$(\beta_1 + \ldots + \beta_k) + (\beta'_1 + \ldots + \beta'_{k'})$$

where the sets $\{\beta_1, \ldots, \beta_k\}$ and $\{\beta'_1, \ldots, \beta'_{k'}\}$ each contain distinct positive roots. Thus we are looking for such expressions which are elements of ℓQ. Since we are assuming $\ell > 2d$ this can only occur if all weights are zero, that is we are in degree zero part of $\Lambda_\epsilon \otimes \Lambda'_\epsilon$. Since the polynomial algebra is generated in degree two the only non-zero points on the first page of the spectral sequence (3.11) have even degree so, in particular, it collapses on the first page. Thus we have an isomorphism of graded vector spaces

$$\mathrm{Gr}(\mathrm{Ext}^*_T(\mathbb{C},\mathbb{C})) \cong S(V^*_{2N}) \tag{3.12}$$

since the only surviving part of the left-hand side of (3.11) is $\mathbb{C} \otimes S(V^*_{2N})$.

By Corollary 2.18 the natural trangression

$$\tau : V^*_{2N} \cong \mathrm{Ext}^1_{S(V^*_{2N})}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ext}^2_{B_{2N}}(\mathbb{C},\mathbb{C})$$

has image central in $\operatorname{Ext}^*_{B_{2N}}(\mathbb{C},\mathbb{C})$. This map is also a U^0_ϵ-module map, see Lemma 4.9. Thus, since V^*_{2N} has trivial action,

$$\operatorname{im}\tau \subseteq (\operatorname{Ext}^2_{B_{2N}}(\mathbb{C},\mathbb{C}))^{U^0_\epsilon} \cong \operatorname{Ext}^2_T(\mathbb{C},\mathbb{C})\,.$$

Therefore the subalgebra, R, generated by this image is a quotient of $S(V^*_{2N})$. We also have $\mathrm{Gr}R \subseteq S(V^*_{2N})$ by (3.12). Proposition 3.3, however, shows this is an isomorphism so $S(V^*_{2N}) \cong \operatorname{Ext}^*_T(\mathbb{C},\mathbb{C})$ as required.

We can also prove a result on the cohomology of finite dimensional $\mathcal{O}_\epsilon[G]$-modules.

Theorem 3.8. *Suppose ℓ satisfies Hypothesis 1 and Hypothesis 7. For any finite dimensional $\mathcal{O}_\epsilon[G]$-module M, $\operatorname{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},M)$ is a finitely generated $\operatorname{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$-module.*

Proof. Again by Lemma 1.32 it is enough to prove this for T. By Lemma 3.6 we have

$$\operatorname{Ext}^*_T(\mathbb{C},M) \cong \operatorname{Ext}^*_S(\mathbb{C},M)^{U^0_\epsilon}\,.$$

Therefore, by Lemma 3.5 we have a spectral sequence

$$E^*_1(M) = (\operatorname{Ext}^{p+q}_{\mathrm{Gr}S}(\mathbb{C},\mathbb{C})_{(p)} \otimes M)^{U^0_\epsilon} \Longrightarrow \operatorname{Ext}^{p+q}_T(\mathbb{C},M)\,. \tag{3.13}$$

By Lemmas 2.19 and 3.4 this spectral sequence is compatible with right $\operatorname{Ext}^*_T(\mathbb{C},\mathbb{C})$-module structure. By Proposition 3.3, the spectral sequence (3.13) becomes

$$(\Lambda_\epsilon \otimes \Lambda'_\epsilon \otimes M)^{U^0_\epsilon} \otimes S(V^*_{2N}) \Longrightarrow \operatorname{Ext}^*_T(\mathbb{C},M)\,.$$

As M is finite dimensional and, by the proof of Theorem 3.7, $E^*_1(\mathbb{C}) \cong S(V^*_{2N})$ it follows that $E^*_1(M)$ is a Noetherian $E^*_1(\mathbb{C})$-module.

We have for $r \gg 0$, $E^*_r(M) \cong E^*_\infty(M)$ so, as a subquotient of $E^*_1(M)$, $E^*_\infty(M)$ is a Noetherian $E^*_1(\mathbb{C}) = E^*_\infty(\mathbb{C})$-module (in the proof of Theorem 3.7 we saw the spectral sequence $E^*_1(\mathbb{C})$ collapses on the first page).

The result now follows from standard commutative algebra.

In the following chapter we'll need a generalisation of the previous results. We begin with a couple of definitions.

Definition 23. Let $\Lambda^{(t)}_\epsilon$ be the subalgebra of Λ_ϵ generated by $e_{\beta_{t+1}}, \dots e_{\beta_N}$. Using the notation following (3.2) let T_t be the subalgebra of B_{2N} generated by $\bar{E}_{\beta_{t+1}}, \dots \bar{E}_{\beta_N}$.

Lemma 3.9. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 1. Then $(\Lambda^{(t)}_\epsilon)^{U^0_\epsilon} = \mathbb{C}.1$.*

Proof. For ℓ a prime number this is [47, Lemma 12.10]. This is adapted to general ℓ satsifying the hypotheses in the proof of [39, Theorem 2.5].

Corollary 3.10. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 1.*
(i) There is a graded algebra isomorphism

$$\mathrm{Ext}^*_{T_t}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}[X_{t+1},\dots,X_N]\otimes\Lambda^{(t)}_\epsilon,$$

where $deg(X_i) = 2$ and $deg(e_{\beta_i}) = 1$ for all $t+1 \le i \le N$. Moreover e_{β_i} has U^0_ϵ-weight $-\beta_i$ and X_i has trivial U^0_ϵ-weight.
(ii) For $0 \le j \le N$ there is an isomorphism of graded algebras

$$\mathrm{Ext}^*_{U_w(1)}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}[X_1,\dots X_j]$$

where $j = \ell(w)$ and $deg(X_i) = 2$ for $1 \le i \le j$.

Proof. Part (i) is of course just a minor modification of Proposition 3.3. Part (ii) will now follow using the arguments of Theorem 3.7 if we can show that the cohomology ring $\mathrm{Ext}^*_{U_w(1)}(\mathbb{C},\mathbb{C})$ is commutative. We prove this by downward induction on $\ell(w)$, the case $\ell(w) = N$ being known by Proposition 2.7, since $U_{w_0}(1) \cong U_\epsilon^{\geq 0}$ is a Hopf algebra.

Let $w \in W$ and fix a reduced expression for w,

$$w = s_{i_1}\dots s_{i_j}.$$

We can extend this reduced expression to a reduced expression for w_0,

$$w_0 = s_{i_1}\dots s_{i_j}s_{i_{j+1}}\dots s_{i_N}.$$

For t between j and N let $w_t = s_{i_1}\dots s_{i_t}$ and let U_t equal $U_{w_t}(1)$ and let $A_t = A_{w_t}(1)$.

The discussion following Lemma 1.11 shows that A_{w_t} has a filtration induced from $U_\epsilon^{\geq 0}$. Moreover this induces a filtration on A_t since we have factored out homogeneous central elements as in the discussion following (3.2). Let S_t be the associated graded ring of $A_t(1)$. It's clear that the rings S_t are truncated quantised polynomial rings and that the triple $(k[\bar{E}_{\beta_{t+1}}], S_{t+1}, S_t)$ satisfies the conditions of Lemma 2.15. Therefore we see that

$$\mathrm{res} : \mathrm{Ext}^*_{S_{t+1}}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ext}^*_{S_t}(\mathbb{C},\mathbb{C})$$

is onto. By Lemma 2.20 there is a map of spectral sequences

$$\begin{array}{ccc} E_1^{p,q} = \mathrm{Ext}^{p+q}_{S_{t+1}}(\mathbb{C},\mathbb{C})^{U^0_\epsilon}_{(p)} & \Longrightarrow & \mathrm{Ext}^{p+q}_{U_{t+1}}(\mathbb{C},\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ {}'E_1^{p,q} = \mathrm{Ext}^{p+q}_{S_t}(\mathbb{C},\mathbb{C})^{U^0_\epsilon}_{(p)} & \Longrightarrow & \mathrm{Ext}^{p+q}_{U_t}(\mathbb{C},\mathbb{C}). \end{array}$$

By Lemma 3.9 the bound $\ell > h$ ensures that $\Lambda_\epsilon^{(t)}$ has no non-trivial U_ϵ^0-fixed points. So using (an appropriate version of) part (i) and following the proof of Theorem 3.7 these spectral sequences must collapse. Therefore

$$\text{res} : \text{Ext}^*_{U_{t+1}}(\mathbb{C},\mathbb{C}) \longrightarrow \text{Ext}^*_{U_t}(\mathbb{C},\mathbb{C})$$

is onto. Since $\text{Ext}^*_{U_{t+1}}(\mathbb{C},\mathbb{C})$ is commutative by induction we are done.

Remark 3.11. Letting $w = w_0$ we obtain the following theorem, due to Ginzburg and Kumar.

Theorem 3.12. *[39, Theorem 2.5] Suppose $\ell > h$ and ℓ satisfies Hypothesis 1. Then* $\text{Ext}^{odd}_{U_\epsilon^{\geq 0}}(\mathbb{C},\mathbb{C}) = 0$ *and there is a graded algebra isomorphism*

$$\text{Ext}^{2*}_{U_\epsilon^{\geq 0}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}[X_1,\ldots,X_N].$$

3.2 An improved bound.

The previous section determined the ring $\text{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ for sufficiently large ℓ, showing in particular that it is a finitely generated commutative algebra. In this section we show that $\text{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ continues to be a finitely generated graded commutative algebra under less restrictive conditions on ℓ. The same arguments will also work for $\text{Ext}^*_{U_\epsilon^{\geq 0}}(\mathbb{C},\mathbb{C})$.

Lemma 3.13. *Let $\alpha,\beta,\gamma \in \Delta^+$, not all equal. Define c_i by*

$$\alpha+\beta+\gamma = \sum_{i=1}^{r} c_i\alpha_i.$$

Then if $\ell \geq 5$ there exists an integer $1 \leq j \leq r$ such ℓ does not divide c_j. Moreover if G consists only of components of type A and D the same is true for $\ell = 3$.

Proof. It is clear that it is enough to prove the lemma for G simple. We proceed by case-by-case analysis using [12, Planches I-IX].

Type A: In this case the positive roots have the following form:

$$\sum_{k=i}^{j} \alpha_k \quad 1 \leq i \leq j \leq r. \tag{3.14}$$

We let $i_0 = i_0(\alpha,\beta,\gamma)$ be the least i for which $c_i \neq 0$ and let $j_0 = j_0(\alpha,\beta,\gamma)$ be the greatest j for which $c_j \neq 0$. Since α,β,γ are not all equal it follows from (3.14) that one of c_{i_0} and c_{j_0} must be less than 3.

Type B: In this case the positive roots have the following forms:

$$\sum_{k=i}^{j} \alpha_k \qquad 1 \le i \le j \le r; \tag{3.15}$$

$$\sum_{k=i}^{j} \alpha_k + 2\sum_{k=j+1}^{r} \alpha_k \qquad 1 \le i \le j < r. \tag{3.16}$$

Observe from (3.15) and (3.16) that $c_1 \le 3$ so if $\ell \ge 5$ we can assume $c_1 = 0$ and therefore reduce to the rank $r-1$ case. This reduces us to the B_2 case which, by the above observation, is clear.

Type C: In this case the positive roots have the following forms:

$$\sum_{k=i}^{j} \alpha_k \qquad 1 \le i \le j < r; \tag{3.17}$$

$$\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r-1} \alpha_k + \alpha_r \qquad 1 \le i \le j \le r. \tag{3.18}$$

Observe from (3.17) and (3.18) that $c_r \le 3$. Hence, for $\ell \ge 5$, we can assume $c_r = 0$ and so reduce to the A_{r-1} case which we have already checked.

Type D: In this case the positive roots have the following forms:

$$\sum_{k=i}^{j} \alpha_k \qquad 1 \le i \le j < r; \tag{3.19}$$

$$\sum_{k=i}^{r-2} \alpha_k + \alpha_r \qquad 1 \le i < r; \tag{3.20}$$

$$\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r-2} \alpha_k + \alpha_{r-1} + \alpha_r \qquad 1 \le i \le j < r \tag{3.21}$$

Observe from (3.19), (3.20) and (3.21) that $c_r \le 3$ so, for $\ell \ge 5$ we can reduce to the case where $c_r = 0$, that is the A_{r-1} which we have already checked.

Suppose now $\ell = 3$. Suppose that $c_r = 3$. Then we can only have roots from (3.20) and (3.21). It is clear that if α, β, γ all come from (3.20) then we can argue as in the type A case. Hence $c_{r-1} \ne 0$ and we can assume that $c_{r-1} = 3$. Thus α, β, γ all come from (3.21). Let $i_0 = i_0(\alpha, \beta, \gamma)$ be the least i such that c_i is non-zero. It's clear that we must have $c_{i_0} = 3$. But now we can assume without loss of generality that $\alpha \le \beta < \gamma$ in the usual dominance ordering in P (the case $\alpha < \beta \le \gamma$ is similar). In this case let $j_0 = j_0(\alpha, \beta, \gamma)$ be the least integer such that the coefficient of α_{j_0} in β and γ differ. Then we must have $c_{j_0} = 4$, a contradiction. Therefore we must have that $c_r = 0$. The result now follows by induction.

Type E_6: We refer the reader to [12, Planche V] for the list of positive roots having a coefficient greater than 2 in their description as a sum of simple roots. We observe from this list that the coefficient of α_6 for any positive root is at most 1. Therefore, for $\ell \ge 5$, we can assume $c_6 = 0$. This reduces us to the D_5 case which we have checked.

Type E_7: As in the E_6 case [12, Planche VI] contains a list of positive roots having a coefficient greater than 2. Again we observe that the coefficient of α_7 for any positive root is at most 1. For $\ell \geq 5$ this allows to reduce to E_6 case which is checked.

Type E_8: From [12, Planche VII] we see that the coefficient of α_8 for any positive root is at most 1 except for the positive root $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. Hence, for $\ell \geq 7$ we can reduce to the E_7 case. For $\ell = 5$ we observe that if $c_8 = 5$ we must have, without loss of generality, that both α and β equal $\tilde{\alpha}$. It follows that $c_2 \geq 6$. So if c_2 is divisible by 5 the coefficient of α_2 in γ must be at least 4. A quick look at [12, Planche VII] shows this is impossible.

Type F_4: From [12, Planche VIII] we see that the coefficient of α_1 for any positive root is at most 1 except for the positive root $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Hence, for $\ell \geq 7$, we can reduce to the C_3 case. If $\ell = 5$ then, without loss of generality, we must have that both α and β equal $\tilde{\alpha}$. Therefore $c_2 \geq 6$ and so the coefficient of α_2 in γ must be at least 4. We see in [12, Planche VIII] that this is impossible.

Type G_2: From [12, Planche IX] we see

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}. \tag{3.22}$$

Hence the coefficient of α_2 is at most 1 except for $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$. Therefore, for $\ell \geq 7$, the lemma follows. If $\ell = 5$ then, without loss of generality, we must have both α and β equal to $\tilde{\alpha}$. We see then that the α_1 coefficient of γ must be at least 4 which, by (3.22), is impossible.

Remark: The bound in Lemma 3.13 for ℓ satisfying Hypothesis 1 is the best possible. Indeed suppose $\ell = 3$. We list triples of positive roots representing α, β and γ which show that, in this situation, the lemma is false.

B_r	$(\alpha_1 + \ldots \alpha_r, \alpha_1 + 2\alpha_2 + \ldots 2\alpha_r, \alpha_1)$
C_r	$(2\alpha_1 + \ldots 2\alpha_{r-1} + \alpha_r, \alpha_1 + \ldots \alpha_r, \alpha_r)$
E_6, E_7 and E_8	$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_4 + \alpha_5)$
F_4	$(\alpha_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)$.

Following Lemma 3.13 we define $c(G) \in \mathbb{N}$ as follows:

$$c(G) = \begin{cases} 3 & \text{if } G \text{ consists of types } A \text{ and } D \text{ only,} \\ 5 & \text{otherwise.} \end{cases}$$

Theorem 3.14. *Assume $\ell \geq c(G)$ and that ℓ satisfies Hypothesis 1. Then $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is a finitely generated, graded commutative $\mathbb{C}$-algebra. Moreover, for any finite dimensional $\mathcal{O}_\epsilon[G]$-module M, $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C}, M)$ is a finitely generated $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$-module.*

Proof. By (3.11) we see, for ℓ satisyfing Hypothesis 1, that we have a multiplicative spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{B_{2N}}(\mathbb{C},\mathbb{C})^{U^0}_{(p)} \cong (\Lambda_\epsilon \otimes \Lambda'_\epsilon \otimes S(V^*_{2N}))^{U^0}_{(p)} \Longrightarrow \mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C}), \tag{3.23}$$

where Λ_ϵ and Λ'_ϵ are quantum exterior algebras with generators $\{f_\beta : \beta \in \Delta^+\}$ and $\{e_\beta : \beta \in \Delta^+\}$ respectively and V_{2N} is a $\mathbb{C}$-vector space of dimension $2N$. Here the elements f_β and e_β have cohomological degree 1 and U^0-weight β whilst the generators of $S(V^*_{2N})$ have cohomological degree 2 and trivial U^0-weight. We claim that the generators, $\{X_1, \dots, X_{2N}\}$, of the algebra $S(V^*_{2N})$ are permanent cycles, that is $0 \neq X_i \in E^{p,q}_\infty$.

By definition the differential of the spectral sequence (3.23) has degree one, that is it raises the cohomological degree of elements by one.

Since e_β and f_β are the only elements of $\mathrm{Ext}^{p+q}_{B_{2N}}(\mathbb{C},\mathbb{C})$ of cohomological degree one it follows from Hypothesis 1 that $E_1^{p,q}$ contains no elements of cohomological degree one. Therefore X_i can never be a coboundary.

The elements of $E_1^{p,q}$ having cohomological degree three have two possible types of weight: α for $\alpha \in \Delta^+$ or $\alpha+\beta+\gamma$ for $\alpha,\beta,\gamma \in \Delta^+$, not all equal. Arguing as above we see that elements of the first type cannot be fixed under the U^0-action, so suppose we are in the second case. Define the integers c_i as follows:

$$\alpha+\beta+\gamma = \sum_{i=1}^r c_i\alpha_i.$$

If an element with this weight is fixed under the U^0-action then ℓ divides c_i for $1 \leq i \leq r$. Indeed K_{ϖ_i} acts on such an element as multiplication by $\epsilon^{(\alpha+\beta+\gamma,\varpi_i)} = \epsilon^{d_i c_i}$. As ℓ is prime to d_i and ϵ is a primitive ℓth root of unity it follows that $\epsilon^{d_i c_i} = 1$ if and only if ℓ divides c_i. By Lemma 3.13 for $\ell \geq c(G)$ this is impossible and so $E_1^{p,q}$ contains no elements of cohomological degree three. Therefore each X_i is a permanent cycle as claimed.

It follows from the multiplicative properties of the spectral sequence that the elements X_i generate an algebra, R, isomorphic to a quotient of $S(V_{2N}^*)$ in $E_\infty^{p,q}$, a noetherian algebra. Moreover the action of R on $E_\infty^{p,q}$ is induced from the action of $S(V_{2N}^*)$ on $E_1^{p,q}$. Since the latter is a finite $S(V_{2N}^*)$-module and $E_\infty^{p,q}$ is subquotient of $E_1^{p,q}$ it follows that $E_\infty^{p,q}$ is a finite R-module, and so a finitely generated algebra.

By (3.23) there is an algebra isomorphism $\mathrm{Gr}(\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})) \cong E_\infty^{*,*}$. It therefore follows that the graded commutative algebra $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is finitely generated. That it is graded commutative follows since $\mathcal{O}_\epsilon[G]$ is a Hopf algebra, [39, Corollary 5.6].

The second statement of the theorem can now be proved in the same manner as Theorem 3.8.

Remark 3.15. Theorem 3.14 does not yield the algebra structure of $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ so will not be of much use to us later. However it does seem plausible that the Krull dimension of $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is $2N$ for $\ell \geq c(G)$. In fact for $\ell > h$ the Krull dimension of $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C},\mathbb{C})$ is $2N$, see Remark 4.16.

3.3 Additional remarks

1. Theorem 3.7 is a simple extension of [39, Theorem 2.5]. We do present a number of details which were either ignored or omitted in [39].

2. Theorem 3.14 adds evidence to the hope that the cohomology ring of a finite dimensional Hopf algebra is finitely generated. As noted in Remark 2.25 it has been proved in [36] that all finite dimensional (commutative) or cocommutative Hopf algebras have a finitely generated cohomology ring. Of course the proof of Theorem 3.14 works equally well for the algebra $U_\epsilon^{\geq 0}$ or indeed for any of the algebras $U_w(1)$ introduced in Definition 13, Section 1.6.

Chapter 4

Complexity of the algebras $\mathcal{O}_\epsilon[G](g)$ and $U_\epsilon^{\geq 0}(b)$

In this chapter we determine the complexity of the algebras $\mathcal{O}_\epsilon[G](g)$ and $U_\epsilon^{\geq 0}(b)$. For $b \in X_{w,e}$ the idea for $U_\epsilon^{\geq 0}(b)$ is simple: introduce "PBW-type" basis elements inductively, using the semisimplicity of $U_w(b)$, known by Corollary 1.30, as a starting point. Each time we introduce a basis element we have a corresponding Lyndon-Hochschild-Serre spectral sequence, allowing us to perform the inductive step. This is a little disingenuous since the spectral sequence takes coefficients in modules related to simple $U_\epsilon^{\geq 0}(b)$-modules, objects which are definitely not trivial (in the augmented algebras sense). Rather like the previous chapter we finesse this point by using the filtrations introduced in Section 1.3 and passing to the associated graded rings where we can prove our results. The results of Section 2.4 then allow us to pull these up to $U_\epsilon^{\geq 0}(b)$. We get our hands on the complexity of $\mathcal{O}_\epsilon[G](g)$ by using the results for $U_\epsilon^{\geq 0}(b)$ and the maps introduced in Theorem 1.32. In both cases we see that the complexity has a very simple and beautiful description in terms of the length function on the Weyl group.

Since we'll be repeatedly using Hypothesis 3 in this chapter let's recall what it says.

Hypothesis 3. *The positive integer ℓ is odd and good.*

4.1 The complexity of $U_\epsilon^{\geq 0}$ factors.

Recall from (1.2) the cell decomposition of B^- given by

$$B^- = \coprod_{u \in W} X_{w,e},$$

where $X_{w,e} = B^+wB^+ \cap B^-$. By Theorem 1.28 if $b, b' \in X_{w,e}$ then there is an isomorphism of algebras

$$U_\epsilon^{\geq 0}(b) \cong U_\epsilon^{\geq 0}(b').$$

Combining the definition of support varieties with Theorem 1.29(ii) and Corollary 2.29 we obtain the following lemma.

Lemma 4.1. *Let $b \in B^-$ and let L be any simple $U_\epsilon^{\geq 0}(b)$-module. Then*

$$\mathrm{cx}(U_\epsilon^{\geq 0}(b)) = \mathrm{cx}_{U_\epsilon^{\geq 0}(b)}(L).$$

Fix an element $w \in W$ and suppose $\ell(w) = t$. Let

$$w = s_{i_1} \dots s_{i_t}$$

be a reduced expression for w. Extend $w \in W$ to w_0 by

$$w_0 = s_{i_1} \dots s_{i_t} s_{i_{t+1}} \dots s_{i_N}. \tag{4.1}$$

Let w_j denote the truncation of w_0 to the first j simple reflections given in (4.1),

$$w_j = s_{i_1} \dots s_{i_j}. \tag{4.2}$$

Recall the definitions, (1.19) and (1.20), of the algebras $A_{w_j}(b)$ and $U_{w_j}(b)$ for $b \in B^-$. For ease of notation let $A_j(b)$ and $U_j(b)$ denote the algebras $A_{w_j}(b)$ and $U_{w_j}(b)$ respectively.

It is clear that, for $0 \leq j \leq N$, $A_j(1) \subset U_j(1)$ is a normal extension such that

$$U_j(1)//A_j(1) \cong U_\epsilon^0$$

as algebras and so it follows that the algebra A_j inherits a U_ϵ^0-action. As a result there is a U_ϵ^0-action on $\mathrm{Ext}^*_{A_j(1)}(\mathbb{C}, \mathbb{C})$ for $0 \leq j \leq N$. By abuse of notation we can describe the action of U_ϵ^0 on cohomology in terms of the root lattice Q. For example we will say $\zeta \in \mathrm{Ext}^*_{A_j(1)}(\mathbb{C}, \mathbb{C})$ has U_ϵ^0-weight $\gamma \in Q$ if

$$K_\lambda.\zeta = \epsilon^{(\gamma,\lambda)}\zeta$$

for all $K_\lambda \in U_\epsilon^0$. Of course, since ϵ is an ℓth root of unity the U_ϵ^0-weight of ζ can be taken to be in $Q_\ell = Q/\ell Q$.

Lemma 4.2. *Assume that $\ell > h$ and that ℓ satisfies hypothesis 1. Then for $0 \leq j \leq N$ the possible U_ϵ^0-weights of $\mathrm{Ext}^*_{A_j(1)}(\mathbb{C}, \mathbb{C})$ have the form*

$$\sum_{k=1}^{j} c_k \beta_k$$

where $c_k \in \{0, -1\}$.

Proof. By Lemma 2.19 there is a spectral sequence

$$\mathrm{Ext}^*_{\mathrm{Gr}A_j(1)}(\mathbb{C}, \mathbb{C}) \Longrightarrow \mathrm{Ext}^*_{A_j(1)}(\mathbb{C}, \mathbb{C}) \tag{4.3}$$

which, by Lemma 3.5 preserves the U_ϵ^0-actions. Arguing as in Corollary 3.10 we see that the U_ϵ^0-weights on the left hand side of (4.3) are of the form $\sum_{k=1}^{j} c_k\beta_k$ where $c_k \in \{0, -1\}$. Therefore only these can be weights on the right hand side of (4.3), as required.

The proof of the following lemma is the same as that for Lemma 3.6.

Lemma 4.3. *Let M be a $U_j(1)$-module for some j, $1 \leq j \leq N$. Then there is a natural isomorphism*

$$\mathrm{Ext}^*_{A_j(1)}(\mathbb{C}, M)^{U_\epsilon^0} \cong \mathrm{Ext}^*_{U_j(1)}(\mathbb{C}, M).$$

Moreover this is an isomorphism of $\mathrm{Ext}^*_{U_j(1)}(\mathbb{C}, \mathbb{C})$-modules.

We now prove a crucial lemma. This is a standard Hopf algebraic result in the case $j = N$; the problem is to extend this to the algebras $U_j(1)$, which are in general not Hopf algebras.

Lemma 4.4. *For $U_\epsilon^{\geq 0}(b)$-modules M and N there is a graded isomorphism*

$$\mathrm{Ext}^*_{U_j(b)}(M, N) \xrightarrow{\sim} \mathrm{Ext}^*_{U_j(1)}(\mathbb{C}, N \otimes M^*),$$

for any j, $0 \leq j \leq N$.

Proof. Since $U_\epsilon^{\geq 0}(1)$ is a free $U_j(1)$-module and $U_\epsilon^{\geq 0}$ is a Hopf algebra we have, by [10, Proposition 3.3.1(i)] and Lemma 1.27, isomorphisms

$$\begin{aligned} \mathrm{Ext}^*_{U_j(1)}(\mathbb{C}, N \otimes M^*) &\cong \mathrm{Ext}^*_{U_\epsilon^{\geq 0}(1)}\left(U_\epsilon^{\geq 0}(1) \otimes_{U_j(1)} \mathbb{C}, N \otimes M^*\right) \\ &\cong \mathrm{Ext}^*_{U_\epsilon^{\geq 0}(b)}\left((U_\epsilon^{\geq 0}(1) \otimes_{U_j(1)} \mathbb{C}) \otimes M, N\right) \end{aligned}$$

and similarly

$$\mathrm{Ext}^*_{U_j(b)}(M, N) \cong \mathrm{Ext}^*_{U_\epsilon^{\geq 0}(b)}\left(U_\epsilon^{\geq 0}(b) \otimes_{U_j(b)} M, N\right).$$

Therefore it is enough to show that $U_\epsilon^{\geq 0}(b) \otimes_{U_j(b)} M$ and $(U_\epsilon^{\geq 0}(1) \otimes_{U_j(1)} \mathbb{C}) \otimes M$ are isomorphic as $U_\epsilon^{\geq 0}(b)$-modules.

Consider first the $U_\epsilon^{\geq 0}$-modules $U_\epsilon^{\geq 0} \otimes M$ and $(U_\epsilon^{\geq 0} \otimes \mathbb{C}) \otimes M$, where, here and elsewhere, an unadorned tensor product signifies $\otimes_{\mathbb{C}}$. For $u, u' \in U_\epsilon^{\geq 0}$ and $m \in M$ we have

$$u.(u' \otimes m) = uu' \otimes m$$

and

$$u.((u' \otimes 1) \otimes m) = \sum_u (u_{(1)}u' \otimes 1) \otimes u_{(2)}.m$$

respectively. There are maps

$$U_\epsilon^{\geq 0} \otimes M \underset{\phi}{\overset{\theta}{\rightleftarrows}} (U_\epsilon^{\geq 0} \otimes \mathbb{C}) \otimes M$$

where

$$\theta(u \otimes m) = \sum_u (u_{(1)} \otimes 1) \otimes u_{(2)}.m,$$

and

$$\phi((u \otimes 1) \otimes m) = \sum_u u_{(1)} \otimes S(u_{(2)}).m.$$

It is an easy exercise to see that θ and ϕ are mutually inverse and are $U_\epsilon^{\geq 0}$-homomorphisms. (For a Hopf algebra such maps can be used to show that the tensor product of a projective module by an arbitrary module is again projective, [64, Theorem 1.9.4].) Moreover, by considering the action of $\mathcal{O}[B^-]$, we see these maps induce $U_\epsilon^{\geq 0}(b)$-isomorphisms

$$U_\epsilon^{\geq 0}(b) \otimes M \underset{\phi}{\overset{\theta}{\rightleftarrows}} (U_\epsilon^{\geq 0}(1) \otimes \mathbb{C}) \otimes M.$$

Therefore there is an induced surjective map

$$\theta : U_\epsilon^{\geq 0}(b) \otimes M \longrightarrow (U_\epsilon^{\geq 0}(1) \otimes_{U_j(1)} \mathbb{C}) \otimes M.$$

We claim that this is $U_j(b)$-balanced, or, in other words, that for all $j' \leq j$

$$\theta(uE_{\beta_{j'}} \otimes m) = \theta(u \otimes E_{\beta_{j'}}.m), \tag{4.4}$$

and for all $\lambda \in P$

$$\theta(uK_\lambda \otimes m) = \theta(u \otimes K_\lambda.m). \tag{4.5}$$

Recall, by Lemma 1.4,

$$\Delta(E_{\beta_{j'}}) = E_{\beta_{j'}} \otimes 1 + K_{\beta_{j'}} \otimes E_{\beta_{j'}} + \sum_v x_v \otimes y_v, \tag{4.6}$$

where $x_v \in U_\epsilon^{\geq 0}$ has U_ϵ^0-weight $\gamma_v \in Q$ where $w_{j'-1}^{-1}(\gamma_v) < 0$, using the notation of (4.2). Since

$$\{\beta \in \Delta^+ : w_{j'-1}^{-1}(\beta) < 0\} = \{\beta_1, \dots \beta_{j'-1}\}$$

we see that x_v is a sum of monomials $E_{\beta_N}^{r_N} \dots E_{\beta_1}^{r_1}$ where there exists $j'' < j'$ such that $r_{j''} > 0$.

By (4.6) we have

$$\theta(E_{\beta_{j'}} \otimes m) = (E_{\beta_{j'}} \otimes_{U_j(1)} 1) \otimes m + (K_{\beta_{j'}} \otimes_{U_j(1)} 1) \otimes E_{\beta_{j'}}.m + \sum_v (x_v \otimes_{U_j(1)} 1) \otimes y_v.m.$$

Clearly $E_{\beta_{j'}} \otimes_{U_j(1)} \mathbb{C} = 0$ and the above comments show that $x_v \otimes_{U_j(1)} \mathbb{C} = 0$ also. Therefore (4.4) follows. Using $\Delta(K_\lambda) = K_\lambda \otimes K_\lambda$ equation (4.5) follows similarly. Therefore there is a surjective map

$$\theta : U_\epsilon^{\geq 0}(b) \otimes_{U_j(b)} M \longrightarrow (U_\epsilon^{\geq 0}(1) \otimes_{U_j(1)} \mathbb{C}) \otimes M. \tag{4.7}$$

Since both sides have the same dimension this is an isomorphism and so we have

$$\operatorname{Ext}^*_{U_j(b)}(M, N) \cong \operatorname{Ext}^1_{U_j(1)}(\mathbb{C}, N \otimes M^*),$$

as required.

Remark 4.5. For this proposition to hold for $U_j^{\leq 0}(1)$ we must adjust the Hopf structure on $U_\epsilon^{\leq 0}(1)$: replace the usual (Δ, ν, S) by $(\tau \circ \Delta, \nu, S^{-1})$, where

$$\tau : U_\epsilon^{\leq 0}(1) \otimes U_\epsilon^{\leq 0}(1) \longrightarrow U_\epsilon^{\leq 0}(1) \otimes U_\epsilon^{\leq 0}(1)$$

is the twist map. This adjustment does not change our final results on complexity, since these depend only on the structure of $U_\epsilon^{\leq 0}(1)$ as an algebra .

Corollary 4.6. *Keep the above notation and suppose $b \in X_{w,e}$, where $w = w_t$. Let L and L' be simple $U_\epsilon^{\geq 0}(b)$-modules. Then*

$$\operatorname{Ext}^q_{U_t(1)}(\mathbb{C}, L' \otimes L^*) = \begin{cases} \mathbb{C} & \text{if } q = 0 \text{ and } L \cong L', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Combine Corollary 1.30 with Lemma 4.4.

Remember $\ell(w) = t$ and continue, as above, with notation following (4.1). Let $U = U_\epsilon^{\geq 0}(1)$ and let $A = A_N(1)$. Recall from Section 1.3 the "t-degree" filtration on $U_\epsilon^{\geq 0}$ (respectively on U_ϵ^+ by restriction), where we have

$$d_t(E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N} K_\lambda) = (k_{t+1}, \dots, k_N) \in \mathbb{N}^{N-t}.$$

It's clear that this filtration passes to a filtration on U (respectively on A) since the generators of the central subalgebra $\mathcal{O}[B^-]$ are homogeneous. We will denote the associated graded algebra by $\mathrm{Gr}^{(t)}U$ (respectively $\mathrm{Gr}^{(t)}A$). The following lemma is the analogue for U of Lemma 1.12.

Lemma 4.7. *Let U and A have the above "t-degree" filtration. Then*
i) The algebra $A_t(1)$ is normal subalgebra of $\mathrm{Gr}^{(t)}A$;
ii) The quotient $\mathrm{Gr}^{(t)}A//A_t(1)$ is isomorphic to the algebra with generators $\bar{E}_{\beta_i}$ for $t+1 \leq i \leq N$ and relations

$$\bar{E}_{\beta_i}^{\ell} = 0,$$

and, for $t+1 \leq i < j \leq N$,

$$\bar{E}_{\beta_i}\bar{E}_{\beta_j} = \epsilon^{(\beta_j,\beta_i)}\bar{E}_{\beta_j}\bar{E}_{\beta_i}.$$

That is, $\mathrm{Gr}^{(t)}A//A_t(1)$ is isomorphic to a truncated skew polynomial ring.
iii) The algebra $A_t(1)$ is a normal subalgebra of $U_t(1)$ and the algebra $\mathrm{Gr}^{(t)}A$ is a normal subalgebra of $\mathrm{Gr}^{(t)}U$.
iv) Both the quotient $U_t(1)//A_t(1)$ and the quotient $\mathrm{Gr}^{(t)}U//\mathrm{Gr}^{(t)}A$ are isomorphic to U_ϵ^0.

We let $T_t \equiv \mathrm{Gr}^{(t)}A//A_t(1)$ be the truncated polynomial ring above. Let V be a $\mathrm{Gr}^{(t)}U$-module. By Lemma 2.11 and Lemma 4.7, parts (i) and (ii), we have a spectral sequence

$$\mathrm{Ext}_{T_t}^p\left(\mathbb{C}, \mathrm{Ext}_{A_t(1)}^q(\mathbb{C}, V)\right) \Longrightarrow \mathrm{Ext}_{\mathrm{Gr}^{(t)}A}^{p+q}(\mathbb{C}, V). \tag{4.8}$$

Lemma 4.8. *Suppose V is a $\mathrm{Gr}^{(t)}U$-module on which E_{β_i} acts trivially for all $i > t$. Then the T_t-action on $\mathrm{Ext}_{A_t(1)}^*(\mathbb{C}, V)$ is trivial.*

Proof. Following Lemma 2.8 the action of E_{β_i} on $\mathrm{Ext}_{A_t(1)}^*(\mathbb{C}, V)$ can be calculated from the map of induced bar resolutions

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{1\otimes d_n} & \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} A_t(1)^{\otimes n+1} & \xrightarrow{1\otimes d_{n-1}} & \cdots & \xrightarrow{1\otimes \eta} & \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_n} & & & & \downarrow{\scriptstyle m} & & \\
\cdots & \xrightarrow{1\otimes d_n} & \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} A_t(1)^{\otimes n+1} & \xrightarrow{1\otimes d_{n-1}} & \cdots & \xrightarrow{1\otimes \eta} & \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} \mathbb{C} & \longrightarrow & 0
\end{array}$$

where $m : \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} \mathbb{C} \longrightarrow \mathrm{Gr}^{(t)}A \otimes_{A_t(1)} \mathbb{C}$ denotes right multiplication by E_{β_i}. Since $A_t(1) \hookrightarrow \mathrm{Gr}^{(t)}A$ is in degree zero, each component of these resolutions inherits a grading under which the boundary maps $1 \otimes d_n$ are homogeneous. Since the map m has positive degree it follows that without loss of generality we may assume that each θ_n has positive degree also.

Let $f \in \mathrm{Hom}_{A_t(1)}\left(A_t(1)^{\otimes n+1}, V\right)$ and define

$$\bar{f} \in \mathrm{Hom}_{\mathrm{Gr}^{(t)}A}\left(\mathrm{Gr}^{(t)}A \otimes_{A_t(1)} A_t(1)^{\otimes n+1}, V\right)$$

by

$$\bar{f}(x \otimes z) = x.f(z).$$

Then we have

$$(E_{\beta_i}.f)(z) = \bar{f}(\theta_n(1 \otimes z)).$$

But $\theta_n(1 \otimes z) = \sum x_i \otimes z_i$ where x_i has positive degree for each i. It follows from the hypothesis on V that we have

$$(E_{\beta_i}.f)(z) = \sum x_i.f(z_i) = 0,$$

as required.

By Lemma 4.7, parts (iii) and (iv), there is an action of U_ϵ^0 on $\mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, V)$, on $\mathrm{Ext}^*_{T_t}\left(\mathbb{C}, \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, V)\right)$ and on $\mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, V)$.

Lemma 4.9. *The spectral sequence (4.8) preserves the U_ϵ^0-action.*

Proof. The construction in [7, Chapter VI, Section 2] describes a double complex for which the associated spectral sequence is (4.8). It is clear that U_ϵ^0-action on this double complex preserves the usual filtration and that the U_ϵ^0-action on the E_2-term coincides with the natural action induced from U_ϵ^0-action on $\mathrm{Hom}_{T_t}\left(\mathbb{C}, \mathrm{Hom}_{A_t(1)}(\mathbb{C}, V)\right)$.

Now let M be a U-module and filter it by $F^i(M) = M$ for all $i \geq 0$. Therefore $\mathrm{Gr}M = F^0(M)$, so, as an abelian group, $\mathrm{Gr}M$ is isomorphic to M. Since the filtration on M is compatible with the t-degree filtration on A it follows that $\mathrm{Gr}M$ is a $\mathrm{Gr}^{(t)}A$-module. For $j > t$, $E_{\beta_j} \in F^iA$ for some $i \geq 1$ so we have that $\bar{E}_{\beta_j}.\mathrm{Gr}M \in F^i(M)/F^{i-1}(M) = 0$. On the other hand any element of $A_t(1)$ is in zeroth degree of filtration of A so it follows that the action of $A_t(1) \subseteq \mathrm{Gr}^{(t)}A$ on $\mathrm{Gr}M$ is just the natural action of $A_t(1)$ on M. This describes the $\mathrm{Gr}^{(t)}A$-module $\mathrm{Gr}M$ which from now on we will denote by $\widetilde{M}$.

Since, for $j > t$, the action of E_{β_j} on $\widetilde{M}$ is trivial Lemma 4.8 applies and so, as a T_t-module, $\mathrm{Ext}^q_{A_t(1)}\left(\mathbb{C}, \widetilde{M}\right)$ is trivial. By Lemma 4.9 and the above, we deduce that there is a spectral sequence

$$(\mathrm{Ext}^p_{T_t}(\mathbb{C},\mathbb{C}) \otimes \mathrm{Ext}^q_{A_t(1)}(\mathbb{C}, M))^{U_\epsilon^0} \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{M}\right)^{U_\epsilon^0}. \tag{4.9}$$

In order to exploit (4.9) we need some new notation and a short lemma. Let L be a U_ϵ^0-module and, for $\beta \in Q$, define

$$L^\beta \equiv \{l \in L : K_\lambda l = \epsilon^{(\beta,\lambda)} l\}.$$

In particular note that $L^0 = L^{U_\epsilon^0}$ and that L^β depends only on the coset of β modulo ℓQ. Since U_ϵ^0 is semisimple each L decomposes into a direct sum of such spaces.

Lemma 4.10. *Let M be a U-module. For $\beta \in Q$ and $1 \leq j \leq N$ there is an isomorphism*

$$\mathrm{Ext}^p_{A_j(1)}(\mathbb{C}, M)^\beta \cong \mathrm{Ext}^p_{A_j(1)}(\mathbb{C}, M \otimes \mathbb{C}_{-\beta})^{U_\epsilon^0}.$$

Proof. Since $U_j(1)$ is a skew group extension of $A_j(1)$ by $U_\epsilon^0 \cong \mathbb{C}(\mathbb{Z}_\ell^r)$ it follows from Lemma 2.10 that there is an isomorphism

$$\mathrm{Ext}^p_{A_j(1)}(\mathbb{C}, M)^\beta \cong \mathrm{Ext}^p_{U_j(1)}\left(\mathbb{C}, {}^\beta M\right)$$

where ${}^\beta M$ means that we take the $U_j(1)$-structure on M twisted by the character $-\beta$. However, it is immediate from the comultiplication on $U_\epsilon^{\geq 0}$ that ${}^\beta M \cong M \otimes \mathbb{C}_{-\beta}$. The result follows from Lemma 4.3.

Proposition 4.11. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let L be a simple $U_\epsilon^{\geq 0}(b)$-module. Then, in the above notation, $\mathbb{C}[X_{t+1}, \dots, X_N] \subseteq \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C},\mathbb{C})^{U_\epsilon^0}$ and $\mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right)^{U_\epsilon^0}$ is a finitely generated, faithful module over $\mathbb{C}[X_{t+1}, \dots, X_N]$.*

Proof. Combining the spectral sequence (4.9) with Corollary 3.10(i) we find we have a spectral sequence

$$(\mathbb{C}[X_{t+1}, \dots, X_N] \otimes \Lambda_q^{(t)} \otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, L \otimes L^*))^{U_\epsilon^0} \Longrightarrow \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right)^{U_\epsilon^0}. \tag{4.10}$$

By Corollary 3.10(i) each X_i has trivial U_ϵ^0-action so the left hand side of (4.10) becomes

$$\mathbb{C}[X_{t+1}, \dots, X_N] \otimes \left(\bigoplus_{\beta \in Q} (\Lambda_q^{(t)})^\beta \otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, L \otimes L^*)^{-\beta}\right). \tag{4.11}$$

By Lemma 4.10 we have an isomorphism

$$\mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, L\otimes L^*)^{-\beta} \cong \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, L\otimes L^*\otimes \mathbb{C}_\beta)^{U_\epsilon^0}.$$

By Lemma 4.3 and Corollary 4.6 we have that

$$\mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, L\otimes L^*\otimes \mathbb{C}_\beta)^{U_\epsilon^0} = \begin{cases} \mathbb{C} & \text{if } q=0 \text{ and } \mathbb{C}_{-\beta}\otimes L\cong L, \\ 0 & \text{otherwise.}\end{cases} \tag{4.12}$$

Therefore combining (4.11) and (4.12) the spectral sequence (4.10) is concentrated in the $q=0$ column and so, in particular, collapses. Thus there is an isomorphism

$$\mathrm{Ext}^*_{T_t}\left(\mathbb{C}, \mathrm{Hom}_{A_t(1)}(\mathbb{C}, L\otimes L^*)\right)^{U_\epsilon^0} \cong \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L\otimes L^*}\right)^{U_\epsilon^0} \tag{4.13}$$

given by inflation.

Consider the spectral sequence:

$$E_2^{**} = \mathrm{Ext}^*_{T_t}(\mathbb{C},\mathbb{C})\otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C},\mathbb{C}) \Longrightarrow \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C},\mathbb{C}). \tag{4.14}$$

By Corollary 3.10(i), the left hand side of this spectral sequence becomes

$$\mathbb{C}[X_{t+1},\dots,X_N]\otimes \Lambda_q^{(t)}\otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C},\mathbb{C}).$$

By Lemma 2.15 and Corollary 3.10(i) the inflation map of (4.14)

$$\mathbb{C}[X_{t+1},\dots,X_N]\otimes \Lambda_q^{(t)} \cong \mathrm{Ext}^*_{T_t}(\mathbb{C},\mathbb{C}) \longrightarrow \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C},\mathbb{C}) \tag{4.15}$$

is injective.

In particular, taking fixed points, we have that the inflation map

$$\mathbb{C}[X_{t+1},\dots,X_N] \cong \mathrm{Ext}^*_{T_t}(\mathbb{C},\mathbb{C})^{U_\epsilon^0} \longrightarrow \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C},\mathbb{C})^{U_\epsilon^0}$$

is injective.

By (4.11) and (4.12) we have that $\mathrm{Ext}^*_{T_t}\left(\mathbb{C}, \mathrm{Hom}_{A_t(1)}(\mathbb{C}, L\otimes L^*)\right)$ is a finite module over the algebra $\mathrm{Ext}^*_{T_t}(\mathbb{C},\mathbb{C})^{U_\epsilon^0} = \mathbb{C}[X_{t+1},\dots,X_N]$. Since, by Lemma 2.14, inflation preserves cohomological module structure the result follows from the previous two paragraphs.

The following proposition allows us to describe the complexity of the algebra $U_\epsilon^{\geq 0}(b)$.

Proposition 4.12. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$ where $\ell(w) = t$ and let L be a simple $U_\epsilon^{\geq 0}(b)$-module. Then $\mathbb{C}[X_{t+1}, \dots, X_N]$ is a subalgebra of $\mathrm{Ext}^*_{U_\epsilon^{\geq 0}}(\mathbb{C}, \mathbb{C})$ and, as a module over $\mathbb{C}[X_{t+1}, \dots, X_N]$, $\mathrm{Ext}^*_{U_\epsilon^{\geq 0}}(\mathbb{C}, L \otimes L^*)$ is finitely generated and faithful.*

Proof. Consider the spectral sequence (4.9) with $M = \mathbb{C}$:

$$(\mathrm{Ext}^p_{T_t}(\mathbb{C}, \mathbb{C}) \otimes \mathrm{Ext}^q_{A_t(1)}(\mathbb{C}, \mathbb{C}))^{U_\epsilon^0} \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0}. \tag{4.16}$$

By Corollary 3.10(i), the left hand side of this spectral sequence becomes

$$\mathbb{C}[X_{t+1}, \dots, X_N] \otimes (\Lambda_q^{(t)} \otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, \mathbb{C}))^{U_\epsilon^0}.$$

By Lemma 4.2 the U_ϵ^0-weights of $\mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, \mathbb{C})$ are of the form $\sum_{k=1}^t c_k \beta_k$ where $c_k \in \{0, -1\}$, and so it follows that the U_ϵ^0-weights of $\Lambda_q^{(t)} \otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, \mathbb{C})$ are of the form $\sum_{k=1}^N c_k \beta_k$ where $c_k \in \{0, -1\}$. Since $\ell > h$ it follows from Lemma 3.9 that there is a graded isomorphism

$$(\Lambda_q^{(t)} \otimes \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, \mathbb{C}))^{U_\epsilon^0} \cong \mathrm{Ext}^*_{A_t(1)}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0} \cong \mathrm{Ext}^*_{U_t(1)}(\mathbb{C}, \mathbb{C}).$$

Using Corollary 3.10(ii) the spectral sequence (4.16) becomes

$$\mathbb{C}[X_1, \dots, X_N] \Longrightarrow \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0},$$

where the degree of X_i is 2. In particular, since this is concentrated in even degrees it must collapse.

By Lemma 2.19 we have a spectral sequence, $E(M)$, for any finite dimensional $U_\epsilon^{\geq 0}(1)$-module, M,

$$E_1^{p,q}(M) = \mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{M}\right)_{(p)} \Longrightarrow \mathrm{Ext}^{p+q}_{A_N(1)}(\mathbb{C}, M).$$

Since, by Lemma 3.5, this commutes with action of U_ϵ^0, we obtain the following spectral sequence by taking fixed points and letting $M = \mathbb{C}$,

$$\mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0}_{(p)} \Longrightarrow \mathrm{Ext}^{p+q}_A(\mathbb{C}, \mathbb{C})^{U_\epsilon^0} \cong \mathrm{Ext}^{p+q}_{U_\epsilon^{\geq 0}(1)}(\mathbb{C}, \mathbb{C}).$$

By the above $\mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0}$ is concentrated in even degrees, so this spectral sequence must collapse. Proposition 4.11 shows that $\mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right)^{U_\epsilon^0}$ is a finitely generated module over a subalgebra $\mathbb{C}[X_{t+1}, \dots, X_N]$ of $\mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C})^{U_\epsilon^0}$. Using $E(L \otimes L^*)^{U_\epsilon^0}$, this immediately implies that $\mathrm{Ext}^*_{U_\epsilon^{\geq 0}(1)}(\mathbb{C}, L \otimes L^*)$ is a finitely generated $\mathbb{C}[X_{t+1}, \dots, X_N]$-module.

Consider the (cohomological) degree zero part of the spectral sequence

$$E_1^{p,q}(L \otimes L^*) = \mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right)_{(p)} \Longrightarrow \mathrm{Ext}^{p+q}_{A_N(1)}(\mathbb{C}, L \otimes L^*). \tag{4.17}$$

By (4.8) and Lemma 4.8 $\mathrm{Hom}_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right) \cong \mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*)$. Let $\beta \in P$. Using Lemma 4.4 and Lemma 4.10 and the fact that, by Theorem 1.29(i), L is a simple $U_t(b)$-module we have

$$\begin{aligned}
\mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*)^\beta &\cong \mathrm{Hom}_{U_t(1)}(\mathbb{C}, L \otimes L^* \otimes \mathbb{C}_{-\beta}) \\
&\cong \mathrm{Hom}_{U_t(b)}(\mathbb{C}_\beta \otimes L, L) \\
&\cong \mathrm{Hom}_{U_\epsilon^{\geq 0}(b)}(\mathbb{C}_\beta \otimes L, L) \\
&\cong \mathrm{Hom}_{A_N(1)}(\mathbb{C}, L \otimes L^*)^\beta .
\end{aligned}$$

Since $\mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*)$ is the sum of its U_ϵ^0-weight spaces it follows that every element of $\mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*)$ is a permanent cycle in (4.17).

Taking fixed points of (4.17) we have

$$\mathrm{Ext}^{p+q}_{\mathrm{Gr}^{(t)}A}\left(\mathbb{C}, \widetilde{L \otimes L^*}\right)_{(p)}^{U_\epsilon^0} \Longrightarrow \mathrm{Ext}^{p+q}_{U_\epsilon^{\geq 0}(1)}(\mathbb{C}, L \otimes L^*).$$

Combining (4.13) with Corollary 3.10(i) the left hand side of this becomes

$$(\mathbb{C}[X_{t+1}, \dots, X_N] \otimes \Lambda_q^{(t)} \otimes \mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*))^{U_\epsilon^0}. \tag{4.18}$$

By Corollary 3.10(i) and (4.15) we have an embedding

$$\mathbb{C}[X_{t+1}, \dots, X_N] \otimes \Lambda_q^{(t)} \subseteq \mathrm{Ext}^*_{\mathrm{Gr}^{(t)}A}(\mathbb{C}, \mathbb{C}).$$

Let $x \otimes f \in ((\mathbb{C}[X_{t+1}, \dots, X_N] \otimes \Lambda_q^{(t)}) \otimes \mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*))^{U_\epsilon^0}$. Since $E(L \otimes L^*)^{U_\epsilon^0}$ is an $E(\mathbb{C})^{U_\epsilon^0}$-module and $d_r(f) = 0$ for all r we have that

$$d_r(x \otimes f) = d_r(x) \otimes f \tag{4.19}$$

where the left d_r is taken in $E_r(L \otimes L^*)^{U_\epsilon^0}$ and the right d_r in $E_r(\mathbb{C})$.

Claim: Fix $r \in \mathbb{N}$ and let $\iota : \mathbb{C} \to L \otimes L^*$ be the natural inclusion. Suppose $p_s(X)$ is a non-zero homogenous polynomial of degree s in $E_r(\mathbb{C})^{U_\epsilon^0} = \mathbb{C}[X_{t+1}, \dots X_N]$. Then, in (4.17), taking note of (4.18), $0 \neq p_s(X) \otimes 1 \otimes \iota \in E_r(L \otimes L^*)^{U_\epsilon^0}$.

The case $r = 1$ is clear. Suppose the claim holds for $r' < r$. Let f_i be a basis of $\mathrm{Hom}_{A_t(1)}(\mathbb{C}, L \otimes L^*)$ such that $f_1 = \iota$. Suppose, for a contradiction, that $p_s(X) \otimes 1 \otimes \iota = 0 \in E_r(L \otimes L^*)^{U_\epsilon^0}$. Then, for some $\sum x_i \otimes f_i \in E_{r-1}(L \otimes L^*)^{U_\epsilon^0}$, we must have by (4.19)

$$\begin{aligned} p_s(X) \otimes 1 \otimes \iota &= d_{r-1}(\sum x_i \otimes f_i) \\ &= \sum d_{r-1}(x_i) \otimes f_i. \end{aligned}$$

We claim that $p_s(X) \otimes 1 \otimes \iota = d_{r-1}(x_1) \otimes \iota$. Given this we have that $(p_s(X) \otimes 1 - d_{r-1}(x_1)) \otimes \iota = 0$. Since x_1 must have trivial U_ϵ^0-weight we have that $x_1 \in \mathbb{C}[X_{t+1}, \dots, X_N]$. As $d_{r-1}(x_1)$ has odd cohomological degree in $E_{r-1}(\mathbb{C})$ and has trivial U_ϵ^0-action we must have $d_{r-1}(x_1) = 0$. Therefore $p_s(X) \otimes 1 \otimes \iota = 0$ in $E_{r-1}(L \otimes L^*)^{U_\epsilon^0}$, contradicting the induction hypothesis.

Thus it remains only to show that if $a_1 \otimes \iota = \sum_{i>1} a_i \otimes f_i$ in $E_{r-1}(L \otimes L^*)^{U_\epsilon^0}$ then $a_1 \otimes \iota = 0$. In $E_{r-2}(L \otimes L^*)^{U_\epsilon^0}$ we must have

$$a_1 \otimes \iota - \sum_{i>1} a_i \otimes f_i = d_{r-2}(b_1) \otimes \iota + \sum_{i>1} d_{r-2}(b_i) \otimes f_i.$$

Hence, by induction, $a_1 \otimes \iota = d_{r-2}(b_1 \otimes \iota)$, as required. This completes the claim.

Since, for each r, $E_r(L \otimes L^*)^{U_\epsilon^0}$ is a graded $E_r(\mathbb{C})^{U_\epsilon^0}$-module it follows that each page $E_r(L \otimes L^*)^{U_\epsilon^0}$ is a faithful $E_r(C)^{U_\epsilon^0} = \mathbb{C}[X_{t+1}, \dots, X_N]$-module. Thus the same is true of the page $E_\infty(L \otimes L^*)^{U_\epsilon^0}$ and so, lifting to $\mathrm{Ext}^*_{A_N(1)}(\mathbb{C}, L \otimes L^*)^{U_\epsilon^0} \cong \mathrm{Ext}^*_{U_\epsilon^{\geq 0}}(\mathbb{C}, L \otimes L^*)$, the proposition follows.

As a consequence we can describe the complexity of the finite dimensional algebras $U_\epsilon^{\geq 0}(b)$ for any $b \in B^-$.

Theorem 4.13. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$. Then* $\mathrm{cx}(U_\epsilon^{\geq 0}(b)) = N - \ell(w)$.

Proof. Just combine Theorem 2.28 and Proposition 4.12 with Lemma 4.1.

4.2 The complexity of $\mathcal{O}_\epsilon[G]$ factors.

We want to use the arguments of the previous section to deduce the complexity of the finite dimensional factors of the quantised function algebras. To this end fix reduced expressions for $w_1, w_2 \in W$

$$w_1 = s_{i_1} \dots s_{i_t}, \quad w_2 = s_{i'_1} \dots s_{i'_{t'}}.$$

Twisting by the torus T if necessary, let $b_+ \in X_{e,w_2}$ and $b_- \in X_{w_1,e}$ be unipotent and set $g = b_+b_- \in X_{w_1,w_2} \cap B^+B^-$. By Proposition 1.32(ii) we have an embedding

$$\gamma_g : \mathcal{O}_\epsilon[G](g) \longrightarrow U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-),$$

whose image is a skew group ring extension of $U^-(b_+) \otimes U^+(b_-)$ by $(\mathbb{Z}/\ell\mathbb{Z})^r \cong U_\epsilon^0$. Moreover $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$ is a skew group ring extension of the image of γ_g by $(\mathbb{Z}/\ell\mathbb{Z})^r \cong U_\epsilon^0$.

Proposition 4.14. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $b_+ \in X_{e,w_2}$ and $b_- \in X_{w_1,e}$ be as above. Then* $\mathrm{cx}(U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)) = 2N - \ell(w_1) - \ell(w_2)$.

Proof. Let M be a simple $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$-module. The arguments of Section 4.1 can be applied to $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$ verbatim with only the following adjustments: we must consider the normal chain of algebras

$$A_{w_1,w_2}(1) = A_{t,t'}(1) \subset A_{t+1,t'}(1) \subset \cdots \subset A_{N,t'}(1) \subset A_{N,t'+1}(1) \subset \cdots \subset A_{N,N}(1).$$

and we must construct the filtration of $A_{N,N}(1)$ and $U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$ in the obvious way. The Hopf structure of $U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$ allows us to use Section 2.6, yielding the result.

We can now calculate the complexity of the algebras $\mathcal{O}_\epsilon[G](g)$.

Theorem 4.15. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $g \in X_{w_1,w_2}$. Then* $\mathrm{cx}(\mathcal{O}_\epsilon[G](g)) = 2N - \ell(w_1) - \ell(w_2)$.

Proof. Without loss of generality we consider the case $g = b_+b_-$, as above. For $\ell > h$ we have, as in Corollary 3.10(ii),

$$\mathrm{Ext}^*_{U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}[X_1, \dots, X_{2N}].$$

Let M and N be simple $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$-modules. Then, arguing as in Proposition 4.11 and Proposition 4.12, we have $\mathrm{Ext}^*_{U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)}(M, N)$ is finitely generated over $k[X_{t+t'+1}, \dots, X_{2N}]$ (for a suitable, but fixed, ordering of the X_i's) and, if $M = N$, it is moreover faithful.

Since $\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \cong U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$, there is, by Lemma 2.10, a graded isomorphism

$$\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G](g)}(M, M) = \bigoplus_{\beta \in Q_\ell} \mathrm{Ext}^*_{U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)}\left(M, {}^\beta M\right). \tag{4.20}$$

The right hand side of (4.20) is a finitely generated, faithful $\mathbb{C}[X_{t+t'+1}, \dots, X_{2N}]$-module.

By [67, Theorem 4.2] M is semisimple as an $\mathrm{Im}(\gamma_g)$-module. Thus it is clear that every simple module of $\mathrm{Im}(\gamma_g)$ can be realised as a direct summand of some simple M. Letting $\{L_1, \dots, L_n\}$ be a complete list of simple $\mathrm{Im}(\gamma_g)$-modules we see that that the rate of growth of $\mathrm{Ext}^*_{\mathrm{Im}(\gamma_g)}(\oplus_i L_i, \oplus_i L_i)$ is $2N - \ell(w_1) - \ell(w_2)$. The result follows from Remark 2.30.

Remark 4.16. It follows from a variant of Lemma 4.1 and Theorem 4.15 that, for $\ell > h$, the algebra $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C}, \mathbb{C})$ has Krull dimension $2N$, as claimed in Remark 3.15.

4.3 Additional remarks

1. Morally speaking, complexity counts nilpotence. In the case of group algebras and Lie algebras this statement can be given precise meaning by introducing *rank varieties.* These are related to support varieties through theorems of Avrunin and Scott, [6], in the group case and Jantzen, [46], in the Lie algebra case. For quantum groups there is no such theory developed (yet). Intuitively, however, the results here lead one to think that similar behaviour is present for quantised function algebras. For example the algebra $\mathcal{O}_\epsilon[G]$ has complexity $2N$, corresponding to the equations $E_{\beta_i}^\ell = 0$ and $F_{\beta_i}^\ell = 0$ for $1 \leq i \leq N$. Theorem 5.9 also adds weight to this philosophy. To set up rank varieties for quantised enveloping algebras at roots of unity would be particularly interesting since, following [70], they may lead to a proof of the Kac-Weisfeiler theorem for quantum groups.

Chapter 5

Representation theory of $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$

Having discovered in Theorems 4.13 and 4.15 the intimate relationship between, on the one hand, the complexity of factors of $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$ and, on the other hand, the length function on the Weyl group we can describe completely the representation type of these algebras now. Indeed combining our results with the discussion in Section 2.5 on representation type and complexity we immediately discover a slew of wild quotients. If this wasn't bad enough we'll show, after some work, that there are no tame quotients. By Theorem 2.23 and Proposition 2.24 tame quotients might have arisen only in the cases of complexity equal to one or two. In the complexity one case it's easy to see that our factors have finite representation type - cohomology will do this for us. The representation type of the algebras having complexity two is more difficult to discover. However, in the case of $\mathcal{O}_\epsilon[G]$, the majority of these algebras are controlled by their centre, more precisely they are on the Azumaya locus. As a result we can explicitly describe them using geometry. It turns out that these algebras have straightforward descriptions and consequently it's no trouble at all to see that they are wild. We can use this new description, together with the embeddings of Theorem 1.32, to describe the representation type of complexity two factors of $U_\epsilon^{\geq 0}$ also. One advantage in approaching these problems geometrically is that we can completely describe the factors of $\mathcal{O}_\epsilon[G]$ having finite representation type. For completeness we also describe the finite representation type factors of $U_\epsilon^{\geq 0}$. This result, although it is really just a computation, seems to point to some interesting connections between factors of different rank Lie algebras. Moreover it really is neccesary to understand these factors well as they arise in studying the complexity two factors of $\mathcal{O}_\epsilon[G]$ not lying on the Azumaya locus.

Since we'll always assume throughout this chapter that ℓ satisfies Hypothesis 3 let's recall what it says.

Hypothesis 3. *The positive integer ℓ is odd and good.*

5.1 Representation type

By Theorem 1.26 the algebras $U_\epsilon^{\geq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$ are Frobenius for $b \in B^-$ and $g \in G$. Therefore we can apply Theorem 2.23 to the calculations made in Theorems 4.13 and 4.15 to obtain the following result.

Theorem 5.1. *Assume that $\ell > h$, the Coxeter number of G, and that ℓ satisfies Hypothesis 3. Let $w, w_1, w_2 \in W$ and let $b \in X_{w,e}$ and $g \in X_{w_1,w_2}$. Then*
a) i) if $\ell(w) \leq N-3$ the algebra $U_\epsilon^{\geq 0}(b)$ is wild;
ii) the algebra $U_\epsilon^{\geq 0}(b)$ is semisimple if and only if w is the longest word of W;
b) i) if $\ell(w_1)+\ell(w_2) \leq 2N-3$ the algebra $\mathcal{O}_\epsilon[G](g)$ is wild;
ii) the algebra $\mathcal{O}_\epsilon[G](g)$ is semisimple if and only if both w_1 and w_2 are the longest word of W.

Remarks 5.2. i) Parts a) ii) and b) ii) of Theorem 5.1 are already known. Indeed they follow from Theorem 1.29 and Theorem 1.34 respectively.
ii) We can calculate the representation type of $U_\epsilon^{\geq 0}$ and $\mathcal{O}_\epsilon[G]$ for any ℓ satisfying Hypothesis 1. Indeed, by [20], $U_\epsilon^{\geq 0}(\mathfrak{sl}_2)$ has finite representation type, whilst, for any other $\mathfrak{g}$, the algebra $U_\epsilon^{\geq}(\mathfrak{g})$ has wild representation type, [21, Proposition 3.3]. Using an argument similar to [21, Proposition 3.3] it is easy to see $\mathcal{O}_\epsilon[SL(2)]$ has wild representation type. In fact this can be proved using Lemma 5.16 and the fact that $\mathbb{C}[X,Y]/(X^\ell, Y^\ell)$ is wild, [76, 1.1(c), 1.2]. Let α_i be any simple root for G and let

$$\zeta_i : SL(2) \longrightarrow G$$

be the corresponding inclusion of algebraic groups. By [25, Proposition 5.1], there is a surjective map

$$\nu_i : \mathcal{O}_\epsilon[G] \longrightarrow \mathcal{O}_\epsilon[SL(2)]$$

which restricts to ζ_i^* on the central subalgebra $\mathcal{O}[G]$. In particular, since $\zeta_i(1) = 1$, we have a surjective map

$$\nu_i : \mathcal{O}_\epsilon[G] \longrightarrow \mathcal{O}_\epsilon[SL(2)],$$

which shows $\mathcal{O}_\epsilon[G]$ is wild for all G.

iii) This is in contrast with the situation for $U_\epsilon(\mathfrak{g})$. Indeed the algebra $U_\epsilon(\mathfrak{sl}_2)$ is tame - its indecomposable representations have been classified in [18]. For higher rank [39, Theorem 3] and Theorem 2.23 show that the algebras $U_\epsilon(\mathfrak{g})$ are all wild.

There still remain several cases not covered by Theorem 5.1, namely $\ell(w) \in \{N - 2, N - 1\}$ for $U_\epsilon^{\geq 0}(b)$ and $\ell(w_1) + \ell(w_2) \in \{2N - 2, 2N - 1\}$ for $\mathcal{O}_\epsilon[G](g)$. By Proposition 2.24, Theorem 4.13 and Theorem 4.15 the algebras $U_\epsilon^{\geq 0}(b)$ for which $\ell(w) = N - 2$ and the algebras $\mathcal{O}_\epsilon[G](g)$ for which $\ell(w_1) + \ell(w_2) = 2N - 2$ have infinite representation type. We'll deal with these in Section 5.4.

Let's consider the algebras $U_\epsilon^{\geq 0}(b)$ for which $\ell(w) = N - 1$ and the algebras $\mathcal{O}_\epsilon[G](g)$ for which $\ell(w_1) + \ell(w_2) = 2N - 1$. That is, we are studying those factor algebras whose complexity is 1. We shall explicitly describe the structure of these algebras in Sections 5.2 and 5.3 but, in the interim, it's easy to uncover enough structure to see that these algebras have finite representation type.

Proposition 5.3. *Suppose $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $w, w_1, w_2 \in W$ be such that $\ell(w) = N - 1$ and $\ell(w_1) + \ell(w_2) = 2N - 1$. Let $b \in X_{w,e}$ and $g \in X_{w_1,w_2}$. Then the algebras $U_\epsilon^{\geq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$ are Nakayama, that is their projective indecomposable modules are uniserial. In particular $U_\epsilon^{\geq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$ have finite representation type.*

Proof. By Theorem 3.11, $\mathrm{Ext}^*_{U_\epsilon^{\geq 0}}(\mathbb{C}, \mathbb{C})$ is a polynomial ring whose generators have cohomological degree two. The same holds for $\mathrm{Ext}^*_{U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}}(\mathbb{C}, \mathbb{C})$. Since restriction

$$\mathrm{Ext}^*_{U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}}(\mathbb{C}, \mathbb{C}) = (\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C}, \mathbb{C}))^{U_\epsilon^0} \hookrightarrow \mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C}, \mathbb{C})$$

is a finite algebra map (cf. proof of Theorem 4.15) we see that $\mathrm{Ext}^*_{\mathcal{O}_\epsilon[G]}(\mathbb{C}, \mathbb{C})$ is finitely generated over a polynomial ring whose generators have cohomological degree two. We deduce, arguing as in [31, Proposition 8.4.4], that the minimal projective resolution of any simple $U_\epsilon^{\geq 0}(b)$-module or any simple $\mathcal{O}_\epsilon[G](g)$-module has periodicity no more than two.

We now follow the proof of [32, Theorem 3.2(i)]. Since the argument is valid for both $U_\epsilon^{\geq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$ we consider the first case only. So let L be a simple $U_\epsilon^{\geq 0}(b)$-module.

As the minimal projective resolution of L has period no more than two we have an exact sequence

$$0 \longrightarrow L \longrightarrow P \longrightarrow P(L) \longrightarrow L \longrightarrow 0,$$

where $P(L)$ is the projective cover of L and P is a projective $U_\epsilon^{\geq 0}(b)$-module. Then $\dim P(L) = \dim P$. Since $U_\epsilon^{\geq 0}(b)$ is a Frobenius algebra, P is also an injective $U_\epsilon^{\geq 0}(b)$-module and so there is an isomorphism $P \cong I(L) \oplus Q$, where $I(L)$ is the injective hull of L. Therefore $\dim I(L) \leq \dim P(L)$. Using the periodicity again we obtain another exact sequence

$$0 \longrightarrow L \longrightarrow I(L) \longrightarrow I \longrightarrow L \longrightarrow 0,$$

where I is an injective $U_\epsilon^{\geq 0}(b)$-module. But I is also projective so there is a surjection $I \longrightarrow P(L)$ and we deduce that $\dim I(L) = \dim(I) \geq \dim P(L)$. Consequently we have an isomorphism $P \cong I(L)$. Therefore P is the projective cover of a simple $U_\epsilon^{\geq 0}(b)$-module, say L'. As we have isomorphisms, for any simple $U_\epsilon^{\geq 0}(b)$-module M,

$$\mathrm{Ext}^1_{U_\epsilon^{\geq 0}(b)}(L, M) \cong \mathrm{Hom}_{U_\epsilon^{\geq 0}(b)}(P, M) \cong \mathrm{Hom}_{U_\epsilon^{\geq 0}(b)}(L', M),$$

we see that

$$\mathrm{Ext}^1_{U_\epsilon^{\geq 0}(b)}(L, M) = \begin{cases} \mathbb{C} & \text{if } M \cong L', \\ 0 & \text{otherwise.} \end{cases}$$

Since L' is completely determined by L this shows that there is exactly one non-split extension of any simple $U_\epsilon^{\geq 0}(b)$-module, completing the first claim.

The second claim follows from [5, Chapter VI, Theorem 2.1].

5.2 Morita theorems for quantised function algebras

Suppose T is an affine Noetherian $\mathbb{C}$-algebra which is finitely generated as a module over its centre $Z = Z(T)$. Then the simple modules of T all have finite dimension over $\mathbb{C}$ and there is a bound on these dimensions, namely the PI-degree of T. Moreover there is always a simple T-module whose dimension equals this bound, [78, Chapter 1].

By Schur's lemma Z acts on simple modules by scalars, so to each simple T-module, S, we can associate the central character $\zeta_S : Z \longrightarrow \mathbb{C}$ defined by these scalars.

Definition 24. The *Azumaya locus of T* is the set

$$\mathcal{A}_T = \{\zeta_S : S \text{ a simple } T\text{-module of maximal dimension}\}.$$

Let S be a simple T-module. If $\zeta_S \in \mathcal{A}_T$ then $\ker(\zeta_S)T$ is a maximal ideal of T, [78, Theorem 1.9.21]. This has a couple of interesting consequences. First, if S and S' are simple T-modules of maximal dimension then $S \cong S'$ if and only if $\zeta_S = \zeta_{S'}$. Secondly, suppose $\mathrm{Ext}^1_T(S, S') \neq 0$ and let $M = \ker(\zeta_S)T$ and $M' = \ker(\zeta_{S'})T$. Then $\mathrm{Ext}^1_T(T/M, T/M') \neq 0$. A basic argument now shows that we must have $M = M'$ and so $S \cong S'$. That is, $\mathrm{Ext}^1_T(S, S') \neq 0$ only if $S \cong S'$.

By Theorem 1.14 $\mathcal{O}_\epsilon[G]$ is a finite module over its centre. Moreover $\mathcal{O}_\epsilon[G]$ is a Noetherian domain by [25, Theorems 7.2 and 7.4], and it's clearly affine. In this case the PI-degree of $\mathcal{O}_\epsilon[G]$, and so the maximal dimension of simple $\mathcal{O}_\epsilon[G]$-modules, is equal to ℓ^N, [25, Corollary 7.3]. The following theorem gives an alternative description of the Azumaya locus of $\mathcal{O}_\epsilon[G]$.

Theorem 5.4. *[13, Theorem 4.5] Suppose ℓ satisfies Hypothesis 3. Then the Azumaya locus of $\mathcal{O}_\epsilon[G]$ is equal to the regular locus of $Z(\mathcal{O}_\epsilon[G])$.*

For $g \in G$ all simple $\mathcal{O}_\epsilon[G](g)$-modules have the same dimension by Theorem 1.34. In particular if $g \in G$ is such that $\mathcal{O}_\epsilon[G](g)$ has a simple $\mathcal{O}_\epsilon[G]$-module of maximal dimension then all simple $\mathcal{O}_\epsilon[G](g)$-modules necessarily have maximal dimension. Abusing notation we'll call such an element of G *Azumaya*.

Recall from Proposition 1.15 the algebra Z_q, central in $\mathcal{O}_\epsilon[G]$ and generated by the elements $b_i^k c_i^{\ell-k}$ for $1 \leq i \leq r$ and $0 \leq k \leq \ell$, for the definition of b_i and c_i see Definition 9. By Proposition 1.15(ii), the intersection $\mathcal{O}[G] \cap Z_q$ is a polynomial ring in $2r$ variables and $Z(\mathcal{O}_\epsilon[G]) \cong \mathcal{O}[G] \otimes_{\mathcal{O}[G]\cap Z_q} Z_q$. We denote $\mathrm{Maxspec} Z_q$ by C_ℓ^r.

Lemma 5.5. *Let $\pi : C_\ell^r \longrightarrow \mathbb{A}^{2r}$ be the surjective morphism induced by the inclusion $\mathcal{O}[G] \cap Z_q \longrightarrow Z_q$ given in Proposition 1.15(ii). Write the points of $\mathbb{A}^{2r}$ as $2r$-tuples $(\beta_1, \dots, \beta_r, \gamma_1, \dots \gamma_r)$. If $p \in \mathbb{A}^{2r}$ is such that $\beta_i = 0 = \gamma_i$ for some i then every point of $\pi^{-1}(p)$ is singular in C_ℓ^r.*

Proof. Following Remark 1.16 it is clear that π factorises as $\pi_1 \times \dots \times \pi_r$ where $\pi_i : \mathrm{Spec} Z_q^i \longrightarrow \mathbb{A}^2 = \{(\beta_i, \gamma_i)\}$. It is therefore enough to prove this for the case $r = 1$.

By Proposition 1.15(i) we have a surjection $\mathbb{C}[Y_0, \dots, Y_\ell] \longrightarrow \mathcal{O}[C_\ell^1]$ and so we can consider C_ℓ^1 as an affine variety embedded in $\mathbb{A}^{\ell+1}$ (with co-ordinate functions $\alpha_1(k)$). Under this identification π takes $(a_0, \dots, a_\ell)$ to (a_0, a_ℓ).

Note that $\pi^{-1}((0,0)) = (0,\ldots,0)$. Indeed the equations

$$\alpha_1(k)\alpha_1(k') = \begin{cases} \alpha_1(0)\alpha_1(k+k') & \text{if } k+k' \leq \ell, \\ \alpha_1(\ell)\alpha_1(k+k'-\ell) & \text{if } k+k' > \ell, \end{cases}$$

show that $a_k^2 = 0$ for all k if $\pi(a_0,\ldots,a_\ell) = (0,0)$.

Define $f_{k,k'} \in \mathbb{C}[Y_0,\ldots,Y_\ell]$ as follows

$$f_{k,k'} = \begin{cases} Y_kY_{k'} - Y_0Y_{k+k'} & \text{if } k+k' \leq \ell, \\ Y_kY_{k'} - Y_\ell Y_{k+k'-\ell} & \text{if } k+k' > \ell. \end{cases}$$

By Proposition 1.15(i) the elements $f_{k,k'}$ generate the kernel of the map $\mathbb{C}[Y_0,\ldots,Y_\ell] \longrightarrow \mathcal{O}[C_\ell^1]$. For $a = (a_0,\ldots,a_\ell) \in C_\ell^1$ we define

$$f^{(1)}_{k,k',a} = \sum_{j=0}^{\ell} \frac{\partial f_{k,k'}}{\partial Y_j}(a)(Y_j - a_j).$$

By definition, [73, 6.4], the tangent space of C_ℓ^1 at a is

$$T_aC_\ell^1 = \{x \in \mathbb{A}^{\ell+1} : f^{(1)}_{k,k',a}(x) = 0 \text{ for } 1 \leq k,k' \leq \ell - 1\}.$$

Since $f_{k,k'}$ is homogeneous of degree two it follows that for all k,k'

$$f^{(1)}_{k,k',(0,\ldots,0)} \equiv 0,$$

which implies that $\dim(T_0C_\ell^1) = \ell + 1$.

Since Z_q is finite over $\mathcal{O}[G] \cap Z_q$, a polynomial ring in two variables, if follows that C_ℓ^1 has dimension 2, [4, Lemma 11.26]. Therefore $(0,\ldots,0) = \pi^{-1}((0,0))$ is a singular point.

Remark 5.6. In fact these are the only singular points in C_ℓ^r, as one can prove by further elementary calculations as above.

Proposition 5.7. *Suppose ℓ satisfies Hypothesis 3. Let $g \in G$ be such that $b_i^\ell(g) = 0 = c_i^\ell(g)$ for some i, $1 \leq i \leq r$. Then g is not an Azumaya point.*

Proof. Using Theorem 5.4 it is enough to show that any maximal ideal of Z lying over $\mathfrak{m}_g$ is singular. Denote Spec$Z(\mathcal{O}_\epsilon[G])$ by Y, the fibre product $G \times_{\mathbb{A}^{2r}} C_\ell^r$.

Claim. Let $\tilde{\pi} : Y \longrightarrow C_\ell^r$ be the projection map. If $x \in C_\ell^r$ is singular then any point of $\tilde{\pi}^{-1}(x)$ is singular in Y.

Proof of claim. For ease of notation let $R = \mathcal{O}[G]$, $S = Z_q \cap \mathcal{O}[G]$ and $T = Z_q$. Thus, by Proposition 1.15 and Remark 1.16, we are considering $R \otimes_S T$ where:

(i) as an S-module T is finitely generated and free;

(ii) the algebra S is smooth;

(iii) all algebras are commutative affine domains.

Let $\mathfrak{m}_T \lhd T$ be the maximal ideal corresponding to $x \in C_\ell^T$, and suppose M lies over $\mathfrak{m}_T$ in $R \otimes_S T$. Define $\mathfrak{m}_R = M \cap R$ and $\mathfrak{m}_S = M \cap S$, maximal ideals of R and S respectively.

We first show that

$$(R \otimes_S T)_M = R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}. \tag{5.1}$$

Note that $M = \mathfrak{m}_R \otimes_S T + R \otimes_S \mathfrak{m}_T$. This means in particular that if $y \in R \setminus \mathfrak{m}_R$ and $z \in T \setminus \mathfrak{m}_T$ then $y \otimes 1$ and $1 \otimes z$ are elements of $R \otimes_S T \setminus M$. So we have an embedding

$$A := R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T} \longrightarrow (R \otimes_S T)_M. \tag{5.2}$$

Let $x \in R \otimes_S T \setminus M$. If A were local then $x \in A \setminus MA$ would be an invertible element, since MA is maximal. Therefore the map (5.2) would be an isomorphism, proving (5.1). Thus it is enough to show that A is a local ring.

Observe that by (i) the algebra $T/\mathfrak{m}_S T$ is finite dimensional. Therefore localising this at $\mathfrak{m}_T$ yields another finite dimensional algebra $T_{\mathfrak{m}_T}/\mathfrak{m}_S T_{\mathfrak{m}_T}$. Nakayama's lemma implies that $T_{\mathfrak{m}_T}$ is finite over $S_{\mathfrak{m}_S}$.

Let $\mathfrak{m} = \mathfrak{m}_R A$. Then

$$A/\mathfrak{m} \cong T_{\mathfrak{m}_T}/\mathfrak{m}_S T_{\mathfrak{m}_T}$$

is finite dimensional, so Nakayama's lemma also implies that A is finite over $R_{\mathfrak{m}_R}$. Therefore $\mathfrak{m}$ is contained in the Jacobson radical of A, [61, Lemma 9.2]. However, since $A/\mathfrak{m}$ is local it follows that A is local, as required. So we have proved (5.1).

To complete the claim we must show that $A = R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}$ has infinite global dimension, [61, Theorem 19.2]. By hypothesis $T_{\mathfrak{m}_T}$ has infinite global dimension.

We have a change of rings spectral sequence, [77, Theorem 11.66],

$$\mathrm{Ext}_A^p\left(\mathbb{C}_A, \mathrm{Ext}_{T_{\mathfrak{m}_T}}^q(A, M)\right) \Longrightarrow \mathrm{Ext}_{T_{\mathfrak{m}_T}}^{p+q}(\mathbb{C}_{\mathfrak{m}_T}, M), \tag{5.3}$$

for any $T_{\mathfrak{m}_T}$-module M.

By Frobenius reciprocity we have

$$\mathrm{Ext}_{T_{\mathfrak{m}_T}}^q(A, M) = \mathrm{Ext}_{T_{\mathfrak{m}_T}}^q\left(R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}, M\right) \cong \mathrm{Ext}_{S_{\mathfrak{m}_S}}^q(R_{\mathfrak{m}_R}, M).$$

As $S_{\mathfrak{m}_S}$ is smooth there exists a natural number Q such that

$$\mathrm{Ext}^q_{S_{\mathfrak{m}_S}}(R_{\mathfrak{m}_R}, M) = 0,$$

for all $q > Q$ and all $S_{\mathfrak{m}_S}$-modules M.

If A were smooth there would be a natural number P such that

$$\mathrm{Ext}^p_A(\mathbb{C}_A, M) = 0,$$

for all $p > P$ and all A-modules M. Then by (5.3) we would have

$$\mathrm{Ext}^n_{T_{\mathfrak{m}_T}}(\mathbb{C}_{\mathfrak{m}_T}, M) = 0,$$

for all $n > P + Q$ and $T_{\mathfrak{m}_T}$-modules M, contradicting the singularity of $T_{\mathfrak{m}_T}$. Thus A must be singular, proving the claim.

The proposition now follows from Lemma 5.5 combined with the claim.

Definition 25. For all $g \in G$ define $Z_g = Z(\mathcal{O}_\epsilon[G])/\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])$.

The following proposition shows us exactly how, for an Azumaya element $g \in G$, the algebra structure of the quotient $\mathcal{O}_\epsilon[G](g)$ reflects the geometry of the fibre over g of the map $Y \longrightarrow G$.

Proposition 5.8. *Suppose ℓ satisfies Hypothesis 3. Then if $g \in G$ is an Azumaya point there is an algebra isomorphism*

$$\mathcal{O}_\epsilon[G](g) \cong Mat_{\ell^N}(Z_g).$$

Proof. For ease of notation let $R = \mathcal{O}_\epsilon[G](g)$. Suppose that

$$Z_g = e_1 Z_g \oplus \cdots \oplus e_t Z_g = Z_1 \oplus \cdots \oplus Z_t$$

is a direct sum of local rings where $\{e_1, \ldots, e_t\}$ is a set of central primitive idempotents. It follows that

$$R = e_1 R \oplus \cdots \oplus e_t R = R_1 \oplus \cdots \oplus R_t.$$

Let m_1 be the maximal ideal of Z_1 and let $S = Z_g \setminus M$ be the multiplicatively closed set of central elements where $M = m_1 \oplus Z_2 \oplus \cdots \oplus Z_t$, a maximal ideal of Z_g. Then we have an isomorphism

$$RS^{-1} \cong \frac{R}{\{r \in R : rs = 0 \text{ for some } s \in S\}}.$$

It's clear that $\{r \in R : rs = 0 \text{ for some } s \in S\} = \mathrm{Ann}_R(e_1) = R_2 \oplus \cdots \oplus R_t$ so $RS^{-1} \cong R_1$. Similarly $ZS^{-1} \cong Z_1$. It is therefore enough to prove that $R_1 \cong Mat_{\ell^N}(Z_1)$.

Let $\widehat{S}$ be the inverse image of S in $Z(\mathcal{O}_\epsilon[G])$. As a $Z(\mathcal{O}_\epsilon[G])\widehat{S}^{-1}$-module $\mathcal{O}_\epsilon[G]\widehat{S}^{-1}$ is free of rank ℓ^{2N}. Indeed this is a formal property of Azumaya algebras, [78, Definition 1.8.35], or can be deduced from the fact that $\mathcal{O}_\epsilon[G]$ is a maximal order of degree ℓ^N, [25, Corollary 7.3, Theorem 7.4], and so the associated reduced trace map splits the inclusion of $Z(\mathcal{O}_\epsilon[G])$ in $\mathcal{O}_\epsilon[G]$, [26, Corollary 4.6]. As a consequence $\dim(Z_1) = \dim(R_1)/\ell^{2N}$.

Since $g \in G$ is an Azumaya point, m_1R_1 is a maximal ideal of R_1 and we have an algebra isomorphism

$$\frac{R_1}{m_1R_1} \cong Mat_{\ell^N}(\mathbb{C}).$$

As m_1R_1 is nilpotent we see that R_1 is a local, finite dimensional algebra with simple right R_1-module V having dimension ℓ^N.

Let $P(V)$ be the R_1-projective cover of V. Then, as right R_1-modules, we have

$$R_1 \cong \bigoplus^{\ell^N} P(V).$$

Therefore there are algebra isomorphisms

$$R_1 \cong End_{R_1}(\bigoplus^{\ell^N} P(V)) \cong Mat_{\ell^N}(End_{R_1}(P(V))).$$

Since Z_1 is central in R_1 and acts on $P(V)$ by multiplication there is an embedding

$$Mat_{\ell^N}(Z_1) \longrightarrow Mat_{\ell^N}(End_{R_1}(P(V)).$$

This now proves the proposition since both sides have the same dimension.

We now get a description of the algebras $\mathcal{O}_\epsilon[G](g)$ for $g \in G$ an Azumaya point.

Theorem 5.9. *Suppose ℓ satisfies Hypothesis 3. Let $\pi : G \times_{\mathbb{A}^{2r}} C_\ell^r \longrightarrow G$ be projection.*
(i) For $g \in G$, every maximal ideal in $\pi^{-1}(g)$ is in the Azumaya locus if and only if one ideal in $\pi^{-1}(g)$ is in the Azumaya locus.
(ii) Let $g \in X_{w_1,w_2}$. Then g is Azumaya if and only if $\ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1) = 2N$.
(iii) Suppose $g \in X_{w_1,w_2}$ is an Azumaya element of G. Then there is an algebra isomorphism

$$\mathcal{O}_\epsilon[G](g) \cong \bigoplus_1^{\ell^{r-t}} Mat_{\ell^N}\left(\frac{\mathbb{C}[X_1,\dots,X_t]}{(X_1^\ell,\dots,X_t^\ell)}\right),$$

where $t = s(w_2^{-1}w_1)$.

Proof. Parts (i) and (ii) are immediate from Theorem 1.34. Suppose now that g is Azumaya, $g \in X_{w_1,w_2}$. By Proposition 5.8 we have an algebra isomorphism

$$\mathcal{O}_\epsilon[G](g) \cong Mat_{\ell^N}(Z_g).$$

So it only remains to calculate Z_g.

We have isomorphisms

$$Z_g = \frac{Z(\mathcal{O}_\epsilon[G])}{\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])} \cong \frac{Z_q}{(\mathfrak{m}_g \cap Z_q)Z_q}.$$

Here $\mathfrak{m}_g \cap Z_q \lhd \mathcal{O}[G] \cap Z_q$ is, in the notation of Lemma 5.5, specified by $b_i^\ell(g) = \beta_i$ and $c_i^\ell(g) = \gamma_i$. Recalling our decomposition of Z_q in Remark 1.16,

$$Z_q = \bigotimes_{i=1}^{r} Z_q^i,$$

we see that Z_g is the tensor product of rings R_i, for $1 \leq i \leq r$, where

$$R_i := \frac{Z_q^i}{(\alpha_i(0) - \beta_i, \alpha_i(\ell) - \gamma_i)Z_q^i}.$$

Let's describe the possible structure of the rings R_i. Since $\mathcal{O}_\epsilon[G](g)$ is Azumaya it follows from Proposition 5.7 that we never have $\beta_i = 0 = \gamma_i$. There are only two cases to consider.

(i) $\beta_i \neq 0 \neq \gamma_i$: in this case we have an algebra isomorphism

$$R_i \cong \frac{\mathbb{C}[X_i]}{(X_i^\ell - 1)} \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}. \tag{5.4}$$

Indeed sending $X_i \longmapsto \alpha_i(1)$ produces an isomorphism

$$\frac{\mathbb{C}[X_i]}{(X_i^\ell - \beta_i^{\ell-1}\gamma_i)} \cong R_i.$$

Since this is a semisimple algebra of dimension ℓ, the isomorphisms in (5.4) are clear.

(ii) $\beta_i \neq 0 = \gamma_i$ (the case $\beta_i = 0 \neq \gamma_i$ is the same by symmetry): in this case we have an algebra isomorphism

$$R_i \cong \frac{\mathbb{C}[X_i]}{(X_i^\ell)}. \tag{5.5}$$

Again sending $X_i \longmapsto \alpha_i(1)$ yields the required isomorphism.

To complete the theorem, recall that, by Theorem 1.34(ii), $\mathcal{O}_\epsilon[G](g)$ has exactly ℓ^{r-t} simple modules. But R_i has exactly ℓ simple modules in Case (i) and a unique simple module in Case (ii). Therefore

$$\begin{aligned} Z_g \cong \bigotimes_{i=1}^{r} R_i &\cong \bigotimes_{i=1}^{t} \frac{\mathbb{C}[X_i]}{(X_i^\ell)} \otimes \bigotimes_{i=t+1}^{r} \frac{\mathbb{C}[X_i]}{(X_i^\ell - 1)} \\ &\cong \bigoplus_{1}^{\ell^{r-t}} \frac{\mathbb{C}[X_1, \dots, X_t]}{(X_1^\ell, \dots, X_t^\ell)}. \end{aligned}$$

Remark 5.10. If $g \in X_{w_1,w_2}$ is an Azumaya element then, by Theorem 5.9(ii), we have $2N = \ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1)$. Thus, by Theorem 5.9(iii), the complexity of $\mathcal{O}_\epsilon[G](g)$ equals $2N - \ell(w_1) - \ell(w_2)$, as shown in Theorem 4.15.

The set $\{w_0 s_i : 1 \leq i \leq r\}$ consists precisely of the elements $w \in W$ for which $\ell(w) = N - 1$. Therefore the following corollary describes all the algebras $\mathcal{O}_\epsilon[G](g)$ which have complexity one, in other words the non-semisimple factors of finite representation type.

Corollary 5.11. *Suppose ℓ satisfies Hypothesis 3. Let $g \in X_{w_0,w}$ or X_{w,w_0} where $w = w_0 s_i$. Then there is an algebra isomorphism*

$$\mathcal{O}_\epsilon[G](g) \cong \bigoplus_{j=1}^{\ell^{r-1}} Mat_{\ell^N}\left(\frac{\mathbb{C}[X]}{(X^\ell)}\right).$$

Proof. This follows from Theorem 5.9 since the point g is Azumaya. Indeed, by Theorem 1.34(iii), the simple $\mathcal{O}_\epsilon[G](g)$-modules have dimension $\ell^{\frac{1}{2}(\ell(w_0)+\ell(w_0 s_i)+s((w_0 s_i)^{-1}w_0))} = \ell^{\frac{1}{2}(N+(N-1)+1)} = \ell^N$, the PI degree of $\mathcal{O}_\epsilon[G]$.

5.3 Morita theorems for quantised Borel subalgebras

Here we describe the factors $U_\epsilon^{\geq 0}(b)$ having complexity one. In this case the rank function, s, makes a crucial appearance. For $w \in W$ and $b \in B^-$ recall the algebras $U_w(b)$ introduced in (1.20), Section 1.6.

Lemma 5.12. *Assume that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$ for some $w \in W$ and let L be a simple $U_\epsilon^{\geq 0}(b)$-module. Then the projective cover of L is the induced module*

$$P(L) = U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L.$$

Using the notation of Section 1.6, all other projective covers of simple $U_\epsilon^{\geq 0}(b)$-modules are given by $\mathbb{C}_\mu \otimes L$ for $\mu \in Q_\ell$.

Proof. The final claim is an immediate consequence of the discussion preceding Theorem 1.29. In particular this means that all projective covers of simple modules have the same dimension.

There is an equality, [10, Section 1.7],

$$\dim(U_\epsilon^{\geq 0}(b)) = \sum_{L \text{ simple}} \dim(L)\dim(P(L)).$$

Thus by the above comments and Theorem 1.29 we have

$$\dim(P(L)) = \ell^{(r+N)-\frac{1}{2}(\ell(w)+s(w))-(r-s(w))} = \ell^{N+\frac{1}{2}(s(w)-\ell(w))}.$$

In particular, since $U_\epsilon^{\geq 0}(b)$ is a free $U_w(b)$-module of rank $\ell^{N-\ell(w)}$ the $U_\epsilon^{\geq 0}(b)$-module $U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L$ has the correct dimension. So it's enough to show that this is projective and has L in its head.

We have a graded isomorphism for N a $U_\epsilon^{\geq 0}(b)$-module

$$\mathrm{Ext}^*_{U_w(b)}(L, N) \cong \mathrm{Ext}^*_{U_\epsilon^{\geq 0}(b)}\left(U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L, N\right). \tag{5.6}$$

By Corollary 1.30 the algebra $U_w(b)$ is semisimple so (5.6) shows that $U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L$ is a projective module whose head includes L, completing the lemma.

For the remainder of this section let $b \in X_{w,e}$ where $w = w_0 s_i$ for some $1 \leq i \leq r$. Let L and L' be simple $U_\epsilon^{\geq 0}(b)$-modules.

Corollary 5.13. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$ be as above and let L and L' be simple $U_\epsilon^{\geq 0}(b)$-modules. Then we have*

$$\mathrm{Ext}^1_{U_\epsilon^{\geq 0}(b)}(L, L') = \begin{cases} \mathbb{C} & \text{if } L' \cong \mathbb{C}_{\beta_N} \otimes L, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 5.12 the projective cover of L is $U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L$. Recall that by (4.7) there is a $U_\epsilon^{\geq 0}(b)$-module isomorphism

$$U_\epsilon^{\geq 0}(b) \otimes_{U_w(b)} L \cong \left(U_\epsilon^{\geq 0} \otimes_{U_w(1)} \mathbb{C}\right) \otimes L.$$

It is clear the set $\{E^i_{\beta_N} \otimes 1 : 0 \leq i \leq \ell - 1\}$ is a basis $U_\epsilon^{\geq 0} \otimes_{U_w(1)} \mathbb{C}$. Moreover, since $E^\ell_{\beta_N} = 0$ in $U_\epsilon^{\geq 0}$, there is a filtration on $U_\epsilon^{\geq 0} \otimes_{U_w(1)} \mathbb{C}$ given by

$$F_j = \mathbb{C}\text{-span}\{E^i_{\beta_N} \otimes 1 : j \leq i \leq \ell - 1\}.$$

Using this we see that there is a non-split extension of $(F_1/F_2) \otimes L$ by L. It is straightforward to check that there is an isomorphism

$$\theta : (F_1/F_2) \otimes L \longrightarrow \mathbb{C}_{\beta_N} \otimes L$$

induced by $\theta((E_{\beta_N} \otimes 1) \otimes x) = 1 \otimes x$ for all $x \in L$. This proves there is a non-split extension of $\mathbb{C}_{\beta_N} \otimes L$ by L. Proposition 5.3 ensures that there is only one such extension and that there are no extensions invloving any other irreducibles.

It's clear we have to decide when $\mathbb{C}_{\beta_N} \otimes L$ is isomorphic to L, but this requires a combinatorial lemma. We have $w_0(\Delta^+) = -\Delta^+$ and since W acts on the bases of the root system of G it follows that $w_0(\Pi) = -\Pi$. Hence $-w_0$ induces a permutation on Π which we will denote by $\sigma \in \Sigma_r$.

For any given root system the fixed points of σ are known. We list these in Table 1 using the notation of [12, Planches I-IX].

System	Rank	Fixed points
A	r odd	$\alpha_{\frac{r+1}{2}}$
	r even	none
B	$r \geq 2$	$\alpha_1, \dots, \alpha_r$
C	$r \geq 3$	$\alpha_1, \dots, \alpha_r$
D	$r \geq 4$, even	$\alpha_1, \dots, \alpha_r$
	$r \geq 5$, odd	$\alpha_1, \dots, \alpha_{r-2}$
E	6	α_2, α_4
	7	$\alpha_1, \dots, \alpha_7$
	8	$\alpha_1, \dots, \alpha_8$
F	4	$\alpha_1, \dots, \alpha_4$
G	2	α_1, α_2

Table 5.1: Fixed points of $-w_0$

It is known that w can fall into one of two distinct classes, [27, 5.3]:

Case i) $s(w_0) = s(w) - 1 = s(s_{\sigma(i)}w_0) - 1$; here we have

$$P^w = P^{w_0} \cap P^{s_{\sigma(i)}}; \tag{5.7}$$

Case ii) $s(w_0) = s(w) + 1 = s(s_{\sigma(i)}w_0) + 1$; here we have

$$P^{w_0} = P^w \cap P^{s_{\sigma(i)}}. \tag{5.8}$$

Lemma 5.14. *Let $\ell(w) = N-1$, so that $w = w_0 s_i$ for some $1 \leq i \leq r$. Let $\beta_N = w(\alpha_i) = \alpha_{\sigma(i)}$ and let L be a simple $U_\epsilon^{\geq 0}(b)$-module. Then*

$$\mathbb{C}_{\beta_N} \otimes L \cong L \iff s(w) = s(w_0) + 1 \iff \sigma(i) \neq i. \tag{5.9}$$

Proof. We first prove

$$s(w) = s(w_0) + 1 \iff \sigma(i) \neq i, \tag{5.10}$$

using a case-by-case analysis of the simple root systems having non-trivial permutation σ. Indeed if σ is trivial w_0 must send α_i to $-\alpha_i$ so it follows that $P^{w_0} = \{0\}$. As a result $s(w_0) = r$ is maximal and so $s(w) = s(w_0) - 1$. From Table 5.1 we conclude that (5.10) holds for types B_r, C_r, D_r (r even), E_7, E_8, F_4 and G_2.

Type A_r: From Table 1 we see that $\sigma(i) = i$ if and only if $i = \frac{r+1}{2}$. Realising the Weyl group of A_r as the permutation group Σ_{r+1}, the longest word can be written as a product of transpositions

$$w_0 = (1 \quad r+1)(2 \quad r)(3 \quad r-1)\ldots \tag{5.11}$$

Moreover the reflection s_i can be written as the transposition $(i \quad i+1)$. If $i = \frac{r+1}{2}$ it is therefore clear that $s(w_0 s_i) = s(w_0) - 1$.

Henceforth assume $i \neq \frac{r+1}{2}$. Since the individual transpositions in (5.11) commute we can adjust $w_0 s_i$ so that

$$w_0 s_i = ((1 \quad r+1)(2 \quad r)\ldots)(i \quad r+2-i)(i+1 \quad r+1-i)(i \quad i+1). \tag{5.12}$$

For ease of notation we define $p, p' \in \Sigma_{r+1}$ as follows:

$$p = (i \quad r+2-i)(i+1 \quad r+1-i); \qquad p' = (i \quad r+2-i)(i+1 \quad r+1-i)(i \quad i+1).$$

For the moment we assume that $i \neq \frac{r}{2}$ and that $i \neq \frac{r+2}{2}$. This ensures that $\ell(p) = 2$ and that $\ell(p') = 3$. From (5.12), it is clear that $s(w_0 s_i) = (s(w_0) - 2) + s(p')$. Therefore it is enough to check that $s(p') = 3$. Since $\ell(p') = 3$ it follows that $s(p')$ equals either 1 or 3. However, p' is not a transposition since it contains at least three distinct integers. This proves (5.10) in this case. A similar argument shows that $s(w_0 s_i) = s(w_0) + 1$ if $i = \frac{r}{2}$ or

$i = \frac{r+2}{2}$.

Type D_r (r odd): By [12, Planche IV, (XI)] $-w_0$ fixes $\alpha_1, \dots, \alpha_{r-2}$ and switches α_{r-1} and α_r. The Cartan matrix shows that

$$\alpha_{r-1} - \alpha_r = 2(\varpi_{r-1} - \varpi_r).$$

From this it follows that

$$P^{w_0} = \mathbb{Z}\text{-span}\{\varpi_{r-1} - \varpi_r\},$$

implying, in particular, that $s(w_0) = r - 1$.

Suppose that $\sigma(i) = i$, that is $i \leq r-2$. Then we have that

$$s_i(\alpha_{r-1} - \alpha_r) = \alpha_{r-1} - \alpha_r$$

which implies $\alpha_{r-1} - \alpha_r \in P^{w_0 s_i} = P^w$. If we were in Case i) then we would have

$$\dim(\mathbb{R} \otimes_{\mathbb{Z}} P^w) < \dim(\mathbb{R} \otimes_{\mathbb{Z}} P^{w_0}) = 1,$$

forcing $P^w = \{0\}$, a contradiction. Hence we are in Case ii) as required.

Suppose now that $\sigma(i) \neq i$ so that $i = r-1$ or $i = r$. By symmetry we assume $i = r-1$ without loss of generality. If we were in Case ii) then, by (5.8), we would have

$$P^w \cap P^{s_{\sigma(r-1)}} = P^w \cap P^{s_r} = \mathbb{Z}\text{-span}\{\varpi_{r-1} - \varpi_r\},$$

a contradiction since $P^{s_r} = \mathbb{Z}\text{-span}\{\varpi_j : 1 \leq j \leq r-1\}$. We are therefore in Case i) as required.

Type E_6: By [12, Planche V, (XI)] $-w_0$ fixes α_2 and α_4 and switches α_1 and α_3 with α_6 and α_5 respectively. It follows that

$$\mathbb{R} \otimes_{\mathbb{Z}} P^{w_0} = \mathbb{R}\text{-span}\{\alpha_1 - \alpha_6, \alpha_3 - \alpha_5\}. \tag{5.13}$$

We see from the Cartan matrix that

$$\begin{aligned} \alpha_1 &= 2\varpi_1 - \varpi_3 & \alpha_6 &= -\varpi_5 + 2\varpi_6 \\ \alpha_3 &= -\varpi_1 + 2\varpi_3 - \varpi_4 & \alpha_5 &= -\varpi_4 + 2\varpi_5 - \varpi_6. \end{aligned}$$

A simple calculation using the above description shows that $\varpi_1 - \varpi_6, \varpi_3 - \varpi_5 \in P^{w_0}$. By (5.13) it follows that

$$P^{w_0} = \mathbb{Z}\text{-span}\{\varpi_1 - \varpi_6, \varpi_3 - \varpi_5\}. \tag{5.14}$$

Suppose that $\sigma(i) = i$, that is $i = 2$ or $i = 4$. If we were in Case i) we would have

$$P^w = P^{s_i} \cap P^{w_0}.$$

Equation (5.14) shows however that $P^{w_0} \subset P^{s_i}$, implying $P^{w_0} = P^w$, a contradiction. Therefore we are in Case ii).

Suppose that $\sigma(i) \neq i$, so $i \in \{1, 3, 5, 6\}$. It follows from (5.14) that $P^{w_0} \not\subseteq P^{s_{\sigma(i)}}$ so we cannot be in Case ii). Therefore, from (5.7), we have

$$P^w = \begin{cases} \mathbb{Z}\text{-span}\{\varpi_3 - \varpi_5\} & i = 1 \text{ or } 6, \\ \mathbb{Z}\text{-span}\{\varpi_1 - \varpi_6\} & i = 3 \text{ or } 5. \end{cases}$$

This is sufficient to prove (5.10) in this, the final instance.

By Theorem 1.29(ii) we have

$$\mathbb{C}_{\beta_N} \otimes L \cong L \iff (\lambda, \beta_N) \in \ell\mathbb{Z} \text{ for all } \lambda \in P^w.$$

If $\sigma(i) = i$ then we have that $\beta_N = \alpha_i$. Moreover we also have that $\alpha_i \in P^w$. Then $(\alpha_i, \alpha_i) = 2 \notin \ell\mathbb{Z}$, proving

$$\mathbb{C}_{\beta_N} \otimes L \cong L \Longrightarrow \sigma(i) \neq i.$$

If $\sigma(i) \neq i$ we have shown that $s(w_0) = s(w) - 1$. Thus we are in Case i) above and we have

$$P^w \subseteq P^{s_{\sigma(i)}}.$$

Therefore we have

$$(\lambda, \beta_N) = (\lambda, \alpha_{\sigma_i}) = 0 \text{ for all } \lambda \in P^w.$$

This shows

$$\sigma(i) \neq i \Longrightarrow \mathbb{C}_{\beta_N} \otimes L \cong L,$$

as required.

We can now describe the algebras $U_\epsilon^{\geq 0}(b)$ having complexity one.

Theorem 5.15. *Assume that $\ell > h$ and that ℓ satisfies Hypothesis 3. Let $b \in X_{w,e}$ where $w = w_0 s_i$ and let σ be as above.*

i) If $\sigma(i) \neq i$ then we have an algebra isomorphism

$$U_\epsilon^{\geq 0}(b) = \bigoplus_{j=1}^{\ell^{r-s(w_0)-1}} Mat_{\ell^{\frac{1}{2}(N+s(w_0))}}\left(\frac{\mathbb{C}[X]}{(X^\ell)}\right).$$

ii) If $\sigma(i) = i$ then we have an algebra isomorphism

$$U_\epsilon^{\geq 0}(b) = \bigoplus_{j=1}^{\ell^{r-s(w_0)}} Mat_{\ell^{\frac{1}{2}(N-2+s(w_0))}}\left(U_\epsilon^{\geq 0}(\mathfrak{sl}_2)\right).$$

Proof. Let $\{L_1, \dots, L_n\}$ be a complete list of non-isomorphic simple $U_\epsilon^{\geq 0}(b)$-modules and let $P(L_j)$ be the projective cover of L_j. By Lemma 5.12 and Theorem 1.29(iii)

$$\dim(P(L_j)) = \ell^{\frac{1}{2}(N+s(w)+1)}.$$

This, together with Theorem 1.29(iii), Corollary 5.13 and Lemma 5.14, describes the composition factors of $P(L_j)$: if $\sigma(i) \neq i$ then there are ℓ composition factors, each isomorphic to L_j; if $\sigma(i) = i$ then the composition factors are $\mathbb{C}_{m\alpha_i} \otimes L$ for $0 \leq m \leq \ell - 1$.

Assume first that we are in Case (i), that is $\sigma(i) \neq i$. Fix a simple module L_j. By Lemma 5.14 $s(w) = s(w_0) + 1$ and so $\dim(P(L_j)) = \ell^{\frac{1}{2}(N+s(w_0)+2)}$. As the composition factors of the uniserial module $P(L_j)$ are all isomorphic to L_j it follows that

$$\dim \mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(L_j)) = \ell. \tag{5.15}$$

Choose an embedding, e, of $P(L_j)/L_j$ in $P(L_j)$ and let $\theta_X \in \mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(L_j))$ be the composition

$$P(L_j) \longrightarrow P(L_j)/L_j \xrightarrow{e} P(L_j).$$

It is clear that, for $0 \leq j \leq \ell$

$$\dim \mathrm{im}(\theta_X^j) = \ell^{\frac{1}{2}(N+s(w_0))}(\ell - j), \tag{5.16}$$

which, together with (5.15), shows that

$$\mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(L_j)) \cong \frac{\mathbb{C}[X]}{(X^\ell)}.$$

The result for Case (i) now follows since the block containing L_j consists of $\dim L_j = \ell^{\frac{1}{2}(N+s(w_0))}$ copies of $P(L_j)$.

On to Case (ii) where $\sigma(i) = i$. Let B be a block of $U_\epsilon^{\geq 0}(b)$ and suppose L belongs to the block B. Let $P(L)$ be the projective cover of L. By Lemma 5.14 $\dim(P(L)) = \ell^{\frac{1}{2}(N+s(w_0))}$ and $\mathbb{C}_{j\alpha_i} \otimes L$ for $0 \leq j \leq \ell - 1$ are precisely the simple modules belonging to B. It follows from Lemma 5.12 that a projective generator of B is

$$P(B) \equiv \bigoplus_{j=0}^{\ell-1} \mathbb{C}_{j\alpha_i} \otimes P(L).$$

Since $C_{j\alpha_i} \otimes L$ appears as a composition factor of $P(B)$ exactly ℓ times for each j it follows that

$$\dim \mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(B)) = \ell^2. \tag{5.17}$$

Choose an embedding, e, of $P(L)/(C_{-\alpha_i} \otimes L)$ in $C_{-\alpha_i} \otimes P(L)$ and define the $U_\epsilon^{\geq 0}(b)$-map, ρ_0 by

$$\rho_0 : P(L) \longrightarrow P(L)/(\mathbb{C}_{-\alpha_i} \otimes L) \xrightarrow{e} \mathbb{C}_{-\alpha_i} \otimes P(L),$$

and, for $0 \leq j \leq \ell - 1$, let

$$\rho_j = 1 \otimes \rho_0 : \mathbb{C}_{-j\alpha_i} \otimes P(L) \longrightarrow \mathbb{C}_{-(j+1)\alpha_i} \otimes P(L).$$

Note that, as in (5.16), we have

$$\dim \mathrm{im}(\rho_j) = \ell^{\frac{1}{2}(N-2+s(w_0))}(\ell - 1).$$

Let π_j denote the projection

$$\pi_j : P(B) \twoheadrightarrow \mathbb{C}_{-j\alpha_i} \otimes P(L),$$

and let ι_j denote the inclusion

$$\iota_j : \mathbb{C}_{-j\alpha_i} \otimes P(L) \hookrightarrow P(B).$$

We define $\theta_K, \theta_E \in \mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(B))$ as

$$\theta_K = \sum_{j=0}^{\ell-1} \epsilon^{-j} \iota_j \circ \pi_j$$

and

$$\theta_E = \sum_{j=0}^{\ell-1} \iota_{j+1} \circ \rho_j \circ \pi_j.$$

Simple calculations show that, for $1 \leq s \leq \ell$

$$\theta_K^s = \sum_{j=0}^{\ell-1} \epsilon^{-sj} \iota_j \circ \pi_j$$

and

$$\theta_E^s = \sum_{j=0}^{\ell-1} \iota_{j+s} \circ \rho_{j+s-1} \circ \cdots \circ \rho_j \circ \pi_j$$

and also that

$$\theta_K \circ \theta_E = \epsilon^{-1} \theta_E \circ \theta_K.$$

In particular, using (5.17), it follows that

$$\theta_E^\ell = 0 \text{ whilst } \theta_E^s \neq 0 \text{ for } 1 \leq s \leq \ell - 1$$

and that

$$\theta_K^\ell = 1.$$

Hence we have a well-defined surjective algebra map

$$\phi : \mathrm{End}_{U_\epsilon^{\geq 0}(b)}(P(B))^{\mathrm{op}} \longrightarrow U_\epsilon^{\geq 0}(\mathfrak{sl}_2)$$

defined by $\phi(\theta_E) = E$ and $\phi(\theta_K) = K$. By (5.17) the map ϕ must be an isomorphism. The result now follows since B consists of $\dim L = \ell^{\frac{1}{2}(N-2+s(w_0))}$ copies of $P(B)$.

5.4 Complexity two factors.

Recall that Theorem 5.1 was unable to describe factors of complexity two. Using the earlier results in this chapter and a useful lemma from the theory of finite dimensional algebras we can now determine their representation type.

Lemma 5.16. *Let S be a finite dimensional algebra over $\mathbb{C}$. Let G be a finite abelian group acting by automorphisms on S. Then S and the skew group algebra $S * G$ have the same representation type.*

Proof. Suppose we have an inclusion of algebras $S \subseteq T$. Suppose further that T has a S-bimodule decomposition $T = S \oplus M$. Then, by [11, Proposition 2], the representation type of S is a lower bound for the representation type of T.

It's clear that $S * G$ is a bimodule direct summand of S. The character group of G, say H, acts naturally on $S * G$ by

$$\chi(sg) = \chi(g)sg,$$

and by [75, Corollary 5.2] the algebras S and $(S * G) * H$ are Morita equivalent (this uses the fact that $|G|$ is invertible in $\mathbb{C}$). Combining this with the previous paragraph yields the lemma.

Theorem 5.17. *Suppose ℓ satisfies Hypothesis 3 and that $\ell > h$. Let $w_1, w_2 \in W$ and suppose $g \in X_{w_1,w_2}$. If $\ell(w_1) + \ell(w_2) = 2N - 2$ then the algebra $\mathcal{O}_\epsilon[G](g)$ is wild.*

Proof. The proof is based on the simple fact that the truncated polynomial algebra

$$\frac{\mathbb{C}[X,Y]}{(X^\ell, Y^\ell)}$$

has wild representation type if $\ell \geq 3$. This is a consequence of [76, 1.1(c), 1.2].

In order that $\ell(w_1) + \ell(w_2) = 2N - 2$ we must have one of the following for $1 \leq i \leq r$ and $1 \leq j \leq r$:

1. $w_1 = w_0 s_i s_j, w_2 = w_0$ where $i \neq j$ (and the symmetric case obtained by exchanging the roles of w_1 and w_2);

2. $w_1 = w_0 s_i, w_2 = w_0 s_j$ where $i \neq j$;

3. $w_1 = w_0 s_i, w_2 = w_0 s_i$.

It follows from Theorem 1.34 that Cases 1 and 2 are Azumaya so, by Theorem 5.9,

$$\mathcal{O}_\epsilon[G](g) \sim_{\mathrm{Mor}} \bigoplus^{\ell^{r-2}} \frac{\mathbb{C}[X,Y]}{(X^\ell, Y^\ell)}.$$

Therefore $\mathcal{O}_\epsilon[G](g)$, and each of its blocks, has wild representation type.

Suppose we are in Case 3. By Theorem 1.32(i) we can assume without loss of generality that there is an embedding

$$\gamma_g : \mathcal{O}_\epsilon[G](g) \longrightarrow U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-),$$

where

$$g = b_+ b_-,$$

for $b_- \in X_{w_0 s_i, e}$ and $b_+ \in X_{e, w_0 s_i}$ unipotent. Moreover $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$ is a skew group extension of the image of γ_g by the group $\mathbb{Z}_\ell^r$. So, by Lemma 5.16, it is enough to show that $U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-)$ is wild. By Theorem 5.15 the algebras $U_\epsilon^{\leq 0}(b_+)$ and $U_\epsilon^{\geq 0}(b_-)$ are isomorphic to direct sums of either

$$\mathrm{Mat}_s\left(\frac{\mathbb{C}[X]}{(X^\ell)}\right),$$

or

$$\mathrm{Mat}_t\left(U_\epsilon^{\geq 0}(\mathfrak{sl}_2)\right).$$

Since $\mathrm{Mat}_s(A) \otimes \mathrm{Mat}_t(B) \cong \mathrm{Mat}_{st}(A \otimes B)$ it therefore suffices to show that the following algebras are wild:

(i) $\frac{\mathbb{C}[X,Y]}{(X^\ell, Y^\ell)}$;

(ii) $U_\epsilon^{\geq 0}(\mathfrak{sl}_2) \otimes \frac{\mathbb{C}[X]}{(X^\ell)}$;

(iii) $U_\epsilon^{\geq 0}(\mathfrak{sl}_2) \otimes U_\epsilon^{\geq 0}(\mathfrak{sl}_2)$.

It's clear, however, that the algebra in (ii) (respectively in (iii)) is a skew group extension of $\mathbb{C}[X,Y]/(X^\ell, Y^\ell)$ by $\mathbb{Z}_\ell$ (respectively by $\mathbb{Z}_\ell^2$). Applying Lemma 5.16 again and the comments at the beginning of this proof shows that these are indeed wild.

Corollary 5.18. *Suppose ℓ satisfies Hypothesis 3. Let $b \in X_{w_0 s_i s_j, e}$. Then $U_\epsilon^{\geq 0}(b)$ has wild representation type.*

Proof. Let $g \in X_{w_0 s_i s_j, w_0}$ be such that $g = b_+ b_-$ where $b_+ \in X_{e, w_0}$ and $b_- \in X_{w_0 s_i s_j, e}$ are unipotent. By Theorem 1.32(i) we have an algebra isomorphism

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \cong U_\epsilon^{\leq 0}(b_+) \otimes U_\epsilon^{\geq 0}(b_-).$$

Since $b_+ \in X_{e, w_0}$ we have, by Remark 5.2(i) and Theorem 1.29, an isomorphism

$$U_\epsilon^{\leq 0}(b_+) \cong \bigoplus^{\ell^{r-s(w_0)}} Mat_{\ell^{\frac{1}{2}(N+s(w_0))}}(\mathbb{C}).$$

Therefore

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \cong \bigoplus^{\ell^{r-s(w_0)}} Mat_{\ell^{\frac{1}{2}(N+s(w_0))}}\left(U_\epsilon^{\geq 0}(b_+)\right).$$

Therefore $U_\epsilon^{\geq 0}(b_+)$ has the same representation type as $\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r$ and hence, by Lemma 5.16, as $\mathcal{O}_\epsilon[G](g)$. Now apply Theorem 5.17.

5.5 Additional remarks

1. Of course for $\ell \leq h$ satisfying Hypothesis 3 the question of the representation type of $\mathcal{O}_\epsilon[G](g)$ is still open except in the extreme cases of $g \in X_{e,e}$ and $g \in X_{w_0,w_0}$: see Remark 5.2.

2. It is well known the finite dimensional modules of a finite dimensional algebra, A, of complexity one have periodic projective resolutions, [31, Proposition 8.4.4]. If S is a simple A-module whose minimal projective resolution has period one then it is immediate that the block of A containing S is Morita equivalent to $\mathbb{C}[X]/(X^2)$. Proposition 5.3 is due to Farnsteiner [32, Theorem 3.2]. However, we can't go much higher since the group algebra of the quaternions over a field of characteristic two has periodicity four, [1, Chapter IV, Theorem 2.9]. For further details in the case of group algebras see [1, IV.6].

3. Theorem 5.4 is considered in [13] for a general class of Hopf algebras finite over affine central sub-Hopf algebras, proving a conjecture of De Concini and Kac, [23, Conjecture 5.2(c)]. The study of the (non-trivial) finite dimensional algebras over Azumaya points of G is an analogue of regular nilpotent representations in the theory of non-restricted representations of reductive Lie algebras in characteristic p. A description of the corresponding finite dimensional algebras, proved using geometric techniques, can be found in [63, Theorem 11].

Chapter 6

Quivers

By Theorem 1.34(iii) and Theorem 1.29(iii) the algebras $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$ are basic, that is their simple modules are one dimensional. Therefore they must be quotients of path algebras of quivers, [10, Proposition 4.1.7]. Moreover to apply results from the theory of finite dimensional algebras it is often necessary to know the underlying quiver and relations explicitly. In this chapter we achieve this for $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$, finding the quivers are basically Cayley graphs for $\mathbb{Z}_\ell^r$. If we wanted we could use these descriptions together with Gabriel's Theorem, [10, Proposition 4.7.1], to prove that $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$ have wild representation type for $G \neq SL(2)$.

6.1 Quivers for $\mathcal{O}_\epsilon[G]$ and $U_\epsilon^{\geq 0}$.

Recall that a quiver Γ is an oriented graph, that is a set of vertices together with a set of arrows between vertices. A path in Γ is either the "trivial" path e_v associated with a vertex v or a sequence $\nu_n \dots \nu_1$ of arrows with $n \geq 1$ where, if $n > 1$, ν_i ends at the vertex where ν_{i+1} starts for $1 \leq i \leq n-1$. As a vector space over $\mathbb{C}$, the path algebra $\mathbb{C}\Gamma$ has a basis consisting of the paths in Γ. Multiplication of the basis elements is given by composition of paths when possible and is defined to be 0 otherwise.

Let P' be a lattice such that $Q \subseteq P' \subseteq P$ and let $\{\rho_1, \dots, \rho_r\}$ be a basis of P'. Define an $r \times r$ matrix $Z_G(P')$ by

$$Z_G(P')_{ij} = (\rho_i, \alpha_j).$$

We define the following quiver $\Gamma_G(P', Z_G(P'))$. The set of vertices is given by elements of the elementary abelian group $\langle T_1, \dots T_r : T_iT_j = T_jT_i, T_i^\ell = 1\rangle$. There are $2r$ arrows ending at each vertex T^c, denoted $A(T^c, \pm i)$ for $1 \leq i \leq r$. The arrows $A(T^c, \pm i)$ have source $T^{c-Z_G(P')u_i}$, where $u_i = (\delta_{i1}, \ldots, \delta_{ir})$.

The path algebra $\mathbb{C}\Gamma_G(P', Z_G(P'))$ is a $\mathbb{C}$-algebra with generators $e_{T^c}, A(T^c, \pm i)$ and relations

$$e_{T^c} e_{T^b} = \delta_{b,c} e_{T^c}, \tag{6.1}$$

$$e_{T^b} A(T^c, \pm i) = \delta_{b,c} A(T^c, \pm i), \tag{6.2}$$

$$A(T^c, \pm i) e_{T^b} = \delta_{b,c-Z_G(P')u_i} A(T^c, \pm i). \tag{6.3}$$

We define $S_G(P', Z_G(P')))$ to be the $\mathbb{C}$-algebra with generators X_i, Y_i, C_i for $1 \leq i \leq r$, subject to the relations

$$C_i C_j = C_j C_i, \quad C_i^{\ell} = 1 \tag{6.4}$$

$$C_i X_j = \epsilon^{(\rho_i, \alpha_j)} X_j C_i, \quad C_i Y_j = \epsilon^{(\rho_i, \alpha_j)} Y_j C_i. \tag{6.5}$$

Lemma 6.1. *There is an algebra isomorphism between the path algebra $\mathbb{C}\Gamma_G(P', Z_G(P'))$ and $S_G(P', Z_G(P'))$.*

Proof. To ease notation we will denote $\mathbb{C}\Gamma_G(P', Z_G(P'))$ by $\mathbb{C}\Gamma$, $S_G(P', Z_G(P'))$ by S and $Z_G(P')$ by Z. To prove the lemma we will construct an algebra isomorphism $\chi : \mathbb{C}\Gamma \longrightarrow S$ and an inverse $\psi : S \longrightarrow \mathbb{C}\Gamma$. This will be done in three steps.

Step 1: Constructing χ. Since $\mathbb{C}\Gamma$ is generated by the paths in Γ of length less than or equal to one it is enough to define χ on these elements and then check it respects the relations imposed by the quiver. Let $\mathfrak{s} = \{(x_1, \ldots, x_r) : 0 \leq x_i < \ell\}$. We define

$$\chi_0(e_{T^c}) = \ell^{-r} \sum_{x \in \mathfrak{s}} \epsilon^{-c.x} C^x$$

$$\chi_1(A(T^c, +i)) = \chi_0(e_{T^c}) X_i$$

$$\chi_1(A(T^c, -i)) = \chi_0(e_{T^c}) Y_i$$

where $C^x = C_1^{x_1} \ldots C_r^{x_r}$. To check this is a well-defined algebra map we must show that the relations (6.1)-(6.3) are preserved by χ. Consider (6.1). We have

$$\begin{aligned} \chi_0(e_{T^b})\chi_0(e_{T^c}) &= \ell^{-2r} \sum_{x,y \in \mathfrak{s}} \epsilon^{-b.x-c.y} C^{x+y} \\ &= \ell^{-2r} \sum_{z \in \mathfrak{s}} \left(\sum_{x \in \mathfrak{s}} \epsilon^{-b.x-c.(z-x)} \right) C^z \\ &= \ell^{-2r} \sum_{z \in \mathfrak{s}} \left(\sum_{x \in \mathfrak{s}} \epsilon^{(-b+c).x} \right) \epsilon^{-c.z} C^z. \end{aligned}$$

If $b = c$ then

$$\sum_{x \in s} \epsilon^{(-b+c).x} = \ell^r. \tag{6.6}$$

On the other hand if $b \neq c$ then there exists i such that $b_i \neq c_i$. Since ϵ is an ℓth root of unity we have

$$\begin{aligned} \sum_{x \in s} \epsilon^{(-b+c)x} &= \sum_{s=0}^{\ell-1} \epsilon^{(-b_i+c_i)s} \lambda \\ &= 0, \end{aligned} \tag{6.7}$$

where

$$\lambda = \sum_{\substack{x \in s \\ x_i \text{fixed}}} \epsilon^{\sum_{j \neq i}(-b_j+c_j)x_j}.$$

Thus we we have

$$\chi_0(e_{T^b})\chi_0(e_{T^c}) = \delta_{b,c}\chi_0(e_{T^c}) = \chi_0(e_{T^b}e_{T^c}). \tag{6.8}$$

Moreover we now see

$$\begin{aligned} \chi(e_{T^b}A(T^c,+i)) &= \chi_0(e_{T^b})\chi_1(A(T^c,+i)) \\ &= \chi_0(e_{T^b})\chi_0(e_{T^c})X_i \\ &= \delta_{b,c}\chi_0(e_{T^c})X_i \qquad \text{by (6.8)} \\ &= \chi(\delta_{b,c}A(T^c,+i)). \end{aligned}$$

showing half of the relations (6.2) are also preserved by χ. Of course, the same calculation covers the other half and similar ones show (6.3) is preserved by χ.

Step 2: Constructing ψ. We construct ψ, the inverse of χ, by specifying it on generators of S and extending it multiplicatively. Define

$$\psi(C^c) = \sum_{x \in s} \epsilon^{c.x} e_{T^x}, \tag{6.9}$$

$$\psi(X_i) = \sum_{x \in s} A(T^x,+i), \tag{6.10}$$

$$\psi(Y_i) = \sum_{x \in s} A(T^x,-i). \tag{6.11}$$

We can check that ψ and χ are mutually inverse. For example we have

$$\begin{aligned}
\psi \circ \chi_0(e_{T^c}) &= \psi\left(\ell^{-r}\sum_{x\in s}\epsilon^{-c.x}C^x\right) \\
&= \ell^{-r}\sum_{x\in s}\epsilon^{-c.x}\sum_{y\in s}\epsilon^{x.y}e_{T^y} \\
&= \ell^{-r}\sum_{y\in s}\left(\sum_{x\in s}\epsilon^{(-c+y).x}\right)e_{T^y} \\
&= \ell^{-r}\sum_{y\in s}\delta_{c,y}\ell^r e_{T^y},
\end{aligned}$$

where the last equality follows from (6.6) and (6.7). The remainder of the calculation proceeds similarly.

Step 3: ψ is well-defined. To finish the proof we must check the defining relations of S, that is (6.4) and (6.5), are respected by ψ. We have

$$\begin{aligned}
\psi(C_iX_j) &= \sum_{x\in s}\epsilon^{u_i.x}e_{T^x}\sum_{y\in s}A(T^y,+j) \\
&= \sum_{y\in s}\epsilon^{y_i}A(T^y,+j) && \text{by (6.2)} \\
&= \epsilon^{(\rho_i,\alpha_j)}\sum_{y\in s}\epsilon^{y_i-u_iZu_j}A(T^y,+j) \\
&= \epsilon^{(\rho_i,\alpha_j)}\sum_{y\in s}A(T^y,+j)\sum_{x\in s}\epsilon^{u_i.x}e_{T^x} && \text{by (6.3)} \\
&= \psi(\epsilon^{(\rho_i,\alpha_j)}X_jC_i).
\end{aligned}$$

The other relations can be checked in the same way.

It is easy to see that, up to isomorphism, $S_G(P', Z_G(P'))$ is independent of the choice of basis for P'. Indeed if $\{\rho_1,\dots,\rho_r\}$ and $\{\rho'_1,\dots,\rho'_r\}$ are both bases of P' then there exists an element $A\in GL_r(\mathbb{Z})$ such that $A\rho_i=\rho'_i$. Let X_i, Y_i and C_i denote the generators of $S_G(P', Z_G(P'))$ and X'_i, Y'_i and C'_i denote the generators of $S_G(P', Z'_G(P'))$. Then the map given by

$$\begin{aligned}
X'_i &\longmapsto X_i, \\
Y'_i &\longmapsto Y_i, \\
C'_i &\longmapsto C_1^{a_{1i}}\dots C_r^{a_{ri}}
\end{aligned}$$

induces an algebra isomorphism between $S_G(P', Z'_G(P'))$ and $S_G(P', Z_G(P'))$. Therefore, in the future, we will write $S_G(P')$ and $\mathbb{C}\Gamma_G(P')$, making no reference to an explicit basis of P'.

Given any vertex, T^c, of $\Gamma_G(P', Z_G(P'))$ we can associate to any elementary vector, u_i, two paths: $A(T^c, +i)$ and $A(T^c, -i)$. Iterating this we see that given any ordered n-tuple of elementary vectors we have 2^n associated paths ending at T^c. We will denote such paths by, for example, $A(T^c, +i_1 - i_2 \ldots + i_n)$. This path would have source $T^{c-u_{i_1}-u_{i_2}-\cdots-u_{i_n}}$. For brevity we shall denote $A(T^c, +i+\ldots+i)$ and $A(T^c, -i-\ldots-i)$, both paths of length n, by $A(T^c, (+i)^n)$ and $A(T^c, (-i)^n)$ respectively.

Fix a reduced expression of w_0 and hence an ordering on Δ^+. Then, for each $\beta \in \Delta^+$, we have

$$E_\beta^\ell = \sum_{t^\beta \in \mathbb{N}^m} c_{t^\beta} E_{i_1}^{t_1^\beta} \ldots E_{i_m}^{t_m^\beta}, \tag{6.12}$$

where $m > 0$, $1 \leq i_k \leq r$ and $c_{t^\beta} \in \mathbb{C}$. In particular, if $\beta = \alpha_j$ for some $1 \leq j \leq r$ we have $E_\beta^\ell = E_j^\ell$. Similarly, we have

$$F_\beta^\ell = \sum_{v^\beta \in \mathbb{N}^n} c'_{v^\beta} F_{j_1}^{v_1^\beta} \ldots F_{j_n}^{v_n^\beta}, \tag{6.13}$$

where $n > 0$, $1 \leq j_k \leq r$ and $c'_{v^\beta} \in \mathbb{C}$.

Theorem 6.2. *Retain the above notation. The algebra $\mathcal{O}_\epsilon[G]$ is the quotient of the path algebra $\mathbb{C}\Gamma_G(P, Z_G(P))$ by the two-sided ideal J generated by*

1. *The elements*

$$\sum_{t^\beta} c_{t^\beta} A(T^x, (+i_1)^{t_1^\beta} \ldots (+i_m)^{t_m^\beta})$$

and

$$\sum_{v^\beta} c'_{v^\beta} A(T^x, (-j_1)^{v_1^\beta} \ldots (-j_n)^{v_n^\beta})$$

for $\beta \in \Delta^+$ and $x \in \mathbb{Z}_\ell^r$;

2. *commuting squares $A(T^x, +i-j) - A(T^x, -j+i)$ for $1 \leq i, j \leq r$ and $x \in \mathbb{Z}_\ell^r$;*

3. *the elements*

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} A(T^x, (\pm i)^{1-a_{ij}-s}(\pm j)(\pm i)^s)$$

for all $i \neq j$ and $x \in \mathbb{Z}_\ell^r$.

Proof. Recall that the fundamental weights $\{\varpi_1, \dots, \varpi_r\}$ are a basis of P. Since $(\varpi_i, \alpha_j) = \delta_{ij} d_i$ it follows from Lemma 1.32(ii) that $\mathcal{O}_\epsilon[G]$ is a factor of $S_G(P)$. We use the isomorphism $\psi : S_G(P) \longrightarrow \mathbb{C}\Gamma_G(P)$ from the proof of Lemma 6.1 to transfer the relations of $\mathcal{O}_\epsilon[G]$ to $\mathbb{C}\Gamma_G(P)$. These relations are:

1. The elements E^ℓ_β and F^ℓ_β equal zero for $\beta \in \Delta^+$;
2. the elements E_i and F_j commute for $1 \leq i, j \leq r$;
3. the quantum Serre relations for $i \neq j$.

From (6.12) and (6.13) we have representations of both E^ℓ_β and F^ℓ_β. Thus, using (6.9)-(6.11), we obtain the relations

$$\sum_{x \in s} \sum_{t\beta} c_{t\beta} A(T^x, (+i_1)^{t_1^\beta} \dots (+i_m)^{t_m^\beta}) = 0, \quad \text{for } \beta \in \Delta^+,$$

$$\sum_{x \in s} \sum_{v\beta} c'_{v\beta} A(T^x, (-j_1)^{v_1^\beta} \dots (-j_n)^{v_n^\beta}) = 0, \quad \text{for } \beta \in \Delta^+,$$

$$\sum_{x \in s} A(T^x, +i-j) - A(T^x, -j+i) = 0, \quad \text{for } 1 \leq i, j \leq r,$$

$$\sum_{x \in s} \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} A(T^x, (\pm i)^{1-a_{ij}-s}(\pm j)(\pm i)^s) = 0, \quad \text{for } i \neq j.$$

Using the idempotents e_{T^c}, however, we recover the relations in the statement of the theorem.

Remarks 6.3. i) If $\beta \in \Delta^+$ is a simple root then the relevant generators (1) in Theorem 6.2 become $A(T^x, (\pm i)^\ell)$ for $x \in \mathbb{Z}_\ell^r$.
ii) At the time of writing we are unaware of explicit formulae for E^ℓ_β in general. However, several cases are known, see [8, Section 2] for type A_r.

Consider the subquiver, $\Gamma_G^+(P', Z_G(P'))$, of $\Gamma_G(P', Z_G(P'))$ having the same vertices as $\Gamma_G(P', Z_G(P'))$ but only the positive arrows, that is the arrows $A(T^c, +i)$. Similarly let $S^+(P', Z_G(P'))$ be the subalgebra of $S(P', Z_G(P'))$ generated by X_i and C_i for $1 \leq i \leq r$. Again it is easy to see that the algebra $S^+(P', Z_G(P'))$ is independent of the choice of basis for P', so we may omit reference to $Z_G(P')$. Moreover, it follows from the proof of Lemma 6.1 that the path algebra $\mathbb{C}\Gamma_G^+(P', Z_G(P'))$ is isomorphic to $S^+(P')$.

Theorem 6.4. *The algebra $U_\epsilon^{\geq 0}(P')$ is the quotient of the path algebra $\mathbb{C}\Gamma_G^+(P')$ by the two-sided ideal $J(P')$ generated by*

1. *The elements*

$$\sum_{t^\beta} c_{t^\beta} A(T^x, (+i_1)^{t_1^\beta} \dots (+i_m)^{t_m^\beta})$$

for $\beta \in \Delta^+$ and $x \in \mathbb{Z}_\ell^r$;

2. *the elements*

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} A(T^x, (+i)^{1-a_{ij}-s}(+j)(+i)^s)$$

for all $i \neq j$ and $x \in \mathbb{Z}_\ell^r$.

Proof. Follow the proof of Theorem 6.2.

Remark 6.5. As a special case of Theorem 6.4, we recover the corrected version of [21, Theorem 2.4], which gives the quiver and relations for $U_\epsilon^{\geq 0}(Q)$ when $\mathfrak{g}$ is simply laced. Indeed let $\mathfrak{g}$ be simply laced. If $P' = Q$ and we take the simple roots as a basis of Q we see $Z_G(Q)$ is just the Cartan matrix of $\mathfrak{g}$. Therefore $\Gamma_G(Q, Z_G(Q))$ is the quiver considered in [21]. In fact the approach given here generalises that of [21].

The quiver $\Gamma_G^+(P')$ can be thought of as a Cayley graph for $\mathbb{Z}_\ell^r$, a perspective which leads, at least heuristically, to the form of the underlying quivers for all the factors of $U_\epsilon^{\geq 0}$. Indeed let $b \in B^-$. It follows immediately from the discussion in Section 1.6 on the transitivity of the action of $\mathrm{Pic}(U_\epsilon^{\geq 0}) \cong \mathbb{Z}_\ell^r$ on the simple $U_\epsilon^{\geq 0}(b)$-modules that the blocks of $U_\epsilon^{\geq 0}$ are permuted transitively by $\mathbb{Z}_\ell^r$. Thus it is enough to consider only one block to uncover the whole structure of $U_\epsilon^{\geq 0}(b)$. So fix a block B. The simple modules of this block B are parametrised by the simple B-modules and the edges by the first Ext groups between the simple modules. However $\mathbb{Z}_\ell^r$ acts on the simples in a such a way that it is enough only to know the non-split extensions of a fixed simple B-module. It follows that the quiver of B is homogeneous under the action of $\mathbb{Z}_\ell^r$. In fact it is possible, with some work, to use the above approach and results of [75] to reduce the problem of describing the quiver to describing the quiver of a basic local algebra and a group action on such a quiver. For example, in the case of $U_\epsilon^{\geq 0}$ the basic algebra is U_ϵ^+ and the group is $\mathbb{Z}_\ell^r$ whilst in the case of $U_\epsilon^{\geq 0}(b)$ with $b \in X_{w_0 s_i, e}$, such that $w_0\alpha_i \neq -\alpha_i$, the basic algebra is $\mathbb{C}[X]/(X^\ell)$ and the group is trivial. Even if one can describe this basic local algebra, or in particular the non-split extensions of its simple module, it still remains to describe the relations on the path algebra which will yield B. As the relations of Theorem 6.4 suggest, this seems to be a much more subtle problem.

6.2 Additional remarks

1. As pointed out in Remark 6.5 the approach given here generalises that of [21].

2. The possible underlying quivers for basic finite dimensional Hopf algebras have been described in [43].

Bibliography

[1] A. Adem and R.J. Milgram. *Cohomology of Finite Groups*. Number 309 in Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1994.

[2] J.L. Alperin and L. Evens. Representations, resolutions, and Quillen's dimension theorem. *J. Pure Appl. Algebra*, 22:1–9, 1981.

[3] H.H. Andersen, J.C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, 220:321, 1994.

[4] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing, 1969.

[5] M. Auslander, I. Reiten, and S.O. Smalø. *Representation Theory of Artin Algebras*. Number 36 in Cambridge studies in advanced mathematics. Cambridge University Press, first paperback edition, 1995.

[6] G.S. Avrunin and L.L Scott. Quillen stratification for modules. *Invent. Math.*, 66:277–286, 1982.

[7] D.W. Barnes. Spectral sequence constructors in algebra and topology. *Mem. Amer. Math. Soc.*, 53(317), 1985.

[8] R. Bédard. Root vectors for Hecke algebras and quantized enveloping algebras. *J. Alg.*, 175:575–603, 1995.

[9] A.D. Bell and R. Farnsteiner. On the theory of Frobenius extensions and its applications to Lie superalgebras. *Trans. Amer. Math. Soc.*, 335(1):407–424, 1993.

[10] D.J. Benson. *Representations and Cohomology*. Number 30 (1) in Cambridge studies in advanced mathematics. Cambridge University Press, 1991.

[11] V.M. Bondarenko and Yu.A. Drozd. Representation type of finite groups. *J. Soviet Math.*, 20:2515–2528, 1982.

[12] N. Bourbaki. *Groupes et algèbres de Lie, Chapitres 4,5 et 6.* Éléments de Mathématique. Hermann, 1968.

[13] K.A. Brown and K.R. Goodearl. Homological aspects of noetherian PI Hopf algebras of irreducible modules and maximal dimension. *J. Alg.*, 198(1):240–265, 1997.

[14] N. Cantarini. Mod-p reduction for quantum groups. *J. Algebra*, (to appear).

[15] J.F. Carlson. The complexity and varieties of modules. In *Integral representations and their applications*, number 882 in Springer Lecture Notes in Mathematics, pages 415–422. Springer-Verlag, Berlin/New York, 1981. Oberwolfach 1980.

[16] J.F. Carlson. *Modules and Group Algebras.* Lectures in Mathematics. Birkhäuser, Basel, 1996.

[17] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton Univ. Press, 1956.

[18] V. Chari and A. Premet. Indecomposable restricted representations of quantum sl_2. *Publ. Res. Inst. Math. Sci.*, 30(2):335–352, 1994.

[19] V. Chari and A. Pressley. *A Guide to Quantum Groups.* Cambridge University Press, first paperback edition, 1995.

[20] C. Cibils. A quiver quantum group. *Comm. Math. Phys.*, 157:459–477, 1993.

[21] C. Cibils. Half-quantum groups at roots of unity, path algebras and representation type. *Internat. Math. Res. Notices*, 12:541–553, 1997.

[22] W.W. Crawley-Boevey. On tame algebras and bocses. *Proc. London Math. Soc.*, 56:451–483, 1988.

[23] C. De Concini and V.G. Kac. Representations of quantum groups at roots of 1. In A. Connes et al, editor, *Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory*, number 92 in Progress in Mathematics, pages 471–506. Boston, Birkhäuser, 1990.

[24] C. De Concini, V.G. Kac, and C. Procesi. Some quantum analogues of solvable lie groups. In *Geometry and Analysis*, pages 41–65. Tata Inst. Fund. Res., Bombay, 1992.

[25] C. De Concini and V. Lyubashenko. Quantum function algebras at roots of 1. *Adv. Math.*, 108:205–262, 1994.

[26] C. De Concini and C. Procesi. Quantum groups. Springer Lecture Notes in Mathematics 1565. 31-140.

[27] C. De Concini and C. Procesi. Quantum Schubert cells and representations at roots of 1. In G.I. Lehrer, editor, *Algebraic groups and Lie groups*, number 9 in Australian Math. Soc. Lecture Series. Cambridge University Press, Cambridge, 1997.

[28] V.G. Drinfel'd. Quantum Groups. *Proc. Int. Cong. Math.*, pages 798–820, 1986. Berkeley, 1.

[29] Yu. A. Drozd. Tame and Wild matrix problems. In *Representations and quadratic forms*, pages 39–74. Institute of Mathematics, Academy of Sciences, Ukranian SSR, Kiev, 1979. (in Russian).

[30] B. Enriquez. Le centre des algèbres de coordonnées des groupes quantiques aux racines p^{α}-ièmes de l'unité. *Bull. Soc. Math. France*, 122(4), 1994.

[31] L. Evens. *The Cohomology of Groups*. Oxford mathematical monographs. Oxford Science Publications, 1991.

[32] R. Farnsteiner. Periodicity and representation type of modular Lie algebras. *J. reine angew. Math.*, 464:47–65, 1995.

[33] E.M. Friedlander and B.J. Parshall. Support varieties for restricted Lie algebras. *Invent. Math.*, 86:553–562, 1986.

[34] E.M. Friedlander and B.J. Parshall. Geometry of p-unipotent Lie algebras. *J. Alg.*, 109:25–45, 1987.

[35] E.M. Friedlander and B.J. Parshall. Modular representation theory of Lie algebras. *Amer. J. Math.*, 110(6):1055–1093, 1988.

[36] E.M. Friedlander and A. Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.

[37] M. Gerstenhaber. The cohomology structure of an associative ring. *Ann. of Math.*, 78(2), 267-288 1962.

[38] M. Gerstenhaber and S.D. Schack. Algebraic cohomology and deformation theory. In M. Hazewinkel and M. Gerstenhaber, editors, *Deformation Theory of Algebras and Related Structures and Applications*, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 247, pages 11–264. Kluwer Acad. Publ., Dordrecht, 1988.

[39] V. Ginzburg and S. Kumar. Cohomology of quantum groups at roots of unity. *Duke Math. J.*, 69(1):179–198, 1993.

[40] I. Gordon. Cohomology of quantized function algebras at roots of unity. University of Glasgow preprint series, no. 97/14 (submitted).

[41] I. Gordon. Quantised function algebras at roots of unity, path algebras and representation type. University of Glasgow preprint series, no 97/48 (submitted).

[42] I. Gordon. The representation theory of quantised function algebras. University of Glasgow preprint series, no. 98/14 (submitted).

[43] E.L. Green and Ø. Solberg. Basic Hopf algebras and quantum groups. Preprint.

[44] T.J. Hodges and T. Levasseur. Primitive ideals of $\mathbb{C}_q[SL(3)]$. *Comm. Math. Phys.*, 156:581–605, 1993.

[45] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Number 9 in Graduate Texts in Mathematics. Springer Verlag, 1970.

[46] J.C. Jantzen. Kohomologie von p-Lie-Algebren und nilpotente Elemente. *Abh. Math. Sem. Univ. Hamburg*, 56:191–219, 1986.

[47] J.C. Jantzen. *Representations of Algebraic Groups.* Academic Press, Boston, 1987.

[48] J.C. Jantzen. *Lectures on Quantum Groups.* Number 6 in Graduate Studies in Mathematics. Amer. Math. Soc., Providence, R.I., 1996.

[49] J.C. Jantzen. Representations of Lie algebras in prime characteristic. Preprint, 1997.

[50] M. Jimbo. A q-difference analogue of $U(\mathfrak{g})$ and the Yang Baxter equation. *Lett. Math. Phys.*, 10:63–69, 1985.

[51] A. Joseph. On the prime and primitive spectra of the algebra of functions on a quantum group. *J. Algebra*, 169(2):411–511, 1994.

[52] A. Joseph. *Quantum Groups and Their Primitive Ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1995.

[53] M. Kashiwara and T. Tanisaki. The Kazhdan-Lusztig conjecture for Kac-Moody Lie algebras with negative level. *Duke Math. J.*, 77(1):21–62, 1995.

[54] M. Kashiwara and T. Tanisaki. The Kazhdan-Lusztig conjecture for Kac-Moody Lie algebras with negative level. II. Nonintegral case. *Duke Math. J.*, 84(3):771–813, 1996.

[55] D. Kazhdan and G. Lusztig. Affine Lie algebras and quantum groups. *Internat. Math. Res. Notices*, 2:21–29, 1991. in *Duke Math. J.* 62 (1991).

[56] H.F. Kreimer and M. Takeuchi. Hopf algebras and Galois extensions of an algebra. *Indiana Univ. Math. J.*, 30(5):675–692, 1981.

[57] S.Z. Levendorskii and Ya.S. Soibelman. Algebras of functions on compact quantum groups, Schubert cells and quantum tori. *Comm. Math. Phys.*, 139:141–170, 1991.

[58] G. Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Adv. Math.*, 70:237–249, 1988.

[59] G. Lusztig. Finite dimensional Hopf algebras arising from quantized enveloping algebras. *J. Amer. Soc.*, 3:257–296, 1990.

[60] G. Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35:89–114, 1990.

[61] H. Matsumura. *Commutative Ring Theory.* Number 8 in Cambridge studies in advanced mathematics. Cambridge University Press, 1994.

[62] J.C. McConnell and J.C. Robson. *Noncommutative Noetherian Rings.* J. Wiley and Sons, 1988.

[63] I. Mirković and D. Rumynin. Centers of reduced enveloping algebras. To appear Math. Z.

[64] S. Montgomery. *Hopf Algebras and Their Actions on Rings.* Number 82 in Regional Conference Series in Mathematics. American Mathematical Society, Providence, Rhode Island, 1992.

[65] V. Ostrik. Support varieties for quantum groups. q-alg./97110008.

[66] B.J. Parshall and J.-P. Wang. Cohomology of quantum groups: the quantum dimension. *Can. J. Math.*, 45(6):1276–1298, 1993.

[67] D. Passman. *Infinite Crossed Products.* Academic Press Inc., 1989.

[68] A. Premet. Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture. *Invent. Math.*, 121(1):79–117, 1995.

[69] A. Premet. Complexity of Lie algebra representations and nilpotent elements of the stabilizers of linear forms. Preprint, 1996.

[70] A. Premet. Support varieties of non-restricted modules over Lie algebras of reductive groups. *J. London Math. Soc.*, 55(2):236–250, 1997.

[71] S.B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[72] D. Quillen. The spectrum of an equivariant cohomology ring, I and II. *Ann. of Math.*, 94:549–572, 573–602, 1971.

[73] M. Reid. *Undergraduate Algebraic Geometry.* Number 12 in London Mathematical Society Student Texts. Cambridge University Press, 1990.

[74] J. Rickard. The representation type of self-injective algebras. *Bull. London. Math. Soc.*, 22:540–546, 1990.

[75] I. Rieten and C. Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Alg.*, 92:224–282, 1985.

[76] C.M. Ringel. The representation type of local algebras. In *Representations of algebras*, number 488 in Lecture Notes in Mathematics. Springer Verlag, 1975.

[77] J.J. Rotman. *An Introduction to Homological Algebra.* Academic Press, New York, 1979.

[78] L.H. Rowen. *Polynomial Identities in Ring Theory.* Number 84 in Pure and Applied Mathematics. Academic Press, 1980.

[79] H.J. Schneider. Representation theory of Hopf Galois extensions. *Isr. J. Math.*, 72(1):196–231, 1990.

[80] W.C. Waterhouse. *Introduction to Affine Group Schemes.* Number 66 in Graduate Texts in Mathematics. Springer Verlag, 1979.

[81] C.A. Weibel. *An Introduction to Homological Algebra.* Number 38 in Cambridge studies in advanced mathematics. Cambridge University Press, 1995.