# On the Equilibrium Structure of Some Problems in Game Theory and Economics

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## Declaration

I declare that the work presented in this dissertation it is my own, that due credit has been given whenever required and references to original sources have been given throughout, any omissions or mistakes are mine and should not be attributed to the original authors. Moreover, the results contained here have not been submitted or presented in any other Higher Education institution in the United Kingdom or elsewhere.

Tikz code for the diagrams in this thesis has been generated using ChatGPT. I declare however that the resulting diagrams are accurate and have been verified and compared to plots generated via relevant software.

**Abstract**

The thesis studies equilibrium in a variety of settings relevant to economics and game theory.

Chapter 1 presents structure theorems for optimisation problems. These results are in the spirit of the celebrated theorem by Mertens and Kohlberg concerning the existence of a homeomorphism between the space of games and the graph of the Nash correspondence. In particular, chapter 1A establishes an analogue of the theorem by Mertens and Kohlberg for the case of simultaneous linear programmes. Meanwhile, chapter 1B presents a generalisation to monotone variational inequalities. Specifically, the chapter proves that given a closed translation invariant subset of the space of variational monotone inequalities, there exists a homeomorphism between the set and the restriction of the graph of the correspondence that associates to each variational inequality in the given set its set of solutions. Chapter 1B also introduces a weakening of monotonicity, $\omega$-monotonicity, and an associated structure theorem which is used to derive a generalisation of Kohlberg-Mertens due to Predtetchinski for games with continuous actions and concave differentiable pay-offs.

Chapter 2 establishes that given an algebraic curve on the square $[0,1]^2$ which does not intersect the top or bottom of the square, i.e., does not intersect $\{0,1\} \times [0,1]$, there exists a binary game whose Nash equilibrium set contains a component that projects injectively onto the original curve. Moreover, individual coordinates of the inverse of this projection are affine functions. If, in addition to the previous condition, the curve intersects each fibre $\{t\} \times (0,1)$, for every $t \in [0,1]$, at least once. Then the whole Nash equilibrium set projects injectively onto the curve instead. Several novel examples of these results are also provided. Notably, we show that there exists a binary game whose set of equilibria is equivalent to a set consisting of a horizontal line segment and a parabolic segment meeting at a zero-degree angle. This equivalence solves a conjecture proposed by Levy.

Chapter 3 modifies an equilibrium notion due to Azevedo and Gottlieb for competitive markets with adverse selection. Following Azevedo and Gottlieb, an equilibrium is characterised by firms making no profits from traded contracts; consumers choosing their most preferred option and robustness to certain perturbations. Unlike Azevedo and Gottlieb, where agents are required to behave optimally in the perturbation, we only require that agents behave $\varepsilon$-optimally. Equilibrium is shown to exist under the assumptions that agents' utility is uniformly Lipschitz in price, and uniformly Lipschitz in contract characteristics. Equilibrium is shown to have the same properties as the original definition by Azevedo and Gottlieb; notably, the equilibrium price vector is a Lipschitz function of contract characteristics. A significant advantage however is that the set of equilibria are shown to satisfy natural continuity properties in the set of contract characteristics. This property is shown to not hold for Azevedo-Gottlieb equilibria. More generally, the equilibrium notion is simultaneously restrictive enough to satisfy the regularity properties Azevedo-Gottlieb equilibrium does, while being significantly easier to verify the robustness to perturbation stipulated in their definitions.

# Introduction

## To the thesis

This thesis concerns the rigorous analysis of equilibria and its properties. Formally, in the economics literature, the notion of *equilibrium* depends heavily on the context (historically, the term was adopted by analogy with *mechanical equilibrium*, which refers to the state of a physical system where all forces acting on it cancel each other out). In that regard, in a price-taking environment, equilibrium refers to a situation where prices are such that demand and supply are simultaneously met. Meanwhile, in game theory, equilibrium often refers to *Nash equilibrium*; a state of play where no agent can gain by altering their action. A more suitable terminology would thus be that of *solution concept*, a set of assumptions that characterise *rational behaviour* by agents in the given context (in keeping with tradition, this term will be rarely used). Herein, both equilibrium notions described earlier are indeed known to yield more than one possible solution (often an *odd* number of them). In game theory, this can be attributed, for instance, to Nash equilibria ignoring the sequence of play, or because it does not constrain behaviour enough. Moreover, if some agents have relevant information which is not known to others, equilibria where some actions remain unused may arise because uninformed agents are required to make unsound assumptions about the behaviour of informed ones. In such situations, it is necessary to introduce a *refinement*, a set of additional assumptions on rational behaviour that further restrict the set of possible outcomes. Formally, most refinements rely on introducing small *perturbations* on the parameters of the model (e.g., pay-offs, beliefs, etc.) which rule out some undesirable solutions. Although these issues may not be addressed directly in this thesis, they play a significant role in the background. In that respect, chapter 1 presents formal tools for the study of generic equilibrium properties in a variety of settings. On the other hand, equilibrium multiplicity features heavily in chapter 2, where we construct uncountable Nash equilibrium sets satisfying certain algebraic requirements. Lastly, chapter 3 discusses an equilibrium notion and refinement in the context context of screening markets, in particular insurance markets.

This thesis is divided into 3 chapters. For ease of presentation, chapter 1 is sub-divided into chapters 1A and 1B. Each chapter is as self-contained as possible and contains sections compiling any required mathematical preliminaries.

## To chapter 1

Chapter 1 studies the structure of the *solution correspondence* associated with optimisation problems. Informally, the solution correspondence is the multi-valued mapping that associates to each problem in a given class (suitably parametrised) the set of its solutions. In broad terms, the chapter focuses on both the class of linear problems and the class of non-linear problems. The chapter is thus split into two parts, chapter 1A and chapter 1B, each devoted to one class of problems.

The motivation behind the analysis of the solution correspondence comes from the famous result by Kohlberg and Mertens (see [24]) related to the *Nash correspondence*. Given a game with finite agents where each agent can choose among finite choices; the Nash correspondence returns its set of *mixed Nash equilibria* (so that players are allowed to randomise across their actions). Kohlberg and Mertens proved that the graph of this correspondence is topologically equivalent to the underlying space of games. A game is parametrised by a vector resulting from “flattening” its pay-off matrix. The space of games is thus the set of all such vectors. The theorem of Kohlberg and Mertens provides an important tool in analysing generic properties. In particular, the theorem allows for a simple proof of the fact that almost all games contain an odd number of equilibria. In addition, the theorem is helpful in characterising sets of equilibria that are stable under pay-off perturbation. Moreover, the theorem also plays a role in the computation of equilibria via the global Newton's method (see [19]).

Given the versatility of the Kohlberg-Mertens theorem it seems desirable to obtain generalisations. Chapter 1 tackles this question by first reformulating the problem as the study of the graph of the solution correspondence to *simultaneous linear problems.* A simultaneous linear problem arises when finding best replies, where each agent maximises pay-offs given the actions of others. Pay-off maximisation is equivalent to solving a linear programme where the objective depends on the solution of other, similar, problems. Chapter 1A then re-derives the original result of Kohlberg and Mertens in this abstract setting, while also laying the groundwork for a generalisation. This generalisation is tackled in chapter 1B, where we consider *monotone variational inequalities.* A monotone variational inequality generalises a concave programme by dropping differentiability assumptions. Chapter 1B generalises the result of Kohlberg and Mertens to this setting. Moreover, the chapter also considers a further generalisation to variational inequalities satisfying a weakening of the condition of monotonicity. This weakening is particularly amenable to the study of games with differentiable pay-offs. Indeed, under an additional differentiability assumption, the main result in chapter 1B is shown to be equivalent to a previous result by Predtetchinski (see [34]).

## To chapter 2

Chapter 2 studies the Nash equilibria in *binary games.* A binary game is a game where the number of players is finite but such that each player has two actions available to them. In particular, the chapter contributes to the literature concerning *Nash Universality.* In broad terms, Nash Universality refers to the informal conjecture that any compact non-empty *semi-algebraic set* can be realised as the set of Nash equilibria of some game. A semi-algebraic set refers to a set that can be described by a finite number of polynomial equalities and inequalities.

Chapter 2 shows that given an *algebraic curve* (the set of zeroes of a polynomial in two variables) in the unit square $[0,1]^2$ which does not intersect the top or bottom of the square, i.e., does not intersect $\{0,1\} \times [0,1]$; there exists a binary game containing a component that projects injectively onto the curve. Moreover, the inverse of this projection is such that each component is an affine mapping. If in addition, the intersection between the curve and each fibre $\{t\} \times (0,1)$, $t \in [0,1]$, is non-empty, then the whole Nash equilibrium of the set projects injectively onto the curve. In particular, the chapter constructs a curve consisting of a parabolic segment and a horizontal segment meeting at 0-degree angle. As a result, the curve presents a kink at the point of intersection. This feature is interesting as it implies that the set falls outside the scope of previous construction methods. In addition, the construction in this chapter establishes a conjecture by Levy (see [27]).

The chapter also provides several other novel examples of Nash equilibrium sets that were previously unknown in the literature. We note that the results in the chapter do not rely on machinery from real algebraic geometry. Instead, all proofs are fairly elementary and rely on concepts and approaches from game theory in the spirit of previous work by Chin et al. (see [13]), Levy (see [27]) and, Vigeral and Viossat (see [41]).

## To chapter 3

Chapter 3 focuses on equilibria in markets with adverse selection, insurance markets being a notable example. In particular, the chapter focuses on a setting introduced by Azevedo and Gottlieb (see [5]) but modifies their equilibrium notion. The economy modelled in the chapter consists of both consumers wishing to purchase insurance contracts and firms providing them. Firms do not observe individual consumer characteristics and operate in a competitive market. An equilibrium in the Azevedo-Gottlieb sense then requires that each contract traded in the market makes zero profits (note that this is a stricter condition than firms making no profit at all in aggregate); consumers choose their most preferred contracts; and it is robust to perturbations. By a perturbation, Azevedo and Gottlieb mean an economy with finite contracts where, in addition to the original consumers, a small amount of "behavioural consumers" have been added to each of the finitely many contracts. These behavioural agents wish to purchase the specific contract to which they have been assigned and can do so at no cost to firms.

Under the assumption that marginal willingness to pay for contract characteristics is bounded, equilibrium exists in the Azevedo-Gottlieb model. It enjoys many nice properties, such as equilibrium prices being a Lipschitz continuous function of contract characteristics. It is however, discontinuous with respect to changes on contract characteristics. To see this, consider the following example of an economy consisting of two agents (both equally likely, for simplicity): a consumer whose willingness to pay for insurance is zero, and a second consumer whose willingness to pay for a unit of insurance is 1. Firms can sell a single contract offering a unit of insurance at unit cost. The Azevedo-Gottlieb equilibrium in this economy is the equilibrium where the consumer with unit willingness to pay purchases insurance at unit price, while the other type does not. Note that although the consumer who purchases insurance is indifferent between buying and not buying, robustness demands that they must purchase in equilibrium, since in any perturbation behavioural agents drive the cost of insurance below 1. Now, suppose that firms could sell an additional contract offering an infinitesimal amount of insurance $\delta x$, which costs firms exactly $\delta x$ when provided to the consumer with unit willingness to pay, and nothing when provided to the other type. Then the new Azevedo-Gottlieb equilibrium would be one in which the risky type mixes between purchasing either contract with a probability that does not depend on $\delta x$. Once again, this behaviour occurs because, were the risky type not to purchase the new contract in the perturbation, its price would be driven down to zero, which could make it more preferable. Hence, equilibria do not converge as $\delta x \to 0$.

To recover continuity, we propose that instead of behaving optimally in the perturbation, agents choose their most preferred choice "up to $\varepsilon$" (more precisely, as in trembling-hand or perfect equilibrium notions, we take $\varepsilon \to 0$ along the sequence of perturbations). In this way, the consumer can be made to mix in the original economy, which resolves the continuity issue. Weakening the requirement that agents optimise in the perturbation is also interesting as it can be applied to settings where optimisation may not be possible. Under some mild Lipschitz conditions on consumers' pay-offs, which translate into agents having bounded marginal willingness to pay, as well as the requirement that the marginal disutility from price increases remains both bounded away from zero and finite, the new equilibrium notion can be shown to exist. It also enjoys all the original properties that the Azevedo-Gottlieb equilibrium satisfies, in addition to equilibria being robust when the set of contracts changes (i.e., it satisfies continuity). The chapter also addresses some minor gaps in the Azevedo and Gottlieb's proof of equilibrium existence. The chapter also provides a generalisation of their argument for the existence of equilibria in perturbations using notions from set convergence.

# Chapter 1A

# The Structure of Solution Sets for Optimisation Problems I: Linear Problems

## 1A.1 Introduction

The celebrated theorem due to Kohlberg and Mertens (see theorem 1 in section 3.2 of [24]) has played a central role in the study of equilibrium under pay-off perturbation for finite games, as well as in other related questions. As a result, attempts to broaden its scope have been undertaken. Notably, Pahl has extended Kohlberg and Mertens' result to games where action sets are polytopes (see [32]). Meanwhile, Predtetchinski has extended the result to continuous games where action sets are arbitrary convex sets and pay-offs are differentiable (see [34]). Aside from work by Laraki and co-authors (see [26]), most extensions have been confined to extending the class of games for which the theorem of Kohlberg and Mertens holds. Chapters 1A and 1B take a different approach by reformulating (and extending) Kohlberg and Mertens' result to commonly encountered problems in optimisation.

The present chapter discusses linear programmes (the problem of maximising a linear functional subject to linear constraints), while chapter 1B presents generalisations of these results to a class of non-linear problems with convex constraints. Furthermore, "coupled" or "simultaneous" problems are also discussed. This latter class consists of a finite number of problems each depending on parameters that are determined by the remaining problems. Solving simultaneous problems can thus be viewed as an exercise in optimisation and finding fixed points. Instances of such simultaneous problems are well-known in the theory of games as they arise naturally when describing the Nash equilibria of a particular game. Indeed, if the simultaneous problem consists of linear programmes the results presented here are equivalent to the structure theorem in [32] (see lemma 3.7 and section 5.1 in the paper's appendix). Note that Pahl also discusses robustness and stability of equilibria in such a setting.

If instead the individual problems are variational inequalities the results are related to that of Predtetchinski's extension of the result of Kohlberg and Mertens (see [34] and [35]). Recall that Predtetchinski's work concerns pure Nash equilibria of games with differentiable pay-offs and convex action sets. The extension to mixed strategies remains, to the author's knowledge, open. Note that although references are made to non-linear problems in this introduction a complete discussion is postponed until chapter 1B.

In the context of chapters 1A and 1B, the notion of *structure theorem* refers to theorems concerning the topological equivalence (i.e. the existence of a continuous bijection with a continuous inverse, a *homeomorphism*) between the graph of the solution correspondence (the multi-valued mapping associating to each individual problem the set of its solutions) and the underlying set of possible problems. For reference, when considering linear programmes the underlying space is $\mathbb{R}^n$; the solution correspondence then associates to each $z \in \mathbb{R}^n$ the set of maximisers of

$$\langle z, x \rangle = z_1 \cdot x_1 + \ldots + z_n \cdot x_n$$

over a certain polyhedral set. The main difficulty in establishing these results is that a given problem might have more than one solution. Indeed, if uniqueness of solutions held for every problem then the solution correspondence would in fact be a continuous function and the result would become trivial. In that case the projection mapping provides the desired homeomorphism. More formally,

**Proposition 1A.1.** *Let $X$, $Y$ be two topological spaces and $f : X \longmapsto Y$ a continuous mapping. The set $X$ is homeomorphic to*

$$\{(x, y) \in X \times Y : y = f(x)\}$$

*endowed with the subspace topology inherited from the product topology over $X \times Y$.*

Interestingly, when considering the case of linear programming there exists a dense subset of values for which the correspondence is single valued. This remains true when considering variational inequalities, but does not generalise to coupled simultaneous problems, be it linear or otherwise. It would be of interest to study whether such uniqueness over a dense set gives a sufficient condition for the graph of a correspondence to be homeomorphic to its domain.

Results in this chapter provide a much needed systematic review of known results in the literature, while also introducing new ideas. In particular, the set of simultaneous problems is often identified with some (very large) Euclidean space. When considering simultaneous linear programmes, this identification is only guaranteed if the action sets of players are sufficiently large. Instead, the general approach is to identify the set of simultaneous problems as a subset of a space of continuous functions. This issue is often overlooked in the literature, but plays an important role when considering extensions (see Chapter 1B). In addition, the techniques used in proving structure theorems are often seen as not intuitive, particularly with respect to the choice of projections. While we do not claim to fully resolve this issue, the chapter endeavours to provide some motivation for this. Moreover, analysing the techniques used in a proof helps motivate any generalisations made. Lastly, by formalising a technique from Bich and Fixary [9] (see second half of the proof of theorem 2.2 in section 4.3 in the appendix of [9]), it is possible to establish that the properness of a certain homotopy (see lemma 1A.17) does not depend on properties of the Euclidean space. Note that standard arguments, which usually follow Kohlberg and Mertens' original lines, often rely on the Euclidean norm on the space.

This chapter is structured as follows: section 2 introduces some of the concepts used in the remainder of the chapter formally. In particular, a linear programme is defined as the problem of maximising a linear functional over the embedding of a given polytope in a larger dimensional space. This approach is similar to that in [32] (see section 3 there, particularly subsections 3.1 and 3.2). The main difference being that the description in [32] emphasizes the geometry of the action sets, whereas in the present chapter an algebraic description is favoured instead. Section 3 then introduces the proof of the structure theorem. The argument follows [18] and [20]. This section also reviews some consequences of the formalization of linear programme. Notably the existence of Kuhn-Tucker multipliers being equivalent to the existence of solutions to linear programmes. In addition, an equivalent condition in terms of fixed points is also reviewed. Section 4 introduces the notion of a simultaneous linear programme and extends the characterisation of solutions given in section 2 to this setting. Section 5 then discusses the extension of the structure theorem in section 4 to the simultaneous case. Finally, section 6 discusses further comparisons between the formulation presented here and those in the literature.

For convenience, structure theorems are divided into 2 statements called Kohlberg-Mertens I and II. This nomenclature is to emphasize the connection to their original work; while allowing for a more analytical presentation. Note that in the original paper of Kohlberg and Mertens the theorems appear as a single statement (see theorem 1 in section 3.2 of [24]). Note further that combining theorem 1A.32 and corollary 1A.36.1 results in their original result.

## 1A.2 Preliminaries

Denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^n$ and by $\| \cdot \|_2$ the Euclidean norm induced by it. Moreover, given $x, y \in \mathbb{R}^n$ write $x \leqslant y$ if $x_i \leqslant y_i$ for every $i \in \{1, \dots, n\}$. Lastly, let $\mathcal{M}_{m \times n}$ refer to the set of all real matrices with $m$ rows and $n$ columns.

**Definition 1A.2** (Linear programme)**.** Given a triple $(z, A, b)$ with $z \in \mathbb{R}^n$,

$A \in \mathcal{M}_{m\times n}$ and $b \in \mathbb{R}^m$, a *linear programme* is the problem

$$\max_{x\in\mathbb{R}^n} \langle z, x\rangle \text{ s.t. } Ax = b,\ x \geqslant 0. \tag{1A.1}$$

For a triple $(z, A, b)$ describing a linear programme the set $\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\}$ is referred to as the *feasible set.* Any element $x_0$ of this set is then said to be *feasible.* It is moreover a *solution of the linear programme*, or simply a *solution*, if it satisfies $\langle z, x_0\rangle \geqslant \langle z, y\rangle$ for arbitrary $y$ in the feasible set.

*Remark* 1A.3. Observe that the above definition of linear programme is not the most general. It in fact corresponds to programme in *standard form.* Note further that it contrasts to the preferred form of a linear programme, the *canonical form*, used in applications:

$$\max_{x\in\mathbb{R}^n} \langle z, x\rangle \text{ s.t. } Ax \leqslant b,\ x \geqslant 0. \tag{1A.2}$$

There are two reasons standard form is chosen. First, it turns out that this formulation is more convenient when discussing the simultaneous case (see corollary 1A.27.1 below). Second, recall that a programme in canonical form may be brought into standard form by means of slackness variables (as can any general programme). Indeed, let $s_1, \ldots, s_m$ be defined by

$$s_i + \sum_{j=1}^{n} A_{ij}x_j - b_i = 0,\ \ s_i \geqslant 0. \tag{1A.3}$$

The programme can then be written as

$$\max_{y\in\mathbb{R}^n\times\mathbb{R}^m} \langle (z, 0), y\rangle \text{ s.t. } A'y = b,\ y \geqslant 0. \tag{1A.4}$$

where $y = (x, s) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$A' = \begin{bmatrix} A & I \end{bmatrix}.$$

In this form, the results in this section may be applied to the class (of standard form linear programmes) defined by the set $\{((z, 0), A', b)\}_{z\in\mathbb{R}^n}$. Alternatively, it is possible to mirror the arguments presented here to develop an analogous theory for programmes in canonical form. However, in so far as the feasible set is non-empty and compact the main results of such theory would follow from the general theory presented in chapter 1B.

Given pair $(A, b)$ with $A \in \mathcal{M}_{m\times n}$ and $b \in \mathbb{R}^m$, for the class of linear programmes parametrised by the set $\{(z, A, b)\}_{z\in\mathbb{R}^n}$ to have solutions for arbitrary $z \in \mathbb{R}^n$, it is necessary that the feasible set is non-empty and compact. Conditions for these are provided in the next

**Definition 1A.4** (Admissible Programme)**.** An ordered pair $(A, b)$ with $A \in \mathcal{M}_{m\times n}$ and $b \in \mathbb{R}^m$ is an *admissible programme* if it satisfies the following requirements

(i) $0 < \text{rank(A)} = m \leqslant n$.

(ii) Given $y \in \mathbb{R}^m$ satisfying $A^T y \geqslant 0$ we have $\langle b, y\rangle \geqslant 0$.

(iii) There exists $i \in \{1, \dots, m\}$ such that $b_i > 0$ and $\min_{j\in\{1,\dots,n\}} A_{ij} > 0$.

(iv) Given $x$ in the feasible set, the matrix

$$C(x) = \begin{bmatrix} A \\ -I_{k\times n} \end{bmatrix},$$

where $k$ is the number of components of $x$ such that $x_i = 0$ and

$$(I_{k\times n})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

satisfies rank $C(x) = m + k \leqslant n$.

*Remark* 1A.5*.* Note that if $x_i > 0$ for every $i \in \{1, \dots, n\}$ then $C(x) = A$ by definition. Observe that condition (i) then implies that $C(x)$ is of full rank automatically.

*Remark* 1A.6*.* Condition (i) is standard, it simply amounts to assuming that the set of equations defining the feasible set is minimal, in the sense that all redundant equations have been eliminated.

Condition (ii) along with the lemma of Farkas (see e.g., chapter 2 of [29]) implies that the feasible set is non-empty.

Condition (iii) ensures the feasible set is compact (a formal proof is given in proposition 1A.7). Observe that, geometrically, given $i \in \{1, \dots, m\}$, the equation $\sum_{j=1}^{m} A_{ij}x_j = b_j$ defines a hyperplane on $\mathbb{R}^n$. Condition (iii) imposes the existence of a hyperplane whose normal vector (i.e., the vector $(A_{i1}, \dots, A_{in})$) lies in the positive orthant. For linear programmes, the condition $b_i > 0$ may be omitted from (iii). It is however relevant when considering simultaneous linear programmes (see section 3) as it ensures that the Kohlberg-Mertens decomposition is possible (see corollary 1A.27.1 and corollary 1A.27.1 below). Geometrically, imposing $b_i > 0$ implies that the feasible set does not collapse to just the origin.

Lastly, condition (iv) shows that for every $x$ in the feasible set $C(x)$ has full rank; hence $C(x)$ defines an injective mapping. This condition is placed to guarantee that Kuhn-Tucker multipliers (a precise definition is given in theorem 1A.10 (ii) below) are unique. For a formal proof see 1A.40 in the appendix. Alternatively, see e.g. [42] for a discussion on uniqueness of the multipliers in a broader setting.

**Proposition 1A.7.** *Let $A \in \mathcal{M}_{m\times n}$ satisfy requirement (iii) in definition 1A.4, the set*

$$\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\}$$

*is compact.*

*Proof.* If

$$\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\} = \emptyset$$

there is nothing to prove. Suppose otherwise that the set is non-empty, the fact that is closed is straightforward. To show it is bounded, assume without loss of generality, that $b_1 > 0$ and $A_{1j} > 0$ for $j \in \{1, \dots, n\}$. Let $x$ be any element in the feasible set and $i \in \{1, \dots, n\}$, then

$$0 \leqslant x_i \min_{j\in\{1,\dots,n\}} A_{1j} \leqslant \sum_{j=1}^{n} A_{1j}x_j = b_1. \tag{1A.5}$$

Hence

$$\|x\|_2 \leqslant \frac{\sqrt{n}\cdot b_1}{\min_{j\in\{1,\dots,n\}} A_{1j}}.$$

$\blacksquare$

*Remark* 1A.8. Condition (iii) in definition 1A.4 is sufficient to ensure the compactness of

$$\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\},$$

but it is not necessary. This may be illustrated by the following example:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \tag{1A.6}$$

and $b = [0\ 1]^T$ the set

$$\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\} = \{(1,1)\}$$

is obviously bounded.

## 1A.3 The structure theorem

Recall that given a non-empty, closed and convex $K \subset \mathbb{R}^n$; for a fixed $x \in \mathbb{R}^n$ there exists a unique $y \in K$ that minimises $\|x - \cdot\|_2$ over $K$. Denote this minimiser by $\rho(x, K)$. The *projection mapping onto* $K$, or simply the *projection*, is the mapping $x \mapsto \rho(x, K)$. By a slight abuse of notation this mapping is also denoted by $\rho(\cdot, K)$ throughout. This mapping satisfies:

**Proposition 1A.9.** *Let $K \subset \mathbb{R}^n$ non-empty, closed and convex. Then for any $x_0, x_1 \in \mathbb{R}^n$*

$$\langle x_0 - \rho(x_0, K), y - \rho(x_0, K)\rangle \leqslant 0 \quad \forall y \in K. \tag{1A.7}$$

$$\|\rho(x_0, K) - \rho(x_1, K)\|_2 \leqslant \|x_0 - x_1\|_2. \tag{1A.8}$$

A proof of proposition 1A.9 can be found in, e.g., [22].

Let $\alpha = (A, b)$ be an admissible programme. Accordingly, denote by $K(\alpha)$ its feasible set. Definition 1A.4 implies $K(\alpha)$ is non-empty, convex and compact. The continuity of $x \mapsto \langle z, x\rangle$ implies that the *solution set*

$$E(z,\alpha) = \{x \in K(\alpha) : \langle z, x\rangle \geqslant \langle z, y\rangle \ \forall y \in K(\alpha)\}$$

is non-empty for arbitrary $z \in \mathbb{R}^n$.

Given $z \in \mathbb{R}^n$, the following theorem contains well-known equivalent conditions for $x_0$ to be a solution of the linear programme $(z, \alpha)$. In particular, condition (iii) shows that a requirement for $x_0 \in E(z, \alpha)$ is that $x_0$ must be a fixed point of a certain mapping. The proof of the structure theorem (see 1A.14) rests on this fact. Moreover, this idea underpins most of the work in chapter 1B also.

**Theorem 1A.10.** *Let $z \in \mathbb{R}^n$ and $\alpha = (A, b)$ an admissible programme. The following are equivalent:*

*(i)* $x_0 \in E(z, \alpha)$.

*(ii) There exists a pair* $(\nu, \mu) \in \mathbb{R}^m \times [0, \infty)^n$ *such that* $z = A^T\nu - \mu$ *and* $\langle \mu, x_0\rangle = 0$.

*(iii)* $x_0 = \rho(z + x_0, K(\alpha))$

*Proof.* For the proof of (i)⇒(ii) see e.g. [10].

To show that (ii)⇒(iii), let $y \in K(\alpha) \setminus \{x_0\}$ and observe that

$$\langle z, x_0 - y\rangle = \langle A^T\nu, x_0 - y\rangle - \langle \mu, x_0\rangle + \langle \mu, y\rangle = \langle \nu, A(x_0 - y)\rangle + \langle \mu, y\rangle \geqslant 0. \tag{1A.9}$$

Then

$$\|z + x_0 - y\|_2^2 = \|z\|_2^2 + 2\langle z, x_0 - y\rangle + \|x_0 - y\|_2^2 > \|z\|_2^2. \tag{1A.10}$$

The point $x_0$ is thus the closest to $z + x_0$ and (iii) follows.

To prove that (iii)⇒(i), let $y \in K(\alpha)$ and observe that by proposition 1A.9

$$0 \geqslant \langle z + x_0 - \rho(z + x_0, K(\alpha)), y - \rho(z + x_0, K(\alpha))\rangle = \langle z, y - x_0\rangle. \tag{1A.11}$$

■

*Remark* 1A.11. As mentioned to earlier, the elements of the pair $(\nu, \mu)$ in condition (ii) in theorem 1A.10 are referred to as the *Kuhn-Tucker multipliers.* The mapping $x \mapsto \rho(z + x, K(\alpha))$ appearing in condition (iii) is usually referred to as the *Gul-Pierce-Stachetti mapping* (see [20]) in the economics and game theory literature. Although it appears that it had been introduced earlier in unrelated work by Stampaccia and co-authors. The mapping $x \mapsto \rho(z + x, K(\alpha))$ is also closely related to the *natural mapping* in optimisation, see proposition 1.5.8 in [16].

**Definition 1A.12** (Equilibrium set)**.** For a given an admissible programme $\alpha$ the *equilibrium set* $E(\alpha)$ is the set $\cup_{z \in \mathbb{R}^n}\{z\} \times E(z, \alpha)$.

*Remark* 1A.13. In the form described in definition 1A.12, the equilibrium set $E(\alpha)$ is the graph of the multi-valued mapping $z \mapsto E(z, \alpha)$ viewed as a subset of $\mathbb{R}^{2n}$. In particular, $E(\alpha)$ could be endowed with the usual Euclidean distance. It will be however more convenient employ the following norm:

$$\|(z, x)\|_{E(\alpha)} = \max\{\|z\|_2, \|x\|_2\}, \tag{1A.12}$$

where $z \in \mathbb{R}^n$ and $x \in E(z, \alpha)$. This convention is adopted when considering other product spaces.

**Theorem 1A.14** (Kohlberg-Mertens I)**.** *For a given admissible programme $\alpha$ there exists a homeomorphism $\psi : \mathbb{R}^n \longmapsto E(\alpha)$.*

*Proof.* Define the Lipschitz continuous mappings $\varphi : (z, x) \in E(\alpha) \mapsto z + x$ and $\psi : z \in \mathbb{R}^n \mapsto (z - \rho(z, K(\alpha)), \rho(z, K(\alpha)))$. First, it is necessary to show that $\psi$ is well-defined, i.e. $\psi(\mathbb{R}^n) = E(\alpha)$. Fix $z_0 \in \mathbb{R}^n$ and write $z = z_0 - \rho(z_0, K(\alpha))$ and $x = \rho(z_0, K(\alpha))$, then

$$\rho(z + x, K(\alpha)) = \rho(z_0, K(\alpha)) = x. \tag{1A.13}$$

Theorem 1A.10 (iii) implies that $x \in E(z, \alpha)$, thus

$$\psi(z_0) = (z, x) \in E(\alpha).$$

This proves $\psi(\mathbb{R}^n) \subset E(\alpha)$. The reverse inclusion follows since $\varphi$ is a right-inverse of $\psi$, i.e.,

$$(\psi \circ \varphi) = \mathrm{id}_{E(\alpha)}. \tag{1A.14}$$

To establish this observe that

$$(\psi \circ \varphi)(z, x) = \psi(z+x) = (z+x-\rho(z+x, K(\alpha)), \rho(z+x, K(\alpha))). \quad (1A.15)$$

Since $x \in E(z, \alpha)$ theorem 1A.10 (iii) implies $\rho(z+x, K(\alpha)) = x$, which yields equation (1A.14).

To complete the proof, it remains to show that $\varphi$ is a left-inverse of $\psi$. Note therefore that,

$$(\varphi \circ \psi)(z) = \varphi(z - \rho(z, K(\alpha)), \rho(z, K(\alpha))) = z. \quad (1A.16)$$

■

*Remark* 1A.15. The uniqueness of the Kuhn-Tucker multipliers (in the form of condition (iv)) in definition 1A.4 was employed in [18] in a setting where it holds naturally. In the interest of consistency, it has been kept here. Nevertheless, it can be seen from the proofs of theorems 1A.10 and 1A.14 that uniqueness is in fact superfluous and might be dropped. As already specified, only the equivalence of conditions (i) and (iii) as stated in theorem 1A.10 is needed.

Now, recall that two continuous mappings $f, g : X \longmapsto Y$ are *homotopic*, $f \simeq g$, if there exists a continuous mapping $H : X \times [0,1] \longmapsto Y$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.

Recall that given a set $X$ endowed with a topology $\mathcal{T}$, the *one-point compactification* of the topological space $(X, \mathcal{T})$ is the topological space $(X \cup \{\infty\}, \mathcal{T}_\infty)$ where $S \in \mathcal{T}_\infty$ if either $S \in \mathcal{T}$, or $S = X \setminus A$, with $A$ compact.

*Remark* 1A.16. A given continuous mapping $f : X \longmapsto Y$ can be extended to a continuous mapping $f : X \cup \{\infty\} \longmapsto Y \cup \{\infty\}$ if it is *proper*. That is, if $f^{-1}(A)$ is compact whenever $A \subset Y$ is compact.

The following lemma generalises an observation due to Bich and Fixary (see proof of theorem 2.2 in section 4.3. in [9]). This is a surprisingly powerful tool that simplifies many of the proofs concerning the homotopies between homeomorphisms and projections (see e.g. theorem 1A.18).

**Lemma 1A.17.** *Given a topological space $C$ and $K$, a compact topological* *space. Let*

$$\eta : [0,1] \times C \times K \longmapsto [0,1] \times C \times K$$

*be a homeomorphism of the form* $\eta : (t,x,y) \mapsto (t, F(t,x,y), y)$*. For any* $X \subset C \times K$ *closed, the homotopy* $H : [0,1] \times X \longmapsto C$ *defined by*

$$H = F|_{[0,1]\times X} = (\pi_C \circ \eta)|_{[0,1]\times X}$$

*is proper.*

Note that in the above statement $\pi_C$ denotes the projection onto the set $C$ and $F|_{[0,1]\times X}$ denotes the restriction of $F$ to the set $[0,1] \times X$.

*Proof of lemma 1A.17.* Observe that it follows from the definition of $\eta$ that $\nu = \eta^{-1}$ must be of the form $\eta : (t,x,y) \mapsto (t, G(t,x,y), y)$.

Let $A \subset C$ be a compact set. Then

$$H^{-1}(A) = F^{-1}(A) \cap ([0,1]\times X) = \eta^{-1}([0,1]\times A \times K) \cap ([0,1]\times X). \quad (1A.17)$$

But $\eta^{-1}([0,1] \times A \times K) = \nu([0,1] \times A \times K)$. The continuity of $\nu$ implies $\nu([0,1] \times A \times K)$ is compact since $[0,1] \times A \times K$ is compact. Meanwhile, $[0,1] \times X$ is closed. $H^{-1}(A)$ is thus compact. ■

**Theorem 1A.18.** *Let* $\varphi = \psi^{-1}$*, where* $\psi$ *is the homeomorphism in theorem 1A.14, then* $\varphi \simeq \pi$*, where* $\pi : (z,x) \in E(\alpha) \mapsto z$*. Moreover, this homotopy can be continuously extended to the one-point compactification* $E(\alpha) \cup \{\infty\}$*.*

*Proof.* Define $H : [0,1] \times E(\alpha) \longmapsto \mathbb{R}^n$ by

$$H(t,(z,x)) = t\varphi(z,x) + (1-t)\pi(z,x) = z + tx. \quad (1A.18)$$

$H$ is continuous and can be extended to a continuous mapping $F$ over $[0,1] \times \mathbb{R}^n \times K(\alpha)$. The mapping $\eta(t,z,x) \mapsto (t, z+tx, x)$ is continuous and has $\nu(t,z,x) \mapsto (t, z-tx, x)$ as inverse. Lemma 1A.17 implies $H$ is proper. ■

**Corollary 1A.18.1** (Kohlberg-Mertens II)**.** *The homeomorphism* $\psi$ *in theorem 1A.14 satisfies* $\pi \circ \psi \simeq id_{\mathbb{R}^n}$*. Moreover, this homotopy can be continuously extended to the one-point compactification* $\mathbb{R}^n \cup \{\infty\}$*.*

*Proof.* Let $\xi : (t, z) \mapsto (t, \psi(z))$. Observe that $\xi$ is a homeomorphism and hence it is proper. The corollary follows by setting $\hat{H} = H \circ \xi$ where $H$ is the proper homotopy in the previous theorem. ■

## 1A.4 Simultaneous linear programmes

This section introduces the notion of simultaneous linear programme formally. Moreover, a decomposition that generalises the one appearing in Mertens and Kohlberg is presented.

Simultaneous linear programmes arise naturally when computing best replies in finite games. Indeed, the results in this section amount to a small extension of the result by Kohlberg-Mertens (see theorem 1 in section 3.2. in [24]) due to Pahl (see lemma 3.7 and associated proof in the appendix of [32]). Differences with this latter piece are mostly in terms of formulation. Pahl's formulation views the objects of interest as games, whilst here they are viewed as "coupled" linear optimisation problems. This position allows for better generalisation in the next chapter. Moreover, Pahl's formulation places more emphasis on the geometrical aspect of the problem, whereas the discussion here describes those geometrical objects as solutions to a set of algebraic equations and inequalities.

### 1A.4.1 The sets $\Lambda$ and $D\Lambda$

Some notation needs to be introduced first. Recall that a mapping

$$T : \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p} \longmapsto \mathbb{R}$$

is said to be *multi-linear*, if for every $i \in \{1, \ldots, p\}$:

$$\begin{aligned} T(x_1, \ldots, x_i + y_i, \ldots, x_p) &= T(x_1, \ldots, x_i, \ldots, x_p) \\ &+ T(x_1, \ldots, y_i, \ldots, x_p) \end{aligned} \tag{1A.19}$$

and

$$T(x_1, \ldots, a \cdot x_i, \ldots, x_p) = a \cdot T(x_1, \ldots, x_i, \ldots, x_p) \tag{1A.20}$$

for arbitrary $x_i, y_i \in \mathbb{R}^{n_i}$ and $a \in \mathbb{R}$. Recall that $\oplus$ denotes the *direct sum*.

*Remark* 1A.19. Regarding notation, it is convenient to adopt the following convention: given $i \in \{1, \dots, p\}$ and a mapping

$$f : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_p} \longmapsto \mathbb{R},$$

we write

$$f(y_i, x_{-i}) = f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_p). \tag{1A.21}$$

For instance, equation (1A.19) takes the form

$$T(x_i + y_i, x_{-i}) = T(x_i, x_{-i}) + T(y_i, x_{-i}). \tag{1A.22}$$

In addition, since direct sums of spaces are being considered, components of a particular vector are denoted by greek letter super-scripts. So that e.g. $x_k^\mu$ refers to the the $\mu^{\text{th}}$-component of the $i^{\text{th}}$ vector in $x = (x_1, \dots, x_p)$.

The set of multi-linear mappings and its image under a certain linear mapping play a central role in formalising the notion of simultaneous linear programmes. In the interest of clarity, both of these are defined precisely next:

**Definition 1A.20** (Set of multi-linear mappings)**.** Let $(n_1, \dots, n_p) \in \mathbb{N}^p$, the set $\Lambda(n_1, \dots, n_p)$ denotes the *set of all multi-linear mappings*

$$T : \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p} \longmapsto \mathbb{R}$$

*Remark* 1A.21. Observe that in the absence of ambiguity, $\Lambda(n_1, \dots, n_p)$ may be shortened as simply $\Lambda$.

*Remark* 1A.22. Given $(n_1, \dots, n_p) \in \mathbb{N}^p$ in order to describe the set of simultaneous linear programmes, it is convenient to consider the image of the bijection $(T_1, \dots, T_p) \mapsto (D_{x_1}T_1, \dots, D_{x_p}T_p)$, where for $i \in \{1, \dots, p\}$:

$$D_{x_i} = \left( \frac{\partial}{\partial x_i^1}, \dots, \frac{\partial}{\partial x_i^{n_i}} \right)$$

and $T_i \in \Lambda(n_1, \dots, n_p)$ for each $i$. Since repeated references to the image of $\Lambda(n_1, \dots, n_p)$ under this bijection are made in the sequel, denote it by $D\Lambda(n_1, \dots, n_p)$. Note that, if it is clear from context, the dependence on

$(n_1, \ldots, n_p)$ may be omitted.

*Remark* 1A.23. Observe that an element $z \in D\Lambda$ maps

$$\mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

to itself. Moreover, for each $i \in \{1, \ldots, p\}$ $z_i$ is independent of $x_i$ and it satisfies

$$z_i(x_{-i} + y_{-i}) = z_i(x_{-i}) + z_i(y_{-i}), \tag{1A.23}$$

$$z_i(a \cdot x_{-i}) = a \cdot z_i(x_{-i}). \tag{1A.24}$$

Here $z_i(x_{-i}, y_{-i}) \in \mathbb{R}^{n_i}$ and $a \in \mathbb{R}$.

### 1A.4.2 Properties of simultaneous linear programmes

**Definition 1A.24** (Simultaneous linear programme)**.** Fix a $2p$-tuple

$$(\mathbf{z}, \boldsymbol{\alpha}) = (z_1, \ldots, z_p, \alpha_1, \ldots, \alpha_p)$$

where $\mathbf{z} \in D\Lambda(n_1, \ldots, n_p)$ and each $\alpha_i$ is an admissible programme. A *$p-$simultaneous linear programme*, or simply a *simultaneous linear programme*, is the problem

$$\max_{x_1 \in K(\alpha_1)} \langle z_1(x_2, \ldots, x_p), x_1 \rangle$$

$$\vdots$$

$$\max_{x_i \in K(\alpha_i)} \langle z_i(x_{-i}), x_i \rangle$$

$$\vdots$$

$$\max_{x_p \in K(\alpha_p)} \langle z_p(x_1, \ldots, x_{p-1}), x_p \rangle$$

An element $(x_1, \ldots, x_p) \in K(\alpha_1) \times \ldots \times K(\alpha_p)$ is a *solution to the simultaneous linear programme* (abbreviated to a *solution* whenever is clear from the context) if for every $i \in \{1, \ldots, p\}$ it satisfies

$$\langle z_i(x_{-i}), x_i - y_i \rangle \geqslant 0 \tag{1A.25}$$

for arbitrary $y_i \in K(\alpha_i)$. The set $E(\mathbf{z}, \boldsymbol{\alpha})$ denotes the *set of all solutions to the simultaneous linear programme.*

*Remark* 1A.25. Observe that $(x_1, \ldots, x_p) \in E(\mathbf{z}, \boldsymbol{\alpha})$ if, and only, if $x_i \in E(z_i(x_{-i}), \alpha_i)$ for each $i$.

As a consequence of remark 1A.25,

**Proposition 1A.26.** *Let $(\mathbf{z}, \boldsymbol{\alpha})$ be a simultaneous linear programme a point $x = (x_1, \ldots, x_p) \in K(\alpha_1) \times \ldots \times K(\alpha_p)$ is a solution if, and only if, it satisfies the following system of non-linear equations:*

$$\begin{gathered}
\rho(x_1 + z_1(x_{-1}), K(\alpha_1)) = x_1 \\
\vdots \\
\rho(x_p + z_1(x_{-p}), K(\alpha_p)) = x_p
\end{gathered}$$

*Proof.* Let $x \in K(\alpha_1) \times \ldots \times K(\alpha_p)$. Theorem 1A.10 implies that

$$\rho(x_i + z_i(x_{-i}), K(\alpha_i)) = x_i. \tag{1A.26}$$

if, and only if, $x_i \in E(z_i(x_{-i}), \alpha_i)$. But this relation is satisfied for every $i$ if, and only if, $x \in E(\mathbf{z}, \boldsymbol{\alpha})$. ■

Observe that the set of non-linear equations in proposition 1A.26 define a fixed point problem for a continuous mapping whose individual component are given by the left-hand side of the equations in proposition 1A.26. This mapping is, in fact, the original GPS mapping (see [20]). Since each $\alpha_i$ in $\boldsymbol{\alpha}$ is admissible $K(\alpha_i)$; is non-empty, convex and compact. Brouwer fixed point theorem implies that $E(\mathbf{z}, \boldsymbol{\alpha}) \neq \emptyset$. The *equilibrium* set can now be defined as

$$E(\boldsymbol{\alpha}) = \bigcup_{\mathbf{z} \in D\Lambda} \{\mathbf{z}\} \times E(\mathbf{z}, \boldsymbol{\alpha}). \tag{1A.27}$$

When proving the structure theorem, it is in fact more convenient to decompose the set $D\Lambda$ as a direct sum of two smaller subspaces. This idea dates back to the original proof of Mertens and Kohlberg. Moreover, this decomposition is in itself interesting as it will serve to motivate some of the extensions in the next chapter.

Let $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$, recall that the *hyper-plane* $H(v,t)$ is the set

$$\{x \in \mathbb{R}^n : \langle v, x \rangle = t\}.$$

**Lemma 1A.27.** *Let $v_1 \in \mathbb{R}^{n_1}, \ldots, v_p \in \mathbb{R}^{n_p}$ all non-zero and $t_1, \ldots, t_p \in \mathbb{R}$ satisfying $|t_1 \cdot \ldots \cdot t_p| > 0$. For every $T \in \Lambda(n_1, \ldots, n_p)$ and $i \in \{1, \ldots, p\}$ there exists a unique pair $(\overline{T}, \tilde{T}) \in \mathbb{R}^{n_i} \oplus \Lambda(n_1, \ldots, n_p)$ such that*

$$T(x_i, x_{-i}) = \langle \overline{T}, x_i \rangle + \tilde{T}(x_i, x_{-i}) \tag{1A.28}$$

*whenever $(x_i, x_{-i}) \in H(v_1, t_1) \times \ldots \times H(v_p, t_p)$. Additionally,*

$$\tilde{T}(x_i, v_{-i}) = \tilde{T}(v_1, \ldots, v_{i-1}, x_i, v_{i+1}, \ldots, v_p) = 0. \tag{1A.29}$$

*Proof.* Define $\overline{T}_M \in \Lambda$ as follows:

$$\overline{T}_M : (x_i, x_i) \mapsto T(x_i, v_{-i}) \prod_{j \neq i} \frac{\langle v_j, x_j \rangle}{\|v_j\|_2^2}; \tag{1A.30}$$

while $\tilde{T} = T - \overline{T}_M$. Moreover, let $\overline{T} \in \mathbb{R}^{n_i}$ be the vector with components $\prod_{j \neq i}^{p} \frac{t_j}{\|v_j\|_2^2} \cdot T(e^\nu, v_{-i})$. Here $e^\nu$ ($\nu \in \{1, \ldots, n_i\}$) is the $\nu^{\text{th}}$ vector in the canonical basis of $\mathbb{R}^{n_i}$.

By construction, if $(x_i, x_{-i}) \in H(v_1, t_1) \times \ldots \times H(v_p, t_p)$

$$\overline{T}_M(x_i, x_{-i}) = \prod_{j \neq i}^{p} \frac{t_j}{\|v_j\|_2^2} \sum_{\nu=1}^{n_i} x_i^\nu \cdot T(e^\nu, v_{-i}) = \langle \overline{T}, x_i \rangle. \tag{1A.31}$$

In addition, given $x_i \in \mathbb{R}^{n_i}$:

$$T(x_i, v_{-i}) = T(x_i, v_{-i}) + \tilde{T}(x_i, v_{-i}). \tag{1A.32}$$

The first part of the lemma follows from equations (1A.31) and (1A.32).

To prove uniqueness, let $(\overline{T}_1, \tilde{T}_1)$ and $(\overline{T}_2, \tilde{T}_2)$ be two pairs satisfying the conditions in the rubric. Fix $x_i \in H(v_i, t_i)$, by assumption

$$\langle \overline{T}_1, x_i \rangle + \tilde{T}_1(x_i, v_{-i}) = \langle \overline{T}_2, x_i \rangle + \tilde{T}_2(x_i, v_{-i}). \tag{1A.33}$$

Recall that $\tilde{T}_1(x_i, v_{-i}) = \tilde{T}_2(x_i, v_{-i}) = 0$, thus

$$\langle \overline{T}_1 - \overline{T}_2, x_i \rangle = 0. \tag{1A.34}$$

Now, let $u_1, \ldots, u_{\nu_i - 1}$ be an orthonormal set of vectors satisfying $\langle u_k, v_i \rangle = 0$ for every $k \in \{1, \ldots, n_i - 1\}$. Fix such $k$ and observe that $u_k + \frac{t_i v_i}{\|v_i\|_2^2} \in H(v_i, t_i)$; hence

$$0 = \left\langle \overline{T}_1 - \overline{T}_2, u_k + \frac{t_i v_i}{\|v_i\|_2^2} \right\rangle = \langle \overline{T}_1 - \overline{T}_2, u_k \rangle, \tag{1A.35}$$

since $\frac{t_i v_i}{\|v_i\|_2^2} \in H(v_i, t_i)$ as well. Equation (1A.35) yields $\overline{T}_1 - \overline{T}_2 = \lambda v_i$. By the same reasoning as before however,

$$0 = \frac{t_i}{\|v_i\|_2^2} \langle \overline{T}_1 - \overline{T}_2, v_i \rangle = \lambda \cdot t_i. \tag{1A.36}$$

Hence, $\overline{T}_1 = \overline{T}_2$. Consequently, $\tilde{T}_1(x_i, x_{-i}) = \tilde{T}_2(x_i, x_{-i})$ if $(x_i, x_{-i}) \in H(v_1, t_1) \times \ldots \times H(v_p, t_p)$.

Now, for each $j \in \{1, \ldots, p\}$, let $u_{j_1}, \ldots, u_{j_{\nu_j - 1}}$ be an orthonormal set of vectors satisfying $\langle u_k, v_j \rangle = 0$ for every $k \in \{j_1, \ldots, j_{n_i - 1}\}$. Fix such $k$ and observe that $u_k + \frac{t_i v_i}{\|v_i\|_2^2} \in H(v_i, t_i)$. Choose $j \in \{1, \ldots, p\}$, $k \in \{j_1, \ldots, j_{n_i - 1\}}$ and

$$x_{-j} \in H(v_1, t_1) \times \ldots \times H(v_{j-1}, t_{j-1}) \times H(v_{j+1}, t_{j+1}) \times \ldots \times H(v_p, t_p).$$

By the same argument that yielded equation (1A.35), it follows that

$$\tilde{T}_1(u_{j_k}, x_{-j}) = \tilde{T}_2(u_{j_k}, x_{-j}). \tag{1A.37}$$

Applying this reasoning iteratively, it may be concluded that:

$$\tilde{T}_1(u_{1_k}, \ldots, u_{p_k}) = \tilde{T}_2(u_{1_k}, \ldots, u_{p_k}). \tag{1A.38}$$

Moreover, a similar reasoning establishes this equality with some of the $u$'s replaced by $v$'s. The equality $\tilde{T}_1 = \tilde{T}_2$ is thus proven. ■

**Corollary 1A.27.1.** *Let $v_1 \in \mathbb{R}^{n_1}, \ldots, v_p \in \mathbb{R}^{n_p}$ all non-zero, and*

$$t_1, \ldots, t_p \in \mathbb{R}$$

*satisfying $|t_1 \cdot \ldots \cdot t_p| > 0$. For every $\mathbf{z} \in D\Lambda(n_1, \ldots, n_p)$ there exists a unique pair $(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) \in \mathbb{R}^N \oplus D\Lambda(n_1, \ldots, n_p)$ $(N = n_1 + \ldots + n_p)$ such that for each $i \in \{1, \ldots, p\}$:*

$$z_i(x_{-i}) = \overline{z}_i + \tilde{z}(x_{-i}) \tag{1A.39}$$

*whenever*

$$x_{-i} \in H(v_1, t_1) \times \ldots \times H(v_{i-1}, t_{i-1}) \times H(v_{i+1}, t_{i+1}) \times \ldots \times H(v_p, t_p).$$

*Additionally,*

$$\tilde{z}_i(v_{-i}) = \tilde{z}_i(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p) = 0. \tag{1A.40}$$

*Proof.* Fix $i \in \{1, \ldots, p\}$ and consider the linear mapping $T : (x_i, x_{-i}) \mapsto \langle z_i(x_{-i}), x_i \rangle$. The previous lemma implies

$$\langle z_i(x_{-i}), x_i \rangle = \langle \overline{z}_i, x_i \rangle + \langle \tilde{z}(x_{-i}), x_i \rangle \tag{1A.41}$$

for a unique pair $(\overline{z}_i, \tilde{z}_i) \in \mathbb{R}^{n_i} \oplus D\Lambda(n_1, \ldots, n_p)$ and $x \in H(v_1, t_1) \times \ldots \times H(v_p, t_p)$. The corollary now follows by differentiating both sides. ■

*Remark* 1A.28. Corollary 1A.27.1 generalises the original decomposition due to Mertens and Kohlberg (see proof of theorem 1 in section 3.2. of [24]). With reference to their original context, the linear mapping $\langle z_i(x_{-i}), x_i \rangle$ corresponds to the pay-off of agent $i$ when actions are $(x_i, x_{-i})$. The original idea in Mertens-Kohlberg is then to decompose this pay-off into a term depending only the actions of agent $i$ (corresponding to $\overline{z}_i$ here) and a term that vanishes when the remaining players mix across their actions with equal probability (corresponding to $\tilde{z}_i$ here). Note that in Kohlberg and Mertens' setting each set $K(\alpha_i)$ corresponds to the simplex formed by all possible mixtures of agent $i$'s actions. The corresponding $v_i$ is then the vector whose entries are all 1.

For the remainder of this section, fix a collection of $p$ admissible programmes $\boldsymbol{\alpha}$. Since instead of working with the set $D\Lambda$ we wish to work with the decomposition in corollary 1A.27.1. Identifying each $\mathbf{z}$ with its decomposition, the following homeomorphism is then required:

$$E(\boldsymbol{\alpha}) = \bigcup_{\mathbf{z}\in D\Lambda} \{\mathbf{z}\} \times E(\mathbf{z}, \boldsymbol{\alpha}) \cong \bigcup_{(\overline{\mathbf{z}},\tilde{\mathbf{z}})\in\mathbb{R}^N\oplus D\Lambda_0} \{(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\} \times E((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), \boldsymbol{\alpha}), \quad \text{(1A.42)}$$

where

$$D\Lambda_0 = \{\mathbf{z} \in D\Lambda : z_i(v_{-i}) = 0 \;\forall\, i \in \{1, \dots, p\}\};$$

here $v_i$ refers to the normal vector of some given negatively tilted hyperplane (see remark 1A.6), while $N = n_1 + \ldots + n_p$.

*Remark* 1A.29. Corollary 1A.27.1 implies that there is a bijection between the two sets in equation (1A.42). To prove continuity, define the following norm over $D\Lambda$,

$$\|\mathbf{z}\|_{D\Lambda} = \max\{\|z_1\|_\infty, \dots, \|z_p\|_\infty\}. \quad \text{(1A.43)}$$

Recall that

$$\|z_i\|_\infty = \sup_{x_{-i}\in K(\alpha_1)\times\ldots\times K(\alpha_{i-1})\times K(\alpha_{i+1})\times\ldots\times K(\alpha_p)} \|z_i(x_{-i})\|_2.$$

*Remark* 1A.30. Observe that the norm defined here corresponds to the uniform norm when viewing elements of $D\Lambda$ as continuous functions. Given that elements of $D\Lambda$ are linear mappings, this choice may not appear natural. Indeed, it is a choice not made in the literature. Nonetheless, this norm is natural when considering general functions, as we do in chapter 1B. Moreover, under some assumptions this choice is equivalent to norms used in the literature. A further discussion is given in section 5 below.

**Proposition 1A.31.** *Let* $\|(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\| = \max\{\|\overline{\mathbf{z}}\|_2, \|\tilde{\mathbf{z}}\|_{D\Lambda}\}$. *There exists constants* $C_1, C_2 > 0$ *depending solely on the choice of admissible programmes*

$\alpha_1, \ldots, \alpha_p$ *such that*

$$C_1\|(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\| \leqslant \|\mathbf{z}\|_{D\Lambda} \leqslant C_2\|(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\|. \tag{1A.44}$$

*Proof.* Let $x \in K(\alpha_1) \times \ldots \times K(\alpha_p)$ and fix $i \in \{1, \ldots, p\}$ then

$$\|z_i(x_{-i})\|_2 \leqslant \|\overline{z}\|_2 + \|\tilde{z}_i(x_{-i})\|_2 \leqslant 2\|(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\|. \tag{1A.45}$$

Let $v_1 \in \mathbb{R}^{n_1}, \ldots, v_p \in \mathbb{R}^{n_p}$ and $\ldots, t_p \in \mathbb{R}$ be as in corollary 1A.27.1, then:

$$\|\overline{z}_i\|_2 \leqslant \frac{|t_i|}{\|v_i\|_2}\|z_i(v_{-i})\|_2 \leqslant C_2\|\mathbf{z}\|_{D\Lambda}. \tag{1A.46}$$

Hence,

$$\|\tilde{z}_i(x_{-i})\|_2 \leqslant \|\overline{z}_i\|_2 + \|z_i(x_{-i})\|_2 \leqslant C_2\|\mathbf{z}\|_{D\Lambda}. \tag{1A.47}$$

The result follows by combining equations (1A.45) and (1A.47). ∎

Proposition 1A.31 yields the homeomorphism in equation (1A.42).

## 1A.5 The structure theorem for simultaneous linear programmes

**Theorem 1A.32** (Kohlberg-Mertens I)**.** *Let $\alpha_1, \ldots, \alpha_p$ be admissible programmes. There exists a homeomorphism*

$$\psi : \mathbb{R}^N \oplus D\Lambda_0(n_1, \ldots, n_p) \longmapsto \bigcup_{(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) \in \mathbb{R}^N \oplus D\Lambda_0(n_1, \ldots, n_p)} \{(\overline{\mathbf{z}}, \tilde{\mathbf{z}})\} \times E((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), \boldsymbol{\alpha}).$$

*Recall that here $N = n_1 + \ldots + n_p$.*

*Remark* 1A.33. The main idea of the proof is to apply the decomposition in corollary 1A.27.1 to view the simultaneous programme $(\mathbf{z}, \boldsymbol{\alpha})$ as the fixed point problem

$$x_i = \rho(\overline{z}_i + \tilde{z}_i(x_{-i}) + x_i, K(\alpha_i)) \tag{1A.48}$$

For $i \in \{1, \ldots, p\}$. In this form, all the dependency on variables $x_j$ (where $j \neq i$) lies in $\tilde{z}_i$. Since these are fixed whenever $(x_1, \ldots, x_p)$ is a solution the proof proceeds roughly as for the linear programme case.

*Proof of theorem 1A.32.* For simplicity, notation is abused slightly by making the identifications

$$\mathbb{R}^N = \mathbb{R}^{n_1+\ldots+n_p} \equiv \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

and

$$E(\boldsymbol{\alpha}) \equiv \bigcup_{(\overline{\mathbf{z}},\tilde{\mathbf{z}})\in\mathbb{R}^N\oplus D\Lambda_0(n_1,\ldots,n_p)} \{(\overline{\mathbf{z}},\tilde{\mathbf{z}})\} \times E((\overline{\mathbf{z}},\tilde{\mathbf{z}}),\boldsymbol{\alpha}).$$

Let

$$\varphi : ((\overline{\mathbf{z}},\tilde{\mathbf{z}}),x) \mapsto (\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x, \tilde{\mathbf{z}}) \in \mathbb{R}^N \oplus D\Lambda_0,$$

where for each $i \in \{1,\ldots,p\}$, $\tilde{\mathbf{z}}(x)_i = \tilde{z}_i(x_{-i}) \in \mathbb{R}^{n_i}$. Similarly, define

$$\begin{aligned}\psi : (\overline{\mathbf{z}},\tilde{\mathbf{z}}) &\mapsto ((\overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}), \tilde{\mathbf{z}}), \boldsymbol{\rho}(\overline{\mathbf{z}})) \\ &\in (\mathbb{R}^N \oplus D\Lambda_0) \times (K(\alpha_1) \times \ldots \times K(\alpha_p).\end{aligned}$$

Here $\boldsymbol{\rho}(\overline{\mathbf{z}}) \in \mathbb{R}^N$ satisfies $\boldsymbol{\rho}(\overline{\mathbf{z}})_i = \rho(\overline{z}_i, K(\alpha_i)) \in \mathbb{R}^{n_i}$, for each $i \in \{1,\ldots,p\}$.

Observe that $\varphi$ is well-defined. To prove that $\psi$ is also well-defined, it is shown that $\psi(\mathbb{R}^N \oplus D\Lambda_0) \subset E(\boldsymbol{\alpha})$. To that end, fix $i \in \{1,\ldots,p\}$ and write

$$(\hat{x}_1,\ldots,\hat{x}_p) = (\rho(\overline{z}_1, K(\alpha_1)),\ldots,\rho(\overline{z}_p, K(\alpha_p))).$$

Observe that

$$\rho(\overline{z}_i - \tilde{z}_i(\hat{x}_{-i}) - \hat{x}_i + \tilde{z}_i(\hat{x}_{-i}) + \hat{x}_i, K(\alpha_i)) = \rho(\overline{z}_i, K(\alpha_i)) = \hat{x}_i. \quad (1A.49)$$

Hence, $(\hat{x}_1,\ldots,\hat{x}_p)$ satisfies the system of non-linear equations in proposition 1A.26 with $\overline{z}_j - \tilde{z}_j(\hat{x}_{-i}) - \hat{x}_j + \tilde{z}_j$ in place of just $z_j$. As a result, $(\hat{x}_1,\ldots,\hat{x}_p) \in E((\overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}), \tilde{\mathbf{z}}), \boldsymbol{\alpha})$. Hence, $\psi(\mathbb{R}^N \oplus D\Lambda_0) \subset E(\boldsymbol{\alpha})$ as required. The other inclusion follows by proving that $\varphi$ is a left-inverse of $\psi$. To establish this, choose some $((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) \in E(\boldsymbol{\alpha})$ then

$$\begin{aligned}(\psi \circ \varphi)((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) = \psi(\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x, \tilde{\mathbf{z}}) = \\ ((\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x - \boldsymbol{\rho}(\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x) \\ -\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x), \tilde{\mathbf{z}}), \boldsymbol{\rho}(\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x)).\end{aligned} \tag{1A.50}$$

Since $x \in E((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), \boldsymbol{\alpha})$, it follows from proposition 1A.26 that for every $i \in \{1, \ldots, p\}$:

$$x_i = \rho(\overline{z}_i + \tilde{z}_i(x_{-i}) + x_i, K(\alpha_i)). \tag{1A.51}$$

Hence,

$$\boldsymbol{\rho}(\overline{\mathbf{z}} + \tilde{\mathbf{z}}(x) + x) = x. \tag{1A.52}$$

Thus, $(\psi \circ \varphi)((\overline{z}, \tilde{z}), x) = ((\overline{z}, \tilde{z}), x)$. All that remains is to prove that $\varphi$ is a right-inverse as well. To keep the discussion succinct, proofs of the continuity of $\varphi$ and $\psi$ are given below in propositions 1A.34 and 1A.35.

Let $(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) \in \mathbb{R}^N \oplus D\Lambda_0$ then

$$\begin{aligned}(\varphi \circ \psi)(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) &= \varphi((\overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}), \tilde{\mathbf{z}}), \boldsymbol{\rho}(\overline{\mathbf{z}})) \\ &= (\overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}) + \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) + \boldsymbol{\rho}(\overline{\mathbf{z}}), \tilde{\mathbf{z}}) \\ &= (\overline{\mathbf{z}}, \tilde{\mathbf{z}}).\end{aligned} \tag{1A.53}$$

■

**Proposition 1A.34.** *The mapping $\psi$ in theorem 1A.32 is continuous.*

*Proof.* Recall that

$$\psi : (\overline{\mathbf{z}}, \tilde{\mathbf{z}}) \mapsto ((\overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}), \tilde{\mathbf{z}}), \boldsymbol{\rho}(\overline{\mathbf{z}})),$$

it is convenient to write $\psi : (\overline{\mathbf{z}}, \tilde{\mathbf{z}}) \mapsto (\psi_1(\overline{\mathbf{z}}, \tilde{\mathbf{z}}), \psi_2(\overline{\mathbf{z}}, \tilde{\mathbf{z}}), \psi_3(\overline{\mathbf{z}}, \tilde{\mathbf{z}}))$ with

$$\begin{aligned}\psi_1(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) &= \overline{\mathbf{z}} - \tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}})) - \boldsymbol{\rho}(\overline{\mathbf{z}}), \\ \psi_2(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) &= \tilde{\mathbf{z}}, \\ \psi_3(\overline{\mathbf{z}}, \tilde{\mathbf{z}}) &= \boldsymbol{\rho}(\overline{\mathbf{z}}).\end{aligned}$$

Observe that $\psi_2$ is continuous by definition; while proposition 1A.9 implies that $\psi_3$ is continuous as well. It remains to prove that $\psi_1$ is continuous. To that end, note that

$$\begin{aligned}\|\psi_1(\overline{\mathbf{z}},\tilde{\mathbf{z}})-\psi_1(\overline{\mathbf{w}},\tilde{\mathbf{w}})\|_2 &\leqslant \|\overline{\mathbf{z}}-\overline{\mathbf{w}}\|_2+\|\boldsymbol{\rho}(\overline{\mathbf{z}})-\boldsymbol{\rho}(\overline{\mathbf{w}})\|_2\\ &+\|\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))-\tilde{\mathbf{w}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))\|_2+\|\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}}))-\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))\|_2.\end{aligned} \tag{1A.54}$$

Now, proposition 1A.9 implies that

$$\|\boldsymbol{\rho}(\overline{\mathbf{z}})-\boldsymbol{\rho}(\overline{\mathbf{w}})\|_2 \leqslant C\|\overline{\mathbf{z}}-\overline{\mathbf{w}}\|_2,$$

for some constant $C>0$ depending only on the choice of $\boldsymbol{\alpha}$. Meanwhile, $\|\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))-\tilde{\mathbf{w}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))\|_2 \leqslant C\|\tilde{\mathbf{z}}-\tilde{\mathbf{w}}\|_{D\Lambda_0}$. Once again, $C>0$ only depends on $\boldsymbol{\alpha}$. Lastly, fix $i\in\{1,\dots,p\}$, since $\tilde{\mathbf{z}}_i$ depends linearly on each separate coordinate. There exists $M_i>0$ such that

$$\begin{aligned}\|\tilde{z}_i(\boldsymbol{\rho}(\overline{\mathbf{z}}))-\tilde{z}_i(\boldsymbol{\rho}(\overline{\mathbf{w}}))\|_2 &\leqslant M_i\prod_{j\neq i}\|\rho(\overline{z}_j,K(\alpha_j))-\rho(\overline{w}_j,K(\alpha_j))\|_2\\ &\leqslant M_i\prod_{j\neq i}\|\overline{z}_j-\overline{w}_j\|_2.\end{aligned} \tag{1A.55}$$

As a consequence of equation (1A.55), $\|\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{z}}))-\tilde{\mathbf{z}}(\boldsymbol{\rho}(\overline{\mathbf{w}}))\|_2\to 0$ as $\overline{\mathbf{w}}\to\overline{\mathbf{z}}$. These observations along with equation (1A.54) imply:

$$0\leqslant \limsup_{\overline{\mathbf{w}}\to\overline{\mathbf{z}}}\|\psi_1(\overline{\mathbf{z}},\tilde{\mathbf{z}})-\psi_1(\overline{\mathbf{w}},\tilde{\mathbf{w}})\|_2\leqslant 0. \tag{1A.56}$$

From where the proposition now follows. ∎

**Proposition 1A.35.** *The mapping $\varphi=\psi^{-1}$, where $\psi$ is the mapping in theorem 1A.32, is continuous.*

*Proof.* Recall that

$$\varphi:((\overline{\mathbf{z}},\tilde{\mathbf{z}}),x)\mapsto(\overline{\mathbf{z}}+\tilde{\mathbf{z}}(x)+x,\tilde{\mathbf{z}}).$$

As in the proof of proposition 1A.34, let

$$\begin{aligned}\varphi_1((\overline{\mathbf{z}},\tilde{\mathbf{z}}),x) &= \overline{\mathbf{z}}+\tilde{\mathbf{z}}(x)+x,\\ \varphi_2((\overline{\mathbf{z}},\tilde{\mathbf{z}}),x) &= \tilde{\mathbf{z}}.\end{aligned}$$

It is to be shown that both $\varphi_1$ and $\varphi_2$ are continuous. Note that $\varphi_2$ is continuous. To prove the continuity of $\varphi_1$, note that

$$\begin{aligned} \|\varphi_1((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) - \varphi_1((\overline{\mathbf{w}}, \tilde{\mathbf{w}}), y)\|_2 \leqslant \|\overline{\mathbf{z}} - \overline{\mathbf{w}}\|_2 + \|\tilde{\mathbf{z}}(y) - \tilde{\mathbf{w}}(y)\|_2 \\ + \|\tilde{\mathbf{z}}(x) - \tilde{\mathbf{z}}(y)\|_2 + \|x - y\|_2. \end{aligned} \tag{1A.57}$$

Since $\tilde{z}$ is continuous, it follows that

$$\begin{aligned} 0 &\leqslant \limsup_{((\overline{\mathbf{w}}, \tilde{\mathbf{w}}), y) \to (\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x)} \|\varphi_1((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) - \varphi_1((\overline{\mathbf{w}}, \tilde{\mathbf{w}}), y)\|_2 \\ &\leqslant \lim_{y \to x} \|\tilde{\mathbf{z}}(x) - \tilde{\mathbf{z}}(y)\|_2 = 0. \end{aligned} \tag{1A.58}$$

■

To conclude this section, an extension of theorem 1A.18 and its corollary are now discussed. Note that the proofs follows the same reasoning as in theorem 1A.18 and corollary 1A.18.1. In particular, lemma 1A.17 holds in this situation as well. Details are therefore omitted.

**Theorem 1A.36.** *The homeomorphism $\varphi = \psi^{-1} : E(\boldsymbol{\alpha}) \longmapsto \mathbb{R}^N \oplus D\Lambda_0$ satisfies $\varphi \simeq \pi$, where $\pi : ((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) \mapsto (\overline{\mathbf{z}}, \tilde{\mathbf{z}})$. Moreover, this homotopy extends to the one-point compactification of both spaces.*

*Proof.* The desired homotopy is the linear homotopy:

$$\begin{aligned} H((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x, t) &= (1-t)\varphi((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) + t\pi((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) \\ &= (\overline{\mathbf{z}} + (1-t)\tilde{\mathbf{z}}(x) + (1-t)x, \tilde{z}). \end{aligned} \tag{1A.59}$$

■

**Corollary 1A.36.1.** *Let $\psi$ be the homeomorphism in theorem 1A.32. Then $\pi \circ \psi \simeq id_{\mathbb{R}^N \oplus D\Lambda_0}$. Here $\pi : ((\overline{\mathbf{z}}, \tilde{\mathbf{z}}), x) \mapsto (\overline{\mathbf{z}}, \tilde{\mathbf{z}})$.*

## 1A.6 Comparison to the literature

This brief section addresses the main differences between common formulations in the literature and the one presented here. In particular, when discussing theorem 1A.32 and corollary 1A.36.1 it is common to identify the set $D\Lambda(n_1, \ldots, n_p)$ with $\Lambda(n_, \ldots, n_p)^N$, $N = n_1 + \ldots + n_p$, and view this latter set as $\mathbb{R}^{Np}$ (see e.g. [24]). This identification is not made in this chapter as it is not guaranteed under the assumptions in definition 1A.4. In addition, viewing elements of $D\Lambda(n_1, \ldots, n_p)$ as continuous functions is consistent with the notation in chapter 1B, where general continuous functions are considered. In fact, the results in this chapter are particular cases of the results in chapter 1B.

In order to retrieve the standard formulation, the following proposition is needed

**Proposition 1A.37.** *Let $N = n_1 + \ldots + n_p \in \mathbb{N}$. Given*

$$K_1 \times \ldots \times K_p \subset \mathbb{R}^N$$

*be a non-empty compact set. Suppose that for $i \in \{1, \ldots, p\}$ Span$(K_i) = \mathbb{R}^{n_i}$ then the mapping $T \mapsto D_{x_i}T$ is a homeomorphism between $(\Lambda(n_1, \ldots, n_p), \| \cdot \|_\infty)$ and $(D_{x_i}(\Lambda(n_1, \ldots, n_p)), \| \cdot \|_\infty)$.*

Recall that

$$\text{Span}(S) = \{a_1 v_1 + \ldots + a_k v_k : k \in \mathbb{N};\ v_1, \ldots, v_k \in S;\ a_1, \ldots, a_k \in \mathbb{R}\}.$$

*Remark* 1A.38. Proposition 1A.37 is not true if $\text{Span}(K_i) \neq \mathbb{R}^{n_i}$. Indeed, let $N = n_1 = 2$, so that $\Lambda$ is the dual space of $\mathbb{R}^2$. Now, let $K = \{0\} \times [0,1] \subset \mathbb{R}^2$ and consider the sequence $\{T_n\}_{n=1}^\infty = \{(\cos(\frac{n-1}{n}\frac{\pi}{2}), \sin(\frac{n-1}{n}\frac{\pi}{2}))\}_{n=1}^\infty$. Viewing the sequence $\{T_n\}_{n=1}^\infty$ as mappings $x \mapsto \langle T_n, x\rangle$; then, since $\|T_n\|_\infty = \cos(\frac{n-1}{n}\frac{\pi}{2}) \to 0$, $T_n \to 0$. Nevertheless, $D_{x_1}T_n \not\to 0$.

*Proof.* Recall that for $T \in \Lambda(n_1, \ldots, n_p)$,

$$(D_{x_i}T)^\nu = T(e^\nu, \cdot, )$$

$\nu \in \{1, \ldots, n_i\}$.

Now, let $T \in \Lambda(n_1, \ldots, n_p)$

$$|T(x_i, x_{-i})| = |\langle (D_{x_i}T)(x_{-i}), x_i\rangle| \leqslant \|x_i\|_2 \ \|(D_{x_i}T)(x_{-i})\|_2. \tag{1A.60}$$

Since $K$ is compact,

$$|T(x_i, x_{-i})| \leqslant \sup_{x \in K} \|x_i\|_2 \, \|(D_{x_i} T)\|_2. \tag{1A.61}$$

Thus $\|T\|_\infty \leqslant \sup_{x\in K} \|x_i\|_2 \, \|(D_{x_i}T)\|_2$. It follows that $D_{x_i}^{-1}$ is continuous. To prove that $D_{x_i}$ is continuous let $u_1, \ldots, u_{n_i}$ be a basis for $\mathrm{Span}(K_i) = \mathbb{R}^{n_i}$. By applying a linear transformation if necessary, it may be assumed that this basis is the canonical one. Then

$$\|(D_{x_i}T)^\nu(x_{-i})| = |T(e^\nu, x_{-i})| \leqslant \|T\|_\infty. \tag{1A.62}$$

Hence,

$$\|D_{x_i}T(x_{-i})\|_2 \leqslant \sqrt{n_i}\, \|T\|_\infty. \tag{1A.63}$$

It follows form this last equation that $D_{x_i}$ is continuous. Since it is a bijection the result follows. ■

*Remark* 1A.39. In the original formulation of Kohlberg and Mertens each $K(\alpha_i)$ corresponds to some simplex so the assumptions of proposition 1A.37 are fulfilled. In that case the following chain of homeomorphism holds:

$$\mathbb{R}^{p\cdot(n_1+\ldots+n_p)} \cong \Lambda(n_1, \ldots, n_p) \cong D\Lambda(n_1, \ldots, n_p).$$

Naturally, these extend to the relevant equilibrium sets.

## Appendix: Uniqueness of the Kuhn-Tucker multipliers

**Proposition 1A.40.** *Let $\alpha = (A, b)$ be an admissible programme. The condition that for any $z \in \mathbb{R}^n$ and arbitrary $x \in E(z, \alpha)$ the Kuhn-Tucker multipliers are unique is equivalent to the following requirement: given $x$ in the feasible set $K(\alpha)$ the matrix*

$$C(x) = \begin{bmatrix} A \\ -I_{k\times n} \end{bmatrix},$$

*where $k$ is the number of components of $x$ such that $x_i = 0$ and*

$$(I_{k\times n})_{ij} = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases},$$

*is of full-rank.*

*Proof.* Assume that the conclusion held first. Given $z \in \mathbb{R}^n$ and $x \in E(z, \alpha)$, take any two pairs of Kuhn-Tucker multipliers (see condition (ii) in theorem 1A.10) $(\nu, \mu)$, $(\bar{\nu}, \bar{\mu}) \in \mathbb{R}^m \times ([0, \infty))^n$. By definition, $\langle \mu, x\rangle = \langle \bar{\mu}, x\rangle = 0$ and

$$z = A^T\nu - \mu = A^T\bar{\nu} - \bar{\mu}. \tag{1A.64}$$

Without loss of generality, it may be assumed that $x_i = 0$ for $i \in \{1, \dots, k\}$ where $0 < k \leqslant n$ and $x_i \neq 0$ otherwise. It follows that $\bar{\mu}_i = \mu_i = 0$ for $\{k + 1, \dots, n\}$. Now, let $\chi \in \mathbb{R}^k$ (resp. $\bar{\chi} \in \mathbb{R}^k$) be the vector with components $\chi_i = \mu_i$ (resp. $\bar{\chi}_i = \bar{\mu}_i$) where $i \in \{1, \dots, k\}$. Equation (1A.64) can be rewritten as

$$0 = A^T(\nu - \bar{\nu}) - \sum_{i=1}^{k}(\chi_i - \bar{\chi}_i)e_i, \tag{1A.65}$$

where $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^k$. Applying the definition of $C(x)$, equation (1A.65) becomes

$$0 = C(x)^T \begin{bmatrix} \nu - \bar{\nu} \\ \chi - \bar{\chi} \end{bmatrix}. \tag{1A.66}$$

By assumption rank $C(x)^T = m + k$ (recall $m =$ rank $A$). Hence, $\nu = \bar{\nu}$ and $\chi = \bar{\chi}$. The Kuhn-Tucker multipliers are thus unique.

Suppose now by way of contradiction that for every $z \in \mathbb{R}^n$ and every $x \in K(\alpha)$, the associated Kuhn-Tucker multipliers were unique; but that, for some $y \in K(\alpha)$, $C(y)$ were not of full rank. In other words, rank $C(y) < m + k$, where $k$ is the number of vanishing components of $y$. Once again, it may be assumed that these are the first $k$ components.

Let $\nu = 0$ and $\mu = (e, 0) \in [0, \infty)^k \times \mathbb{R}^{n-k}$ (here $e \in \mathbb{R}^k$ is the vector whose components are all 1) and set $\zeta = -\mu$. By construction:

$$\zeta = A^T \nu - \mu = C(y)^T \begin{bmatrix} 0 \\ e \end{bmatrix} \tag{1A.67}$$

and $\langle \mu, y \rangle = 0$. Theorem 1A.10 establishes that $y \in E(\zeta, \alpha)$.

Since rank $C(y) < m + k$, there exists $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^k$ which do not vanish simultaneously and satisfy

$$C(y)^T \begin{bmatrix} u \\ w \end{bmatrix} = 0.$$

Choose $N \in \mathbb{N}$ satisfying $N > -\min\limits_{i \in \{1,\dots,k\}} w_i$. Then

$$1 + \frac{w_i}{N} \geqslant 1 + \frac{1}{N} \min_{i \in \{1,\dots,k\}} w_i > 0. \tag{1A.68}$$

Consequently,

$$\zeta = C(y)^T \begin{bmatrix} \frac{1}{N} v \\ e + \frac{1}{N} w \end{bmatrix}. \tag{1A.69}$$

This last equation contradicts the assumption of uniqueness of the Kuhn-Tucker multipliers. ■

# Chapter 1B

# The Structure of Solution Sets for Optimisation Problems II: Non-Linear Problems

## 1B.1 Introduction

This chapter extends the discussion in chapter 1A. While chapter 1A was confined to problems with linear objectives subject to linear constraints, chapter 1B considers non-linear objectives under general convex constraints. Accordingly, uncoupled problems in this chapter take the form of *monotone variational inequalities*, while simultaneous problems satisfy a weakening of this requirement, $\omega$-*monotonicity*. The notion of monotonicity is a well-known property (see e.g. [16] section 2.3 or chapter 11 in [25] for detailed accounts), $\omega$-monotonicity appears to be new however.

Several arguments may be put forward for considering "non-linear problems" as monotone variational inequalities. First, this class of problems can be regarded as a natural generalisation of both concave programmes and linear programmes. Instances of such problems are therefore common in applications. In addition, the class of monotone problems is well-behaved: if a monotone variational inequality has multiple solutions, then an arbitrarily small perturbation can be added in such a way that the resulting problem has a unique solution. The assumption of monotonicity is nevertheless restrictive. In particular, save for very particular exceptions, the problem of determining Nash equilibria does not fulfil the assumption of monotonicity. More generally, saddle point problems fail to satisfy this assumption.

The assumption of monotonicity can be relaxed in the context of simultaneous variational inequalities. In such situation, $\omega-$monotonicity is introduced. Motivated by the theory of games, $\omega-$monotonicity requires that a generalised version of the Kohlberg-Mertens decomposition holds (see corollary 1A.27.1 in chapter 1A. Alternatively, refer to the proof of theorem 1 in [24]). This generalisation amounts to requiring each component of the simultaneous variational inequality can be written as a sum of two components, one monotone and another generic. For instance, a mapping $\Phi : [a_1, b_1] \times [a_2, b_2] \mapsto \mathbb{R}^2$ is $\omega-$monotone if for a given pair $(x_1, x_2)$, $\Phi(x_1, x_2) = (\overline{\omega}_1(x_1), \overline{\omega}_2(x_2)) + \tilde{\omega}(x_1, x_2)$, where each $\omega_i$ is monotone.

Results in this chapter are closely related to Predtetchinski's extension of the result of Kohlberg and Mertens (see [34] and [35]). Indeed, proofs presented in this chapter are very much inspired by his. One important difference however lies in the choice of topology, results in this chapter are given in the uniform norm over the relevant convex set. This is a natural choice since the mappings considered are all continuous. Moreover, unlike Predtetchinski's arguments, we make explicit use of the underlying equivalence between solving variational inequalities and finding the fixed point of a certain map. This characterisation is interesting as it may serve as a stepping stone towards a global index theory. Index theories attach certain integers to isolated solutions (or sets of solutions) which can provide insights as to their behaviour against perturbation. More generally, results in chapter 1A and 1B can be viewed as foundational as they provide technical tools to the study of "generic" or "typical" properties of the objects in consideration.

Lastly, our analysis of variational inequalities is also related to [26]. Rida and co-authors impose no monotonicity assumptions (or indeed any regularity assumptions beyond continuity) and their paper considers a general

Hilbert space $\mathcal{H}$. However, the structure theorems they obtained appear to be limited to the collection

$$\{\Phi + h : h \in \mathcal{H}\}$$

where $\Phi$ is any given continuous mapping over a non-empty, closed and convex set $K \subset \mathcal{H}$. Results in this chapter consider instead a collection of continuous mappings satisfying a regularity assumption (monotonicity). Moreover, even though we framed our results in the context of finite Euclidean spaces, the results generalise easily to arbitrary Hilbert spaces.

This chapter is structured as follows: section 2 defines variational inequalities and simultaneous variational inequalities. Specifically, subsection 2.1 presents variational inequalities as generalisations of concave programmes. Some properties, such as existence of solutions are also discussed. The familiar reader can skip this subsection without prejudice to their understanding of the remainder of the chapter. Subsection 2.2 introduces simultaneous variational inequalities and their properties; existence is once again a notable issue. Section 3 discusses both regularity assumptions in detail, with subsection 3.1 focusing on monotonicity and subsection 3.2, on $\omega-$monotonicity. Notable results in this section include the observation that the set of monotone variational inequalities forms a cone, as well as results providing necessary and sufficient conditions under which a continuous mapping may be $\omega-$monotone. Section 4 then establishes the structure theorem for monotone mappings, while section 5 presents its extension to $\omega-$monotone mappings. Section 6 provides applications by using the structure theorems in sections 4 and 5 to derive a version of the structure theorem (theorem 1) in [34]. Lastly, section 7 summarises the main results and provides some potential extensions.

Lastly, the diagram in figure 1B.1 provides a visualisation of how each optimisation problem in chapters 1A and 1B relate to each other. In particular, arrows represent generalisations.

$$
\begin{tikzcd}
 & \text{Linear Programmes} \arrow[dl] \arrow[dr] & \\
\text{Monotone VI} \arrow[dr] & & \text{Simultaneous LP} \arrow[dl] \\
 & \text{Simultaneous VI} &
\end{tikzcd}
$$

Figure 1B.1: The logical connection between different optimisation problems

## 1B.2 Preliminaries

This section defines variational inequalities and simultaneous variational inequalities formally. The existence of solutions is also recalled.

### 1B.2.1 Variational inequalities

Given $X \subset \mathbb{R}^n$, denote by $X^\circ$ its *interior* and by $\overline{X}$ its *closure*. Lastly, $\partial X$ refers to the *boundary of* $X$. Moreover, recall that $f|_A$ denotes the *restriction* of the mapping $f$ to $A$.

**Definition 1B.1** (Variational Inequality)**.** Given a pair $(\Phi, K)$ with $\Phi : K \longmapsto \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ non-empty, closed and convex a *variational inequality* is the problem of finding $x \in K$ such that

$$\langle \Phi(x), x - y \rangle \geqslant 0 \ \ \forall y \in K. \tag{1B.1}$$

A point $x \in K$ satisfying equation (1B.1) is a *solution to the variational inequality* (or simply a *solution*). For a pair $(\Phi, K)$ as above the *set of solutions* is denoted by $E(\Phi, K)$.

*Remark* 1B.2. Observe that $K$ might be a lower-dimensional set and thus $K^\circ = \emptyset$. Recall that if $K^\circ = \emptyset$ then $\Phi$ is said to be continuous if there exists an open neighbourhood $U$ of $K$ and a continuous function $\hat{\Phi} : U \longmapsto \mathbb{R}^n$ satisfying $\hat{\Phi}|_K = \Phi$.

*Remark* 1B.3. Observe that if $K = K(\alpha)$ for some admissible programme and $\Phi : x \mapsto z$ for some $z \in \mathbb{R}^n$ the resulting variational inequality reduces to a linear programme.

*Example(s)* 1B.1. The simplest example of a variational inequality is given by the pair $(\Phi, [0,1])$, with $\Phi : [0,1] \longmapsto \mathbb{R}$ continuous and satisfying $\Phi(0) > 0 > \Phi(1)$. The variational inequality

$$\Phi(x) \cdot (x - y) \geqslant 0 \ \forall y \in [0,1] \tag{1B.2}$$

is equivalent to the problem of finding the zeroes of $\Phi$.

*Example(s)* 1B.2. Given a compact set $K \subset \mathbb{R}^n$ such that $K^\circ \neq \emptyset$ and a smooth concave function $\phi : K \mapsto \mathbb{R}$ the problem of finding $x \in K$ such that $\phi(x) \geqslant \phi(y)$ for $y \in K \setminus \{x\}$ is equivalent to the variational inequality given by the pair $(\nabla\phi, K)$.

*Remark* 1B.4. The above example illustrates that variational inequalities can be regarded as generalisation of concave programmes where the assumption of $\Phi$ being integrable is dropped. On the other hand, remark 1B.3 provides an alternative interpretation, variational inequalities generalise linear programmes by allowing the vector $z$ to depend on points of $K$. Nonetheless, variational inequalities cover a large class of problems, a comprehensive list can be found in chapter 11 of [25]. Moreover, important applications of the theory of variational inequalities can be found in chapter 12 of [25] or chapter 1 in [16].

The analogy between variational inequalities and concave programmes can be pushed further:

**Proposition 1B.5.** *Given $x \in E(\Phi, K)$ such that $x \in K^\circ$ then $\Phi(x) = 0$.*

A proof of this result may be found in, e.g., corollary 3.2 in [22].

*Remark* 1B.6. Let $U \subset \mathbb{R}^n$ be open and convex. Integrability conditions ensuring that the variational inequality $(\Phi, \overline{U})$ is equivalent to a non-linear programme are discussed in theorem 1.3.1 of [16].

Existence of solutions to a variational inequality is discussed now. Recall that for a given closed convex set $K$ $\rho(\cdot, K)$ denotes the *projection on the set* $K$, i.e. the mapping that associates to each $x \in \mathbb{R}^n$ its closest point in $K$.

**Theorem 1B.7.** *Let $K \subset \mathbb{R}^n$ be a compact, convex non-empty set and $\Phi$ a continuous mapping. The variational inequality $(\Phi, K)$ satisfies $E(\Phi, K) \neq \emptyset$. Moreover $x \in E(\Phi, K)$ if, and only if, $x = \rho(x + \lambda\Phi(x), K)$ for any given $\lambda > 0$.*

For a proof of the "if, and only if," part see proposition 11.13 in [25]. Under this equivalence, existence then follows by a standard application of Brouwer's fixed point theorem. Alternatively, see theorem 3.1 in [22]. Note that both texts consider a minimisation problem as opposed to a maximisation one as it is done here, the notation is therefore slightly different.

*Remark* 1B.8. Theorem 1B.7 generalises theorem 1A.10 in chapter 1A to variational inequalities. Crucially, both theorems characterise solutions of an optimisation problem as fixed points of a well-behaved mapping (for variational inequalities the mapping is $f : x \mapsto \rho(x + \lambda\Phi(x), K)$). As it was the case in chapter 1A, this representation is what underpins the proof of the structure theorem in section 4.

*Remark* 1B.9. The appearance of $\lambda > 0$ in the above statement is due to the fact that if $x \in E(\Phi, K)$ then $x \in E(\lambda\Phi, K)$ for arbitrary $\lambda > 0$. For $x \in K^\circ$ this is clear since $\Phi(x)$ vanishes. If, on the other hand, $x \in \partial K$, then $x \in E(\Phi, K)$ if, and only if, $\Phi(x)$ is the normal vector to a supporting hyperplane. Hyperplanes are of course only defined up such scaling.

### 1B.2.2 Simultaneous variational inequalities

**Definition 1B.10** (Simultaneous Variational Inequality)**.** Given a $(p+1)$-tuple $(\Phi_1, \ldots, \Phi_p, K)$ with

$$K = K_1 \times \ldots \times K_p \subset \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

where each $K_i$ is non-empty, closed and convex; and each $\Phi_i : K \longmapsto \mathbb{R}^{n_i}$. A *simultaneous variational inequality* is the problem of finding $x =$

$(x_1, \ldots, x_p) \in K$ such that the following variational inequalities are satisfied at once:

$$\begin{aligned}
\langle \Phi_1(x_1, \ldots, x_p), x_1 - y_1 \rangle &\geqslant 0 \ \forall y_1 \in K_1, \\
&\vdots \\
\langle \Phi_p(x_1, \ldots, x_p), x_p - y_p \rangle &\geqslant 0 \ \forall y_p \in K_p.
\end{aligned}$$

A $p$-tuple $x = (x_1, \ldots, x_p)$ satisfying the previous inequalities is said to be a *solution to the simultaneous variational inequality problem* or simply a *solution* if there is no ambiguity. Given a $(p+1)$-tuple $(\Phi_1, \ldots, \Phi_p, K)$ satisfying the conditions in definition 1B.10 the set of solutions is denoted by $E(\Phi_1, \ldots, \Phi_p, K)$.

*Remark* 1B.11. Simultaneous variational inequalities arise naturally in the context of game theory when computing best responses. In such setting a solution to the simultaneous system corresponds to a Nash equilibrium. Interestingly, just as variational inequalities may be regarded as generalisations of non-linear programmes, simultaneous variational inequalities can therefore be regarded as generalising best responses.

**Proposition 1B.12.** *Let $(\Phi_1, \ldots, \Phi_p, K)$ be $(p+1)$-tuple where*

$$K = K_1 \times \ldots \times K_p \subset \mathbb{R}^{n_i} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

*with each $K_i \subset \mathbb{R}^{n_i}$ non-empty, convex and compact. Moreover, each $\Phi_i : K \longmapsto \mathbb{R}^{n_i}$ is continuous. Then*

$$E(\Phi_1, \ldots, \Phi_p, K) \neq \emptyset$$

*and it coincides with the set of solutions to the following non-linear system of equations:*

$$\begin{aligned}
\rho(\Phi_1(x_1, \ldots, x_p) + x_1, K_1) &= x_1, \\
&\vdots \\
\rho(\Phi_p(x_1, \ldots, x_p) + x_p, K_p) &= x_p.
\end{aligned}$$

The proof of proposition 1B.12 follows closely that of theorem 1B.7. Indeed, the existence of solutions to the non-linear system is a consequence of Brouwer's fixed point theorem. The equivalence part then follows from theorem 1B.7. An alternative argument is provided by

**Proposition 1B.13.** *Let* $(\Phi_1, \ldots, \Phi_p, K)$*, where* $K = K_1 \times \ldots \times K_p$*, satisfy the assumptions in definition 1B.10 and consider the mapping* $\Phi : K \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$ *defined by* $\Phi : x \mapsto (\Phi_1(x), \ldots, \Phi_p(x))$*. The fact that* $x \in E(\Phi_1, \ldots, \Phi_p, K)$ *is equivalent to* $x \in E(\Phi, K)$*.*

*Proof.* Suppose that

$$x = (x_1, \ldots, x_p) \in K = K_1 \times \ldots \times K_p$$

satisfies

$$\begin{aligned} \langle \Phi_1(x_1, \ldots, x_p), x_1 - y_1 \rangle &\geqslant 0 \ \forall y_1 \in K_1, \\ &\vdots \\ \langle \Phi_p(x_1, \ldots, x_p), x_p - y_p \rangle &\geqslant 0 \ \forall y_p \in K_p. \end{aligned}$$

Then for a given $y \in K$,

$$\langle \Phi(x), x - y \rangle = \sum_{i=1}^{p} \langle \Phi_i(x), x_i - y_i \rangle \geqslant 0. \tag{1B.3}$$

To prove the converse, let $x \in E(\Phi, K)$ and $i \in \{1, \ldots, p\}$. Now, take

$$y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p)$$

where $y_i \in K_i$ is arbitrary. Thus

$$0 \leqslant \langle \Phi(x), x - y \rangle = \sum_{j=1}^{p} \langle \Phi_j(x), x_j - y_j \rangle = \langle \Phi_i(x), x_i - y_i \rangle. \tag{1B.4}$$

Clearly, $x \in E(\Phi_1, \ldots, \Phi_p, K)$. ∎

Assuming proposition 1B.13, proposition 1B.12 is then a direct consequence of the fact that

$$\rho(x, K) = (\rho(x_1, K_1), \ldots, \rho(x_p, K_p)). \tag{1B.5}$$

## 1B.3 Monotonicity assumptions

The previous section imposed no regularity requirements beyond the continuity of the mapping $\Phi$. Additional assumptions are introduced now. In particular, *monotonicity* is discussed. This assumption is commonly made in elementary discussions of optimisation problems as it serves as an analogue of concavity (see e.g. sections 2.3 and 2.4 of [16]). Monotonicity is however fairly restrictive. To ameliorate this situation, a mild relaxation, *$\omega$-monotonicity*, is introduced in subsection 3.2. In effect, this assumption extends monotonicity to simultaneous variational inequalities.

### 1B.3.1 Monotonicity

**Definition 1B.14** (Monotone mapping)**.** A mapping $\Phi : K \longmapsto \mathbb{R}^n$ is *monotone* if it satisfies

$$\langle \Phi(x) - \Phi(y), x - y \rangle \leqslant 0 \ \forall x, y \in K. \tag{1B.6}$$

If the inequality is strict whenever $x \neq y$ then the mapping is said to be *strictly monotone.*

*Remark* 1B.15. A necessary condition for the mapping $\Phi : K \longmapsto \mathbb{R}^n$ to be monotone is that for every $i \in \{1, \ldots, n\}$ and any given choice of

$$x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n$$

the mapping

$$\Phi_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$$

be decreasing. This condition is not sufficient however. For instance, let $\Phi : (x_1, x_2) \mapsto (x_2, x_1)$. Since $\Phi_i$ does not depend on $x_i$, $\Phi_i$ is decreasing.

Nevertheless,

$$\langle \Phi(1,1) - \Phi(0,0), (1,1) \rangle = 2 > 0.$$

When $\Phi$ is differentiable monotonicity is equivalent to the derivative of $\Phi$, $D\Phi$, being negative semi-definite:

**Proposition 1B.16.** *Let $U \subset \mathbb{R}^n$ be open and convex. A given differentiable mapping $\Phi : U \longmapsto \mathbb{R}^n$ is monotone if, and only if, $D\Phi$ is negative semi-definite.*

A detailed proof of this result is omitted. For a sketch, observe that the implication that monotonicity implies $D\Phi$ is negative semi-definite follows from the definition of derivative. The reverse implication can be obtained by a direct application of the fundamental theorem of calculus.

*Remark* 1B.17. If the condition in proposition 1B.16 is strengthened to $D\Phi$ being negative definite, then it follows that $\Phi$ is strictly monotone; but the converse ceases to hold (see e.g. proposition 11.1 in [25], or proposition 2.3.2 in [16]).

The class of strictly monotone mappings is of interest since it enjoys particularly nice properties:

**Proposition 1B.18.** *Consider the variational inequality given by $(\Phi, K)$ with $\Phi$ continuous and strictly monotone and $K$ a non-empty closed set of $\mathbb{R}^n$, then $|E(\Phi, K)| \leqslant 1$.*

For a proof see e.g. proposition 11.14 in [25].

*Remark* 1B.19. Let $K \neq \emptyset$ be both convex and compact. In addition, let $\Phi$ continuous over $K$ and strictly monotone. Theorem 1B.7 together with proposition 1B.18 imply that the variational inequality given by $(\Phi, K)$ has a unique solution.

Proposition 1B.18 implies that $E(\Phi, K)$ is a convex set whenever $\Phi$ is strictly monotone. This is a general fact:

**Proposition 1B.20.** *Let $K$ be a closed convex set. Given a continuous monotone mapping $\Phi : K \longmapsto \mathbb{R}^n$, the set $E(\Phi, K)$ is convex.*

*Remark* 1B.21. For a proof see theorem 2.3.5 (a) in [16]. Note that theorem 2.3.5 is stated for the weaker condition of *pseudo-monotonicity* (see definition 2.3.1 (a) in [16]), which monotone mappings in particular satisfy.

Given a non-empty, compact and convex $K \subset \mathbb{R}^n$, label by $C(K;\mathbb{R}^n)$ the set of all continuous mappings from $K$ to $\mathbb{R}^n$ endowed with the $L^\infty$ norm,

$$\|\Phi\|_\infty = \max_{x\in K} \|\Phi(x)\|_2. \tag{1B.7}$$

The subset $\mathcal{F}(K) \subset C(K;\mathbb{R}^n)$ refers to the set of all continuous mappings that are monotone as well.

*Remark* 1B.22. Although the choice of the Euclidean norm in equation (1B.7) is natural, an equivalent norm may be obtained under an alternative choice of norm over $\mathbb{R}^n$ (e.g., any $L^p$ norm with $p \in [1,\infty]$).

Let $V$ be a topological vector space, a closed subset $C$ of $V$ is a *cone* if $0 \in C$ and $x + \alpha y \in C$ whenever $x, y \in C$ and $\alpha \geqslant 0$.

The set $\mathcal{F}(K)$ is not a vector space, it is instead a cone:

**Proposition 1B.23.** *Let $K \subset \mathbb{R}^n$ non-empty, convex and compact. The set $\mathcal{F}(K)$ is a cone. Moreover, the class of strictly monotone continuous mappings is dense in $\mathcal{F}(K)$.*

*Proof.* Fix $\alpha \geqslant 0$ and $\Phi_1, \Phi_2 \in \mathcal{F}(K)$; that $\Phi_1 + \alpha\Phi_2 \in \mathcal{F}(K)$ follows directly from the definition. Moreover, the mapping $\Phi : K \longmapsto \{0\}$ is in $\mathcal{F}(K)$.

To prove that the set is closed consider a sequence of monotone mappings $\{\Phi_n\}_{n=1}^\infty$ converging uniformly to a certain $\Phi$. Since $\mathcal{F}(K) \subset C(K;\mathbb{R}^n)$, $\Phi$ is continuous. Now fix $x, y \in K$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\|\Phi - \Phi_n\|_\infty < \frac{\varepsilon}{2(\operatorname{diam}(K) + 1)}, \tag{1B.8}$$

whenever $n > n_0$. Let $n = n_0 + 1$ then an application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
\langle \Phi(x)-\Phi(y), x-y\rangle \leqslant \|\Phi(x)-\Phi_n(x)\|_2 \cdot \|x-y\|_2 + \\
\langle \Phi_n(x)-\Phi_n(y), x-y\rangle + \|\Phi(y)-\Phi_n(y)\|_2 \cdot \|x-y\|_2 \leqslant \\
2\|\Phi-\Phi_n\|_\infty \cdot \operatorname{diam}(K) + \langle \Phi_n(x)-\Phi_n(y), x-y\rangle < \varepsilon.
\end{aligned}
\tag{1B.9}
$$

Since $\varepsilon$ was arbitrary we can conclude that

$$\langle \Phi(x)-\Phi(y), x-y\rangle \leqslant 0. \tag{1B.10}$$

Thus $\Phi \in \mathcal{F}(K)$.

Finally, to show that strictly monotone mappings are dense in $\mathcal{F}(K)$, observe that for a given $\Phi \in \mathcal{F}(K)$ the sequence $\{\Phi - \frac{x}{n}\}_{n=1}^{\infty}$ consists solely of strictly monotone mappings; indeed

$$\left\langle \Phi(x) - \frac{x}{n} - \Phi(y) + \frac{y}{n}, x-y \right\rangle = \langle \Phi(x)-\Phi(y), x-y\rangle - \frac{1}{n^2}\|x-y\|_2^2 < 0. \tag{1B.11}$$

In addition, it converges uniformly to $\Phi$. ■

*Remark* 1B.24. The class of monotone variational inequalities is sufficiently large to encompass linear problems considered in chapter 1A. More formally, the mapping $T : \mathbb{R}^n \longmapsto \mathcal{F}(K)$ defined by identifying $z \in \mathbb{R}^n$ with the constant mapping $\Phi_z : K \longmapsto \{z\}$ is an isometric embedding. Indeed, for such choice of $\Phi$

$$\|\Phi_z\|_\infty = \|z\|_2. \tag{1B.12}$$

In what follows, notation is abused by writing $z$ in place of the constant mapping $\Phi_z$. Observe that with this identification,

$$\|z\|_\infty = \|z\|_2, \tag{1B.13}$$

where $\|\cdot\|_\infty$ denotes the $L^\infty$ norm over $C(K;\mathbb{R}^n)$ and not the maximum norm.

### 1B.3.2 $\omega$-monotonicity

A mild relaxation of monotonicity is now introduced. The assumption is particularly suitable in the context of computing best replies to a certain class of games.

**Definition 1B.25** ($\omega-$monotone mapping)**.** A mapping

$$\Phi : K_1 \times \ldots \times K_p \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p},$$

where each $K_i \subset \mathbb{R}^{n_i}$ non-empty, is $\omega-$*monotone* if there exists mappings

$$\tilde{\omega}, \overline{\omega} : K_1 \times \ldots \times K_p \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

satisfying

(i) $\Phi(x_1, \ldots, x_p) = \overline{\omega}(x_1, \ldots, x_p) + \tilde{\omega}(x_1, \ldots, x_p)$ for any $(x_1, \ldots, x_p) \in K_1 \times \ldots \times K_p$.

(ii) For every $i \in \{1, \ldots, p\}$, $\overline{\omega}_i : K_i \longmapsto \mathbb{R}^{n_i}$ and satisfies

$$\langle \overline{\omega}_i(x_i) - \overline{\omega}_i(y_i), x_i - y_i \rangle \leqslant 0$$

for arbitrary $x_i, y_i \in K_i$.

(iii) There exists $y \in K_1 \times \ldots \times K_p$ such that for every $i \in \{1, \ldots, p\}$

$$\tilde{\omega}_i(y_1, \ldots, y_{i-1}, \cdot, y_{i+1}, \ldots, y_p) = 0.$$

Given an $\omega-$monotone mapping $\Phi$, a pair $(\overline{\omega}, \tilde{\omega})$ satisfying conditions (i)-(iii) in definition 1B.25 is referred to as an $\omega-$*decomposition rooted at* $y$ *of* $\Phi$ (or simply as an $\omega-$*decomposition of* $\Phi$), where $y$ is the element of $K_1 \times \ldots \times K_p$ in condition (iii).

*Remark* 1B.26. Definition 1B.25 states that a mapping $\Phi$ defined over a product space is $\omega-$monotone if it can be decomposed as the sum of two mappings $\overline{\omega}$ and $\tilde{\omega}$, where each component $\overline{\omega}_i$ of $\overline{\omega}$ is a monotone mapping that depends on $x_i$ only. Furthermore, each $\tilde{\omega}_i$ vanishes over

$$y_1 \times \ldots \times y_{i-1} \times K_i \times y_{i+1} \times \ldots \times y_p,$$

with $(y_1, \ldots, y_p) \in K_1 \times \ldots \times K_p$. This last condition is needed to ensure that certain well-behaved mappings arising from simultaneous variational inequalities belong to the class of $\omega$-monotone mappings (see 1B.29 below).

*Remark* 1B.27. Observe that $\omega$-monotonicity generalises the decomposition in 1A.27.1 in chapter 1A, which was itself a generalisation of the original decomposition from Mertens and Kohlberg (see proof of theorem 1 in [24]). In particular, as suggested by the notation, $\overline{\omega}$ (resp. $\tilde{\omega}$) corresponds to $\overline{\mathbf{z}}$ (resp. $\tilde{\mathbf{z}}$) in 1A.27.1.

*Remark* 1B.28. Observe that given $K' \subset \mathbb{R}^n$, letting $K_1 = K'$ as well as $K_2 = \ldots = K_p = \{0\}$ definition 1B.25 can be seen to reduce to the definition of monotonicity by taking $\tilde{\omega} : K_1 \times \ldots \times K_p \mapsto \{0\}$.

To simplify notation, recall the convention adopted in chapter 1A:

$$f(y_i, x_{-i}) = f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p), \tag{1B.14}$$

where $f : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_p} \longmapsto \mathbb{R}$. Moreover, it is convenient to extend this notation: given sets $X_1, \ldots, X_p$ write $X_{-i}$ in place of

$$X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_p.$$

An element of this product can then be written as $x_{-i}$

It is now proven that a mapping $\Phi : K_1 \times \ldots \times K_p \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$ with the property that for every $i \in \{1, \ldots, p\}$ the mapping $\Phi_i(\cdot, x_{-i})$ is monotone for all $x_{-i} \in K_{-i}$ is $\omega$-monotone.

**Proposition 1B.29.** *Suppose that $\Phi : K_1 \times \ldots \times K_p \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$, where each $K_i \subset \mathbb{R}^{n_i}$ is non-empty, has the following property: for every $i \in \{1, \ldots, p\}$ the mapping given by*

$$\Phi_i(\cdot, x_{-i}) : K_i \longmapsto \mathbb{R}^{n_i}$$

*is monotone for every possible choice of $x_{-i} \in K_{-i}$. Then given $y \in K_1 \times \ldots \times K_p$ $\Phi$ has an $\omega$-decomposition rooted at $y$.*

*Proof.* Fix $y \in K_1 \times \ldots \times K_p$ and define

$$\begin{aligned} \overline{\omega}_i(x_i) &= \Phi_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_p), && (1B.15) \\ \tilde{\omega}_i(x) &= \Phi_i(x) - \overline{\omega}_i(x_i). && (1B.16) \end{aligned}$$

The assumption on $\Phi$ implies that each $\overline{\omega}_i$ is monotone in their respective domain. Condition (ii) in definition 1B.25 is therefore satisfied. Moreover,

$$\tilde{\omega}_i(y_1, \ldots, y_{i-1}, \cdot, y_{i+1}, \ldots, y_p) = 0.$$

■

**Proposition 1B.30.** *Let $\Phi$ be as in proposition 1B.29. Given*

$$y \in K_1 \times \ldots \times K_p,$$

*the corresponding $\omega$-decomposition rooted at $y$ of $\Phi$ is unique.*

*Proof.* Fix $y \in K_1 \times \ldots \times K_p$ and, by way of contradiction, suppose that for some $\Phi$ satisfying the assumption in the statement of proposition 1B.29 two distinct $\omega$-decompositions rooted at $y$, represented by the pairs $(\overline{\omega}, \tilde{\omega})$ and $(\overline{\zeta}, \tilde{\zeta})$ respectively, existed. Condition (i) in definition 1B.25 implies

$$\Phi = \overline{\omega} + \tilde{\omega} = \overline{\zeta} + \tilde{\zeta}. \tag{1B.17}$$

Hence,

$$\overline{\omega} - \overline{\zeta} = -(\tilde{\omega} - \tilde{\zeta}). \tag{1B.18}$$

Let $i \in \{1, \ldots, p\}$ and choose $x_i \in K_i$, condition (iii) in definition 1B.25 implies that

$$\begin{aligned} \overline{\omega}_i(x_i) - \overline{\zeta}_i(x_i) = -(&\overline{\omega}_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, x_p) - \\ &\overline{\zeta}_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, x_p)) = 0. \end{aligned} \tag{1B.19}$$

Consequently, $\overline{\omega}_i = \overline{\zeta}_i$. But since $i$ was arbitrary, $\overline{\omega} = \overline{\zeta}$, which results in a contradiction. ■

*Remark* 1B.31. Given $K_1 \times \ldots \times K_p \neq \emptyset$. Propositions 1B.29 and 1B.30 together imply that for every $y \in K_1 \times \ldots \times K_p$ there exists a bijection between the set of mappings $\Phi$ with the property that for every $i \in \{1, \dots, p\}$ the mapping given by

$$\Phi_i(\cdot, x_{-i}) : K_i \longmapsto \mathbb{R}^{n_i}$$

is monotone for every possible choice of $x_{-i} \in K_{-i}$, and the set of pairs $(\overline{\omega}, \tilde{\omega})$ satisfying conditions (i)-(iii) in definition 1B.25.

*Remark* 1B.32. If a non-monotone mapping $\Phi$ satisfies for every given $i \in \{1, \dots, p\}$ the mapping

$$\Phi_i(\cdot, x_{-i}) : K_i \longmapsto \mathbb{R}^{n_i}$$

is monotone for any choice of $x_{-i} \in K_{-i}$, the set $E(\Phi, K)$ need not be convex. This is illustrated by the example below.

*Example(s)* 1B.3. Let $\Phi : [0,1]^2 \longmapsto \mathbb{R}^2$ be defined by $\Phi : (x_1, x_2) \mapsto (x_2, x_1)$. $\Phi$ is non-monotone. However, both $\Phi(\cdot, x_2) = x_2$ and $\Phi(x_1, \cdot) = x_1$ are monotone since they are constant mappings. By applying proposition 1B.13 solutions to the variational inequality $\langle \Phi(x), x - y \rangle \geqslant 0$ for arbitrary $y \in [0,1]$ coincide with the solutions to the simultaneous variational inequality

$$x_2(x_1 - y_1) \geqslant 0 \, \forall \, y_1 \in [0,1], \tag{1B.20}$$

$$x_1(x_2 - y_2) \geqslant 0 \, \forall \, y_2 \in [0,1]. \tag{1B.21}$$

From where it can be seen that $E(\Phi, [0,1]^2) = \{(0,0), (1,1)\}$, a disconnected (and hence non convex) set.

Given $K_1 \subset \mathbb{R}^{n_1}, \dots, K_p \subset \mathbb{R}^{n_p}$ all convex, compact and non-empty. Call $\mathcal{M}(K_1, \dots, K_p) \subset C(K_1; \mathbb{R}^{n_1}) \oplus \dots \oplus C(K_p; \mathbb{R}^{n_p})$ the set of all mappings that satisfy the condition in proposition 1B.29. Namely, for all $i \in \{1, \dots, p\}$ the mapping

$$\Phi_i(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_p) : K_i \longmapsto \mathbb{R}^{n_i}$$

is monotone for every $x_1 \in K_1, \dots, x_p \in K_p$. Note that the space

$$C(K_1; \mathbb{R}^{n_1}) \oplus \dots \oplus C(K_p; \mathbb{R}^{n_p})$$

is assumed to be endowed with the metric

$$\|(\Phi_1, \dots, \Phi_p)\|_{\mathcal{M}} = \max\{\|\Phi_1\|_{C(K_1; \mathbb{R}^{n_1})}, \dots, \|\Phi_p\|_{C(K_p; \mathbb{R}^{n_p})}\}, \tag{1B.22}$$

where $\|\cdot\|_{C(K_i;\mathbb{R}^{n_i})}$ is the $L^\infty$ norm in the respective space. Norms are spelled out here for clarity. In the sequel we resume to writing $\|\cdot\|_\infty$ to ease notation.

*Remark* 1B.33. It might be objected that the following norm

$$\|(\Phi_1,\ldots,\Phi_p)\|_1 = \|\Phi_1\|_{C(K_1;\mathbb{R}^{n_1})} + \ldots + \|\Phi_p\|_{C(K_p;\mathbb{R}^{n_p})} \tag{1B.23}$$

provides a more natural choice when defining a norm over $\mathcal{M}$. The choice of the "max norm" is motivated by previous choices on chapter 1A.

Suppose that $\{K_i\}_{i=1}^p$ is a finite collection of non-empty, convex and compact sets satisfying $K_i \subset \mathbb{R}^{n_i}$. Fix now $y_0 \in K_1 \times \ldots \times K_p$ and denote by

$$\Omega_{y_0}(K_1,\ldots,K_p) \subset (\mathcal{F}(K_1)\oplus\ldots\oplus\mathcal{F}(K_p))\oplus(C(K_1;\mathbb{R}^{n_1})\oplus\ldots\oplus C(K_p;\mathbb{R}^{n_p}))$$

the set of all possible pairs $(\overline{\omega},\tilde{\omega})$ that satisfy conditions (ii) and (iii) in definition 1B.25 where the $y$ in condition (iii) satisfies $y = y_0$. Moreover, endow this space with the norm

$$\|(\overline{\omega},\tilde{\omega})\|_{\Omega_{y_0}(K_1,\ldots,K_p)} = \max\{\|\overline{\omega}\|_{\mathcal{M}}, \|\tilde{\omega}\|_{\mathcal{M}}\}. \tag{1B.24}$$

**Proposition 1B.34.** *Let $K = K_1 \times \ldots K_p \subset \mathbb{R}^n$ where each $K_i \subset \mathbb{R}^{n_i}$ us non-empty, convex and compact (so that, in particular, $\mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p} = \mathbb{R}^n$). Given $\Phi \in C(K;\mathbb{R}^n)$ having $\omega$-decomposition $(\overline{\omega},\tilde{\omega})$ rooted at $y \in K$. There exists constants $C_1, C_2 > 0$ such that*

$$C_1\|(\overline{\omega},\tilde{\omega})\|_{\Omega_y(K_1,\ldots,K_p)} \leqslant \|\Phi\|_\infty \leqslant C_2\|(\overline{\omega},\tilde{\omega})\|_{\Omega_y(K_1,\ldots,K_p)}. \tag{1B.25}$$

*Moreover, these constants only depend on $p$.*

*Proof.* Fix $i \in \{1,\ldots,p\}$ and choose $x_i \in K_i$, then

$$\begin{aligned} \|\overline{\omega}_i(x_i)\|_2 &= \|\overline{\omega}_i(x_i) + \tilde{\omega}_i(y_1,\ldots,y_{i-1},x_i,y_{i+1},\ldots,y_p)\|_2 \\ &= \|\Phi_i(y_1,\ldots,y_{i-1},x_i,y_{i+1},\ldots,y_p)\|_2 \leqslant \|\Phi\|_\infty. \end{aligned} \tag{1B.26}$$

It follows, that $\|\overline{\omega}\|_{\mathcal{M}} \leqslant \|\Phi\|_\infty$. On the other hand,

$$\|\tilde{\omega}_i(x_i,x_{-i})\|_2 \leqslant \|\Phi_i(x_i,x_{-i})\|_2 + \|\overline{\omega}_i(x_i)\|_2 \leqslant 2\|\Phi\|_\infty. \tag{1B.27}$$

Hence, $\|\tilde{\omega}\|_{\mathcal{M}} \leqslant 2\|\Phi\|_\infty$. From where it follows at once, that

$$\frac{1}{2}\|(\overline{\omega}, \tilde{\omega})\|_{\Omega_y(K_1,\ldots,K_p)} \leqslant \|\Phi\|_\infty. \tag{1B.28}$$

Now, given $x \in K$,

$$\|\Phi_i(x)\|_2 \leqslant \|\overline{\omega}_i(x_i)\|_2 + \|\tilde{\omega}_i(x)\|_2 \leqslant \|\overline{\omega}_i\|_\infty + \|\tilde{\omega}_i\|_\infty \leqslant 2\|(\overline{\omega}, \tilde{\omega})\|_{\Omega(K_1,\ldots,K_p)}. \tag{1B.29}$$

This relation implies $\|(\Phi_1, \ldots, \Phi_p)\|_{\mathcal{M}} \leqslant 2\|(\overline{\omega}, \tilde{\omega})\|_{\Omega_y(K_1,\ldots,K_p)}$. But since $\Phi$ can be identified with the $p$-tuple $(\Phi_1, \ldots, \Phi_p)$ and

$$\|\Phi\|_\infty \leqslant \sqrt{p}\, \|(\Phi_1, \ldots, \Phi_p)\|_{\mathcal{M}} \tag{1B.30}$$

the result follows. ■

**Corollary 1B.34.1.** *Let $K_1 \subset \mathbb{R}^{n_1}, \ldots, K_p \subset \mathbb{R}^{n_p}$ all non-empty, convex and compact. Fix now some $y \in K_1 \times \ldots \times K_p$, the sets $\mathcal{M}(K_1, \ldots, K_p)$ is homeomorphic to a closed subset of $\Omega_y(K_1, \ldots, K_p)$.*

*Remark* 1B.35. By way of a final remark, the spaces $\Omega_y(K_1, \ldots, K_p)$ and $\mathcal{M}(K_1, \ldots, K_p)$ are both cones. The proof involves virtually the same steps as for the space $\mathcal{F}(K)$ and it is therefore omitted.

## 1B.4 The structure theorem for monotone variational inequalities

This section establishes the structure theorem for the class of monotone variational inequalities.

Recall that given two mappings $f$ and $g$, their *composition* is denoted by $f \circ g$.

**Definition 1B.36** (Equilibrium set)**.** Given $K \subset \mathbb{R}^n$ non-empty, convex and compact, the *equilibrium set* $E(K)$ is the set

$$E(K) = \bigcup_{\Phi \in \mathcal{F}(K)} \{\Phi\} \times E(\Phi, K).$$

*Remark* 1B.37. The set $E(K)$ corresponds to the graph of the multi-valued mapping that associates to each $\Phi \in \mathcal{F}(K)$ its solutions.

**Theorem 1B.38.** *Given a non-empty convex compact $K \subset \mathbb{R}^n$ there exists a homeomorphism $\psi : \mathcal{F}(K) \longmapsto E(K)$. Moreover, $\psi$ satisfies $\pi \circ \psi \simeq id_{\mathcal{F}(K)}$, where $\pi : (\Phi, \bar{x}) \in E(K) \mapsto \Phi$. Moreover, this homotopy extends to the one-point compactification $\mathcal{F}(K) \cup \{\infty\}$.*

*Remark* 1B.39. The first part of theorem 1B.38 corresponds to theorem 1A.14 in chapter 1A, while the second corresponds to corollary 1A.18.1.

*Remark* 1B.40. For a given $K \subset \mathbb{R}^n$ non-empty, convex and compact, theorem 1B.7 implies that $E(\Phi, K) \neq \emptyset$ for all $\Phi \in \mathcal{F}(K)$. The set $E(K)$ is therefore well-defined. Moreover, by definition, $E(K) \subset C(K; \mathbb{R}^n) \times K$. Since this latter space has a product structure, the norm inherited by $E(K)$ is

$$\|(\Phi, z)\| = \max\{\|\Phi\|_\infty, \|z\|_2\}. \tag{1B.31}$$

Rather than proving theorem 1B.38, a generalisation is proven instead. To that end, $\mathcal{A} \subset \mathcal{F}(K)$ is said to be *translation invariant* if $\mathcal{A} + \mathbb{R}^n \subset \mathcal{A}$, where

$$\mathcal{A} + \mathbb{R}^n = \{\Phi + z : z \in \mathbb{R}^n, \Phi \in \mathcal{A}\}.$$

*Remark* 1B.41. More rigorously, let $T : z \mapsto \Phi_z$, where $\Phi_z(x) = z$ for every $x \in K$. The embedding $T$ defines an action of the group $\mathbb{R}^n$ over $\mathcal{F}(K)$, a set is translation invariant if it is an invariant subset for this action.

*Remark* 1B.42. Translation invariance is analogous to the *stability assumption* introduced in [9] in the context of polynomial games (see assumption 2 in page 511 of [9]). Translation invariance moreover includes the type of objects studied in [26]. Note that, unlike Rida and co-authors, the additional requirement of monotonicity is assumed here. Lastly, with respect to the results of chapter 1A, the set of linear programmes parametrised by the vector $z$ is a translation invariant set.

**Theorem 1B.43.** *For $\mathcal{A} \subset \mathcal{F}(K)$ closed and translation-invariant, let*

$$E(K; \mathcal{A}) = \bigcup_{\Phi \in \mathcal{A}} \{\Phi\} \times E(\Phi, K).$$

*There exists a homeomorphism $\psi : \mathcal{A} \longmapsto E(K; \mathcal{A})$. Moreover, this homeo-*

*morphism satisfies $\pi \circ \psi \simeq id_{\mathcal{A}}$ and this homotopy extends to the one-point compactification of $\mathcal{A}$.*

This result relies on the following

**Lemma 1B.44.** *Let $K \subset \mathbb{R}^n$ be non-empty, convex and compact. Given $\Phi \in \mathcal{F}(K)$ the function $x \mapsto \rho(\Phi(x), K)$ has a unique fixed point. Moreover, the mapping $X : \mathcal{F}(K) \longmapsto K$ that associates to each $\Phi \in \mathcal{F}(K)$ the fixed point of $\rho(\Phi(\cdot), K)$ is continuous.*

*Proof.* The existence of the fixed point follows from Brouwer's fixed point theorem. To show uniqueness, let $\Psi(x) = \Phi(x) - x$. The mapping $\Psi$ is strictly monotone, which follows by applying the same argument as in proposition 1B.23. Proposition 1B.18 yields the existence of a unique $x_0$ such that

$$x_0 = \rho(\Psi(x_0) + x_0, K) = \rho(\Phi(x_0), K). \tag{1B.32}$$

The continuity of $X$ follows from lemma 1B.45 below. ∎

**Lemma 1B.45.** *Given $K \subset \mathbb{R}^n$ compact, let $\{f_n\}_{n=1}^{\infty} \subset C(K; \mathbb{R}^n)$ be a sequence converging to $f$ uniformly. Moreover, suppose that for each $f_n$ there exists a unique $x_n \in K$ such that $f_n(x_n) = 0$. Lastly, if $f(x) = 0$ is satisfied by a single $x \in K$, then $x_n \to x$.*

*Proof.* Let $\{x_n\}_{n=1}^{\infty}$ be the sequence whose $n^{\text{th}}$ term is the unique zero of $f_n$. Since $K$ is compact, there exists $y \in K$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying $x_{n_k} \to y$.

Now, since $f_n \to f$ uniformly, $f_{n_k}(x_{n_k}) \to f(y)$. Recall that $f(x_{n_k}) = 0$ for every $k \in \mathbb{N}$, thus $f(y) = 0$. By assumption, $f$ has a unique zero at $x$, hence $y = x$. Moreover, suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ contained a cluster point $z \neq x$, the same argument would yield $z = x$, a contradiction. Hence, $x$ is the unique cluster point of $\{x_n\}_{n=1}^{\infty}$. Since $K$ is compact, $x_n \to x$. ∎

*Proof of theorem 1B.43.* Let $X : \mathcal{F}(K) \longmapsto K$ be the mapping that associates to each $\Phi \in K$ the element $x \in K$ satisfying $x = \rho(\Phi(x), K)$.

Define the mappings $\varphi : (\Phi, z) \in E(K;\mathcal{A}) \mapsto \Phi + z$ and $\psi : \Phi \in \mathcal{A} \mapsto (\Phi - X(\Phi), X(\Phi))$. Observe that for a fixed $z \in K$ the mapping $\Phi + z$ is monotone whenever $\Phi$ is.

It is established first that $\varphi$ and $\psi$ are well-defined. Note that translation invariance implies that $\varphi(E(K;\mathcal{A})) \subset \mathcal{A}$; hence $\varphi$ is well-defined. Moreover, it is Lipschitz continuous by definition. On the other hand, to prove that $\psi$ is well-defined fix $\Phi \in \mathcal{F}(K)$ and write $X(\Phi) = \bar{x}$. The definition of $X$ yields

$$\bar{x} = \rho(\Phi(\bar{x}), K) = \rho((\Phi(\bar{x}) - \bar{x}) + \bar{x}, K). \tag{1B.33}$$

It follows from the characterisation given in theorem 1B.7 that $\bar{x} \in E(\Phi - \bar{x}, K)$. Hence, $\psi(\mathcal{A}) \subset E(K;\mathcal{A})$ as required. The reverse inclusion holds as well, since $\varphi$ is well-defined the proof is formally identical to the argument that $\varphi$ is a right inverse of $\psi$. This argument is given below.

Let $(\Phi, z) \in E(K;\mathcal{A})$, observe that

$$(\psi \circ \varphi)(\Phi, z) = \psi(\Phi + z) = (\Phi + z - X(\Phi + z), X(\Phi + z)). \tag{1B.34}$$

All that remains is to show that $X(\Phi + z) = z$; write $\Psi(\cdot) = \Phi(\cdot) + z$ and note that $z \in E(\Phi, K)$ therefore, by theorem 1B.7,

$$z = \rho(\Phi(z) + z, K) = \rho(\Psi(z), K). \tag{1B.35}$$

This implies $z$ is a fixed point of the mapping $\rho(\Psi(\cdot), K)$; but this mapping has a single fixed point. Hence, $X(\Phi + z) = z$ and, consequently, $\psi \circ \varphi = \mathrm{id}_{E(K)}$.

In fact, $\varphi$ is a left inverse of $\psi$:

$$(\varphi \circ \psi)(\Phi) = \varphi(\Phi - X(\Phi), X(\Phi)) = \Phi. \tag{1B.36}$$

Since $\varphi$ and $\psi$ are inverses and $\varphi$ is continuous. The existence of a homeomorphism between $\mathcal{A}$ and $E(K;\mathcal{A})$ would be proven if $\psi$ were continuous. To that end, let $\{\Phi\}_{n=1}^{\infty} \subset \mathcal{A}$ be a sequence converging uniformly to $\Phi$. For any $n \in \mathbb{N}$,

$$\|\psi(\Phi) - \psi(\Phi_n)\| \leqslant \|\Phi - \Phi_n\|_\infty + 2\|X(\Phi) - X(\Phi_n)\|_2, \tag{1B.37}$$

$$\limsup_{n\to\infty} \|\psi(\Phi) - \psi(\Phi_n)\| \leqslant 2 \limsup_{n\to\infty} \|X(\Phi) - X(\Phi_n)\|_2 = 0 \tag{1B.38}$$

by the continuity of $X$. $\psi$ is thus continuous, concluding the proof of the first part of the theorem.

For the second part, define the homotopy $H : [0,1] \times E(K;\mathcal{A}) \longmapsto \mathcal{A}$ by

$$H(t,(\Phi,z)) = t\varphi(\Phi,z) + (1-t)\pi(\Phi,z) = \Phi + tz, \tag{1B.39}$$

here $\pi : (\Phi, z) \mapsto \Phi$. Since $\mathcal{A}$ is closed, lemma 1A.17 implies $H$ is proper. The remainder of the proof follows the exact same lines as for the proof of 1A.36 and its cororllary, further details are therefore omitted. ■

*Remark* 1B.46. An alternative, less elegant, proof for the last part of theorem 1B.43 uses the theorem of Arzelà and Ascoli to establish that $H$ is a proper homotopy.

## 1B.5 A generalisation to simultaneous variational inequalities

In this section, a generalisation of theorem 1B.43 to simultaneous variational inequalities is presented.

Given a collection $\{K_i\}_{i=1}^p$, where each $K_i \subset \mathbb{R}^{n_i}$ is non-empty, convex and compact. Recall that

$$\mathcal{M}(K_1,\dots,K_p) = \{\Phi \in C(K;\mathbb{R}^N) : \ \Phi_i(\cdot, x_{-i}) \text{ monotone } \forall x_{-i} \in K_{-i}\},$$

where $K = K_1 \times \ldots \times K_p$ and

$$\mathbb{R}^N = \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}.$$

Moreover, given $y \in K$ recall that corollary 1B.34.1 implies that there exists a homeomorphism $F_y : \mathcal{M}(K_1, \ldots, K_p) \longmapsto \Omega_y(K_1, \ldots, K_p)$, where this latter set denotes the set of all pairs $(\overline{\omega}, \tilde{\omega})$ that satisfy conditions (ii) and (iii) in definition 1B.25.

**Definition 1B.47** (Equilibrium set)**.** Let $\{K_i\}_{i=1}^p$, where each $K_i \subset \mathbb{R}^{n_i}$ is non-empty, convex and compact. Given $\mathcal{A} \subset \mathcal{M}(K_1, \ldots, K_p)$ translational invariant. For a given choice of $y \in K = K_1 \times \ldots \times K_p$ the *equilibrium set* $E_y(K_1, \ldots, K_p; \mathcal{A})$ is given by

$$E_y(K_1, \ldots, K_p; \mathcal{A}) = \bigcup_{(\overline{\omega}, \tilde{\omega}) \in F_y(\mathcal{M}(K_1, \ldots, K_p))} \{(\overline{\omega}, \tilde{\omega})\} \times E((\overline{\omega}, \tilde{\omega}), K), \quad (1B.40)$$

where $E((\overline{\omega}, \tilde{\omega}), K)$ is the set of solutions to the simultaneous variational inequality given by $(\overline{\omega}_1 + \tilde{\omega}_1, \ldots, \overline{\omega}_p + \tilde{\omega}_p, K_1 \times \ldots \times K_p)$.

*Remark* 1B.48*.* Observe that in the context of simultaneous variational inequalities, $\mathcal{A} \subset \mathcal{M}(K_1, \ldots, K_p)$ being translational invariant implies $(\Phi_1 + z_1, \ldots, \Phi_p + z_p) \in \mathcal{A}$ whenever $\Phi = (\Phi_1, \ldots, \Phi_p) \in \mathcal{A}$ and $(z_1, \ldots, z_p) \in \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$.

*Remark* 1B.49*.* Note that proposition 1B.12 implies that

$$E((\overline{\omega}, \tilde{\omega}), K_1 \ldots, K_p) \neq \emptyset$$

is non-empty. As a result, $E_y(K_1, \ldots, K_p; \mathcal{A})$ is well-defined. It is the restriction of the graph of the multi-valued mapping that associates to each simultaneous variational inequality (viewed in terms of its unique $\omega$-decomposition rooted at $y \in K$) the set of its solutions to $\mathcal{A}$.

The set $E_y(K_1, \ldots, K_p; \mathcal{A})$ is endowed with the norm

$$\|((\overline{\omega}, \tilde{\omega}), z)\| = \max\{\|(\overline{\omega}, \tilde{\omega})\|_{\Omega_y(K_1, \ldots, K_p)}, \|z\|_2\}.$$

The analogue of theorem 1B.43 can now be stated:

**Theorem 1B.50.** *Let $\{K_i\}_{i=1}^p$, where each $K_i \subset \mathbb{R}^{n_i}$ is non-empty, convex and compact. Given $\mathcal{A} \subset \mathcal{M}(K_1, \ldots, K_p)$ closed and translational invariant, and a choice of $y \in K_1 \times \ldots \times K_p$, there exists a homeomorphism $\psi$ from a subset of $\Omega_y(K_1, \ldots, K_p)$ to $E_y(K_1, \ldots, K_p; \mathcal{A})$. Furthermore, $\pi \circ \psi \simeq$*

*$id_{\Omega_y(K_1,\ldots,K_p)}$, where $\pi$ is the projection over $\Omega_y(K_1,\ldots,K_p)$. Moreover, this homotopy extends to the one-point compactification of the given subset.*

*Remark* 1B.51. Recall that $D\Lambda$ denotes the image of $\Lambda^p$ ($\Lambda$ referring to the space of multi-linear mappings) under the bijection

$$(T_1,\ldots,T_p) \mapsto (D_{x_1}T_1,\ldots,D_{x_p}T_p).$$

Since the set $D\Lambda$ is translation invariant, theorem 1B.50 reduces to theorem 1A.32 and corollary 1A.36.1 in the particular case where the collection $\{K_i\}_{i=1}^p$ consists of polytopes, and $\mathcal{A} = D\Lambda$. Moreover, if each polytope in the collection is in fact a simplex then theorem 1B.50 reduces to the original result due to Mertens and Kohlberg.

*Remark* 1B.52. Observe that in the particular case where $\{K_i\}_{i=1}^p$ satisfies $K_1 = K'$ and $K_i = \{0\}$ theorem 1B.50 reduces to theorem 1B.43.

A small technical lemma is required before providing a proof of theorem 1B.50.

**Lemma 1B.53.** *Let $K_1 \subset \mathbb{R}^{n_1},\ldots,K_p \subset \mathbb{R}^{n_p}$; where each subset is non-empty, convex and compact. Given $y \in K_1 \ldots K_p$, denote by*

$$F_y : \mathcal{M}(K_1,\ldots,K_P) \longmapsto \Omega_y(K_1,\ldots,K_p)$$

*the homeomorphism in proposition 1B.29. The set $F_y(\mathcal{A})$ is translation invariant whenever $\mathcal{A} \subset \mathcal{M}(K_1,\ldots,K_p)$ is translation invariant.*

*Remark* 1B.54. Formally, in the context of lemma 1B.53, translational invariance of $F_y(\mathcal{A})$ implies that for every $z \in \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$ and any given pair $(\overline{\omega},\tilde{\omega}) \in \mathcal{A}$, $(\overline{\omega},\tilde{\omega}) + (z,0) \in F_y(\mathcal{A})$. Observe that the last relation holds since $z$ can be viewed as a monotone mapping and therefore the pair $(z,0)$ represents an $\omega$-decomposition rooted at $y$, for any $y \in K_1 \times \ldots \times K_p$.

*Proof of lemma 1B.53.* Recall that by proposition 1B.29, the homeomorphism $F_y : \mathcal{M}(K_1,\ldots,K_p) \longmapsto \Omega_y(K_1,\ldots,K_p)$ satisfies

$$\begin{aligned} F_y : (\Phi_1,\ldots,\Phi_p) \mapsto\ & (\Phi_1(\cdot,y_{-1}),\ldots,\Phi_p(\cdot,y_{-p}), \\ & \Phi_1 - \Phi_1(\cdot,y_{-1}),\ldots,\Phi_p - \Phi(\cdot,y_{-p})), \\ F_y^{-1} : (\overline{\omega}_1,\ldots,\overline{\omega}_p,\tilde{\omega}_1,\ldots,\tilde{\omega}_p) \mapsto\ & (\overline{\omega}_1 + \tilde{\omega}_1,\ldots,\overline{\omega}_p + \tilde{\omega}_p). \end{aligned}$$

Fix $z \in \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$ and $(\overline{\omega}, \tilde{\omega}) = F_y(\Phi)$, where $\Phi \in \mathcal{A}$. By definition, $(\overline{\omega}, \tilde{\omega}) + (z, 0) = (\overline{\omega} + z, \tilde{\omega})$. Meanwhile,

$$F_y^{-1}(\overline{\omega} + z, \tilde{\omega}) = \overline{\omega} + \tilde{\omega} + z = \Phi + z. \tag{1B.41}$$

Now, since $\mathcal{A}$ is translation invariant, $\Phi + z \in \mathcal{A}$. This last relation yields the lemma. ∎

*Proof of theorem 1B.50.* The argument follows the same lines as for theorem 1B.43 and the structure theorems in chapter 1A. In particular, the first step is to construct the homeomorphism $\psi$ and its putative inverse $\varphi$. The construction is presented below. For brevity, proofs of continuity of these mappings is postponed to lemmas 1B.55 and 1B.56 below. In addition, the latter part of the theorem concerning the homotopy $\pi \circ \psi \simeq \mathrm{id}_{\Omega_y(K_1, \ldots, K_p)}$ is only sketched.

Let

$$X : \mathcal{F}(K_1) \oplus \ldots \oplus \mathcal{F}(K_p) \longmapsto K_1 \times \ldots \times K_p$$

be the mapping defined by

$$X : \overline{\omega} \mapsto (X_i(\overline{\omega}_i), \ldots, X_p(\overline{\omega}_p)), \tag{1B.42}$$

where for each $i \in \{1, \ldots, p\}$, $X_i$ is the mapping defined in 1B.44. Recall that $X_i$ maps $\Phi_i \in \mathcal{F}(K_i)$ to the unique fixed point of $\rho(\Phi(\cdot), K_i)$. Moreover, let $F_y$ be the homeomorphism in proposition 1B.29.

Fix $y \in K_1 \times \ldots \times K_p$ and let $\varphi : ((\overline{\omega}, \tilde{\omega}), z) \in E_y(K_1, \ldots, K_p; \mathcal{A}) \mapsto (\overline{\omega} + \tilde{\omega}(z) + z, \tilde{\omega})$ and $\psi : (\overline{\omega}, \tilde{\omega}) \in F_y(\mathcal{A}) \mapsto ((\overline{\omega} - \tilde{\omega}(X(\overline{\omega})) - X(\overline{\omega}), \tilde{\omega}), X(\overline{\omega}))$.

Observe that for any $z \in K_1 \times \ldots \times K_p$, $\tilde{\omega}(z) + z$ is a constant vector; lemma 1B.53 implies that $\varphi$ is well-defined. More precisely, $\varphi(E_y(K_1, \ldots, K_p; \mathcal{A})) \subset F_y(\mathcal{A})$. Moreover, $\varphi$ satisfies

$$(\varphi \circ \psi)(\overline{\omega}, \tilde{\omega}) = \varphi((\overline{\omega} - \tilde{\omega}(X(\overline{\omega})) - X(\overline{\omega}), \tilde{\omega}), X(\overline{\omega})) = (\overline{\omega}, \tilde{\omega}). \tag{1B.43}$$

Hence, $\varphi$ is a left-inverse of $\psi$. All that remains is to show that $\psi$ is well defined and has $\varphi$ as a right-inverse.

Given $(\overline{\omega}, \tilde{\omega}) \in F_y(\mathcal{A})$ and a choice of $i \in \{1, \ldots, p\}$; let $x = X(\overline{\omega})$ and observe that

$$x_i = \rho(\overline{\omega}_i(x), K_i) = \rho((\overline{\omega}_i(x) - \tilde{\omega}_i(x) - x_i) + \tilde{\omega}_i(x) + x_i, K_i). \tag{1B.44}$$

proposition 1B.12 implies that

$$x \in E(K_1, \ldots, K_p, (\overline{\omega} - \tilde{\omega}(X(\overline{\omega})) - X(\overline{\omega}), \tilde{\omega})).$$

Hence,

$$\psi(\Omega_y(K_1, \ldots, K_p)) \subset E_y(K_1, \ldots, K_p; \mathcal{A}).$$

Consequently, $\psi$ is well-defined. To finish the construction, note that

$$(\psi \circ \varphi)((\overline{\omega}, \tilde{\omega}), z) = \psi(\overline{\omega} + \tilde{\omega}(z) + z, \tilde{\omega}) = ((\overline{\omega} + \tilde{\omega}(z) + z - \tilde{\omega}(y) - y, \tilde{\omega}), y), \tag{1B.45}$$

where $y = X(\overline{\omega} + \tilde{\omega}(z) + z)$. Since $z$ is a solution to the variational inequality given by $(K_1, \ldots, K_p, (\overline{\omega}, \tilde{\omega}))$ proposition 1B.12 implies that for $i \in \{1, \ldots, p\}$:

$$z_i = \rho(\overline{\omega}_i(z_i) + \tilde{\omega}_i(z) + z_i, K_i) \tag{1B.46}$$

However, $y_i$ is the only fixed point of $\rho(\overline{\omega}_i(\cdot) + \tilde{\omega}_i(z) + z_i, K_i)$ which means $z_i = y_i$. Hence, $\varphi$ is a right-inverse as well, which concludes the first part of the proof.

For the second part of the proof, recall that $\pi : ((\overline{\omega}, \tilde{\omega}), z) \mapsto (\overline{\omega}, \tilde{\omega})$ and consider the linear homotopy $H(t, \cdot) = (1 - t)\varphi + t\pi$. Since $\mathcal{A}$ is closed, lemma 1A.17 implies this homotopy is proper. By composing, $H$ with the homeomorphism $\xi : (t, (\overline{\omega}, \tilde{\omega})) \mapsto (t, \psi(\overline{\omega}, \tilde{\omega}))$ the result follows. ■

**Lemma 1B.55.** *The mapping $\varphi$ in theorem 1B.50 is continuous.*

*Proof.* Let $\{((\overline{\omega}_n, \tilde{\omega}_n), z_n)\}_{n=1}^{\infty} \subset E_y(K_1, \ldots, K_p; \mathcal{A})$ be a sequence converging to $((\overline{\omega}, \tilde{\omega}), z)$. Observe that for $i \in \{1, \ldots, p\}$

$$\begin{aligned}\|\overline{\omega}^i + \tilde{\omega}^i(z) + z^i - \overline{\omega}^i_n - \tilde{\omega}^i_n(z_n) - z^i_n\|_\infty &\leqslant \|\overline{\omega} - \overline{\omega}_n\|_{\mathcal{M}} + \|\tilde{\omega} - \tilde{\omega}_n\|_{\mathcal{M}} \\ &+ \max_{1 \leqslant i \leqslant p} \|\tilde{\omega}^i(z_n) - \tilde{\omega}^i(z)\|_2 + \|z - z_n\|_2.\end{aligned} \tag{1B.47}$$

To avoid notational clutter superscripts in this proof refer to vector components, while subscripts refer to terms of the sequence.

Fix $\varepsilon > 0$. The continuity of $\tilde{\omega}_i$ implies that there exists $n_0 \in \mathbb{N}$ such that $\max_{1\leqslant i\leqslant p} \|\tilde{\omega}^i(z_n) - \tilde{\omega}^i(z)\|_2 < \varepsilon$ for $n > n_0$. Since $((\overline{\omega}_n, \tilde{\omega}_n), z_n) \to ((\overline{\omega}, \tilde{\omega}), z)$, equation (1B.47) implies that

$$\limsup_{n\to\infty} \|\varphi((\overline{\omega}, \tilde{\omega}), z)) - \varphi((\overline{\omega}_n, \tilde{\omega}_n), z_n))\|_{\Omega_y(K_1,\ldots,K_p)} \leqslant \varepsilon. \tag{1B.48}$$

The arbitrariness of $\varepsilon$ concludes the proof. ∎

**Lemma 1B.56.** *The mapping $\psi$ in theorem 1B.43 is continuous.*

*Proof.* Fix $(\overline{\alpha}, \tilde{\alpha}) \in F_y(\mathcal{A})$ and let $\varepsilon > 0$ be given. Since $K_1 \times \ldots \times K_p$ is compact, $\tilde{\alpha}_i$ is uniformly continuous. There exists $\hat{\delta}_i > 0$ such that if $x, z \in K_i$ fulfil $\|x - z\|_2 < \hat{\delta}_i$ then $\|\tilde{\alpha}_i(x) - \tilde{\alpha}_i(z)\|_2 < \frac{\varepsilon}{4}$.

Set

$$\eta = \frac{1}{p} \min\left\{\hat{\delta}_1, \ldots, \hat{\delta}_p, \frac{\varepsilon}{4}\right\}. \tag{1B.49}$$

Recall that for every $i \in \{1, \ldots, p\}$ the mapping $X_i : \mathcal{F}(K_i) \longmapsto K_i$ sending $\overline{\omega}_i$ to the unique fixed point of $\rho(\omega_i(\cdot), K_i)$ is continuous. There exists $\delta_i > 0$ satisfying $\|\overline{\alpha}_i - \overline{\omega}_i\|_\infty < \delta_i$ implies

$$\|X_i(\overline{\alpha}_i) - X_i(\overline{\omega}_i)\|_2 < \eta \tag{1B.50}$$

whenever $\|\overline{\alpha}_i - \overline{\omega}_i\|_\infty < \delta_i$. Here

Now, let

$$\delta = \min\left\{\delta_1, \ldots, \delta_p, \frac{\varepsilon}{4}\right\}$$

and fix $(\overline{\omega}, \tilde{\omega}) \in F_y(\mathcal{A})$ such that

$$\|(\overline{\omega}, \tilde{\omega}) - (\overline{\alpha}, \tilde{\alpha})\|_{\Omega_y(K_1,\ldots,K_p)} < \delta. \tag{1B.51}$$

To keep notation simple, set $x = X(\overline{\omega})$ and $a = X(\overline{\alpha})$. In this notation,

$$\|x_i - a_i\|_2 < \eta \ \forall i \in \{1, \ldots, p\}. \tag{1B.52}$$

Observe that

$$\|\overline{\omega}_i - \overline{\alpha}_i\|_\infty \leqslant \|\overline{\omega} - \overline{\alpha}\|_{\mathcal{M}} < \delta \leqslant \delta_i \tag{1B.53}$$

holds for any given $i \in \{1, \ldots, p\}$. As a result,

$$\|x - a\|_2 \leqslant \sum_{i=1}^{p} \|x_i - a_i\|_2 < p\eta \leqslant \hat{\delta}_i. \tag{1B.54}$$

This in turn yields, $\|\tilde{\alpha}_i(x) - \tilde{\alpha}_i(a)\|_2 < \frac{\varepsilon}{4}$. From this observation

$$\begin{aligned} \|\overline{\omega}_i - \tilde{\omega}_i(x) - x_i - \overline{\alpha}_i - \tilde{\alpha}_i(a) - a_i\|_\infty &\leqslant \|\overline{\alpha} - \overline{\omega}\|_{\mathcal{M}} + \|\tilde{\alpha} - \tilde{\omega}\|_{\mathcal{M}} \\ &+ \max_{1 \leqslant i \leqslant p} \|\tilde{\alpha}_i(x) - \tilde{\alpha}_i(a)\|_2 + \|x - a\|_2 < \varepsilon. \end{aligned} \tag{1B.55}$$

Moreover,

$$\|\tilde{\omega} - \tilde{\alpha}\| < \delta \leqslant \frac{\varepsilon}{4} \tag{1B.56}$$

and $\|x - a\|_2 < p\eta \leqslant \frac{\varepsilon}{4}$, we arrive to

$$\|\psi(\overline{\omega}, \tilde{\omega}) - \psi(\overline{\alpha}, \tilde{\alpha})\| < \varepsilon. \tag{1B.57}$$

■

## 1B.6 An application

As an interesting application of the structure theorem for variational inequalities, theorem 1B.50, we derive a form of Predetchinski's structure theorem (see theorem 1 [34]). This structure theorem generalises the structure theorem of Kohlberg and Mertens to the case where action sets are general convex and compact sets, and payoffs are differentiable and concave functions of an agent's own actions (for every fixed combination of other agents' actions). The section also discusses how theorem 1B.43 can be used to derive a structure theorem for concave programmes. The arguments for both results are virtually identical, only the proof of theorem 1 in [34] is presented.

**Definition 1B.57** (Concave game)**.** Let $\{U_i\}_{i=1}^{p}$ where for each $i$, $U_i \subset \mathbb{R}^{n_i}$ be non-empty, open and bounded. A *concave game* refers to a continuous mapping

$$u : \overline{U}_1 \times \ldots \times \overline{U}_p \longmapsto \mathbb{R}^{n_1+\ldots+n_p}$$

that satisfies for every $i \in \{1, \ldots, p\}$ the mapping $u_i(\cdot, x_{-i}) : \overline{U}_i \longmapsto \mathbb{R}$ is differentiable and concave for every choice of $x_{-i} \in \overline{U}_{-i}$.

*Remark* 1B.58. Note that definition 1B.57 includes the case where agents mix over a finite set of actions. Such case corresponds to the concave game where each $U_i$ is the interior of a simplex and functions $u_i$ are sums of products of distinct monomials of degree 1.

Given $\{U_i\}_{i=1}^p$ where for each $i$, $U_i \subset \mathbb{R}^{n_i}$ be non-empty, open and bounded, denote by $\Gamma(U_1, \ldots, U_p)$ (or for simplicity, just by $\Gamma$) the set of all concave games. This set can be naturally endowed with the norm

$$\|u\|_\Gamma = \max_{1 \leqslant i \leqslant p} \{\max_{x \in K} |u_i(x)| + \max_{x \in K} \|D_{x_i} u_i(x)\|_2\}, \tag{1B.58}$$

where, for brevity, $K = \overline{U}_1 \times \ldots \times \overline{U}_p$ and $D_{x_i}$ refers to differentiation with respect to the coordinates comprising $x_i$ i.e.,

$$D_{x_i} f = \left( \frac{\partial f}{\partial x_i^1}, \ldots, \frac{\partial f}{\partial x_i^{n_i}} \right).$$

Here superscripts refer to individual coordinates of the vector $x_i \in \mathbb{R}^{n_i}$.

*Remark* 1B.59. Note that for each $i \in \{1, \ldots, p\}$, the norm $\max_{x \in K} |u_i(x)| + \max_{x \in K} \|D_{x_i} u_i(x)\|_2$ in equation (1B.58) roughtly corresponds to the $C^1$ norm on the space. Note that in the particular case where $p = 1$, the norm indeed reduces to the $C^1$ norm.

Given $u, v \in \Gamma$ term them *equivalent*, $u \sim v$, if for every $i \in \{1, \ldots, p\}$

$$u_i(x) - v_i(x) = g_i(x_{-i})$$

for some continuous function $g_i$. Denote by $[u]$ the equivalence class of $u \in \Gamma$. The set of all such equivalent classes is denoted by $\Gamma \backslash \sim$. This set is endowed with the quotient norm

$$\|[u]\|_{\Gamma\backslash\sim} = \inf_{g \in G} \|u + g\|_\Gamma, \tag{1B.59}$$

where $G$ denotes the set of all continuous functions

$$g : \overline{U}_1 \times \ldots \times \overline{U}_p \longmapsto \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_p}$$

such that $g_i(x) = g_i(x_{-i})$ for every $i \in \{1, \ldots, p\}$ (i.e., each component $g_i$ is independent of $x_i$).

*Remark* 1B.60. If $p = 1$ then $u \sim v$ if the $u - v = c \in \mathbb{R}$. Hence, $G$ is the space of all constant functions and

$$\|[u]\|_{\Gamma\backslash\sim} = \inf_{c\in\mathbb{R}} \|u + c\|_\Gamma. \tag{1B.60}$$

Let $U_1, \ldots, U_p$ be all non-empty, open and bounded. Moreover, for each $i$ $U_i \subset \mathbb{R}^{n_i}$. Given $u \in \Gamma(\overline{U}_1, \ldots, \overline{U}_p)$ recall that the set of *Nash equilibria of* $u$, $\mathcal{E}(u, \overline{U}_1 \times \ldots \times \overline{U}_p)$, is the set of points $x \in \overline{U}_1 \times \ldots \times \overline{U}_p$ such that for every $i \in \{1, \ldots, p\}$,

$$u_i(x_i, x_{-i}) \geqslant u_i(y_i, x_{-i}) \ \ \forall y_i \in \overline{U}_i. \tag{1B.61}$$

*Remark* 1B.61. Observe that the fact that $u_i(\cdot, x_{-i})$ is differentiable and concave implies that $x \in \overline{U}_1 \times \ldots \times \overline{U}_p$ if, and only if, for every $i \in \{1, \ldots, p\}$,

$$\langle D_{x_i} u_i(x_i, x_{-i}), x_i - y_i \rangle \geqslant 0$$

for every $y_i \in \overline{U}_i$. In particular, $x$ must be a solution of the simultaneous variational inequality given by $((D_{x_1}u_1, \ldots, D_{x_p}u_p), \overline{U}_1 \times \ldots \times \overline{U}_p)$. Note that this condition further implies that if $v = u + g$, with $g \in G$, then $\mathcal{E}(u, \overline{U}_1 \times \ldots \times \overline{U}_p) = \mathcal{E}(v, \overline{U}_1 \times \ldots \times \overline{U}_p)$. Hence, there is no ambiguity in speaking of *the Nash equilibrium set of* $[u]$, $\mathcal{E}([u], \overline{U}_1 \times \ldots \times \overline{U}_p)$.

**Definition 1B.62** (Equilibrium set)**.** $\{U_i\}_{i=1}^p$ where for each $i$, $U_i \subset \mathbb{R}^{n_i}$ be non-empty, open and bounded. The set

$$\mathcal{E}(\overline{U}_1, \ldots, \overline{U}_p) = \bigcup_{u \in \Gamma(\overline{U}_1, \ldots, \overline{U}_p)} \{[u]\} \times \mathcal{E}([u], \overline{U}_1 \times \ldots \times \overline{U}_p)$$

is *equilibrium set.*

*Remark* 1B.63. If $p = 1$ the set $\mathcal{E}([u], \overline{U}_1)$ is the set of all global maxima of concave functions in $[u]$. The equilibrium set $\mathcal{E}(\overline{U}_1)$ can then be construed as the graph of the multi-valued mapping that associates to each concave programme its set of solutions.

The set $\mathcal{E}(\overline{U}_1, \ldots, \overline{U}_p)$ is endowed with the norm

$$\|([u], x)\| = \max\{\|[u]\|_{\Gamma\backslash\sim}, \|x\|_2\} \tag{1B.62}$$

**Theorem 1B.64.** *Given $U_1, \ldots, U_p$ be all non-empty, open and bounded; where for each $i$ $U_i \subset \mathbb{R}^{n_i}$. There exists a homeomorphism $\bar{\psi}$ between $\Gamma(U_1, \ldots, U_p)\backslash \sim$ and $\mathcal{E}(U_1, \ldots, U_p)$. This homeomorphism satisfies $\bar{\pi} \circ \bar{\psi} \simeq id_{\Gamma(U_1,\ldots,U_p)\backslash\sim}$, where $\bar{\pi} : ([u], x) \mapsto [u]$. Moreover, this homotopy extends to the one-point compactification of $\Gamma(U_1, \ldots, U_p)\backslash \sim$.*

*Proof.* Let $T$ the linear mapping given by

$$T : (u_1, \ldots, u_p) \mapsto (D_{x_1}u_1, \ldots, D_{x_p}u_p).$$

Observe that the zero set of $T$ is precisely $G$. Hence, $T$ defines a linear isomorphism between $\Gamma\backslash \sim$ and a subset of $\mathcal{M}(\overline{U}_1, \ldots, \overline{U}_p)$. It is in fact a homeomorphism. To prove this claim, it suffices to prove that each component of $T$ is continuous. Fix $i \in \{1, \ldots, p\}$ accordingly. Given $u_i$ note that

$$\|D_{x_i}u_i\|_\infty \leqslant \|u_i\|_\infty + \|D_{x_i}u_i\|_\infty. \tag{1B.63}$$

If instead $v_i$, with $u_i - v_i = g_i$ independent of $x_i$, had been chosen:

$$\|D_{x_i}u_i\|_\infty = \|D_{x_i}v_i\|_\infty \leqslant \|u_i + g_i\|_\infty + \|D_{x_i}u_i\|_\infty. \tag{1B.64}$$

Hence,

$$\|D_{x_i}u_i\|_\infty \leqslant \inf_{g_i}\{\|u_i + g_i\|_\infty + \|D_{x_i}u_i\|_\infty\} \tag{1B.65}$$

On the other hand, fix an arbitrary $y \in U_1 \times \ldots \times U_p$ and let $u_i$ be chosen then

$$u_i(x_i, x_{-i}) - u(y_i, x_{-i}) = \int_0^1 \langle D_{x_i}u((1-t)x_i + ty_i, x_{-i}), (x_i - y_i)\rangle dt, \tag{1B.66}$$

for arbitrary $x \in \overline{U}_1 \times \ldots \times \overline{U}_p$. Applying the Cauchy-Schawrz inequality,

$$|u_i(x_i, x_{-i}) - u(y_i, x_{-i})| \leqslant \|D_{x_i} u_i\|_\infty \|x_i - y_i\|_2 \leqslant \operatorname{diam}(U_i)\|Du\|_\infty. \quad (1B.67)$$

Recall that $\operatorname{diam}(U_i) = \sup_{x,y \in U_i} d(x, y) < \infty$ since each $U_i$ is bounded.

Hence,

$$\begin{aligned} \inf_{g_i}\{\|u_i + g_i\|_\infty + \|D_{x_i} u_i\|_\infty\} &\leqslant \|u_i - u_i(y_i, \cdot)\|_\infty + \|D_{x_i} u_i\|_\infty \\ &\leqslant (\operatorname{diam}(U_i) + 1)\|D_{x_i} u_i\|_\infty \end{aligned} \quad (1B.68)$$

Together, equations (1B.65) and (1B.68) imply $T_i$ is continuous.

Theorem 1B.50 implies there exists a homeomorphism $\psi$ between $T(\Gamma\backslash \sim)$ and $\mathcal{E}(\overline{U}_1, \ldots, \overline{U}_p)$. Since $T$ carries $\Gamma\backslash \sim$ homeomorphically into its image the first part of the proposition follows by setting $\overline{\psi} = \psi \circ T$.

To establish the second part, let $\bar{\xi} = T^{-1} \circ \xi \circ T$ with $\xi = \pi_{\mathcal{M}} \circ \psi$, where $\pi_{\mathcal{M}}$ is the projection onto $\mathcal{M}(\overline{U}_1, \ldots, \overline{U}_p)$. By theorem 1B.50 $\xi$ is homotopic to the identity by a linear homotopy $H$. Now, define the homotopy $\bar{H}$ : $[0, 1] \times (\Gamma\backslash \sim) \longmapsto \Gamma\backslash \sim$ by $\bar{H}(t, \cdot) = T^{-1} \circ H(t, \cdot) \circ T$ for every $t \in [0, 1]$. Clearly, these maps are all continuous by composition as it is $\bar{H}$. Moreover,

$$\bar{H}(0, \cdot) = T^{-1} \circ H(0, \cdot) \circ T = T^{-1} \circ \operatorname{id}_{\mathcal{M}(\overline{U}_1, \ldots, \overline{U}_p)} \circ T = \operatorname{id}_{\Gamma\backslash\sim} \quad (1B.69)$$

and

$$\bar{H}(1, \cdot) = T^{-1} \circ H(1, \cdot) \circ T = T^{-1} \circ \xi \circ T = \bar{\xi}. \quad (1B.70)$$

To conclude the proof, let $T([u]) = \Phi \in \mathcal{M}(\overline{U}_1, \ldots, \overline{U}_p)$ and observe that given $x \in \overline{U}_1, \ldots, \overline{U}_p)$,

$$T^{-1} \circ \pi_{\mathcal{M}}(\Phi, x) = T^{-1}(\Phi) = [u] = \bar{\pi}([u], x). \quad (1B.71)$$

Equation (1B.70) yields

$$\bar{H}(1, \cdot) = \bar{\pi} \circ \bar{\psi}. \quad (1B.72)$$

■

**Corollary 1B.64.1.** *Given $U \subset \mathbb{R}^n$ be non-empty, open and bounded.*

*There exists a homeomorphism $\bar{\psi}$ between*

$$\Gamma\backslash \sim = \{[u+c]_{c\in\mathbb{R}} : u \in C^1(\overline{U};\mathbb{R}) : u \text{ concave}\}$$

*and $\mathcal{E}(U)$. This homeomorphism satisfies $\bar{\pi}\circ\bar{\psi} \simeq id_{\Gamma\backslash\sim}$, where $\bar{\pi} : ([u], x) \mapsto [u]$. Moreover, this homotopy extends to the one-point compactification of $\Gamma\backslash \sim$.*

*Remark* 1B.65. Recall that in the context of corollary 1B.64.1, $u \sim v$ if $u - v \in \mathbb{R}$. To make this relation explicit, the set of equivalence classes has been denoted by $[u+c]_{c\in\mathbb{R}}$ as opposed to just $[u]$.

*Remark* 1B.66. As may be seen from the proof of theorem 1B.64, by assuming theorem 1B.64 it is possible to derive a particular instance of theorem 1B.50. Indeed, in such case both theorems are logically equivalent.

## 1B.7 Conclusion

The results in chapters 1A and 1B together demonstrate that, for a large proportion of optimisation problems, the graph of the solution correspondence is homeomorphic to the underlying set of problems. Notably, the result encompasses both linear and non-linear convex optimisation problems, as well as coupled optimisation problems.

Interestingly, the method of proof draws on the connection between these problems and finding fixed points of certain maps. Such connection is important as it may lead to index or degree theories being applied to the class of problems introduced here. Importantly, an explicit fixed point characterisation of solutions was absent from the original argument by Predtetchinski (see proof of proposition 1 and the discussion prior to it in [34]).

There are several avenues to extend the work presented in chapters 1A and 1B. As mentioned earlier, the arguments presented here do not rely on the structure of $\mathbb{R}^N$ and thus extend to general Hilbert spaces. It is however unclear whether they remain valid in the context of Banach spaces. A more difficult question is whether it is possible to prove structure theorems for general topological vector spaces. Note that in that regard, for games where players have a continuum of pure actions mixed strategies are measures over the relevant space.

A potentially simple extension is to prove that the homotopy in the structure theorem is in fact an *isotopy* (i.e., for every $t \in [0,1]$, $H(t,\cdot)$ is a homeomorphism onto its image). This result holds in the context of theorem 1B.38 (see theorem 2 in [34]). For the original setting of finite games this was proven by Demichelis and Germano (see [15]).

Although common in most applications, the assumption of compactness seems quite restrictive (particularly when considering extensions to Hilbert spaces). An interesting question is thus whether an extension theorem holds when the class of mappings $\Phi$ approach 0 fast enough.

Lastly, when considering simultaneous problems, it has been assumed that the sets

$$K_1, \ldots, K_p$$

are independent on the choice of $x$. This assumption rules out an interesting class of problems where $K_i$ might be a well defined function of $x_{-i}$. This "generalised games" arise naturally in the context of general equilibrium theory, where for a given agent the budget set depends on the given prices, which might be determined by the actions of the remaining agents.

# Chapter 2

# Realising Planar Curves as Nash Equilibria

## 2.1 Introduction

The most fundamental concept in game theory is that of Nash equilibrium. Such equilibrium reflects a state of play where no player would gain from choosing an action different from the one they are playing. If the number of players is finite and each player has a finite set of actions available, then finding Nash equilibria amounts to solving a system of polynomial inequalities. Since one may be interested in understanding the structure and richness of equilibrium sets, it is natural to inquire whether the converse is true. Namely, if the points of a set $B$ coincide with the solutions to a given system of polynomial inequalities (a *semi-algebraic set*), is it possible to find players and actions such that the set of Nash equilibria of the resulting game coincides with $B$? In this form the problem is very difficult. On one hand, there is no natural constraint on player numbers, nor there is information about individual player's actions. On the other, if the set $B$ is not given explicitly, but it is instead described by inequalities only, then $B = \emptyset$ is possible while Nash equilibria always exists.

Rather than addressing the difficulties in the previous paragraph, consider the following simplification: Let $B$ be a semi-algebraic set, does there exist a game whose set of Nash equilibria is *equivalent to $B$*? The main difficulty with this formulation being the lack of a suitable notion of equivalence. Indeed, several notions have been considered in the literature so far. Motivated by similar problems in combinatorial geometry, Datta has proven that the set of all completely mixed Nash equilibria (i.e. equilibria where all players randomise across all of their actions) is *stably isomorphic* to the set of solutions of a given system of polynomial equalities (see theorem 2 in [14]). Following Datta, a semi-algebraic set $A \subset \mathbb{R}^n$ is stably isomorphic to another semi-algebraic set $B \subset \mathbb{R}^m$ if there exists a homeomorphism $f : A \times \mathbb{R}^k \longmapsto B$, for some $k \geqslant 0$, whose graph is also a semi-algebraic set of $\mathbb{R}^{n+m+k}$. The requirement that players must randomise across all their actions limits Datta's results in two directions. First, the resulting theorem provides no information about additional equilibria. Second, the result excludes a number of simple sets such as straight line segments.

An alternative approach was taken by Balkenborg and Vermeulen. Specifically, Balkenborg and Vermeulen have shown that given an arbitrary simplicial complex (a set consisting of the union of simplices of different dimensions), there exists a game whose set of Nash equilibria is homeomorphic to the simplicial complex (see theorem 3.1 in [6] and theorem 2.1 in [7]). Since, by a result due to Hironaka, any semi-algebraic set is homeomorphic to a simplicial complex, it follows that there exists a game whose set of Nash equilibria is homeomorphic to any given semi-algebraic set. Observe that this notion of equivalence preserves the topology of the sets involved. Nonetheless, topological equivalence ignores the underlying algebraic structure of the problem: two semi-algebraic sets may be topologically equivalent, but the inequalities defining one set might not reduce to the inequalities defining the other. Furthermore, homeomorphism may not preserve the behaviour of tangent vectors at given points.

Lastly, a parallel strand of the literature has adopted projection as the equivalent notion (see [27] and [41]). This is motivated by the Tarski-Seidenberg theorem, which states that the projection of a semi-algebraic set onto a lower dimensional space also results into a semi-algebraic set (see theorem 2.9 below). In particular, parallel results by Levy and, Vigeral and Viossat show that for a given semi-algebraic set $B$ there exists a game whose Nash equilibrium satisfies the following property: the set $B$ corresponds to the equilibrium actions of the first $N$ players; i.e., the projection of the equilibrium set to the first $N$ coordinates is $B$ (see theorem 3.1 in [27] or proposition 1 in [41]). A main difficulty with projection is, of course, the fact that it might not be an injective mapping. Indeed, Levy provides an example (see example 1 in [27]) of a game with 3 players where the third player receives pay-off 0 independently of the actions of the others. By projecting on the equilibrium actions of the first two players, the set of Nash equilibria projects onto the diagonal of the square $[0, 1]^2$. The resulting projection is however not injective.

This chapter considers an alternative, more restrictive, notion of equivalence: given semi-algebraic sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, where $m < n$; $A$ is said to be *isomorphic* to $B$ if $A$ projects injectively onto $B$ and, in addition, every coordinate of the resulting inverse mapping is given by affine functions. Observe that in the particular case that $A$ and $B$ are isomorphic algebraic sets then this notion of isomorphism implies that $A$ and $B$ are *algebraically isomorphic*. Recall that two algebraic sets are algebraically isomorphic if there exists a bijection between them whose coordinates are given by polynomials (the projection in our case) and, in addition, the coordinates of its inverse are polynomial functions as well. If two algebraic sets are algebraically isomorphic, they are in particular homeomorphic. Importantly however algebraic isomorphism not only preserves the topology of the sets, but also its algebraic structure (i.e., the equations defining one set reduce to the equations defining the other and vice versa). This latter fact is relevant since most semi-algebraic sets considered in this chapter result from the intersection of a given algebraic curve with the square $[0, 1]^2$.

In regards to the different equivalence notions in the literature, note that there are largely logically disconnected from one another. Naturally, projections need not be injective in general and thus do define homeomorphisms. On the other hand, if two semi-algebraic sets are stably isomorphic they need not be homeomorphic. This is because stable isomorphism allows one of the sets to be “stretched” before defining the homeomorphism between the two sets. Stable homeomorphism does require however that the inequalities defining one set reduce to the inequalities defining the other, while this fact need not hold for arbitrary homeomorphisms. The notion of isomorphism in the previous paragraph does however bridge several of these notions. Since the projections are continuous, the requirement that the coordinates of the inverse are given by affine functions implies that an isomorphism in the sense of the previous paragraph is a homeomorphism. Moreover, if the sets involved are algebraic sets then it can be seen to also be a stable isomorphism. Since most sets in this chapter result from intersecting algebraic sets with hypercubes, it can be concluded that if such sets are isomorphic then they are also stably isomorphic.

The main result in this chapter can now be stated: let $A$ be a semi-algebraic set resulting from intersecting the square $[0,1]^2$ with an algebraic curve. Suppose that $A$ intersects neither the top or bottom of the square (i.e., $A \cap ([0,1] \times \{0,1\}) = \emptyset$), then there exists a game where each player has two actions whose Nash equilibrium set can be partitioned into two disjoint sets (one of which may be empty); the first of these sets is isomorphic (i.e., this set projects injectively onto $A$ and the coordinates of its inverse are given by affine functions) to the original curve. The other set is either empty or consisting of a line segment.

To provide some motivation for the main result, consider the following example:

$$Y = \left\{(t,x) \in \mathbb{R}^2 : \left(\frac{1}{2} - x\right) \cdot \left(x - \frac{t^2+2}{4}\right) = 0\right\} \cap [0,1]^2.$$

Observe that the set $Y$ is the union of the graph of the parabola

$$x = \frac{t^2+2}{4}$$

and the straight line $x = \frac{1}{2}$. It is therefore a semi-algebraic set. Suppose that there existed some polynomial $P$ such that

$$Y = \{(t,x) \in \mathbb{R}^2 : P(t,x) = 0\}.$$

Then it must be the case that $P(T_1, T_2) = (T_2 - \frac{1}{2})Q(T_1, T_2)$, but this would imply $(2, \frac{1}{2}) \in Y$. The set $Y$ is therefore not algebraic. Notably, it also presents a kink at $(0, \frac{1}{2})$ where both curve segments meet at a zero angle. The curve $Y$ is thus not differentiable there.

It is proven in section 4 (see proposition 2.25) that there exists an 11-player game where each player has two actions each whose set of Nash equilibria is given by the union of the sets

$$X_0 = \left\{ \left( t, t, t, t, t, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) : t \in [0,1] \right\}$$

and

$$X_1 = \left\{ \left( t, t, t, t, t, \frac{t^2+4}{8}, \frac{t^2+4}{8}, \right.\right.$$
$$\left.\left. \frac{t^2+4}{8}, \frac{t^2+4}{8}, \frac{t^2+2}{4}, \frac{t^2+2}{4} \right) : t \in [0,1] \right\}.$$

The set $Y$ can now be seen to parametrise $X_0 \cup X_1$. Furthermore, by considering the first and second-to-last coordinates, $Y$ is the projection of $X_0 \cup X_1$. The two sets are thus isomorphic.

The fact that $Y$ fails to be differentiable at $(0, \frac{1}{2})$ is important. Previous results in the literature did not encompass semi-algebraic sets with such kinks. In that regard, observe that $Y$ cannot be constructed using Datta's approach since it is not an algebraic set. Moreover, it can be seen from the definition of $X_0 \cup X_1$ that there exist equilibria where the first 5 players do not randomise. Furthermore, since homeomorphism need not preserve the structure of the tangent space at a given point, $Y$ cannot be constructed following the approach by Balkenborg and Vermeulen. More specifically, the approach followed by Balkenborg and Vermeulen would rely on triangulating the given semi-algebraic set (see proof of theorem 6.1 in [6]). Because the line and parabola segments defining $Y$ meet at a zero angle $Y$ at $(0, \frac{1}{2})$, any such triangulation will result in either a single segment or a collection of several segments where no pair of segments forms a zero angle with another. Moreover, since the next step in Balkenborg and Vermeulen's construction is to embed the triangulation into a certain collection of edges of a hypercube, the angle of the two edges corresponding to the kink would in fact be 90°.

The question of whether it was possible to construct a game whose set of Nash equilibria projects injectively was first proposed by Levy (see page 441 in [27]). The aforementioned 11-player game answers this question affirmatively. More generally, results in this chapter generalise this conjecture to a broader class of sets.

This chapter is organised as follows: section 1 provides formal definitions to some of the concepts discussed in the previous paragraph. Section 2 introduces some lemmas that are needed in the proof of the main result. Section 3 discusses the construction of the example in the previous paragraph. This construction is then extended to more general curves in section 4, where the main result is proven. Further examples of the main result are provided in section 5. Lastly, section 6 summarises the main results and concludes the chapter.

## 2.2 Preliminaries

Denote by $|X|$ the number of elements of the set $X$; i.e., its *cardinality*. Given a set $X$ satisfying $0 < |X| < \infty$, let $\Delta(X)$ be the set of all probability measures over $X$. Recall that

$$\Delta(X) \cong \Delta_{|X|-1} = \left\{ x \in \mathbb{R}^{|X|} : \sum_{i=1}^{|X|} x_i = 1, \;\; x_i \geqslant 0 \;\; \forall i \in \{1, \ldots, |X|\} \right\}.$$

This identification is employed throughout this chapter.

Given sets $A_1, \ldots, A_n$, each satisfying $0 < |A_i| < \infty$, a multi-affine mapping $G : \Delta(A_1) \times \ldots \times \Delta(A_n) \longmapsto \mathbb{R}^n$ is said to be an *n-game* $G$. For brevity, and in the absence of ambiguity, a given *n-game* $G$ is often referred to simply as a *game*.

*Remark* 2.1. Observe that the i$^{\text{th}}$ component $G_i$ of a given game $G$ is of the form

$$G_i(x_1, \ldots, x_n) = \sum_{a_1 \in A_1} \ldots \sum_{a_n \in A_n} G_i(a_1, \ldots, a_n) \prod_{j=1}^{n} x_j(a_j), \tag{2.1}$$

for some coefficients $G_i(a_1, \ldots, a_n) \in \mathbb{R}$. These coefficients are determined through the following observation:

$$G_i(\delta_{a_1}, \ldots, \delta_{a_n}) = G_i(a_1, \ldots, a_n), \tag{2.2}$$

where $\delta_{a_i}$ is the *delta measure*,

$$\delta_{a_i}(a) = \begin{cases} 1 & \text{if } a = a_i, \\ 0 & \text{if } a \neq a_i. \end{cases} \tag{2.3}$$

In view of remark 2.1 and to keep notation simple, the identification $a_i \equiv \delta_{a_i}$ is made in the remaining of the chapter. Moreover, it is convenient to write

$$\begin{gathered} G_i(x_1, \dots, x_{i-1}, \hat{a}_i, x_{i+1}, \dots, x_n) = \\ G_i(x_1, \dots, x_{i-1}, \delta_{\hat{a}_i}, x_{i+1}, \dots, x_n) = \\ \sum_{a_1 \in A_1} \cdots \sum_{a_{i-1} \in A_{i-1}} \sum_{a_{i+1} \in A_{i+1}} \cdots \sum_{a_n \in A_n} G_i(a_1, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, a_n) \prod_{j \neq i}^{n} x_j(a_j) \end{gathered} \tag{2.4}$$

*Remark* 2.2. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product over $\mathbb{R}^n$ and observe that the components of given 2-game $G$ are of the form $G_i(x_1, x_2) = \langle x_1, U_i x_2 \rangle$, where $U_i$ is a matrix satisfying $U_i(a_1, a_2) = G_i(a_1, a_2)$ for every pair $(a_1, a_2) \in A_1 \times A_2$. It is customary to represent such game $G$ in terms the matrices $U_1$ and $U_2$. Moreover, abusing notation slightly, the matrices $U_1$ are $U_2$ are labelled $G_1$ are $G_2$ respectively. This representation is used in the sequel (see e.g. proposition 2.12).

Given non-empty finite sets $A_1, \dots, A_n$, denote by $\Gamma(A_1, \dots, A_n)$ the set of all $n-$games. For the purpose of this chapter, it suffices to restrict the analysis to a particular class of games only:

**Definition 2.3** (Binary game)**.** Let $A_1 = \ldots = A_n = \{0, 1\}$ a multi-affine mapping $G : \Delta(A_1) \times \ldots \times \Delta(A_n) \longmapsto \mathbb{R}^n$ is said to be an $n-$*binary* (or simply *binary*) game.

*Remark* 2.4. Since $\Delta(\{0, 1\}) \cong [0, 1]$, it is possible to identify each probability measure $x_i \in \Delta(A_i)$ with a single point $x \in [0, 1]$ through the mapping $x_i \mapsto x_i(1)$. By a slight abuse of notation, denote the point $x = x_i(1)$ by $x_i$. Under this identification, a binary game $G$ is viewed as a mapping $G : [0, 1]^n \longmapsto \mathbb{R}^n$ where each component is of the form

$$G_i(x_1,\ldots,x_n) = \sum_{a_1\in\{0,1\}} \ldots \sum_{a_n\in\{0,1\}} G_i(a_1,\ldots,a_n) \prod_{j\notin Z} x_j \prod_{j\in Z}(1-x_j). \quad (2.5)$$

Here $Z = \{i \in \{1,\ldots,n\} : a_i = 0\}$. Observe that upon expanding the left hand side of equation (2.5), the value of $G_i(x_1,\ldots,x_n)$ is a polynomial whose monomials are all of the form $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_{m-1}} \cdot x_{i_m}$ with $x_{i_k} \neq x_{i_l}$ for $k \neq l$.

The set of all $n-$binary games is denoted by $\mathcal{B}_n$.

Recall the following

**Definition 2.5** (Nash equilibrium)**.** Given a game $G \in \Gamma(A_1,\ldots,A_n)$, $x \in \Delta(A_1) \times \ldots \times \Delta(A_n)$ is a *Nash equilibrium* if for every $i \in \{1,\ldots,n\}$

$$G_i(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) \geqslant G_i(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n) \quad (2.6)$$

holds for arbitrary $y_i \in \Delta(A_i)$.

*Remark* 2.6. In the interest of brevity, the following abbreviation

$$(y_i, x_{-i}) = (x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)$$

is standard. In particular, under this nomenclature equation (2.6) becomes

$$G_i(x) = G_i(x_i, x_{-i}) \geqslant G_i(y_i, x_{-i}). \quad (2.7)$$

**Theorem 2.7** (Nash)**.** *Let $A_1,\ldots,A_n$ be sets with $0 < |A_i| < \infty$ for every $i$. For every game $G \in \Gamma(A_1,\ldots,A_n)$ there exist $x \in \Delta(A_1) \times \ldots \times \Delta(A_n)$ satisfying the conditions given by* (2.6)*.*

For a proof of Nash's theorem see e.g. theorem 2.2.3 in [17].

Given $G \in \Gamma(A_1,\ldots,A_n)$ denote by $E(G) \neq \emptyset$ the set of the Nash equilibria of $G$. From the point of view of algebraic geometry, the set $E(G) \subset \mathbb{R}^{|A_1|+\ldots+|A_n|}$ belongs to a well-studied class of subsets of this Euclidean space. The relevant class is introduced below:

Denote by $\mathbb{R}[X]$ the set of all polynomials in the variable $X$ with real coefficients. More generally, $\mathbb{R}[X_1, \dots, X_k]$ represents the set of all real polynomials in variables $X_1, \dots, X_k$. A set $A \subset \mathbb{R}^k$ is then said to be *semi-algebraic* if it can be written as a finite combination of unions and intersections of sets of the form

$$\{x \in \mathbb{R}^k : P(x) \leqslant 0\} \text{ and } \{x \in \mathbb{R}^k : Q(x) < 0\}$$

with $P, Q \in \mathbb{R}[X_1, \dots, X_k]$. If the inequalities defining the semi-algebraic set are in fact equalities, the resulting semi-algebraic set is said to be *algebraic*.

*Remark* 2.8. It follows from the definition that the collection of all semi-algebraic sets of $\mathbb{R}^n$ is closed under unions and taking the complement (i.e. it forms an *algebra* of sets). Moreover, it is also closed under projection onto a smaller subspace.

**Theorem 2.9** (Tarski-Seidenberg)**.** *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set and denote by $\pi_K$ the projection onto the coordinates $K \subset \{1, \dots, n\}$. Then $\pi_K(A) \subset \mathbb{R}^{|K|}$ is also semi-algebraic.*

A proof of this result may be found in e.g. theorem 2.2.1 in [11].

*Remark* 2.10. The theorem of Tarski-Seidenberg does not hold if inequalities are replaced with equalities throughout. Consider for instance the set

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}; \tag{2.8}$$

then $\pi_1(A) = [-1, 1]$, which is not an algebraic set. It is however semi-algebraic since

$$\pi_1(A) = \{x \in \mathbb{R} : -(1 + x) \leqslant 0\} \cap \{x \in \mathbb{R} : x - 1 \leqslant 0\}. \tag{2.9}$$

**Proposition 2.11.** *Let $A_1, \dots, A_n$ be non-empty finite sets. Given*

$$G \in \Gamma(A_1, \dots, A_n),$$

*$E(G)$ is a semi-algebraic set.*

The proof of this result is fairly elementary and it is provided for completeness.

*Proof of proposition 2.11.* It suffices to show that the condition $x \in E(G)$ can be written in terms of equalities and inequalities of polynomials exclusively. Indeed, for $i \in \{1, \ldots, n\}$

$$\begin{aligned}
\sum_{a_i \in A_i} x_i(a_i) &= 1, && (2.10)\\
-x_i(a_i) &\leqslant 0 \; \forall a_i \in A_i, && (2.11)\\
G_i(a_i, x_{-i}) - G_i(x_i, x_{-i}) &\leqslant 0 \; \forall a_i \in A_i. && (2.12)
\end{aligned}$$

The last of these inequalities being equivalent to definition of Nash equilibrium whilst the first two characterise each $x_i$ as a probability. ■

Proposition 2.11 establishes, in broad terms, that the Nash equilibrium set of an arbitrary game is a non-empty compact semi-algebraic set. A natural question is whether the converse holds true. Namely, given $B \subset \mathbb{R}^d$ non-empty, compact and semi-algebraic; can non-empty finite subsets $A_1, \ldots, A_n$ be found such that $B = E(G)$ for some $G \in \Gamma(A_1, \ldots, A_n)$? This problem appears to be notably hard since no restrictions are placed on either $n$ or the cardinality of each set $A_i$. A related (and somewhat simpler) problem is the following: given non-empty finite subsets $A_1, \ldots, A_n$ determine for which semi-algebraic subsets of

$$\Delta(A_1) \times \ldots \times \Delta(A_n)$$

there exists $G \in \Gamma(A_1, \ldots, A_n)$ such that $E(G)$ coincides with the given semi-algebraic set. A complete classification is known when $n \in \{1, 2\}$. If $n \geqslant 3$ then, by applying tools from real algebraic geometry, it is possible, for example, to obtain an upper bound on the maximum number of connected components of $E(G)$ (see corollary 2.2 and 2.3 in [27]). Consequently, if the number of components of a semi-algebraic subset

$$B \subset \Delta(A_1) \times \ldots \times \Delta(A_n)$$

exceeds this bound it cannot be the set of Nash equilibria of any given $G \in \Gamma(A_1, \ldots, A_n)$. This is just one way in which the algebraic constraints impose restrictions on the complexity of the set, beyond merely bounding the number of components.

Given the difficulties in formulating a converse of proposition 2.11, much of the literature has focused instead on the following problem: given $B \subset \mathbb{R}^d$ non-empty, compact and semi-algebraic does there exist sets $A_1, \ldots, A_n$ ($0 < |A_i| < \infty$ for each $i$) such that $B$ is "equivalent" (in some suitable sense) to $E(G)$ for some $G \in \Gamma(A_1, \ldots, A_n)$. In particular, the result is known to hold under weak notions of equivalence (e.g. up to homeomorphism or projection, see [27], [41], [6], [7] and [14]). The choice of homeomorphism (see [6] and [7]) is motivated by a desire to preserve the Euclidean topology inherited by the semi-algebraic set $B$. Meanwhile, equivalence under projection (see [27] and [41]) can be understood as a converse to the theorem of Tarski and Seidenberg. In this chapter, an alternative notion of equivalence, which conserves the algebraic structure of $B$, is considered instead.

## 2.3 Two ancillary results

In this section two results that constitute the backbone of most of the latter constructions are presented. The first is due to Levy (see example 1 in [27]) and the second is a generalisation of a previous result due to Chin et al. (see example in pages 3 and 4 of [13]).

**Proposition 2.12** (Levy)**.** *Given $q \in \mathbb{R}$ let $H[q]$ be the $2-$binary game defined by the matrices*

$$H_1[q] = \begin{bmatrix} -1 & 4q-1 \\ 3-4q & -1 \end{bmatrix}; \quad H_2[q] = \begin{bmatrix} 1 & 4q-3 \\ 1-4q & 1 \end{bmatrix}. \tag{2.13}$$

*This game satisfies,*

$$E(H[q]) = \begin{cases} (0,1) & \textit{if} \quad q < 0, \\ \{(0,w) : w \in [0,1]\} & \textit{if} \quad q = 0, \\ (q,q) & \textit{if} \quad 0 < q < 1, \\ \{(1,w) : w \in [0,1]\} & \textit{if} \quad q = 1, \\ (1,0) & \textit{if} \quad q > 1. \end{cases} \tag{2.14}$$

A proof of this result amounts to simple calculations and it is thus omitted.

For a discussion of this result see section 3.1 in [27].

**Theorem 2.13.** *Let $N \in 2\mathbb{N}+1$, there exists a game $C_N \in \mathcal{B}_N$ such that*

$$E(C_N) = \{(t, \dots, t) : t \in [0,1]\}. \tag{2.15}$$

*Remark* 2.14. The case $N = 3$ corresponds to the original example by Chin and co-authors (see example in pages 3 and 4 in [13]).

*Remark* 2.15. From a geometrical point of view, theorem 2.13 states that for every $N \geqslant 3$ odd it is possible to construct a game whose Nash equilibrium set corresponds exactly with the diagonal joining the vertex $(0, \dots, 0)$ and the vertex $(1, \dots, 1)$ of the hypercube $[0,1]^N$.

In view of remark 2.15 and for the remainder of this section, let $X$ denote the diagonal of the hypercube $[0,1]^N$ connecting $(0, \dots, 0)$ and $(1, \dots, 1)$; i.e.

$$X = \{(t, \dots, t) : t \in [0,1]\}.$$

*Remark* 2.16. The proof of theorem 2.13 uses a similar approach used to the one followed in section 4 of [12]. The argument is not particularly difficult, but it does involve some subtleties. In the interest of clarity, it is split into several lemmas.

In preparation for proof of theorem 2.13, fix $N \in 2\mathbb{N}+1$ and let $G \in \mathcal{B}_N$ be defined as follows: for each $i \in \{1, \dots, n\}$ set

$$G_i(a_i, x_{-i}) = (1 - 2a_i) \cdot (x_{i-1} - x_{i+1}), \tag{2.16}$$

where addition over the index $i$ is to be understood modulo $N$, so that in particular $0 \equiv N$ and $1 \equiv N+1$. It is to be proven that $G = C_N$.

**Lemma 2.17.** *Given $N \in 2\mathbb{N}+1$, let $G \in \mathcal{B}_N$ be the game with components given by equation* (2.16)*. Suppose that for some $x \in E(G)$, $k \in \{1, \dots, N\}$ is such that $x_k = 0$ and $x_{k+1} < 1$. Then $x_{k+2} = 0$ and $x_{k+1} \geqslant x_{k+3}$.*

*Remark* 2.18. Recall that arithmetic operations are defined modulo $N \in 2\mathbb{N}+1$. In particular, $x_k = x_{k+N}$ for every $k \in \{0, \dots, N-1\}$. Furthermore, given $k \in \{0, \dots, N-1\}$ $x_{N+1-k} = x_1$. This is used in the proof.

*Proof.* By applying the transformation in the latter part of remark 2.18, it may be assumed without loss of generality that $k = 1$. It is to be shown that $x_3 = 0$ and $x_2 \geqslant x_4$. Observe that

$$G_2(x_1, x_2, x_3) = -x_3(1 - 2x_2). \tag{2.17}$$

If, by way of contradiction, $x_3 > 0$ then $x \in E(C_N)$ implies

$$G_2(x_1, x_2, x_3) = x_3(2x_2 - 1) \geqslant 0 = G_2\left(x_1, \frac{1}{2}, x_3\right). \tag{2.18}$$

Hence, $x_2 \in [\frac{1}{2}, 1)$, which further implies

$$0 < x_3 = G_2(x_1, 1, x_3) \leqslant G_2(x_1, x_2, x_3) < x_3, \tag{2.19}$$

which results in a contradiction. Consequently, $x_3 = 0$ must hold. As a result, the fact that $x \in E(C_N)$ yields

$$0 = G_3\left(x_2, \frac{1}{2}, x_4\right) \leqslant G_3(x_2, 0, x_4) = x_2 - x_4. \tag{2.20}$$

The second part of the conclusion thus follows. ∎

**Lemma 2.19.** *Given $N \in 2\mathbb{N}+1$, let $G \in \mathcal{B}_N$ be the game with components given by equation* (2.16)*. Suppose that for some $x \in E(G)$, there existed $k \in \{1, \ldots, N\}$ such that $x_k = 1$ and $x_{k+1} > 0$. Then $x_{k+2} = 1$ and $x_{k+1} \leqslant x_{k+3}$.*

The proof of this lemma is virtually identical to the proof of lemma 2.17 and it is thus omitted.

**Proposition 2.20.** *Given $N \in 2\mathbb{N} + 1$, let $G \in \mathcal{B}_N$ be the game with components given by equation* (2.16)*. Suppose that there existed $x \in E(G)$ such that $x_1 \in \{0, 1\}$ and $(-1)^{x_1}(x_N - x_2) > 0$. Then for every*

$$k \in \{1, \ldots, N\} \cap (2\mathbb{N} + 1),$$

*$x_k = x_1$ and $(-1)^{x_k}(x_{k-1} - x_{k+1}) \geqslant 0$.*

*Proof.* For definiteness assume $x_1 = 0$ (the other case follows by very similar argumentation). Let

$$A \subset \{1, \dots, N\} \cap (2\mathbb{N} + 1)$$

be the set of all $k$ for which the statement in the rubric held. Observe that $A \neq \emptyset$ since $k = 1$ is in $A$ by assumption. Proceed by induction, let $k_0$ be such that $1 \leqslant k \leqslant k_0$ is in $A$. Then the following chain holds

$$1 \geqslant x_N > x_2 \geqslant x_4 \geqslant \dots \geqslant x_{k_0-1} \geqslant x_{k_0+1}.$$

Lemma 2.19 implies that $k_0 + 2$ must also be in $A$. ■

**Corollary 2.20.1.** *There does not exist $x \in E(G)$ and $k \in \{1, \dots, N\}$ such that $x_k \in \{0, 1\}$ and $(-1)^{x_k}(x_{k-1} - x_{k+1}) > 0$.*

*Proof.* By way of contradiction, suppose that such $x \in E(G)$ existed. Recall that by remark 2.18, it might be assumed without loss of generality that $k = 1$. Moreover, for definiteness, let $x_1 = 0$ (the other case follows by the same argument). Proposition 2.20 implies

$$x_1 = x_N = 0 > x_2 \geqslant 0,$$

an obvious contradiction. ■

The above results can now be used to establish theorem 2.13:

*Proof of theorem 2.13.* Recall that

$$X = \{(t, \dots, t) : t \in [0, 1]\}$$

and let $G \in \mathcal{B}_N$ be the game whose $i^{\text{th}}$ component is

$$G_i(a_i, x_{-i}) = (1 - 2a_i) \cdot (x_{i-1} - x_{i+1}).$$

Given $x \in X$ and $k \in \{1, \dots, N\}$,

$$G_k(x_{k-1}, \cdot, x_{k+1}) = 0 \tag{2.21}$$

thus $x \in E(G)$. Suppose now to arrive to a contradiction, that there existed some $x \in E(G) \setminus X$. The oddness of $N$ implies the existence of an index $k \in \{1, \dots, N\}$ such that $x_{k-1} \neq x_{k+1}$ (see lemma 2.21 below). Consider first the case $x_{k-1} > x_{k+1}$. Since $x$ is a Nash equilibrium,

$$\begin{aligned}(1-2x_k)(x_{k-1}-x_{k+1}) &= G_k(x_{k+1}, x_k, x_{k-1}) \\ &\geqslant G_k(x_{k+1}, 0, x_{k-1}) = x_{k-1} - x_{k+1}.\end{aligned} \tag{2.22}$$

It follows that $x_k \leqslant 0$ and, consequently, $x_k = 0$. Observe that this conclusion contradicts corollary 2.20.1. If, on the other hand, the inequality $x_{k-1} < x_{k+1}$ held instead then it would imply $x_k = 1$, which contradicts corollary 2.20.1 once again. This latest observation rules out both cases, it must be concluded that $\in E(G) \setminus X = \emptyset$. ■

**Lemma 2.21.** *Suppose $N \in 2\mathbb{N}+1$ and $x \in [0,1]^N$ is such that for every $k \in \{1, \dots, N\}$ $x_{k-1} = x_{k+1}$, then $x \in X$.*

*Proof.* Observe that by assumption, odd-indexed variables take identical values, whilst all even-indexed ones also take the same value. Since $x_0 = x_N$ with $N \in 2\mathbb{N}+1$, the result follows. ■

*Remark* 2.22. Lemma 2.21 does not hold whenever $N$ is even. In that case, all even indexed variables are equal to each other, while all the odd indexed variables coincide. The values of these two collections might however be different. This implies that the set $E(G)$, where $G \in \mathcal{B}_N$ is the game whose $i^{\text{th}}$ component is

$$G_i(a_i, x_{-i}) = (1-2a_i) \cdot (x_{i-1} - x_{i+1}),$$

is a $2-$dimensional plane that contains the diagonal $X$. The proof of this result follows closely the arguments in this section.

## 2.4 A first example

This section provides a novel example of 11-binary game whose Nash equilibrium set projects injectively onto a kinked semi-algebraic set of the unit square. Before discussing the example it is necessary to introduce some notation. The conventions employed here follow those in [27] (see e.g., theorems 3.2 and 3.3 as well as the examples in section 3.1).

Consider a partition of the set $\{1, \dots, N\} \subset \mathbb{N}$ into disjoint subsets $\alpha$ and $\beta$. Let $\alpha_i$, where the index $i$ runs from 1 to $|\alpha|$, label a generic element of $\alpha$. Similarly, $\beta_i$ denotes a generic element of $\beta$.

Fix now non-empty finite subsets $A_1, \dots, A_N$, using the partition in the previous paragraph an arbitrary point

$$(x_1, \dots, x_N) \in \Delta(A_1) \times \dots \times \Delta(A_N)$$

can now be written as $(x^\alpha, x^\beta) = (x^{\alpha_1}, \dots, x^{\alpha_n}, x^{\beta_1}, \dots, x^{\beta_m})$, where $|\alpha| = n$ and $|\beta| = m$.

For a game $G \in \Gamma(A_1, \dots, A_N)$ and $(x^\alpha, y^\beta) \in \Delta(A_1) \times \dots \times \Delta(A_N)$ write

$$G^{\alpha_i}[y^\beta](x^\alpha) = G_k(x^{\alpha_1}, \dots, x^{\alpha_n}, y^{\beta_1}, \dots, y^{\beta_m}), \tag{2.23}$$

where it is assumed that $\alpha_i = k$. Whenever it is clear from context the dependence on $x^\alpha$ might be omitted. Moreover, if $G_k$ is in fact independent of $y^\beta$ then the term in square brackets is omitted.

Fix now some $y^\beta \in \Delta(A_{\beta_1}) \times \dots \times \Delta(A_{\beta_m})$ let $G^\alpha[y^\beta] : \Delta(A_{\alpha_1}) \times \dots \times \Delta(A_{\alpha_n}) \longmapsto \mathbb{R}^n$ be the game whose individual components are given by equation (2.23).

**Proposition 2.23.** *Let $A_1, \dots, A_N$ be non-empty finite subsets. Given a partition of $\{1, \dots, N\}$ into subsets $\alpha$ and $\beta$, $(x^\alpha, x^\beta) \in \Delta(A_1) \times \dots \times \Delta(A_N)$ is a Nash equilibrium of $G \in \Gamma(A_1, \dots, A_N)$ if, and only if, $x^\alpha$ is a Nash equilibrium of $G^\alpha[x^\beta]$ and $x^\beta$ is an equilibrium of $G^\beta[x^\alpha]$.*

Proposition 2.23 amounts to a re-statement of the definition of Nash equilibrium. It however plays a central role in the subsequent construction.

*Remark* 2.24. In the sequel, partitions of the set $\{1, \dots, N\} \subset \mathbb{N}$ into several disjoint subsets are considered. A simple inductive argument extends the conventions (as well as proposition 2.23) to such situation.

In the interest of clarifying the notation and to give an overview of the main techniques in this chapter, consider the following

**Proposition 2.25.** *Consider the embedding of the semi-algebraic*

$$Y = \left\{(t,x) \in \mathbb{R}^2 : \left(\frac{1}{2} - x\right) \cdot \left(x - \frac{t^2+2}{4}\right) = 0\right\} \cap [0,1]^2$$

*onto* $\mathbb{R}^{11}$ *whose image is given by* $X_0 \cup X_1$*, where*

$$X_0 = \left\{\left(t,t,t,t,t,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) : t \in [0,1]\right\},$$

$$X_1 = \left\{\left(t,t,t,t,t,\tfrac{t^2+4}{8},\tfrac{t^2+4}{8},\tfrac{t^2+4}{8},\tfrac{t^2+4}{8},\tfrac{t^2+2}{4},\tfrac{t^2+2}{4}\right) : t \in [0,1]\right\}.$$

*Then there exists* $G \in \mathcal{B}_{11}$ *such that* $E(G) = X_0 \cup X_1$.

*Remark* 2.26. Observe that if $X_0$ and $X_1$ are the sets defined in the rubric of proposition 2.25 then,

$$X_0 \cap X_1 = \left\{\left(0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right\}.$$

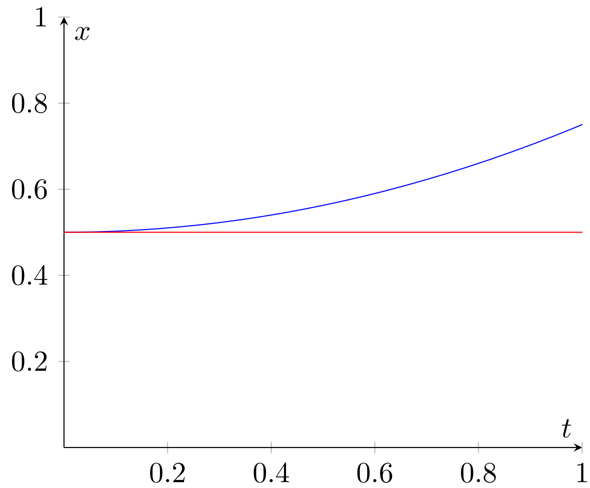


Figure 2.1: An illustration of the curve realised by proposition 2.25. The red line is given by $x = \frac{1}{2}$, while the blue line is a segment of the parabola given by $x = \frac{t^2+2}{4}$

*Remark* 2.27. Aside from illustrating the main techniques in the chapter,

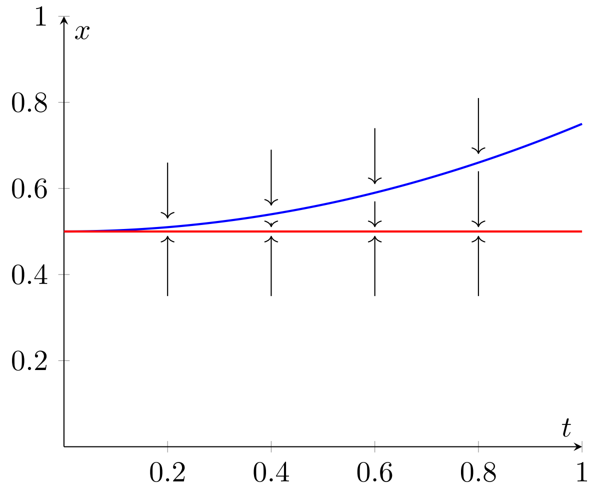


Figure 2.2: An illustration of the dynamics involved in the construction of the example of proposition 2.25. Notably, the parabolic segment is unstable.

proposition 2.25 is interesting in its own right: the set

$$\left\{(t,x) \in \mathbb{R}^2 : \left(\frac{1}{2} - x\right) \cdot \left(x - \frac{t^2+2}{4}\right) = 0\right\} \cap [0,1]^2$$

being semi-algebraic, as opposed to merely algebraic, falls outside the scope of the results in [14]. Furthermore, the notion of universality (homeomorphism) in theorem 6.1 in [6] does not adequately capture the behaviour close to $(0, \frac{1}{2})$. Lastly, this set can be regarded as resulting from projecting $E(G)$, where $G \in \mathcal{B}_{11}$ is the game constructed in proposition 2.25, onto the plane

$$x^{\alpha_2} = \ldots = x^{\alpha_5} = x^{\beta_1} = \ldots = x^{\beta_4} = x^{\gamma_2} = 0.$$

The fact that this projection is injective solves affirmatively a question proposed by Levy (see page 441 in [27]). Theorem 2.35 shows that this fact holds more generally.

*Remark* 2.28. Heuristically, the main idea of the proof of proposition 2.25 is to begin by grouping the 11 players into three groups denoted $\alpha$ (containing 5 players), $\beta$ (containing 4 players, divided into two even subgroups) and $\gamma$ (containing 2 players). Players in the group $\alpha$ play a $C_5$ game, where $C_5$ is as in theorem 2.13. Meanwhile, each subgroup in $\beta$ plays a $H$ game (see proposition 2.12), where the parameter $q$ depends on the actions of players in $\gamma$. Lastly, the two players in $\gamma$ play a $H$ game where the parameter $q$ depends on the actions of players in $\alpha$ and $\beta$. The main idea is that if the first player in the group $\gamma$, $\gamma_1$, takes action 1 with probability $x^{\gamma_1}$, then their action affects pay-offs of players in $\beta$, which in turn affect $\gamma_1$'s pay-offs. This loop recurs until equilibrium is reached, which must lie as a result in either the parabolic segment

$$X_1 = \left\{ \left( t, t, t, t, t, \frac{t^2+4}{8}, \frac{t^2+4}{8}, \frac{t^2+4}{8}, \frac{t^2+4}{8}, \frac{t^2+2}{4}, \frac{t^2+2}{4} \right) : t \in [0,1] \right\}$$

or the line segment

$$X_0 = \left\{ \left( t, t, t, t, t, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) : t \in [0,1] \right\}.$$

The projected dynamics are illustrated in figure 2.2. Interestingly, since for $t \in [0,1]$, the polynomial

$$P(T_2) = \left( \frac{1}{2} - T_2 \right) \cdot \left( T_2 - \frac{t^2+2}{4} \right)^2$$

has a double root at $x = \frac{t^2+2}{4}$ and a single root at $x = \frac{1}{2}$, equilibria on the parabolic segment have saddle-point like instability, while equilibria on the line segment are stable under these dynamics.

*Proof of proposition 2.25.* Denote by $Q \in \mathbb{R}[T_1, \ldots, T_8]$ the following polynomial

$$\begin{gathered} Q(T_1, \ldots, T_8) = \frac{4T_5 - 1}{2} + \left( \frac{1}{2} - \frac{4T_6 - 1}{2} \right) \times \\ \left( \frac{4T_7 - 1}{2} - \frac{1}{2} - \frac{T_1 \cdot T_2}{4} \right) \left( \frac{4T_8 - 1}{2} - \frac{1}{2} - \frac{T_3 \cdot T_4}{4} \right). \end{gathered} \tag{2.24}$$

Observe that upon expanding the product, the polynomial $Q$ can be seen to contain monomials of the form $T_{i_1} \cdot T_{i_2} \cdot \ldots \cdot T_{i_k}$, where the indices $i_1, \ldots, i_k$ are all distinct. In light of remark 2.4, the polynomial $Q$ can be used to define a binary game.

Partition $\{1,\dots,11\}$ as follows:

$$\{1,\dots,5\}\cup\{6,7\}\cup\{8,9\}\cup\{10,11\}.$$

Using this partition, let $G\in\mathcal{B}_{11}$ be defined as follows

$$\begin{aligned}
G^{\alpha_1,\dots,\alpha_5} &= C_5^{\alpha_1,\dots,\alpha_5}, &(2.25)\\
G^{\beta_1,\beta_2} &= H^{\beta_1,\beta_2}\left[\frac{1+2x^{\gamma_1}}{4}\right](x^{\beta_1},x^{\beta_2}), &(2.26)\\
G^{\beta_3,\beta_4} &= H^{\beta_3,\beta_4}\left[\frac{1+2x^{\gamma_1}}{4}\right](x^{\beta_3},x^{\beta_4}), &(2.27)\\
G^{\gamma_1,\gamma_2} &= H^{\gamma_1,\gamma_2}[Q(x^{\alpha_1},\dots,x^{\alpha_4},x^{\beta_1},\dots,x^{\beta_4})](x^{\gamma_1},x^{\gamma_2}). &(2.28)
\end{aligned}$$

Where, by a slight abuse of notation, $G^{\alpha_1,\dots,\alpha_5}$ is identified with the game in theorem 2.13 in the particular case $N=5$. This identification is valid since the only indices involved are $1,\dots,5$. Employing a similar notational abuse, for $i\in\{1,3\}$ the game $G^{\beta_i,\beta_{i+1}}$ is identified with the game $H$ in proposition 2.12. A similar identification holds for $G^{\gamma_1,\gamma_2}$.

Suppose first that $x\in X_0\cup X_1$. By a slight abuse of notation, re-label the coordinates of $x$ using the elements of the partition $\alpha\cup\beta\cup\gamma$. Hence,

$$\begin{aligned}
x^{\alpha_1} &= \ldots = x^{\alpha_5} = t\in[0,1], &(2.29)\\
x^{\beta_1} &= \ldots = x^{\beta_4} = \frac{1+2x^{\gamma_1}}{4}. &(2.30)
\end{aligned}$$

Hence,

$$Q(x^{\alpha_1},\dots,x^{\alpha_4},x^{\beta_1},\dots,x^{\beta_4}) = x^{\gamma_1}+\left(\frac{1}{2}-x^{\gamma_1}\right)\left(\frac{1}{2}+\frac{t^2}{4}-x^{\gamma_1}\right)^2. \quad (2.31)$$

Since $x^{\gamma_1}\in\{\frac{1}{2},\frac{1}{2}+\frac{t^2}{4}\}$ it follows that

$$Q(x^{\alpha_1},\dots,x^{\alpha_4},x^{\beta_1},\dots,x^{\beta_4}) = x^{\gamma_1} = x^{\gamma_2}. \quad (2.32)$$

Equations (2.29), (2.30) and (2.32) along with propositions 2.23 and 2.12, and theorem 2.13 imply that $x\in E(G)$.

Suppose next that $x\in E(G)$ satisfies $0<x^{\gamma_1}<1$, proposition 2.12 implies that (2.32) holds. Moreover, since $x\in E(G)$ it follows by proposition

2.23 that $x^{\alpha_1}, \ldots, x^{\alpha_5}$ forms an equilibrium of $C_5$ and thus equation (2.29) holds as well. Similarly, $x^{\beta_1}, x^{\beta_2}$ must coincide with the unique equilibrium of $H^{\beta_1,\beta_2}\left[\frac{1+2x^{\gamma_1}}{4}\right]$ while $x^{\beta_3}, x^{\beta_4}$ coincides with the only equilibrium of $H^{\beta_3,\beta_4}\left[\frac{1+2x^{\gamma_1}}{4}\right]$. Equation (2.30) is therefore satisfied. Combining these results,

$$x^{\gamma_1} = x^{\gamma_1} + \left(\frac{1}{2} - x^{\gamma_1}\right)\left(\frac{1}{2} + \frac{t^2}{4} - x^{\gamma_1}\right)^2; \tag{2.33}$$

which yields $x^{\gamma_1} \in \{\frac{1}{2}, \frac{1}{2} + \frac{t^2}{4}\}$, and, consequently, $x \in X_0 \cup X_1$. All that remains is to show is that there exists no equilibria with $x^{\gamma_1} \in \{0, 1\}$.

Suppose by way of contradiction that there existed an equilibrium satisfying $x^{\gamma_1} = 0$, proposition 2.12 would then imply that $Q \leqslant 0$. However, following the same logic as in the previous paragraph, $x^{\alpha_1} = \ldots = x^{\alpha_5} = t \in [0, 1]$ and

$$x^{\beta_1} = \ldots = x^{\beta_4} = \frac{1}{4}.$$

These equalities in turn imply

$$0 \geqslant Q(x^{\alpha_1}, \ldots, x^{\alpha_4}, x^{\beta_1}, \ldots, x^{\beta_4}) = \frac{1}{2}\left(\frac{1}{2} + \frac{t^2}{4}\right)^2 > 0, \tag{2.34}$$

which results in the desired contradiction. Suppose that $x^{\gamma_1} = 1$ instead. Proposition 2.12 now implies that $Q \geqslant 1$ must hold. But since $x^{\alpha_1} = \ldots = x^{\alpha_5} = t \in [0, 1]$ and

$$x^{\beta_1} = \ldots = x^{\beta_4} = \frac{3}{4};$$

the following holds

$$1 \leqslant Q(x^{\alpha_1}, \ldots, x^{\alpha_4}, x^{\beta_1}, \ldots, x^{\beta_4}) = 1 - \frac{1}{2}\left(\frac{1}{2} - \frac{t^2}{4}\right)^2 \leqslant \frac{7}{8}. \tag{2.35}$$

Resulting in another contradiction. No equilibria with $x^{\gamma_1} \in \{0, 1\}$ must exist as a result. ∎

*Remark* 2.29. Examining the construction in the proof, the partition set $\{1, \ldots, 11\}$ into four subsets $\alpha = \{1, \ldots, 5\}$, $\beta = \{6, 7\}$, and $\beta' = \{8, 9\}$ $\gamma = \{10, 11\}$ can be justified as follows: in equilibrium, equations indexed by $\alpha$ correspond to the abscissa of a given point on the set. On the other hand, equations index by $\gamma$ refer to the ordinate of the point. Meanwhile, the role of equations index by $\beta$ is to equate variables $T_5, \ldots, T_9$ to ensure that the overall number of variables collapses to only 2.

## 2.5 Equilibrium sets as algebraic curves

Using similar arguments as in the previous section, the result in the previous section is generalised to a broader class of planar semi-algebraic sets. Moreover, the existence of a binary game whose equilibrium set contains a component *isomorphic*, in a sense to be specified, to an algebraic curve is established.

Given $P_1, \ldots, P_n \in \mathbb{R}[T_1, \ldots, T_m]$, let $V(P_1, \ldots, P_n)$ denote the algebraic set defined by the polynomials $P_1, \ldots, P_n$; i.e.,

$$V(P_1, \ldots, P_n) = \bigcap_{i=1}^{n} \{(x_1, \ldots, x_m) \in \mathbb{R}^m : P_i(x_1, \ldots, x_m) = 0\}.$$

In the particular case $P \in \mathbb{R}[T_1, T_2]$ the set

$$V(P) = \{(x_1, x_2) \in \mathbb{R}^2 : P(x_1, x_2) = 0\}$$

is referred to as an *algebraic curve*.

*Remark* 2.30. Observe that since $V(0) = \mathbb{R}^2$ and $V(1) = \emptyset$ the above definition implies that both $\mathbb{R}^2$ and $\emptyset$ are algebraic curves.

**Definition 2.31** (Isomorphic sets)**.** Given $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+k\}$, where $n, k \in \mathbb{N}$; denote by $\pi_{i_1,\ldots,i_n}$ the projection

$$\pi_{i_1,\ldots,i_n} : (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \mapsto (x_{i_1}, \ldots, x_{i_n}).$$

A semi-algebraic set $A \subset \mathbb{R}^{n+k}$ is said to be *isomorphic to* $B \subset \mathbb{R}^n$ if the following conditions are satisfied:

(i) There exists $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+k\}$ such that $B = \pi_{i_1,\ldots,i_n}(A)$.

(ii) The projection $\pi_{i_1,\ldots,i_n} : A \longmapsto B$ is injective.

(iii) The inverse mapping $\varphi = \pi^{-1}_{i_1,\ldots,i_n}$ has the following property: given $j \in \{i_1, \ldots, i_n\}$, $\varphi_j(x_{i_1}, \ldots, x_{i_n}) = a_j \cdot x_{i_l} + b_j$, where $l \in \{1, \ldots, n\}$.

*Remark* 2.32. Definition 2.31 implies that a semi-algebraic set $A$ is isomorphic to $B$, if $A$ projects injectively onto $B$ (note that $B$ must lie in a subspace of the ambient space containing $A$). Moreover, the inverse of the projection must be such that each of its coordinates is an affine function.

Let $A \subset \mathbb{R}^{n+k}$ and $B \subset \mathbb{R}^{n+k}$ be two given semi-algebraic sets. A mapping $\varphi : A \longmapsto B$ satisfying conditions (i) through to (iii) in definition 2.31 is said to be an *isomorphism.*

*Remark* 2.33. Naturally, if two semi-algebraic sets are isomorphic, the projection in definition 2.31 is an isomorphism between the two sets.

*Remark* 2.34. Note the notion of isomorphism in definition 2.31 is not standard. In the particular case that the sets $A$ and $B$ are algebraic sets, then the projection in definition 2.31 defines an *isomorphism of algebraic sets* (see section 2 in chapter 1 of [40]). Recall that $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ algebraic are *isomorphic* (in this latter sense) if there $B$ if there exists a *regular* bijective mapping $f = (f_1, \ldots, f_m) : A \longmapsto B$ whose inverse is also regular. Here regular means that for every $i \in \{1, \ldots, m\}$ the mapping $f_i$ satisfies $f_i(x) = F(x)$, for some $F \in \mathbb{R}[T_1, \ldots, T_n]$ and all $x \in A$.

**Theorem 2.35.** *Let $P \in \mathbb{R}[T_1, T_2]$ be such that*

$$\max_{0 \leqslant t \leqslant 1} P(t, 1) < 0 < \min_{0 \leqslant t \leqslant 1} P(t, 0). \tag{2.36}$$

*There exists $G \in \mathcal{B}_N$, for a certain $N \in \mathbb{N}$ such that $E(G)$ is isomorphic to $V(P) \cap [0,1]^2$ (i.e., $E(G)$ projects onto $V(P) \cap [0,1]^2$ injectively and each component of its inverse is an affine function).*

**Corollary 2.35.1.** *Let $P \in \mathbb{R}[T_1, T_2]$ be such that $V(P) \cap [0,1]^2 \subset [0,1] \times (0,1)$. There exists $N \in \mathbb{N}$ and $G \in \mathcal{B}_N$ such that $E(G) = W \cup X$, where $W$ projects injectively onto $V(P) \cap [0,1]^2 \subset [0,1] \times (0,1)$ and $X$ is a line segment separated from $W$. Furthermore, the inverse of the projection of $W$ is such that each of its components is an affine function. More formally, $E(G)$ has the following properties:*

*(i) $E(G) = \varphi(V(P) \cap [0,1]^2) \cup X$, where $\varphi$ is an isomorphism and $X$ is a line segment.*

(ii) *There exists a compact set $K$ such that $K \cap X = \emptyset$ and whose interior contains $\varphi(V(P) \cap [0,1]^2)$.*

*Remark* 2.36. Observe that an immediate consequence of theorem 2.35 is that $E(G)$ projects homeomorphically onto $V(P) \cap [0,1]^2$. It is worth emphasising however that the construction used in the proof of theorem 2.35 preserves not only the topology of $V(P) \cap [0,1]^2$ but its geometric and algebraic structure.

*Remark* 2.37. Note that, despite being a corollary, corollary 2.35.1 is the main result in the chapter. To provide some understanding, consider the following polynomial

$$P[T_1, T_2] = \left(T_1 - \frac{1}{2}\right)^2 + \left(T_2 - \frac{1}{2}\right)^2 - \frac{1}{18}.$$

Then $V(P) \subset (0,1)^2$ is the circle centred at $(\frac{1}{2}, \frac{1}{2})$ with radius $\frac{\sqrt{2}}{6}$. Consequently, there exists fibres $\{t\} \times [0,1]$, with $t \in [0,1]$, which do not intersect $V(P)$. Thus $P$ does not fulfil the assumptions of theorem 2.35. It is however possible to modify $V(P)$ by adding a line segment parallel to the $t$-axis in such a way that the assumptions of theorem 2.35 are satisfied. By placing the line at a sufficient distance from $V(P)$, the requirements of theorem 2.35 are satisfied, while preserving $V(P)$.

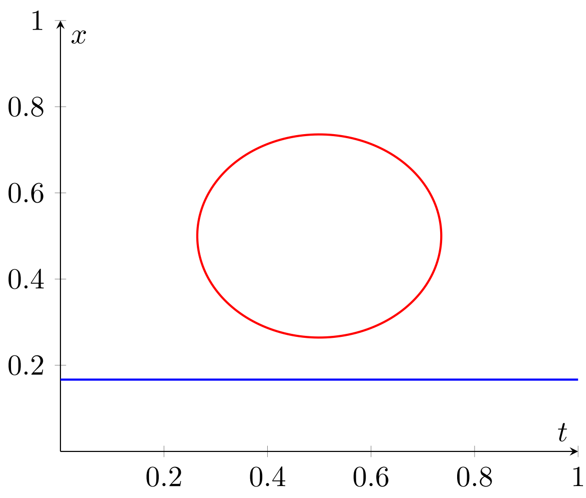


Figure 2.3: Graph of the set in remark 2.37.

*Proof of corollary 2.35.1.* Since $V(P) \subset [0,1] \times (0,1)$ there exist $\varepsilon \in (0,1)$

such that $V(P) \subset [0,1] \times [\varepsilon, 1-\varepsilon]$ (see lemma 2.43 below).

Let $Q(t,x) = \left(\frac{\varepsilon}{2} - x\right) P(t,x)^2$ and $t \in [0,1]$, it follows

$$Q(t,0) = \frac{\varepsilon}{2} P(t,0)^2 > 0, \tag{2.37}$$

$$Q(t,1) = -\left(1 - \frac{\varepsilon}{2}\right) P(t,1)^2 < 0 \tag{2.38}$$

Thus condition (2.36) is fulfilled and $N \in \mathbb{N}$ exists such that $G \in \mathcal{B}_N$ can be found satisfying $\psi(V(Q) \cap [0,1]^2) = E(G)$, where $\psi$ is an isomorphism.

Since $V(Q) = V(P) \cup \left([0,1] \times \left\{\frac{\varepsilon}{2}\right\}\right)$ and $V(P) \cap \left([0,1] \times \left\{\frac{\varepsilon}{2}\right\}\right) = \emptyset$ the result follows by restricting $\varphi$ to $V(P) \cap [0,1]^2$; i.e., $\varphi = \psi|_{V(P)}$. ■

*Remark* 2.38. Observe corollary 2.35.1 imposes no requirement on $V(P)$ beyond not intersecting the set $[0,1] \times \{0,1\}$. In particular, $V(P) \cap [0,1]^2 = \emptyset$ is allowed. A consequence of this observation is that the result in corollary 2.35.1 cannot be improved further.

Before providing a proof of theorem 2.35 two lemmas are needed. For brevity, refer to a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ as a *multi-linear* polynomial if every monomial it contains is of the form $\prod_{k=1}^{l} x_{i_k}$ with $i_1 < i_2 < \ldots < i_{l-1} < i_l$ and $0 \leqslant l \leqslant \deg(P)$ ($\deg(P)$ denotes the *degree* of the polynomial).

*Remark* 2.39. Recall that by remark 2.4, for a given binary game $G \in \mathcal{B}_n$, each component $G_i$ is a multi-linear polynomial.

**Lemma 2.40.** *Let $P \in \mathbb{R}[T,S]$, there exist $\hat{P} \in \mathbb{R}[T_1, \ldots, T_n, S_1, \ldots, S_m]$ multi-linear such that $\hat{P}(t, \ldots, t, s, \ldots, s) = P(t,s)$.*

*Remark* 2.41. The proof of lemma 2.40 is fairly natural as it amounts to transforming a monomial $T^k S^l$ into the monomial $\prod_{p=1}^{k} T_{i_p} \cdot \prod_{q=1}^{l} S_{i_q}$. The argument is presented fully as it provides an upper bound for the number of $n$, the number of $T$ variables, and $m$, the number of $S$ variables.

*Proof of lemma 2.40.* Consider first the case that $P$ contains only $T$, then

$$P(T) = \sum_{n=0}^{N} a_n T^n \tag{2.39}$$

and the desired multi-linear polynomial is

$$\hat{P}(T_1, \ldots, T_{N_0}) = a_0 + \sum_{i=k}^{N} a_k \prod_{i=\mathsf{t}_{k-1}+1}^{\mathsf{t}_k} T_i, \tag{2.40}$$

where

$$\mathsf{t}_k = \frac{k(k+1)}{2}$$

is the $k^{\text{th}}$ triangular number. An analogous result holds if $P$ contains $S$ exclusively.

Consider now the case that $P(T,S)$ is of the form

$$P(T,S) = a(S)T^n \tag{2.41}$$

where $a(Y)$ is of degree at least 1 and $n > 0$. In this instance,

$$\hat{P}(T_1, \ldots, T_n, S_1, \ldots, S_M) = \hat{a}(S_1, \ldots, S_M) \prod_{i=1}^{n} T_i, \tag{2.42}$$

where $\hat{a}$ is constructed using the steps in the previous case.

Observe that since an arbitrary $P \in \mathbb{R}[T,S]$ can be written as

$$P(X,Y) = P_0(T) + \sum_{k=1}^{N} a(S)T^k + P_1(S) \tag{2.43}$$

the existence the desired multi-linear polynomial follows by repeated application of the previous two cases. ■

*Remark* 2.42. Given $P \in \mathbb{R}[T,S]$ there might be several multi-linear polynomials that satisfy the condition in the previous lemma. In particular, consider the following polynomial

$$P(X,Y) = X^2 + 2XY + Y^2 = (X+Y)^2. \tag{2.44}$$

Lemma 2.40 furnishes us with the polynomial

$$\hat{P}(X_1, X_2, X_3, Y_1, Y_2, Y_3) = X_1X_2 + 2X_3Y_1 + Y_2Y_3; \tag{2.45}$$

while the smaller polynomial

$$\hat{Q}(X_1, X_2, Y_2, Y_3) = (X_1 + Y_1) \cdot (X_2 + Y_2) \tag{2.46}$$

also fulfils the desired requirements. In fact, the construction given in the proof of lemma 2.40 generates a multi-linear polynomial containing the largest possible number of monomials.

**Lemma 2.43.** *Let $P \in \mathbb{R}[T_1, T_2]$ be such that $V(P) \cap [0,1]^2 \subset [0,1] \times (0,1)$. There exists $\varepsilon \in (0,1)$ such that $V(P) \cap [0,1]^2 \subset [0,1] \times [\varepsilon, 1-\varepsilon]$.*

*Remark* 2.44. Lemma 2.43 follows by applying the *tube lemma* (see lemma 5.8 and, for a proof, step 1 in page 168 in [31]) to either $\{0\} \times [0,1]$ or $\{1\} \times [0,1]$.

*Remark* 2.45. Let $P \in \mathbb{R}[T_1, T_2]$ satisfy the assumptions in theorem 2.35. An application of the intermediate value theorem along with the condition

$$\max_{0 \leqslant t \leqslant 1} P(t,1) < 0 < \min_{0 \leqslant t \leqslant 1} P(t,0). \tag{2.47}$$

shows that given $t \in [0,1]$, the polynomial $P(t, \cdot) \in \mathbb{R}[X]$ has a root $x_0 \in (0,1)$. Geometrically, this implies that for every $t \in [0,1]$, $V(P) \cap (\{t\} \times [0,1]) \neq \emptyset$. Note that this places a significant constraint on the number of possible choices for the polynomial $P$. Notably, this condition may not be fulfilled if $P$ contains even powers only.

*Remark* 2.46. Observe that a consequence of corollary 2.35.1 is that there exist binary games whose set of Nash equilibria contains an arbitrary number of isolated points.

*Proof of theorem 2.35.* By lemma 2.40 there exists a multi-linear polynomial $\hat{P} \in \mathbb{R}[T_1, \ldots, T_n, S_1, \ldots, S_m]$ such that

$$P(t,x) = \hat{P}(t, \ldots, t, x, \ldots, x).$$

Define $Q \in \mathbb{R}[T_1, \ldots, T_n, S_0, S_1, \ldots, S_m]$ as follows

$$\begin{gathered} Q(T_1, \ldots, T_n, S_0, S_1, \ldots, S_m) = \frac{4S_0 - 1}{2} + \\ \hat{P}\left(T_1, \ldots, T_n, \frac{4S_1 - 1}{2}, \ldots, \frac{4S_m - 1}{2}\right). \end{gathered} \tag{2.48}$$

Moreover, define

$$N_0 = \inf\{k \in 2\mathbb{N}+1 : n \leqslant k\}, \tag{2.49}$$
$$N_1 = \inf\{k \in 2\mathbb{N} : m+1 \leqslant k\}. \tag{2.50}$$

Set $N = N_0 + N_1 + 2$ and let $G : [0,1]^N \longmapsto \mathbb{R}^N$ be a binary game as follows

$$G^{\alpha_1,\ldots,\alpha_{N_0}} = C_{N_0}^{\alpha_1,\ldots,\alpha_{N_0}}, \tag{2.51}$$
$$G^{\beta_1,\beta_2} = H^{\beta_1,\beta_2}\left[\frac{1+2x^{\gamma_1}}{4}\right](x^{\beta_1}, x^{\beta_2}), \tag{2.52}$$
$$\vdots \tag{2.53}$$
$$G^{\beta_{N_1-1},\beta_{N_1}} = H^{\beta_3,\beta_4}\left[\frac{1+2x^{\gamma_1}}{4}\right](x^{\beta_{N_1-1}}, x^{\beta_{N_1}}), \tag{2.54}$$
$$G^{\gamma_1,\gamma_2} = H^{\gamma_1,\gamma_2}[Q(x^{\alpha_1},\ldots,x^{\alpha_n},x^{\beta_1},\ldots,x^{\beta_{m+1}})](x_1^{\gamma},x_2^{\gamma}). \tag{2.55}$$

As in proposition 2.25, by a slight abuse of notation the game $G^{\alpha_1,\ldots,\alpha_{N_0}}$ is identified with the game in theorem 2.13. A similar abuse of notation is employed for the game $G^{\beta_i,\beta_{i+1}}$ which is identified with the game $H$ (see proposition 2.12) on the indices $\beta$. Moreover, since $Q$ is a multi-linear polynomial, the game $H^{\gamma_1,\gamma_2}[Q]$ is well-defined as it contains only monomials that are products of distinct variables.

Now, let

$$X = \left\{\left(t,\ldots,t,\frac{2x+1}{4},\ldots,\frac{2x+1}{4},x,x\right) : (t,x) \in [0,1]^2 \text{ and } P(t,x) = 0\right\}.$$

Observe that to each point $(t,x) \in V(P) \cap [0,1]^2$ there corresponds a unique point of $y \in X$. Moreover, the coordinates of $y$ are given by affine functions of $t$ and $x$. On the other hand, $V(P) \cap [0,1]^2$ results from projecting $X$ onto the plane defined by the first and second-to-last coordinates. Since such projection is injective, the sets $X$ and $V(P) \cap [0,1]^2$ are isomorphic.

It is to be shown now that $E(G) = X$. Assume first that $(x^\alpha, x^\beta, x^\gamma) \in X$ (note that, as it was done in proposition 2.12, by a slight abuse of notation coordinates of $x \in X$ are labelled using the partition $\alpha \cup \beta \cup \gamma$). Consequently,

$$x^{\alpha_1} = \ldots = x^{\alpha_{N_0}} = t \in [0, 1], \tag{2.56}$$
$$x^{\beta_1} = \ldots = x^{\beta_{N_1}} = \frac{1 + 2x^{\gamma_1}}{4}. \tag{2.57}$$

It follows from theorem 2.13 that $x^\alpha$ is a Nash equilibrium of $C_{N_0}^{\alpha_1,\ldots,\alpha_{N_0}}$. Moreover, proposition 2.12 implies that for $i \in \{1, \ldots, N_1 - 1\}$, $(x^{\beta_i}, x^{\beta_{i+1}})$ is a Nash equilibrium of $H^{\beta_i,\beta_{i+1}}\left[\frac{1+2x^{\gamma_1}}{4}\right]$. In addition, equations (2.56) and (2.57) imply

$$\begin{aligned} Q(x^{\alpha_1}, \ldots, x^{\alpha_n}, x^{\beta_1}, \ldots, x^{\beta_{m+1}}) &= x^{\gamma_1} + \hat{P}(t, \ldots, t, x^{\gamma_1}, \ldots, x^{\gamma_1}) \\ &= x^{\gamma_1} + P(t, x^{\gamma_1}) = x^{\gamma_1} = x^{\gamma_2}. \end{aligned} \tag{2.58}$$

Since lemma 2.43 implies that $x^{\gamma_1} \in (0, 1)$, proposition 2.12 implies $(x^{\gamma_1}, x^{\gamma_2})$ is a Nash equilibrium of $H^{\gamma_1,\gamma_2}[Q]$. Proposition 2.23 now yields $(x^\alpha, x^\beta, x^\gamma) \in E(G)$.

The reasoning in the paragraph above yields $X \subset E(G)$. To prove the reverse inclusion, consider $(x^\alpha, x^\beta, x^\gamma) \in E(G)$ with $0 < x^{\gamma_1} < 1$. Proposition 2.12 implies that

$$Q(x^{\alpha_1}, \ldots, x^{\alpha_n}, x^{\beta_1}, \ldots, x^{\beta_{m+1}}) = x^{\gamma_1} = x^{\gamma_2} \tag{2.59}$$

holds. Moreover, since $(x^\alpha, x^\beta, x^\gamma)$ is a Nash equilibrium it follows by proposition 2.23 that $x^{\alpha_1}, \ldots, x^{\alpha_{N_0}}$ forms an equilibrium of $C_{N_0}$ and thus equation (2.56) holds as well. Similarly, for every $i \in \{1, \ldots, N_1 - 1\}$, $(x^{\beta_i}, x^{\beta_{i+1}})$ is the only equilibrium of $H^{\beta_i,\beta_{i+1}}\left[\frac{1+2x^{\gamma_1}}{4}\right]$. Equation (2.57) is therefore satisfied. Combining these results, equation (2.59) becomes

$$x^{\gamma_1} = Q\left(t, \ldots, t, \frac{1 + 2x^{\gamma_1}}{4}, \ldots, \frac{1 + 2x^{\gamma_1}}{4}\right) = x^{\gamma_1} + P(t, x^{\gamma_1}), \tag{2.60}$$

where the last equation follows from the definition of $Q$ given by equation (2.55). Equation (2.60) yields $P(t, x^{\gamma_1}) = 0$ and, consequently, $x \in X$.

All that remains is to prove that no equilibria with $x^{\gamma_1} \in \{0, 1\}$ exists. To that effect, suppose by way of contradiction, that there existed an equilibrium satisfying $x^{\gamma_1} = 0$, proposition 2.12 would then imply that $Q \leqslant 0$.

However, by analogous arguments to those in the previous paragraphs, it must be the case that $x^{\alpha_1} = \ldots = x^{\alpha_{N_0}} = t \in [0,1]$ and

$$x^{\beta_1} = \ldots = x^{\beta_{N_1}} = \frac{1}{4}.$$

This implies the following chain of inequalities

$$0 \geqslant Q\left(t, \ldots, t, \frac{1}{4}, \ldots, \frac{1}{4}\right) = P(t,0) \geqslant \min_{0 \leqslant t \leqslant 1} P(t,0) > 0. \tag{2.61}$$

A contradiction. Suppose instead that $x^{\gamma_1} = 1$, proposition 2.12 now implies that $Q \geqslant 1$ must hold. But since $x^{\alpha_1} = \ldots = x^{\alpha_{N_0}} = t \in [0,1]$ and

$$x^{\beta_1} = \ldots = x^{\beta_{N_1}} = \frac{3}{4};$$

the following chain of inequalities must hold

$$1 \leqslant Q\left(t, \ldots, t, \frac{3}{4}, \ldots, \frac{3}{4}\right) = 1 + P(t,1) \leqslant 1 + \max_{0 \leqslant t \leqslant 1} P(t,1) < 1. \tag{2.62}$$

Since this last chain yields another contradiction, it follows that no equilibria with $x^{\gamma_1} \in \{0,1\}$ can exist. ■

## 2.6 Further examples

This section introduces additional examples that have not been covered in the literature so far. These examples all follow almost at once from theorem 2.35.

### 2.6.1 Three simple examples

In this subsection examples of cubic curves that can be realised as the set of Nash equilibria of certain binary games are presented. The choice of examples is motivated by two main reasons. First, all three examples contain *singular points*; these points all lie in the interior of the relevant hypercube. Second, the fact that each polynomial is of degree 3 in the second variable ensures that the condition

$$\max_{0\leqslant t\leqslant 1} P(t,1) < 0 < \min_{0\leqslant t\leqslant 1} P(t,0)$$

is satisfied.

*Example(s)* 2.1. Let $P \in \mathbb{R}[T_1, T_2]$ denote the polynomial

$$P(T_1,T_2) = \left(2T_1 - \frac{1}{2}\right)^2 - \left(2T_2 - \frac{1}{2}\right)^3.$$

Observe that for arbitrary $t \in [0,1]$,

$$P(t,0) = \left(2t - \frac{1}{2}\right)^2 + \frac{1}{8} > 0, \tag{2.63}$$

$$P(t,1) = \left(2t - \frac{1}{2}\right)^2 - \frac{27}{8} < 0. \tag{2.64}$$

Hence, $P$ satisfies the hypotheses in theorem 2.35. There exists a binary game $G$ such that $E(G)$ is isomorphic to $V(P)\cap[0,1]^2$. Recall that this means that $E(G)$ projects injectively onto $V(P)\cap[0,1]^2$ and, moreover, each component of the inverse mapping of this projection is an affine function.

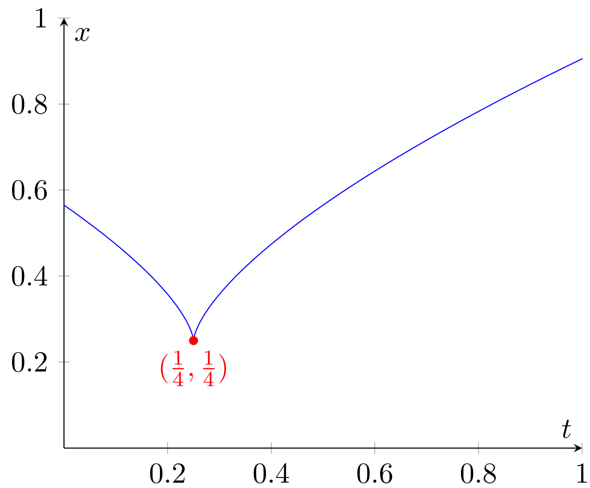


Figure 2.4: Graph of $V(P)\cap[0,1]^2$ where $P$ is as in example 2.1.

*Remark* 2.47. Let

$$P(T_1,T_2) = \left(2T_1 - \frac{1}{2}\right)^2 - \left(2T_2 - \frac{1}{2}\right)^3$$

be as in example 2.1. Observe that at the point $(\frac{1}{4}, \frac{1}{4}) \in V(P) \cap [0,1]^2$ both partial derivatives $\dfrac{\partial P}{\partial t}$ and $\dfrac{\partial P}{\partial x}$ vanish. Indeed, as can be seen from figure 2.4, the curve has a *cusp singularity* at $(\frac{1}{4}, \frac{1}{4})$ and tangent vectors are not well-defined there. Since $E(G)$ is isomorphic to this set, $E(G)$ has a cusp located at the image of $(\frac{1}{4}, \frac{1}{4})$ under the relevant isomorphism.

*Example(s)* 2.2. Let $Q \in \mathbb{R}[T_1, T_2]$ denote the polynomial

$$Q(T_1, T_2) = \left(T_1 - \frac{7}{10}\right)^2 - 17\left(T_2 - \frac{1}{2}\right)^2 \left(T_2 - \frac{3}{10}\right).$$

Observe that for arbitrary $t \in [0,1]$,

$$Q(t,0) = \left(t - \frac{7}{10}\right)^2 + \frac{51}{40} > 0, \tag{2.65}$$

$$Q(t,1) = \left(t - \frac{7}{10}\right)^2 - \frac{119}{40} < 0. \tag{2.66}$$

Hence, $Q$ satisfies the hypotheses in theorem 2.35. There exists a binary game $G$ such that $E(G)$is isomorphic to $V(P) \cap [0,1]^2$. Recall that this means that $E(G)$ projects injectively onto $V(P) \cap [0,1]^2$ and, moreover, each component of the inverse mapping of this projection is an affine function.

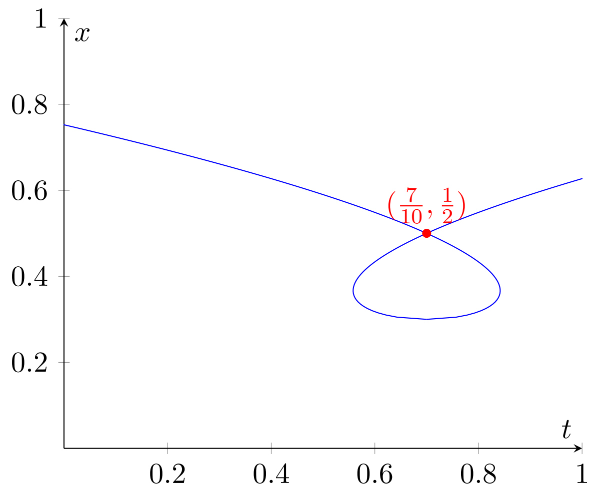


Figure 2.5: Graph of $V(Q) \cap [0,1]^2$ where $Q$ is as in example 2.2.

*Remark* 2.48. Let

$$Q(T_1, T_2) = \left(T_1 - \frac{7}{10}\right)^2 - 17\left(T_2 - \frac{1}{2}\right)^2\left(T_2 - \frac{3}{10}\right)$$

be the polynomial in example 2.2. As may seen from figure 2.5, the curve self-intersects at the point $(\frac{7}{10}, \frac{1}{2}) \in V(P) \cap [0,1]^2$; the curve has a *node* at this point. Since $E(G)$ is isomorphic to this set, $E(G)$ also self-intersects. The point of self-intersection being located at the image of $(\frac{7}{10}, \frac{1}{2})$ under the isomorphism.

*Example(s)* 2.3. Let $F \in \mathbb{R}[T_1, T_2]$ denote the polynomial

$$F(T_1, T_2) = (4T_1 - 1)^2 + \left(4T_2 - \frac{1}{3}\right)^2 - \left(4T_2 - \frac{1}{3}\right)^3.$$

Observe that for arbitrary $t \in [0,1]$,

$$F(t,0) = (4t-1)^2 + \frac{4}{27} > 0, \tag{2.67}$$

$$F(t,1) = (4t-1)^2 - \frac{968}{27} < 0. \tag{2.68}$$

Hence, $F$ satisfies the hypotheses in theorem 2.35. There exists a binary game $G$ such that $E(G)$ is isomorphic to $V(P) \cap [0,1]^2$. Recall that this means that $E(G)$ projects injectively onto $V(P) \cap [0,1]^2$ and, moreover, each component of the inverse mapping of this projection is an affine function.

*Remark* 2.49. As can be seen from figure 2.6, if

$$F(T_1, T_2) = (4T_1 - 1)^2 + \left(4T_2 - \frac{1}{3}\right)^2 - \left(4T_2 - \frac{1}{3}\right)^3,$$

$V(F)$ can be split into two components, a smooth curve and the isolated point $(\frac{1}{4}, \frac{1}{12})$. The existence of an isolated point due to both partial derivatives $\dfrac{\partial P}{\partial t}$ and $\dfrac{\partial P}{\partial x}$ vanishing at $(\frac{1}{4}, \frac{1}{12})$.

*Remark* 2.50. Observe that since examples 2.1 and 2.2 contain singular points, the structure of these singular points is not preserved under the homeomorphism defined in [6]. Additionally, examples in this section all satisfy

$$V(P) \cap (\{0,1\} \times [0,1]) \neq \emptyset,$$

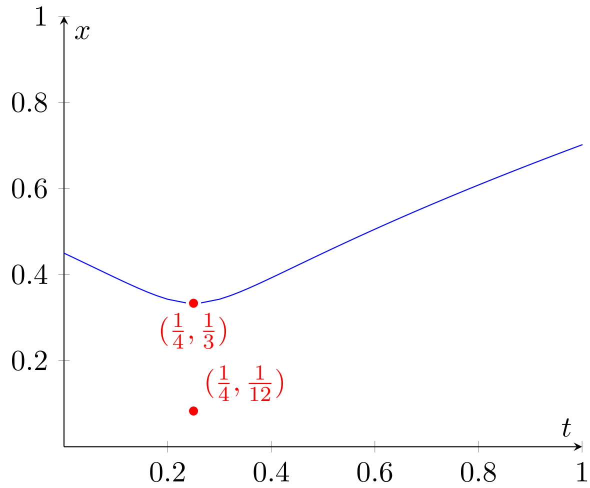


Figure 2.6: Graph of $V(F) \cap [0,1]^2$ where $F$ is as in example 2.3.

suggesting these constructions are not captured by theorem 6 in [14].

### 2.6.2 Rational exponents

In this subsection examples are provided to illustrate that theorem 2.35 enables a generalisation of proposition 2.25 to sets where the quadratic branch is replaced by a branch consisting of a curve with a rational exponent (see example 2.4 below). Moreover, example 2.4 also serves to illustrate the limitations of theorem 2.35 when the polynomials involved are of even degree.

*Example(s)* 2.4. By considering the polynomial $P(T_1, T_2) = T_1^n - (\frac{4T_2-1}{2})^m$ with $m \in 2\mathbb{N} - 1$ it follows from theorem 2.35 that there exists a binary game $G$ such that $E(G)$ is isomorphic to the following set

$$V(P) = \left\{ \left( t, \frac{2\sqrt[m]{t^n} + 1}{4} \right) : t \in [0,1] \right\}. \tag{2.69}$$

If $m \in 2\mathbb{N}$ instead then corollary 2.35.1 implies that there exists a binary game $G$ having a component that coincides with the image under an isomorphism of the set $V(P)$. An instance of such curve is illustrated below in figure 2.7.

*Remark* 2.51. The need for the additional line segment noted in example 2.4 when $m \in 2\mathbb{N}$ reflects a more general fact: given a compact semi-algebraic

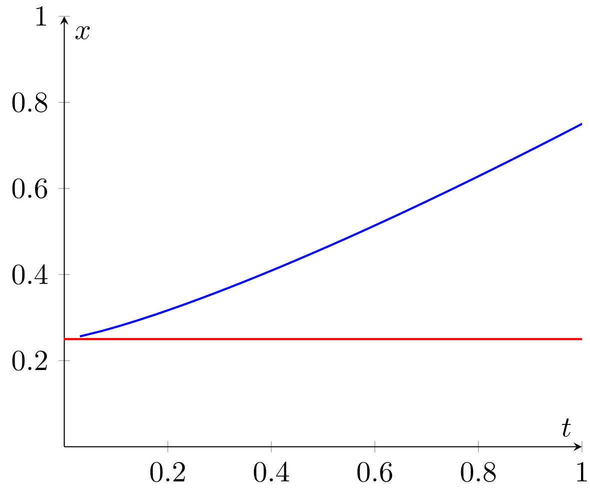


Figure 2.7: $V(P) \cap [0,1]^2$ where $P(T_1, T_2) = (\frac{1}{2} - T_2)\left(T_1^5 - (\frac{4T_2-1}{2})^4\right)$. The rational-exponent curve is illustrated in blue.

curve $A = V(P) \cap [0,1]^2$ satisfying $A \subset (0,1)^2$, it is not possible to infer the existence of a game $G \in \mathcal{B}_N$ (for some $N \in \mathbb{N}$) whose set of Nash equilibria $E(G)$ is isomorphic to $A$ by means of theorem 2.35 alone since $A \cap (\{t\} \times [0,1]$ might be empty for some $t \in [0,1]$. This is often the case when the polynomial describing $A$ has even degree. An application of corollary 2.35.1 would imply the existence of a game $G$ with a component isomorphic to $A$ however. Nonetheless, Bubelis (see section 4 in [12]) has shown that there exists a binary game whose Nash equilibria is isomorphic to the circle

$$V(P) = \left\{(t,x) \in \mathbb{R}^2 : \left(t - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2 - \frac{1}{18} = 0\right\} \subset (0,1)^2.$$

### 2.6.3 Curves with several branches

An extension of proposition 2.25 to the case where the semi-algebraic set contains several branches, as opposed to just two, can now be considered:

**Proposition 2.52.** *Fix $x_0 \in (0,1)$ and let $P \in \mathbb{R}[T_1, T_2]$ be of the form*

$$P[T_1, T_2] = \prod_{k=1}^{n} (x_0 + \beta_k T_1^k - T_2)^2 \cdot \prod_{l=1}^{m} (x_0 - \beta_l T_1^l - T_2)$$

*be such that:*

*(i)* $\beta_l < x_0\ \forall l \in \{1, \dots, m\}$*;*

*(ii)* $m \in 2\mathbb{N} - 1$*;*

*(iii)* $\beta_k < 1 - x_0\ \forall k \in \{1, \dots, n\}$*.*

*Then there exists* $N \in \mathbb{N}$ *such that for some* $G \in \mathcal{B}_N$ *the set* $E(G)$ *is isomorphic to the set* $V(P) \cap [0,1]^2$*.*

*Remark* 2.53. Observe that the set $V(P)$, where

$$P[T_1, T_2] = \prod_{k=1}^{n}(x_0 + \beta_k T_1^k - T_2)^2 \cdot \prod_{l=1}^{m}(x_0 - \beta_l T_1^l - T_2)$$

corresponds to the union of $n + m$ graphs of functions of the form $x = x_0 \pm \beta_k t^k$. Note that the graphs all meet at $(0, x_0)$.

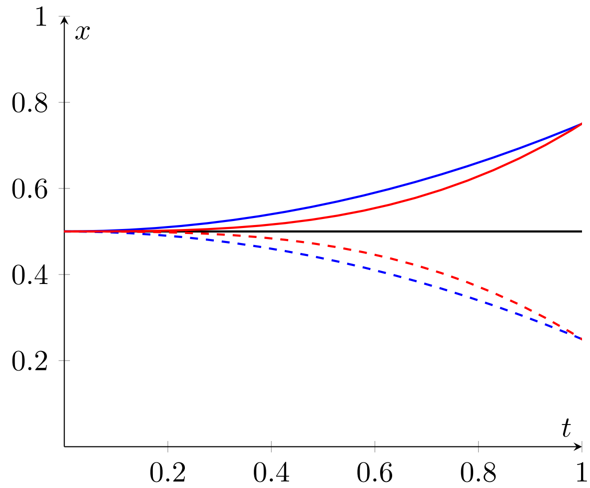


Figure 2.8: An illustration of a curve that fulfils the conditions of proposition 2.52. The blue curves are the parabolas $x = \frac{1}{2} \pm \frac{t^2}{2}$; while red curves correspond to the cubics $x = \frac{1}{2} \pm \frac{t^3}{2}$. Lastly, the solid black line is given by $x = \frac{1}{2}$.

*Proof of proposition 2.52.* As in the examples, it suffices to check that the conditions of theorem 2.35 are fulfilled. Accordingly, observe that

$$P(t,0) = \prod_{k=1}^{n}(x_0+\beta_k t^k)^2 \cdot \prod_{l=1}^{m}(x_0-\beta_l t^l) > x_0^{2n} \cdot \prod_{l=1}^{m}(x_0-\beta_l) > 0. \quad (2.70)$$

Choose now $k \in \{1,\dots,n\}$ and observe that since $\beta_k < 1-x_0$; $\sqrt[k]{\frac{1-x_0}{\beta_k}} > 1$, which corresponds to the unique real root of $x_0 - 1 + \beta_k t^k$. As a result,

$$|x_0 - 1 + \beta_k t^k| > 0$$

for all $t \in [0,1]$. It follows that

$$\begin{aligned} P(t,1) &= \prod_{k=1}^{n}(x_0+\beta_k t^k - 1)^2 \cdot \prod_{l=1}^{m}(x_0-\beta_l t^l - 1) \\ &< \prod_{k=1}^{n}(x_0+\beta_k t^k - 1)^2 \cdot (x_0-1)^m < 0. \end{aligned} \quad (2.71)$$

The last inequality being a consequence of the oddness of $m$. Equations (2.70) and (2.71) together imply that the conditions theorem 2.35 are fulfilled. Theorem 2.35 yields the result. ■

## 2.7 Conclusion

This chapter has shown that given an algebraic curve on the square $[0,1]^2$ which intersects neither the top nor the bottom of the square, there exists an isomorphic to a subset of the set of Nash equilibria of a game where players have two actions each. The set of Nash equilibria of the game contains at most another set separated from the isomorphic subset. The result is used to generate a number of novel examples of sets of Nash equilibrium sets. In particular, the main result in the chapter can be interpreted in the context of the *Universality of Nash equilibria.* The broad conjecture that any given compact semi-algebraic set can be realised (in some sense) as the set of equilibria of a certain game. In this regard, the main result shows that components of Nash equilibrium sets are universal in the class of algebraic curves on $[0,1]^2$. Universality understood here in terms of the new notion of isomorphism introduced. Notably, this notion of isomorphism is the natural one, as it preserves the algebraic structure of the sets involved.

Although the constructions here have been confined to planar curves it would be interesting to examine whether it is possible to extend them to broader algebraic and semi-algebraic sets. In particular, whether similar results hold for a single polynomial containing more than two variables seems like a natural extension of the present work. On a different line, it could be of interest to understand which equilibrium refinements the sets constructed here do in fact survive. Lastly, when examining the results due to Balkenborg and Vermeulen, a fairly natural question, which partially motivated the present chapter, is whether their notion of equivalence can be strengthen so that the mapping the set (as well as its inverse) are not only continuous, but have continuous derivatives as well. Results in this chapter provide particular examples where this conjecture holds. It would however be interesting to examine whether this fact remains true for a larger class of semi-algebraic sets.

# Chapter 3

# A Coarsening of a Limit Equilibrium Concept in Screening Markets

## 3.1 Introduction

In a recent work, Azevedo and Gottlieb (see [5]) have introduced a model of competitive screening markets. Their model extends existing models in the empirical insurance market literature as well as well as theoretical ones, notably the famous Rothschild-Stiglitz one (see [36]).

Azevedo and Gottlieb model can be viewed as a formal economy consisting of an arbitrary number of insurance firms serving a consumer population. Notably, few constraints on contract characteristics beyond compactness of the set of contracts are imposed. Azevedo and Gottlieb then introduce a natural equilibrium notion, *weak equilibrium* (alternatively, *proto-equilibrium* here). In a weak equilibrium, consumers purchase their most preferred contract, while individual contracts break even. In particular, if a contract is purchased by a non-zero mass of consumers, then is priced at its expected cost conditional on buyers' characteristics.

The definition of weak equilibrium places no constraints on those contracts that remain unsold in equilibrium. As a consequence, a given economy may contain a large number of weak equilibrium; some with unreasonable properties. This can be illustrated by means of an example, consider an economy where firms sell a single contract at unit cost. Suppose that consumers have a willingness to pay for the contract of either 0 or 2. Suppose further, that both types are equally likely. The scenario where firms price the contract at 3 and no consumer makes a purchase satisfies the requirements of a weak equilibrium. More strikingly, there exists weak equilibria where both agent types purchase the contract because the option of not purchasing is priced above the cost of not doing so.

To rule out such solutions, Azevedo and Gottlieb introduce a refinement to their weak equilibrium solution concept. An *equilibrium* is a weak equilibrium whose price and consumer choices are robust under certain perturbations. In the above example, a perturbation amounts to adding a vanishingly small mass of fictitious consumers purchasing the contract at no cost to the firms. If no consumer were to purchase the contract then fictitious agents would drive the contract price to 0; creating an incentive for consumers with willingness to pay 2 to deviate. The weak equilibrium with no purchase being made is therefore not an equilibrium. By contrast, the weak equilibrium where consumers with a positive willingness to pay buy is an equilibrium. Perturbations can be defined in more general settings by a similar procedure.

In spite of Azevedo and Gottlieb's notion of equilibrium enjoying many amenable properties, it is not continuous when the set of contracts changes. In particular, it is unknown and unclear, whether if a sequence of equilibria defined over economies whose set of contracts shrinks to a set containing the support of a given equilibrium; then the limit of the sequence of equilibria is an equilibrium of the economy with the smaller contract set. This might appear as a technical quirk, but it is particularly relevant in the context of insurance mandates. Government mandates enforce a particular insurance level and can thus be viewed as shrinking the available set of contracts. The effect of mandates on Azevedo-Gottlieb models has been studied by Levy and Veiga (See [28]). Levy and Veiga seem to assume that restricting an equilibrium to a smaller subset results in another equilibrium with similar properties. The observation at the beginning of this paragraph suggests this might not be so.

In order to ensure continuity when the contract set changes, this chapter proposes a coarsening of Azevedo and Gottlieb's equilibrium by allowing consumers to purchase their optimal choice "up to $\varepsilon$" in the perturbation. Heuristically, consumers in the perturbation can be thought of as being compensated an amount $\varepsilon \geqslant 0$ when making a purchase. The motivation for this notion is that the introduction of a mandate induces some consumers to alter their purchases due to their original choices becoming unavailable. Moreover, the influx of new consumers affects the price of the remaining contracts. Prices can either lower due to an influx of less risky agents or raise if the new buyers are riskier than the original buyers. Artificially compensating agents by a small amount $\varepsilon$ ensures that those consumers whose contracts remain available do not alter their choices significantly.

There are also additional reasons why this new notion might be more appealing. First, requiring that consumers $\varepsilon$-optimise only allows for more flexibility when analysing settings where optimal choices might not exist (e.g. because consumer utilities present discontinuities). Although the original assumptions by Azevedo and Gottlieb are retained, the arguments presented here should extend to this more general case. Lastly, it is worth remarking that the limiting process introduced by Azevedo and Gottlieb is fictitious, there appears to be no compelling reason for imposing the condition that consumers make optimal choices in the perturbation. Indeed, requiring that agents $\varepsilon$-optimise instead of precisely optimising appears to be a more natural choice which agrees with approaches taken to solve similar problems.

The chapter is structured as follows. Section 2 below recalls some technical facts from measure theory that are used in the remainder of the chapter. Readers who are familiar with the material may wish to skip this section. Section 3 introduces the formalism and main equilibrium concepts rigorously. The equilibrium concepts in this section are formulated in slightly different way than those in [5]. In particular, we make no formal distinction between economies and perturbations. Nonetheless, both formalism are logically equivalent. A proof of this fact is given in section 4. Section 5 then provides an extended example demonstrating that the new equilibrium concept is distinct from Azevedo and Gottlieb's notion. Moreover, this section also provides an example of a sequence of equilibria in the Azevedo and Gottlieb sense that does not converge to an equilibrium in their sense, but does so to an equilibrium in our sense. Section 6 then shows that, in Azevedo and Gottlieb's nomenclature, any perturbation contains a weak-equilibrium where agents choose their most preferred contract up to $\varepsilon$. Results in this section correspond to those in lemma 1 in the appendix of [5], which establishes the existence of weak equilibria in perturbations. Under some regularity assumptions on consumer's utilities, existence of equilibrium is established in section 7. The main result in this section being analogous to the equilibrium existence theorem (theorem 1) in [5]. Equilibrium properties are discussed in section 8. These properties translate proposition 1 in [5] to the setting in this chapter. Notably, equilibrium prices are Lipschitz, firms make no equilibrium profits from any contract, and consumers behave optimally. Section 9 summarises the main differences between the work of Azevedo and Gottlieb's and our own. In particular, this section highlights differences in the proofs and definitions presented earlier. The chapter concludes with section 10 where continuity of the new equilibrium notion is established.

## 3.2 Preliminaries

This section gives an overview of the topological properties of certain spaces of measures. A detailed account can be found in e.g. [8]. Further ancillary results are incorporated in relevant places in the main text.

Recall that given a set $S$ the set comprising all subsets of $S$ is denoted by $2^S$ while $|S|$ refers to its *cardinality* (the number of its elements). Additionally,

$$A \setminus B = \{a \in A : a \notin B\}.$$

Given a metric space $(M, d)$ denote by $B_r(t)$ the *open of ball of radius $r$ centred at $t \in M$*, i.e. the set

$$B_r(t) = \{s \in M : d(t, s) < r\}.$$

Moreover, recall that given $t \in M$ and $A \subset M$

$$d(t, A) = \inf_{s \in A} d(s, t)$$

is the *distance from $t$ to $A$* and $\sup_{s,t \in A} d(s,t)$ its *diameter* $\text{diam}(A)$. The closure of $A \subset M$ is denoted by $\overline{A}$.

Let $M$ be a metric space and denote by $\mathcal{B}(M)$ the *$\sigma$-algebra* generated by the open sets of $M$, its *Borel $\sigma$-algebra*. Elements of $\mathcal{B}(M)$ are referred to as *Borel sets* accordingly. Recall now that a mapping $\nu : \mathcal{B}(M) \longmapsto \mathbb{R} \cup \{-\infty, \infty\}$ is said to be a *(Borel) signed measure* if it satisfies the following requirements

(i) $\nu(\emptyset) = 0$;

(ii) if $\nu(S) = \infty$ (resp. $\nu(S) = -\infty$) for some $S \in \mathcal{B}(M)$, then $\nu(S') \neq -\infty$ for every $S' \in \mathcal{B}(M)$ (resp. $\nu(S') \neq \infty$ for every $S' \in \mathcal{B}(M)$);

(iii) Given Borel subsets $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$, $\lim_{n\to\infty} \nu(A_n) = \nu(\bigcup_{n=1}^{\infty} A_n)$.

Given a *measure space* $(M, \mathcal{B}(M))$, denote by $\mathfrak{F}(M)$ the *set of finite signed measures* over $M$; i.e. $\nu \in \mathfrak{F}(M)$ implies $\nu(M) \notin \{-\infty, \infty\}$. Recall that a measure $\nu \in \mathfrak{F}(M)$ is positive if $\nu(A) \geqslant 0$ for every $A \in \mathcal{B}(M)$. The set $\Delta(M) \subset \mathfrak{F}(M)$ is the set of all positive measures satisfying $\nu(M) = 1$. In other words, $\Delta(M)$ is the *set of probability measures* over $M$.

The set $\mathfrak{F}(M)$ can be regarded as a normed vector space under the following operations:

$$\begin{aligned}
(\nu_1 + \nu_2)(A) &= \nu_1(A) + \nu_2(A) \ \forall A \in \mathcal{B}(M), \\
(a \cdot \nu)(A) &= a \cdot \nu(A) \ \forall A \in \mathcal{B}(M), a \in \mathbb{R}.
\end{aligned}$$

The zero-vector $\mathbf{0}$ being the measure that vanishes at every Borel set. The relevant norm on $\mathfrak{F}(M)$ is the *total variation norm*

$$\|\nu\| = \nu^+(M) + \nu^-(M),$$

where $\nu^+$ and $\nu^-$ both positive constitute the (unique) Jordan decomposition of $\nu$, see e.g. section 3.1 in [8]; in particular theorem 3.1.1. and corollary 3.1.2. Given a sequence $\{\nu\}_{n=1}^{\infty} \subset \mathfrak{F}(M)$ $\nu_n \Rightarrow \nu$ means that the sequence converges to $\nu$ in this norm.

Convergence in total variation is often too strict (this is particularly relevant when considering sequences of probability of measures); a less restrictive alternative is that of *weak convergence*. Given a metric space $M$, a sequence $\{\nu_n\}_{n=1}^{\infty} \subset \mathfrak{F}(M)$ *converges to* $\nu \in \mathfrak{F}(M)$ *in the weak topology* (or simply *weakly*), denoted by $\nu_n \rightharpoonup \nu$, if for any given bounded continuous mapping $f : M \longmapsto \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{E}_{\nu_n}(f) = \mathbb{E}_{\nu}(f). \tag{3.1}$$

Where, by an abuse of notation, $\mathbb{E}_\nu$ denotes its (measure-theoretic) *integral*, i.e.

$$\mathbb{E}_\nu(f) = \int_M f(t)\nu(dt). \tag{3.2}$$

This abuse of notation is motivated by the fact that most measures considered in the sequel are probability measures, in which case the integral $\int_M f(t)\nu(dt)$ coincides with the *expectation of* $f$ *with respect to* $\nu$, which is conventionally denoted in this way. A basis for the topology of weak convergence is given by sets of the form

$$\left\{\nu \in \mathfrak{F}(M) : \max_{1\leqslant i\leqslant m} \left|\int_M f_i(t)\nu(dt) - \int_M f_i(t)\nu_0(dt)\right| < \delta\right\} \tag{3.3}$$

for a given $\nu_0 \in \mathfrak{F}(M)$, $\delta > 0$ and continuous mappings $f_1, \ldots, f_m$. For a discussion, see e.g. chapter 2 section 6 in [33].

*Remark* 3.1. The notation $\nu_n \rightharpoonup$ (resp. $\nu_n \Rightarrow \nu$) for weak convergence (resp. convergence in total variation) is common, but not universal. Indeed, some authors write $\nu_n \Rightarrow \nu$ for weak convergence instead (see section 6 in chapter 2 in [33]). The choice here made is motivated by the analogous notation in functional analysis. Furthermore, this choice visually encapsulates the idea that convergence in the weak topology is less restrictive than convergence in the total variation norm. The bolder the arrow, the stronger the requirement.

Recall that a *topological vector space* is a vector endowed with a topology in which the operations of vector addition and multiplication by a scalar are continuous. A topological vector space $V$ is *locally convex* if for every $v \in V$ there exists a collection of convex open sets $\{O_i\}_{i \in I}$ containing $v$ such that whenever $U$ is an open set that satisfies $v \in U$, there exists $j \in I$ such that $O_j \subset U$. Moreover, it is *Haussdorf* if given $v_1, v_2 \in V$ there exists open sets $v_1 \in O_1$ and $v_2 \in O_2$ such that $O_1 \cap O_2 = \emptyset$.

**Proposition 3.2.** *Given a metric space $M$ the set of finite Borel signed measures $\mathfrak{F}(M)$ endowed with the weak topology is a locally convex Haussdorf topological vector space.*

For a proof, it suffices to consider the zero vector only. The result then follows by applying the basis given by the sets in equation (3.3) where $\nu_0 = \mathbf{0}$. A detailed proof is omitted.

Since, as already mentioned, most of the sequel concentrates on elements of $\Delta(M)$ the discussion is now specialised to this set. In particular, we are interested in the case where $M$ is a well-behaved metric space. A metric space is thus a *Polish space* if it is *separable* (it contains a countable set $S$ satisfying $\overline{S} = M$) and *completely metrizable* ($(M, d)$ is homeomorphic to some $(M, \hat{d})$ which is a complete metric space). It assumed throughout that $M$ is endowed with the metric $\hat{d}$ that makes it into a complete metric space.

*Example(s)* 3.1. Under the usual metric the open interval $(-1, 1)$ is not complete however it is complete when endowed with the following metric:

$$\hat{d}(x, y) = \left| \tan\left(\frac{\pi x}{2}\right) - \tan\left(\frac{\pi y}{2}\right) \right|.$$

For an arbitrary Polish space compactness in $\Delta(M)$ has simple characterisation due to Prokhorov:

**Theorem 3.3** (Prokhorov)**.** *Let $M$ be a Polish space. A subset $\Omega \subset \Delta(M)$ is compact (in the weak sense) if, and only if, for every $\delta > 0$ there exists $K_\delta \subset M$ compact such that*

$$\sup_{\nu \in \Omega} \nu(M \setminus K_\delta) \leqslant \delta. \tag{3.4}$$

For a proof of this result see theorem 6.7 in chapter 2 of [33]. If $M$ is itself compact then 3.3 implies that $\Delta(M)$ is compact. More generally,

**Theorem 3.4.** *A metric space $M$ is compact if, and only if, $\Delta(M)$ is compact.*

For a proof see theorem 6.4 in chapter 2 of [33].

**Theorem 3.5.** *Let be $M$ be a Polish space and $\nu \in \Delta(M)$. There exists a unique closed set supp$(\nu)$ with the following properties*

*(i)* $\nu(supp(\nu)) = 1$;

*(ii) If $D$ is a closed set satisfying $\nu(D) = 1$, then supp$(\nu) \subset D$;*

*(iii)* $supp(\nu) = \{t \in M : \nu(U) > 0 \,\forall U$ *open s.t.* $t \in U\}$.

A proof may be found in chapter 2 theorem 2.1 in [33].

Given $\nu \in \Delta(M)$ the set supp$(\nu)$ in theorem 3.5 is the *support of* $\nu$.

*Remark* 3.6. It follows from theorem 3.5 that for given a $\nu \in \Delta(M)$

$$\mathrm{supp}(\nu) = \bigcap_{\substack{A \text{ closed} \\ \nu(A)=1}} A = M \setminus \Big( \bigcup_{\substack{B \text{ open} \\ \nu(B)=0}} B \Big).$$

Furthermore, property (iii) can be specialised to a sub-collection of open balls, i.e.

$$\mathrm{supp}(\nu) = \{t \in M : \nu(B_r(t)) > 0 \;\forall r > 0\}.$$

In accordance with remark 3.6 the support of a positive measure $\nu$ is defined by

$$\mathrm{supp}(\nu) = \{t \in M : \nu(B_r(t)) > 0 \;\forall r > 0\}.$$

*Remark* 3.7. Given a Polish space $M$, the notion of support can be extended to an arbitrary $\nu \in \mathfrak{F}(M)$ the interested reader is referred to section 7 in [8] for an in-depth presentation.

If $\{\nu_n\}_{n=1}^{\infty} \subset \Delta(M)$ satisfies $\nu_n \rightharpoonup \nu$ it is possible to establish a connection between their supports. To exhibit this connection, fix a collection of subsets $\{A_n\}_{n=1}^{\infty}$. A point $t \in M$ is a *limit point* of the collection if there exists

$N \in \mathbb{N}$ and sequence $\{t_n\}_{n=N}^{\infty}$ (where for each $n \in \mathbb{N}$, $t_n \in A_n$) such that $t_n \to t$. $t$ is *cluster point* instead if there exists a sequence $\{m_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that with $m_n \to \infty$, and a sequence $\{t_{m_n}\}_{n=1}^{\infty}$, where for each $m_n \in \mathbb{N}$, $t_{m_n} \in A_{m_n}$. The *Kuratowski lower limit of* $A_n$ L.i.$(A_n)$ is then the set of limit points of $\{A_n\}_{n=1}^{\infty}$, while the *Kuratowski upper limit of* $A_n$ L.s.$(A_n)$ is the set of cluster points the collection. Naturally, if

$$\text{L.s.}(A_n) \subset A \subset \text{L.i.}(A_n)$$

then $A$ is said to be the *Kuratowski limit* of $A_n$, and $A_n \to A$ *in the sense of Kuratowski.*

*Remark* 3.8. Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two collections satisfying $A_n \subset B_n$ for all $n \in \mathbb{N}$ their Kuratowski lower limits satisfy L.i.$(A_n) \subset$ L.i.$(B_n)$.

**Proposition 3.9.** *Let $M$ be a Polish space and $\{\nu_n\}_{n=1}^{\infty} \subset \Delta(M)$ satisfy $\nu_n \rightharpoonup \nu$, then*

$$supp(\nu) \subset L.i.(supp(\nu_n)). \tag{3.5}$$

A proof of this result can be found under proposition 5.1.8 in [3].

Let $f : M \longmapsto \mathbb{R}$ be Borel measurable (i.e. $f^{-1}(S) \in \mathcal{B}(M)$ whenever $S \in \mathcal{B}(\mathbb{R})$) and $\nu \in \mathfrak{F}(M)$ recall that the *pushforward of $\nu$ by $f$* is the measure $f_*(\nu)(S) = \nu(f^{-1}(S))$. In particular, if $M = N_1 \times N_2$ and $f = \pi_{N_1}$ (the projection over $M_1$) then $(\pi_{N_1})_*(\nu)$ (resp. $(\pi_{N_2})_*(\nu)$) is the *marginal of $\nu$ over $N_1$ (resp $N_2$).*

Recall that the following *change of variables formula* (see e.g. [8] theorem 3.6.1)

$$\mathbb{E}_{f_*(\nu)}(g) = \mathbb{E}_{\nu}(g \circ f). \tag{3.6}$$

**Proposition 3.10.** *Let $M$ be a compact metric space. Given $\nu_1, \nu_2 \in \Delta(M)$ if $\mathbb{E}_{\nu_1}(f) = \mathbb{E}_{\nu_2}(f)$ for every continuous mapping $f : M \longmapsto \mathbb{R}$, then $\nu_1 = \nu_2$.*

For a proof see theorem 5.9 in [33].

**Corollary 3.10.1.** *Let $M$ and $N$ be two compact metric spaces. Given a sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Delta(M \times N)$ satisfying $(\pi_M)_*(\nu_n) = \lambda$ for every $n \in \mathbb{N}$. If $\nu_n \rightharpoonup \nu$ then $(\pi_M)_*(\nu) = \lambda$.*

*Proof.* Observe that for any continuous mapping $f : M \longmapsto \mathbb{R}$

$$\mathbb{E}_{(\pi_M)_*(\nu)}(f) = \mathbb{E}_{\nu}(f \circ \pi_M) = \lim_{n\to\infty} \mathbb{E}_{\nu_n}(f \circ \pi_M) = \mathbb{E}_{\lambda}(f). \tag{3.7}$$

■

## 3.3 Formalism

This section introduces the model formally. The section also discusses relevant notions such as convergence and equilibrium concepts.

Given two sets $A$ and $B$ the *disjoint union* $A \sqcup B$ is the set

$$A \sqcup B = (A \times \{0\}) \cup (B \times \{1\}).$$

Abusing notation slightly, the set $A \times \{0\}$ (resp. $B \times \{1\}$) is identified with $A$ (resp. $B$). Note that if $A$ and $B$ are Polish spaces then $A \sqcup B$ is in turn a Polish space. Formally,

**Lemma 3.11.** *Let $(M, d_M)$ and $(N, d_N)$ be two compact Polish spaces then there exists a metric $d$ that turns $M \sqcup N$ into a compact Polish space. Moreover, the restriction of $d$ to $M$ (resp. $N$) coincides with $d_M$ (resp. $d_N$).*

*Proof.* Denote by $d_M$ and $d_N$ the metrics in $M$ and $N$ respectively, and let

$$d_0 = 1 + \max\{\mathrm{diam}(M), \mathrm{diam}(N)\}.$$

The desired metric is

$$d(x, y) = \begin{cases} d_M(x, y) & \text{if } x, y \in M, \\ d_N(x, y) & \text{if } x, y \in N, \\ d_0 & \text{if } x \in M \text{ and } y \in N, \text{ or } x \in N \text{ and } y \in M. \end{cases} \tag{3.8}$$

■

This result underpins much of the subsequent discussion. Moreover, It follows from lemma 3.11 that $\mathcal{B}(M \sqcup N)$ is isomorphic to $\mathcal{B}(M) \sqcup \mathcal{B}(N)$.

In the sequel, it is necessary to consider extensions of measures defined over a subset of a Polish space to the whole space. To avoid unnecessary repetition extensions are identified with the measure defined over the subset. More formally, let $M$ be a Polish space and $N \subset M$ viewed as a Polish space. A $\mathcal{B}(N)$-measure $\nu$ can be extended trivially to a $\mathcal{B}(M)$-measure $\hat{\nu}$ by setting $\hat{\nu}(S) = \nu(S \cap N)$. In the interest of brevity, whenever a measure $\nu$ is given over a Polish subspace $N$ of $M$ it is to be understood that $\nu$ is defined over the whole space by means of this extension. No explicit reference to the extension shall be made.

*Remark* 3.12. In more rigorous terms, the above extension may be viewed as the inclusion $\nu \mapsto \hat{\nu}$. This inclusion defines an isomorphism between the space of Borel measures of $N \subset M$ and the space of Borel measures over $M$. Observe that the inclusion mapping $S \in \mathcal{B}(N) \hookrightarrow S \in \mathcal{B}(M)$ also defines an isomorphism of Borel spaces.

Lastly, given a measure $\nu$, the following omission is convenient $\nu(\{t\}) = \nu(t)$. A similar convention is adopted if $t$ is replaced by a pair $(t, s)$.

### 3.3.1 Skeleta and convergence

**Definition 3.13** (Skeleton)**.** Let $X$ and $\Theta$ be two given compact Polish spaces, the measure space

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

is said to be a *skeleton* if $\mu$ and $\eta$ are finite measures satisfying $\mathrm{supp}(\eta) \subset X$ and $\mathrm{supp}(\mu) \subset \Theta$ with $\mu(\Theta) = 1$, while $|\mathrm{supp}(\eta)| < \infty$.

Note that $\mathrm{supp}(\eta) = \emptyset$ is allowed. In that case, $\eta$ is the zero measure $\mathbf{0}$. Since $\mathbf{0} + \nu = \nu$ reference to $\eta$ will be omitted in this case.

*Remark* 3.14. Observe that by lemma 3.11, $\Theta \sqcup X$ is a compact Polish space. Moreover, since for $S \in \mathcal{B}(\Theta \sqcup X)$, $\mu(S) = \mu(S \cap \Theta)$ (recall that $\mathrm{supp}(\mu) \subset \Theta$) and given that every Borel set of $\Theta$ (resp. $X$) can be identified with an element of $\mathcal{B}(\Theta \sqcup X)$; $\mu$ may be regarded, by a slight abuse of notation, as a probability measure on $(\Theta, \mathcal{B}(\Theta))$. By a similar abuse of notation, $\eta$ can be viewed as a finitely supported measure over $(X, \mathcal{B}(X))$.

*Remark* 3.15. The assumption $|\mathrm{supp}(\eta)| < \infty$ in definition 3.13 can be relaxed to this support containing countably many elements, all isolated. Further generalisations (e.g. $\mathrm{supp}(\eta)$ uncountable and compact) might not be possible. This is because we require certain conditional expectations to be defined at individual elements of $X$ and thus $\eta(x) > 0$ is needed for certain elements of $X$ (see below definition 3.22 and equation (3.46)).

Given skeletons

$$\mathfrak{E}_i = (\Theta_i \sqcup X_i, \mathcal{B}(\Theta_i \sqcup X_i), \mu_i + \eta_i)$$

with $i \in \{1, 2\}$ we say that $\mathfrak{E}_1$ is a *bone* of $\mathfrak{E}_2$ if $X_1 \subset X_2$ and $\Theta_1 \subset \Theta_2$.

Recall that given two non-empty closed subsets $A$ and $B$ of a metric space $M$, their *Hausdorff distance* $H(A, B)$ is

$$\max\{\sup_{s\in A} d(s, B), \sup_{t\in B} d(t, A)\}.$$

Accordingly, a collection of closed non-empty subsets $\{A_n\}_{n=1}^{\infty}$ *converges to* $A$ *in the sense of Hausdorff* if $H(A, A_n) \to 0$.

**Theorem 3.16.** *Let $M$ be a compact metric space. A collection $\{A_n\}_{n=1}^{\infty}$ of non-empty closed subsets converges to $A$ in the sense of Kuratowski if, and only if, $H(A, A_n) \to 0$.*

For a proof of this result see theorems 3.82 and 3.93 in [2].

*Remark* 3.17. Observe that as a consequence of theorem 3.16, given a collection $\{A_n\}_{n=1}^{\infty}$ of closed subsets of the Polish space $M$ such that for every $n$, $A_n \subset A$, where $A$ is also a closed subset. Then convergence $H(A_n, A) \to 0$ in this case is equivalent to the following requirement: for every $t \in A$ there exists $N \in \mathbb{N}$ and a sequence $\{t_n\}_{n=N}^{\infty}$ such that $t_n \in A_n$ and $t_n \to t$. Incidentally, this is the definition employed by Azevedo and Gottlieb (see definition 3 part 1 and footnote 20 in [5].)

**Definition 3.18** (Convergence of bones)**.** Given a skeleton

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

a collection of bones of $\mathfrak{E}$, $\{\mathfrak{E}_n\}_{n=1}^{\infty}$, where

$$\mathfrak{E}_n = (\Theta_n \sqcup X_n, \mathcal{B}(\Theta_n \sqcup X_n), \mu_n + \eta_n);$$

*converges to* $\mathfrak{E}$, denoted by $\mathfrak{E}_n \to \mathfrak{E}$, if $X_n \to X$ and $\Theta_n \to \Theta$ in the sense of Hausdorf; while $\mu_n \rightharpoonup \mu$ and $\eta_n \Rightarrow \eta$.

*Remark* 3.19. Note that if a collection of closed subsets $\{A_n\}_{n=1}^{\infty}$ converges to $A$ in the sense of Hausdorff then any sub-collection $\{A_{n_k}\}_{k=1}^{\infty}$ also converges to $A$ in this sense. This implies that if a collection of bones $\{\mathfrak{E}_n\}_{n=1}^{\infty}$ converges to $\mathfrak{E}$ then any sub-collection $\{\mathfrak{E}_{n_k}\}_{k=1}^{\infty}$ converges to $\mathfrak{E}$ as well.

### 3.3.2 Economies

Hitherto, the presentation has focused on the technical background exclusively. In this section, the model itself is introduced. Recall that the model seek to capture a competitive economy where consumers purchase insurance contracts from competing firms. A given insurance contract is modelled by a point $x$ on the compact Polish space $X$; the point $x$ is then thought of as representing all relevant characteristics of the contract (e.g., level of coverage). Similarly, a given consumer is indexed by a point $\theta$ on a compact Polish space $\Theta$, such point $\theta$ then represents the consumer's type. In other words, $\theta$ encodes all relevant characteristics of the consumer (e.g., the risk of default). As in Azevedo and Gottlieb, to ensure equilibria is reasonable, it is necessary to consider an additional class of fictitious or "behavioural consumers". To introduce this class of agents, fix a finite subset of $X$. To each $x$ in this subset associate a behavioural consumer. The behavioural agent associated to $x$ wishes to purchase $x$ alone. Moreover, firms incur no cost when providing $x$ to the associated behavioural agent. By an abuse of notation, the set of behavioural consumers is also denoted by $X$. Lastly, consumers are assumed to be distributed according to $\mu \in \Delta(\Theta)$, while behavioural consumers are distributed according to a positive measure $\eta$ whose support coincides with the finite subset of $X$. Using the notation in the previous subsection, all information about consumers (both behavioural and otherwise) can be seen to be encapsulated in the skeleton

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta).$$

Observe that unlike Azevedo and Gottlieb all consumers, be it behavioural or otherwise, are treated on equal footing here.

For a given consumer $\theta$, the utility derived from purchasing contract $x$ at price $p$ is denoted by $U(x, p, \theta)$. Similarly, the cost of providing contract $x$ to type $\theta$ is denoted by $c(x, \theta)$, which is assumed to be identical for all firms. If behavioural consumers preferences and costs are included then the notation $\hat{U}$ and $\hat{c}$ is employed instead.

To introduce the above notions precisely, consider now the skeleton

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta).$$

Let $U : X \times [0, \infty) \times \Theta \longmapsto \mathbb{R}$ be a continuous mapping and $c : X \times \Theta \longmapsto [0, \infty)$ also continuous. Denote by $\hat{U} : X \times [0, \infty) \times (\Theta \sqcup X) \longmapsto \mathbb{R} \cup \{\infty\}$ the mapping

$$\hat{U}(x, p, \theta) = \begin{cases} U(x, p, \theta) & \text{if } (x, p, \theta) \in X \times [0, \infty) \times \Theta, \\ 0 & \text{if } (x, p, \theta) \in X \times [0, \infty) \times X, \ \ \theta \neq x, \\ \infty & \text{if } (x, p, \theta) \in X \times [0, \infty) \times X, \ \ \theta = x. \end{cases} \tag{3.9}$$

Similarly, let $\hat{c} : X \times (\Theta \sqcup X) \longmapsto [0, \infty)$ denote the extension of $c$ to $\Theta \sqcup X$ defined by

$$\hat{c}(x, \theta) = \begin{cases} c(x, \theta) & \text{if } (x, \theta) \in X \times \Theta, \\ 0 & \text{if } (x, \theta) \in X \times X. \end{cases} \tag{3.10}$$

The triple $(\mathfrak{E}, \hat{U}, \hat{c})$ is said to be an *economy*.

*Remark* 3.20. Recall that in the definition of economy $\eta = \mathbf{0}$ is possible. In that case, $\hat{U}$ (resp. $\hat{c}$) differ from $U$ (resp. $c$) on a set of measure zero. They can therefore be regarded as identical.

*Remark* 3.21. Observe that given $x \in X$, understood as behavioural type, the conditions $\hat{U}(x, p, x) = \infty$ and $\hat{U}(y, p, x) = 0$ for $y \neq x$ formalise the idea that type $x$ purchases contract $x$ exclusively.

For simplicity, given a function $p : X \longmapsto [0, \infty)$ we abuse notation slightly by writing $\hat{U} \circ p$ to denote the function $(x, \theta) \mapsto \hat{U}(x, p(x), \theta)$. An analogous convention is employed with $U$ in place of $\hat{U}$.

Recall that given two $\sigma$-algebras $\mathcal{F}$ and $\mathcal{G}$, the *product $\sigma$-algebra* $\mathcal{F} \otimes \mathcal{G}$ is the $\sigma$-algebra generated by the collection

$$\{A \times B : A \in \mathcal{F}, B \in \mathcal{G}\}.$$

Note that if $\mathcal{F} = \mathcal{B}(M)$ and $\mathcal{G} = \mathcal{B}(N)$ for $M$ and $N$ Polish, then the product $\sigma$-algebra coincides with the $\sigma-$algebra $\mathcal{B}(M \times N)$.

### 3.3.3 Equilibrium concepts

Following Azevedo and Gottlieb, two notions of equilibria are introduced: the first simply requires that agents purchase their most preferred choice up to $\varepsilon > 0$, and that individual contracts are provided at their expected cost to firms. Meanwhile, the second requires, in addition to these two requirements, robustness to certain perturbations.

Denote by $\chi_A$ the *indicator function* of the set $A$, i.e. $\chi_A : x \mapsto \{0, 1\}$ with $\chi_A(x) = 1$ if, and only if, $x \in A$.

**Definition 3.22** ($\varepsilon$-proto-equilibrium)**.** Given an economy $(\mathfrak{E}, \hat{U}, \hat{c})$, with

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta),$$

and $\varepsilon \geqslant 0$, a pair $(p_\varepsilon, \hat{\alpha}_\varepsilon)$ where $p_\varepsilon : X \longmapsto [0, \infty)$ is a continuous mapping and $\hat{\alpha}_\varepsilon$ is a finite positive Borel measure over $X \times (\Theta \sqcup X)$ is an *$\varepsilon$-proto-equilibrium of the economy* $(\mathfrak{E}, \hat{U}, \hat{c})$ if the following conditions are satisfied:

(i) $\mathbb{E}_{\hat{\alpha}_\varepsilon}(\chi_{A \times (\Theta \sqcup X)} \cdot p_\varepsilon) = \mathbb{E}_{\hat{\alpha}_\varepsilon}(\chi_{A \times (\Theta \sqcup X)} \cdot \hat{c}) \ \forall A \in \mathcal{B}(X)$,

(ii) $\hat{\alpha}_\varepsilon(\{(x, \theta) \in X \times (\Theta \sqcup X) : (\hat{U} \circ p_\varepsilon)(x, \theta) + \varepsilon < (\hat{U} \circ p_\varepsilon)(y, \theta)$ for some $y \in X\}) = 0$,

(iii) $\hat{\alpha}_\varepsilon(X \times S) = \mu(S) + \eta(S) \ \forall S \in \mathcal{B}(\Theta \sqcup X)$.

Note that $\varepsilon$ may be 0 in definition 3.22. In such case, we speak of a *proto-equilibrium* as opposed to a "0-proto-equilibrium" and, in this case, it coincides with the definition of weak equilibrium in [5].

*Remark* 3.23. Definition 3.22 formalises the requirements at the beginning of this subsection. In particular, an $\varepsilon$-proto-equilibrium is a pair $(p_\varepsilon, \hat{\alpha}_\varepsilon)$, where $p_\varepsilon$ associates to each contract $x$ its purchase price $p_\varepsilon(x)$. Moreover, $\hat{\alpha}_\varepsilon(\{x\} \times (\Theta \sqcup X))$ can be interpreted as the total mass of consumers (both behavioural and not) purchasing contract $x$. In that regard, condition (i) in definition 3.22 can then be interpreted to mean that the price of providing contract $x$ is the cost of providing the contract conditional on those consumer types (further details are discussed in remark 3.25 below) who purchase it. Condition (ii) on the other hand requires that the mass of consumers who do not behave $\varepsilon$-optimally is 0. Lastly, condition (iii) states that the marginal of $\hat{\alpha}_\varepsilon$ over $X$ can be decomposed as the sum of the measures $\mu$ and $\eta$.

*Remark* 3.24. Observe that If $(p_\varepsilon, \alpha_\varepsilon)$ is an $\varepsilon$-proto equilibrium of a given economy then it is also an $\varepsilon'$-proto equilibrium for any $\varepsilon' > \varepsilon$. In particular, a proto-equilibrium is an $\varepsilon$-proto equilibrium for every $\varepsilon > 0$.

*Remark* 3.25. Note that condition (i) mimics the usual definition of conditional expectation with respect to the $\sigma$-algebra $\mathcal{B}(X) \otimes \{\emptyset, \Theta \sqcup X\}$. It is however more insightful, albeit less formally, to think of it in the following terms: if $x \in X$ is such that $\hat{\alpha}_\varepsilon(\{x\} \times (\Theta \sqcup X)) \neq 0$ (which happens e.g. if $\eta(x) \neq 0$) then

$$p_\varepsilon(x) = \frac{\iint_{\{x\}\times(\Theta\sqcup X)} c(x,\theta)\ \hat{\alpha}_\varepsilon(dx, d\theta)}{\hat{\alpha}_\varepsilon(\{x\} \times (\Theta \sqcup X))}. \tag{3.11}$$

This relation cannot however be written for a generic $x \in X$ since $\hat{\alpha}_\varepsilon(\{x\} \times (\Theta \sqcup X)) = 0$ cannot be ruled out. This is particularly relevant when $\mathrm{supp}(\eta) = \emptyset$.

Given a function $f : S \longmapsto T$ the *restriction* of $f$ to $A \subset S$ is denoted by $f|_A$.

An additional stronger notion of equilibrium is also considered:

**Definition 3.26** (Equilibrium)**.** Given an economy $(\mathfrak{E}, \hat{U}, \hat{c})$, with

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

, a pair $(p, \hat{\alpha})$ where $p : X \longmapsto [0, \infty)$ is continuous and $\hat{\alpha}$ is a finite Borel measure over $X \times (\Theta \sqcup X)$, is said to be an *equilibrium* if there exists $\{\mathfrak{E}_n\}_{n=1}^\infty$

a collection of bones of $\mathfrak{E}$ of the form

$$\mathfrak{E}_n = (\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)$$

where $X_n = \text{supp}(\eta_n)$ (so that, in particular, $|X_n| < \infty$) converging to $\mathfrak{E}$ and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ converging to 0 satisfying the following three conditions:

(i) For every $n \in \mathbb{N}$, the economy $(\mathfrak{E}_n, \hat{U}|_{X_n \times [0,\infty) \times (\Theta \sqcup X_n)}, \hat{c}|_{X_n \times (\Theta \sqcup X_n)})$ has an $\varepsilon_n$-proto-equilibrium $(p_n, \hat{\alpha}_n)$.

(ii) Given $x \in X$ for any sequence $\{x_n\}_{n=1}^{\infty}$ converging to $x$ with $x_n \in X_n$ for every $n$, $p_n(x_n) \to p(x)$.

(iii) The sequence $\{\hat{\alpha}_n\}_{n=1}^{\infty}$ converges weakly to $\hat{\alpha}$.

An equilibrium is an *Azevedo-Gottlieb equilibrium*, or *AG-equilibrium* for short, if the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ satisfies $\varepsilon_n = 0$ for every $n$. In that case, the sequence $\{(p_n, \hat{\alpha}_n)\}_{n=1}^{\infty}$ is a sequence of proto-equilibria.

*Remark* 3.27. Broadly speaking, definition 3.26 simply states that a pair $(p, \alpha)$ is an equilibrium of the economy $(\mathfrak{E}, \hat{U}, \hat{c})$, which contains no behavioural agents, if this economy is the limit of a sequence of economies each containing a small mass of behavioural agents. Moreover, $(p, \alpha)$ is then the limit of a sequence of $\varepsilon$-proto-equilibria of each such economy. It would appear at first that these conditions provide no information on consumer choices and contract prices in the economy $(\mathfrak{E}, \hat{U}, \hat{c})$. It turns out however that, under some regularity assumptions on $\hat{U}$, the notion of convergence in definition 3.26 (see theorem 3.74 below) is enough to ensure that $(p, \alpha)$ is a proto-equilibrium of the economy $(\mathfrak{E}, \hat{U}, \hat{c})$. This in particular implies that the set of agents who do not choose their most preferred contract has mass 0. In addition, the price of contract $x$ is determined by the expected cost to firms conditional on those consumer types who purchase $x$ (whenever this expectation is well-defined). It further follows that $\alpha(\{x\} \times \Theta)$ can then be interpreted as the proportion of agents purchasing contract $x$, while $p$ associates to each contract its price.

## 3.4 Characterisation of equilibrium notions

This section provides an alternative, but equivalent, characterisation of $\varepsilon-$equilibria. This definition agrees more explicitly with the original characterisation of weak-equilibrium due to Azevedo and Gottlieb (see definition 1 in [5]). The new characterisation is also more advantageous when proving the existence of $\varepsilon-$equilibria for a given economy and some $\varepsilon \geqslant 0$.

*Remark* 3.28. Recall that a $\varepsilon$-proto-equilibrium (definition 3.22) formalizes a market where consumers (both behavioural and not) purchase their most preferred contract up to $\varepsilon$, while firms price each contract at their expected cost. Proposition 3.29 below reflects this idea formally. Conversely, if there was a market where behavioural and non-behavioural types behave optimally, and firms charge their expected cost then the resulting situation should be an "equilibrium". Proposition 3.30 establishes that if these assumptions are made, then a $\varepsilon$-proto-equilibrium indeed results.

Given $t \in M$, denote by $\delta_t$ the *Dirac delta measure at* $t$,

$$\delta_t(A) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A, \end{cases}$$

for any given Borel set $A \subset M$.

**Proposition 3.29.** *Let* $(\mathfrak{E}, \hat{U}, \hat{c})$ *be an economy with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

*and* $\varepsilon \geqslant 0$. *If* $(p_\varepsilon, \hat{\alpha}_\varepsilon)$ *is an* $\varepsilon$*-proto-equilibrium there exists a unique pair of positive Borel measures* $\alpha_\varepsilon$ *and* $\overline{\eta}$ *with* $supp(\alpha_\varepsilon) \subset X \times \Theta$ *and* $supp(\overline{\eta}) \subset X \times X$ *such that* $\hat{\alpha}_\varepsilon = \alpha_\varepsilon + \overline{\eta}$. *Moreover, the following holds true:*

*(i)* $\mathbb{E}_{\alpha_\varepsilon + \overline{\eta}}(\chi_{A \times (\Theta \sqcup X)} \cdot p_\varepsilon) = \mathbb{E}_{\alpha_\varepsilon + \overline{\eta}}(\chi_{A \times (\Theta \sqcup X)} \cdot c)\ \forall A \in \mathcal{B}(X)$;

*(ii)* $\alpha_\varepsilon(X \times B) = \mu(B)\ \forall B \in \mathcal{B}(\Theta)$;

*(iii)* $\alpha_\varepsilon(X \times \Theta) = 1$;

*(iv)* $\alpha_\varepsilon(\{(x,\theta) \in X \times \Theta : (U \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (U \circ p_\varepsilon)(y,\theta)\ \forall y \in X\}) = 1$;

*(v)* $\overline{\eta}(x,\theta) = \eta(x)\cdot\delta_x(\theta)$ *if* $x \in$*supp*$(\eta)$ *and* 0 *otherwise.*

*Proof.* Define the finite measures $\alpha_\varepsilon$ and $\overline{\eta}$ by

$$\begin{aligned} \alpha_\varepsilon(S) &= \hat{\alpha}_\varepsilon(S \cap (X \times \Theta)), && (3.12)\\ \overline{\eta}(S) &= \hat{\alpha}_\varepsilon(S \cap (X \times X)) && (3.13)\end{aligned}$$

for an arbitrary $S \in \mathcal{B}(X \times (\Theta \sqcup X))$. Observe that by definition $\mathrm{supp}(\alpha_\varepsilon) \subset X \times \Theta$ and $\mathrm{supp}(\eta) \subset X \times X$. This results in the desired unique decomposition of $\hat{\alpha}_\varepsilon$. Observe that this decomposition implies

$$\mathbb{E}_{\hat{\alpha}_\varepsilon}(\chi_{A\times(\Theta\sqcup X)}\cdot\hat{c}) = \mathbb{E}_{\alpha_\varepsilon}(\chi_{A\times(\Theta\sqcup X)}\cdot\hat{c}) + \mathbb{E}_{\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot\hat{c}) = \mathbb{E}_{\alpha_\varepsilon}(\chi_{A\times(\Theta\sqcup X)}\cdot c), \quad (3.14)$$

for any given $A \in \mathcal{B}(X)$ since $\hat{c}(x,\theta) = 0$ over $X \times X \supset \mathrm{supp}(\overline{\eta})$. This yields (i).

To establish property (ii), let $B \in \mathcal{B}(\Theta) \subset \mathcal{B}(\Theta \sqcup X)$, then since $X \times B \not\subset \mathrm{supp}(\overline{\eta})$ requirement (iii) in definition 3.22 implies

$$\alpha_\varepsilon(X \times B) = \hat{\alpha}_\varepsilon(X \times B) = \mu(B) + \eta(B) = \mu(B). \quad (3.15)$$

Property (iii) in the rubric now follows by letting $B = \Theta$.

To prove property (iv) write

$$A_{\varepsilon,p_\varepsilon} = \{(x,\theta) \in X \times \Theta : (U \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (U \circ p_\varepsilon)(y,\theta)\ \forall y \in X\} \quad (3.16)$$

and, similarly,

$$\hat{A}_{\varepsilon,p_\varepsilon} = \{(x,\theta) \in X \times (\Theta \sqcup X) : (\hat{U} \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (\hat{U} \circ p_\varepsilon)(y,\theta)\ \forall y \in X\}. \quad (3.17)$$

Note that these sets are both Borel since they are closed. This follows from the continuity of $U$ and the fact that $X \times X$ is closed.

Now, $(X \times (\Theta \sqcup X)) \setminus \hat{A}_{\varepsilon,p_\varepsilon}$ is identical to

$$\{(x,\theta) \in X \times (\Theta \sqcup X) : (\hat{U} \circ p_\varepsilon)(x,\theta) + \varepsilon < (\hat{U} \circ p_\varepsilon)(y,\theta) \text{ for some } y \in X\}.$$

Requirement (ii) in definition 3.22 yields therefore,

$$0 \leqslant \alpha_\varepsilon((X \times \Theta) \setminus A_{\varepsilon,p_\varepsilon}) \leqslant \hat{\alpha}_\varepsilon((X \times (\Theta \sqcup X)) \setminus \hat{A}_{\varepsilon,p_\varepsilon}) = 0. \tag{3.18}$$

From which (iv) in the rubric follows.

Lastly, analogous arguments to the ones above yield

$$\overline{\eta}(X \times A) = \eta(A) \;\; \forall A \in \mathcal{B}(X). \tag{3.19}$$

On the other hand since $(\hat{U} \circ p_\varepsilon)|_{X\times X}(x,\theta)$ reaches its maximum when $x = \theta$, where it is infinite, it follows that

$$\overline{\eta}(\{(x,\theta) \in X \times X : x \neq \theta\}) = 0. \tag{3.20}$$

Thus $\text{supp}(\overline{\eta}) = \{(x,\theta) \in X \times \text{supp}(\eta) : \theta = x\}$. Equations (3.19) and (3.20) lead to

$$\overline{\eta}(x,x) = \overline{\eta}(X \times \{x\}) = \eta(x) \tag{3.21}$$

whenever $x \in \text{supp}(\eta)$. Property (v) is thus proven. ■

Proposition 3.29 has a converse;

**Proposition 3.30.** *Given an economy $(\mathfrak{E}, \hat{U}, \hat{c})$ with skeleton*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

*define the measure*

$$\overline{\eta}(x,\theta) = \begin{cases} \eta(x) \cdot \delta_x(\theta) & \textit{if} \;\; x \in \textit{supp}(\eta), \\ 0 & \textit{otherwise}. \end{cases} \tag{3.22}$$

*If for $p_\varepsilon \longmapsto [0,\infty)$ continuous; $\alpha_\varepsilon \in \Delta(X \times (\Theta \sqcup X))$ with $supp(\alpha_\varepsilon) \subset X \times \Theta$; and $\varepsilon \geqslant 0$ the following holds:*

*(i')* $\mathbb{E}_{\alpha_\varepsilon+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot p_\varepsilon) = \mathbb{E}_{\alpha_\varepsilon+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot c) \;\; \forall A \in \mathcal{B}(X)$,

*(ii')* $\alpha_\varepsilon(\{(x,\theta) \in X \times \Theta : (U \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (U \circ p_\varepsilon)(y,\theta) \; \forall y \in X\}) = 1.$

*(iii')* $\alpha_\varepsilon(X \times B) = \mu(B) \;\; \forall B \in \mathcal{B}(\Theta)$,

*then $\alpha_\varepsilon + \overline{\eta}$ is an $\varepsilon$-proto-equilibrium.*

*Proof.* Set $\hat{\alpha}_\varepsilon = \alpha_\varepsilon + \overline{\eta}$. Condition (i) in definition 3.22 now follows from the fact that $\hat{c}$ vanishes over $X \times X \supset \operatorname{supp}(\overline{\eta})$.

As in the proof of proposition 3.29, write

$$\hat{A}_{\varepsilon,p_\varepsilon} = \{(x,\theta) \in X \times (\Theta \sqcup X) : (\hat{U} \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (\hat{U} \circ p_\varepsilon)(y,\theta)\ \forall y \in X\}$$

and

$$A_{\varepsilon,p_\varepsilon} = \{(x,\theta) \in X \times \Theta : (U \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (U \circ p_\varepsilon)(y,\theta)\ \forall y \in X\}.$$

Observe that by definition,

$$\begin{aligned} \operatorname{supp}(\overline{\eta}) &\subset \{(x,\theta) \in X \times X : (\hat{U} \circ p_\varepsilon)(x,\theta) + \varepsilon \geqslant (\hat{U} \circ p_\varepsilon)(y,\theta)\ \forall y \in X\} \\ &\subset \hat{A}_{\varepsilon,p_\varepsilon} \cap (X \times X) \end{aligned} \tag{3.23}$$

Hence, by assumption (ii') in the rubric

$$\begin{aligned} \hat{\alpha}_\varepsilon[(X \times (\Theta \sqcup X)) \setminus \hat{A}_{\varepsilon,p_\varepsilon}] &= \alpha_\varepsilon[(X \times \Theta) \setminus A_{\varepsilon,p_\varepsilon}] \\ &\quad + \overline{\eta}[(X \times X) \cap ((X \times (\Theta \sqcup X)) \setminus \hat{A}_{\varepsilon,p_\varepsilon})] = 0. \end{aligned} \tag{3.24}$$

This yields condition (ii) in definition 3.22.

To establish condition (iii) in definition 3.22, let $S \in \mathcal{B}(\Theta \sqcup X)$ and write $B = S \cap \operatorname{supp}(\eta)$ then

$$\overline{\eta}(X \times S) = \overline{\eta}(\operatorname{supp}(\eta) \times S) = \sum_{x \in \operatorname{supp}(\eta)} \sum_{\theta \in B} \overline{\eta}(x,\theta) = \sum_{\theta \in B} \eta(\theta) = \eta(B). \tag{3.25}$$

Note that the sums are well-defined since $\eta$ (and hence $\overline{\eta}$) is non-zero only over a finite set of points.

Equation (3.25) along with condition (iii') implies for $S \in \mathcal{B}(\Theta \sqcup X)$

$$\hat{\alpha}_\varepsilon(X \times S) = \alpha_\varepsilon(X \times S) + \overline{\eta}(X \times S) = \mu(S) + \eta(B) = \mu(S) + \eta(S). \tag{3.26}$$

$\blacksquare$

*Remark* 3.31. Observe that given and economy $(\mathfrak{E}, \hat{U}, \hat{c})$ the measure $\overline{\eta}$ is determined only by the choice of skeleton. In other words, it is independent of $\varepsilon$ and the functions $\hat{U}$ and $\hat{c}$. Propositions; 3.29 and 3.30 imply that the problem of finding $\varepsilon-$proto-equilibria can be simplified to finding $\alpha_\varepsilon \in \Delta(X \times \Theta)$ and $p_\varepsilon$ satisfying assumptions (i')-(iii').

A simple characterisation of an equilibrium $(\hat{\alpha}, p)$ analogous to the one in propositions 3.29 and 3.30 is not possible. However, it follows from definition 3.26 that if $(\hat{\alpha}, p)$ is an equilibrium of some economy, then $\hat{\alpha}$ may be regarded as an element of $\Delta(X \times \Theta)$. More precisely,

**Proposition 3.32.** *Let $(\mathfrak{E}, \hat{U}, \hat{c})$ be an economy and $(p, \hat{\alpha})$ an equilibrium. Then supp$(\hat{\alpha}) \subset X \times \Theta$ and $\hat{\alpha}(X \times \Theta) = 1$.*

*Proof.* Let $\{\hat{\alpha_n}\}_{n=1}^{\infty}$ be the sequence satisfying condition (iii) in definition 3.26. Note that, in particular, $\hat{\alpha}_n \rightharpoonup \hat{\alpha}$.

Since $X \times X$ is open, $\chi_{X \times X}$ is a continuous function. Then

$$\hat{\alpha}(X \times X) = \lim_{n \to \infty} \mathbb{E}_{\overline{\eta}_n}(\chi_{X \times X}) \leqslant \lim_{n \to \infty} \|\overline{\eta}_n\| = 0. \tag{3.27}$$

Hence since $X \times X$ is also closed, $\text{supp}(\hat{\alpha}) \cap (X \times X) = \emptyset$. On the other hand,

$$\hat{\alpha}(X \times \Theta) = \lim_{n \to \infty} \mathbb{E}_{\alpha_n}(\chi_{X \times \Theta}) = 1. \tag{3.28}$$

$\blacksquare$

Recall that propositions 3.29 and 3.30 imply that any $\varepsilon$-proto equilibrium is of the form $(p_\varepsilon, \alpha_\varepsilon + \overline{\eta})$ where $\overline{\eta}$ only depends on the choice of skeleton. Hence a $\varepsilon$-proto equilibrium is described fully by the mapping $p_\varepsilon$ and the measure $\alpha_\varepsilon$. Consequently, we now speak of the $\varepsilon$-proto equilibrium $(p_\varepsilon, \alpha_\varepsilon)$. This convention (which is in agreement with [5]) is adopted now. Moreover, proposition 3.32 shows that if $(p, \alpha)$ is an equilibrium $\alpha$ can be treated as an element of $\Delta(X \times \Theta)$. These observations indicate that only measures whose support lie in $\Delta(X \times \Theta)$ are of interest. Accordingly, the minor abuse of notation $\alpha_\varepsilon$ (resp. $\alpha$) $\in \Delta(X \times \Theta)$ is adopted.

## 3.5 Two examples

This section consists of two important examples. More precisely, an example of an economy whose set of AG-equilibria is a proper subset of the set of equilibria is constructed (see propositions 3.34 and 3.37). A second example illustrating the fact that a sequence of economies that converges to a given economy (where convergence is defined in a suitable sense) does not guarantee convergence of AG-equilibria, is provided as well.

Despite most arguments in this section requiring, for the most part, only elementary tools, they are still treated as propositions to ease presentation.

To fix notation, let $U : (x, p, \theta) \mapsto x \cdot \theta - p$ and $c : (x, \theta) \mapsto x \cdot \theta$, where $x \in \{0, 1\}$ and $\theta \in \{0, 1\}$. Recall that, in addition, $p \in [0, \infty)$.

Now, let $E_\infty = (\mathfrak{S}, U, c)$ be the economy with skeleton

$$\mathfrak{S} = (\{0, 1\} \sqcup \{0, 1\}, 2^{\{0,1\} \sqcup \{0,1\}}, \mu),$$

where $\mu(\theta = 0) = \mu(\theta = 1) = \frac{1}{2}$. For the remaining of this section, functions $U$ and $c$, as well as the measure $\mu$, are assumed to take these specific forms.

*Remark* 3.33. In the interest of clarity, observe that although $X = \Theta = \{0, 1\}$ in $\mathfrak{S}$, this equality is only formal. In that regard, recall that since $X$ may be interpreted as contract characteristics, while $\Theta$ represents consumers' characteristics, both sets are logically distinct.

### 3.5.1 An equilibrium which is not an AG-equilibrium

For a given set $S$ denote the *identity mapping* by $\mathrm{id}_S$.

**Proposition 3.34.** *Let $p = id_{\{0,1\}}$ and $\alpha \in \Delta(\{0, 1\} \times \{0, 1\})$ be such that*

$$\begin{aligned} \alpha(x = 0, \theta = 0) &= \alpha(x = 1, \theta = 1) = \frac{1}{2} \text{ and} \\ \alpha(x = 0, \theta = 1) &= \alpha(x = 0, \theta = 1) = 0. \end{aligned}$$

*The pair $(p, \alpha)$ is the only AG-equilibrium of $E_\infty$.*

*Proof.* Fix a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive measures measures over $X = \{0,1\}$ satisfying $\eta_n(\{0,1\}) \to 0$. Furthermore, for each $n \in \mathbb{N}$, let $p_n : \{0,1\} \longmapsto [0,\infty)$ satisfy the following equations:

$$\begin{aligned} p_n(0) &= 0, & (3.29) \\ p_n(1) &= \frac{1}{1+2\eta_n(x=1)}. & (3.30) \end{aligned}$$

Observe that if $\alpha \in \Delta(\{0,1\} \times \{0,1\})$ is the measure in the rubric then,

$$p_n(1) = \frac{\mathbb{E}_\alpha(\chi_{\{1\}\times\Theta} \cdot c)}{\alpha(x=1,\theta=1)+\eta_n(x=1)} = \frac{\alpha(x=1,\theta=1)}{\alpha(x=1,\theta=1)+\eta_n(x=1)}. \tag{3.31}$$

A similar relation holds for $p_n(0)$. Furthermore, since $p_n(1) < 1$,

$$\mathrm{supp}(\alpha) = \{(x,\theta) \in \{0,1\}^2 : (U \circ p_n)(x,\theta) > (U \circ p_n)(y,\theta),\ \forall y \neq x\}.$$

The pair $(p_n, \alpha)$ is thus a proto-equilibrium of the economy $(\mathfrak{T}_n, U, c)$, where

$$\mathfrak{T}_n = (\{0,1\} \sqcup \{0,1\}, 2^{\{0,1\}\sqcup\{0,1\}}, \mu + \eta_n).$$

Since $p_n(1) \to 1$, $(\mathrm{id}_{\{0,1\}}, \alpha)$ is an AG-equilibrium of $E_\infty$. Furthermore, since the sequence $\{\eta_n\}_{n=1}^{\infty}$ was arbitrary, it is unique. ∎

*Remark* 3.35. As an aside, observe that even a simple economy such as $E_\infty$ can contain uncountably many proto-equilibria. For instance, the set of all pairs $\{(p_c, \alpha_c)\}_{c>1}$ with $p_c : \{0,1\} \longmapsto \{0,c\}$ satisfying $p(0) = 0$ and $p(1) = c$; while $\alpha \in \Delta(\{0,1\} \times \{0,1\})$ is given by $\alpha_c(x,\theta) = \delta_0(x)\mu(\theta)$ forms a family of proto-equilibria parametrised by $c$.

*Remark* 3.36. Heuristically, the argument in proposition 3.34 has a simple interpretation: a consumer represented by $\theta = 1$ is indifferent between purchasing contract $x = 1$ at price $p = 1$ or not doing so. Once behavioural types are included, the price of contract $x = 1$ is driven below 1, the consumer $\theta = 1$ strictly prefers to purchase the contract in any economy with behavioural agents. This choice is retained in the limit.

**Proposition 3.37.** *Let $\beta \in \Delta(\{0,1\}\times\{0,1\})$ be such that*

$$\begin{aligned}\beta(x=0,\theta=0) &= \frac{1}{2},\\ \beta(x=0,\theta=1)=\beta(x=1,\theta=1) &= \frac{1}{4},\\ \beta(x=1,\theta=0) &= 0.\end{aligned}$$

*The pair $(id_{\{0,1\}},\beta)$ is an equilibrium of $E_\infty$.*

*Proof.* Denote by $\{\eta_n\}_{n=1}^{\infty}$ the sequence of positive measures over $\{0,1\}$ satisfying $\eta_n(0)=\eta_n(1)=\frac{1}{8n-4}$ for $n\in\mathbb{N}$. Moreover, given $n\in\mathbb{N}$ let $p_n:\{0,1\}\longmapsto[0,\infty)$ satisfy the following relations

$$p_n(0) = 0, \tag{3.32}$$
$$p_n(1) = \frac{1}{1+4\eta_n(1)}. \tag{3.33}$$

Let $\beta$ be as in the rubric, then

$$p_n(1)=\frac{\mathbb{E}_\beta(\chi_{\{1\}\times\Theta}\cdot c)}{\beta(x=1,\theta=1)+\eta_n(x=1)}, \tag{3.34}$$

and similarly for $p_n(0)$. Furthermore, notice that since

$$1-p_n(1)=\frac{1}{2n}<\frac{1}{n}, \tag{3.35}$$

$$\beta\left(\left\{(x,\theta)\in\{0,1\}^2:(U\circ p_n)(x,\theta)+\frac{1}{n}>(U\circ p_n)(y,\theta)\ y\neq x\right\}\right)=1,$$

the pair $(p_n,\beta)$ defines a $\frac{1}{n}$-proto-equilibrium of the economy $(\mathfrak{T}_n,U,c)$, where

$$\mathfrak{T}_n=(\{0,1\}\sqcup\{0,1\},2^{\{0,1\}\sqcup\{0,1\}},\mu+\eta_n).$$

Since $p_n(1)\to 1$, the pair $(\mathrm{id}_{\{0,1\}},\beta)$ is an equilibrium of $E_\infty$. ■

*Remark* 3.38. Observe that in the proof of proposition 3.37, if $n>1$ then, in fact,

$$\left\{(x,\theta)\in\{0,1\}^2:(U\circ p_n)(x,\theta)+\frac{1}{n}>(U\circ p_n)(y,\theta)\ y\neq x\right\}=\mathrm{supp}(\beta).$$

*Remark* 3.39. The argument in proposition 3.37 can be understood intuitively as follows: if the mass of behavioural consumers is not too large then the price of contract $x = 1$ is close to 1. In that case, the set of "$\varepsilon$-optimal" choices might include, in addition to purchasing $x = 1$, purchasing $x = 0$ as well. It is then possible for the consumer to randomise across both purchase options. In the limit, this randomisation is retained. Note in that regard that, while both choices are given the same probability in the proof of proposition 3.37, it is possible to construct examples were the probability of purchasing $x = 1$ takes any value in $[0, 1]$.

### 3.5.2 A sequence of AG-equilibria which does not converge to an AG-equilibrium

Propositions 3.34 and 3.37 together provide an example of an economy where the set of AG-equilibria is a strict subset of the set of equilibria. Now, a sequence of $\{(p_N, \beta_N)\}_{N=1}^{\infty}$ AG-equilibria that converges $(p, \beta)$, which is not an AG-equilibrium, but it is an equilibrium in the sense of this chapter is constructed. This follows from the results in the previous subsection, along with proposition 3.41 below. To construct the aforementioned sequence, given $N \in \mathbb{N}$ let $E_N$ be the economy $(\mathfrak{S}_N, U, c)$, where

$$\mathfrak{S}_N = \left(\{0, 1\} \sqcup \left\{0, \frac{1}{N}, 1\right\}, 2^{\{0,1\} \sqcup \left\{0, \frac{1}{N}, 1\right\}}, \mu\right).$$

Note that $U$, $c$ and $\mu$ are as before.

*Remark* 3.40. Observe that in the following construction, as $N$ changes the domain of $p_N$, $\left\{0, \frac{1}{N}, 1\right\}$, changes as well. Nonetheless, the domain itself converges to $\{0, 1\}$ in the sense of Hausdorff. The notion of converge of equilibrium adopted reflects this (see remark 3.44 at the end of this subsection).

**Proposition 3.41.** *Given $N \in \mathbb{N}$, let $\beta_N \in \Delta(\left\{0, \frac{1}{N}, 1\right\} \times \{0, 1\})$ satisfy the* *following relations:*

$$\begin{aligned}
\beta_N(x=0,\theta=0) &= \frac{1}{2};\\
\beta_N(x=1,\theta=1)=\beta_N\left(x=\frac{1}{N},\theta=1\right) &= \frac{1}{4};\\
\beta_N(x=0,\theta=1)=\beta_N\left(x=\frac{1}{N},\theta=0\right)=\beta_N(x=1,\theta=0) &= 0.
\end{aligned}$$

*The pair $(id_{\{0,\frac{1}{N},1\}},\beta_N)$ is then an AG-equilibrium of $E_N$.*

*Remark* 3.42. The main idea behind the proof of proposition 3.41 is to ensure that, once behavioural consumers are introduced, consumer $\theta=1$ is indifferent between purchasing contracts $x=1$ and $x=\frac{1}{N}$. Formally, this implies

$$1-p_n(1)=\frac{1}{N}-p_n\left(\frac{1}{N}\right), \tag{3.36}$$

where $p_n$ is the putative proto-equilibrium price. Moreover, to qualify as a proto-equilibrium, $p_n$ must satisfy

$$p_n(x)=\frac{x\cdot\alpha_n(x,\theta=1)}{\alpha_n(x,\theta=1)+\eta_n(x)},$$

where $\alpha_n$ is a probability supported over each consumer's optimal choices. Equation (3.37) becomes accordingly

$$1-\frac{\alpha_n(x=1,\theta=1)}{\alpha_n(x=1,\theta=1)+\eta_n(1)}=\frac{1}{N}-\frac{\alpha_n(x=\frac{1}{N},\theta=1)}{N\alpha_n(x=\frac{1}{N},\theta=1)+N\eta_n(\frac{1}{N})}. \tag{3.37}$$

Since consumer $\theta=1$ is to be indifferent between purchasing $x=1$ and $x=\frac{1}{N}$; it can that be assumed he purchases one of the two contracts randomly. Furthermore, consumer $\theta=0$ must acquire $x=0$ in a proto-equilibrium, the probability that $\theta=1$ chooses $x=1$ (resp. $x=\frac{1}{N}$) is $\alpha_n(x=1,\theta=1)$ (resp. $\alpha_n(x=\frac{1}{N},\theta=1)$)). In addition, these probabilities must satisfy

$$\alpha_n(x=1,\theta=1)+\alpha_n\left(x=\frac{1}{N},\theta=1\right)=\mu(\theta=1)=\frac{1}{2}. \tag{3.38}$$

Equations (3.37) and (3.38) define a system of two linear equations for the probabilities $\alpha_n(x=1,\theta=1)$ and $\alpha_n\left(x=\frac{1}{N},\theta=1\right)$. The unique solution of this system of equations is

$$\begin{aligned}
2\alpha_n(x=1,\theta=1) &= \frac{N\eta_n(1)+2(N-1)\eta_n(1)\eta_n(\frac{1}{N})}{N\eta_n(1)+\eta_n(\frac{1}{N})}; \\
2\alpha_n\left(x=\frac{1}{N},\theta=1\right) &= \frac{\eta_n(\frac{1}{N})-2(N-1)\eta_n(1)\eta_n(\frac{1}{N})}{N\eta_n(1)+\eta_n(\frac{1}{N})}.
\end{aligned}$$

To verify these relations, note that, upon multiplication by $N$ and rearrangement, equation (3.37) can be rewritten as

$$\frac{N\eta_n(1)}{\alpha_n(x=1,\theta=1)+\eta_n(1)} = \frac{\eta_n(\frac{1}{N})}{\alpha_n(x=\frac{1}{N},\theta=1)+\eta_n(\frac{1}{N})}. \tag{3.39}$$

A further rearrangement yields the equation

$$\begin{aligned}
\eta_n\left(\frac{1}{N}\right)\alpha_n(x=1,\theta=1) - N\eta_n(1)\alpha_n\left(x=\frac{1}{N},\theta=1\right) \\
= (N-1)\eta_n(1)\eta_n\left(\frac{1}{N}\right).
\end{aligned} \tag{3.40}$$

Substituting equation (3.38) into equation (3.40) yields the desired relations.

*Proof of proposition 3.41.* Proceeding as in the previous propositions, let $\{\eta_n\}_{n=N}^{\infty}$ be the sequence of positive measures given by:

$$\begin{aligned}
\eta_n(0)=\eta_n(1) &= \frac{1}{2n}, \\
\eta_n\left(\frac{1}{N}\right) &= \frac{N}{2n}.
\end{aligned}$$

Now, fix $n \in \mathbb{N}$ and let $\alpha_n \in \Delta\left(\left\{0,\frac{1}{N},1\right\}\times\{0,1\}\right)$ with

$$\operatorname{supp}(\alpha_n) = \left\{(0,0),\ (1,1),\ \left(\frac{1}{N},1\right)\right\}$$

be such that

$$\begin{aligned}
\alpha_n(x=0,\theta=0) &= \frac{1}{2};\\
2\alpha_n(x=1,\theta=1) &= \frac{N\eta_n(1)+2(N-1)\eta_n(1)\eta_n(\frac{1}{N})}{N\eta_n(1)+\eta_n(\frac{1}{N})};\\
2\alpha_n\left(x=\frac{1}{N},\theta=1\right) &= \frac{\eta_n(\frac{1}{N})-2(N-1)\eta_n(1)\eta_n(\frac{1}{N})}{N\eta_n(1)+\eta_n(\frac{1}{N})}.
\end{aligned}$$

Observe that since $n \geqslant N$, $\frac{N-1}{n} < 1$ holds. This latter inequality implies

$$\eta_n\left(\frac{1}{N}\right) - 2(N-1)\eta_n(1)\eta_n\left(\frac{1}{N}\right) > 0.$$

Thus $\alpha_n(x,\theta=1) \in (0,1)$ for $x \in \{\frac{1}{N},1\}$.

Lastly, let $p_n : \{0,\frac{1}{N},1\} \longmapsto [0,\infty)$ satisfy the relations

$$\begin{aligned}
p_n(0) &= 0,\\
N\cdot p_n\left(\frac{1}{N}\right) &= \frac{\alpha_n(x=\frac{1}{N},\theta=1)}{\alpha_n(x=\frac{1}{N},\theta=1)+\eta_n(\frac{1}{N})},\\
p_n(1) &= \frac{\alpha_n(x=1,\theta=1)}{\alpha_n(x=1,\theta=1)+\eta_n(1)}.
\end{aligned}$$

Observe that since $\frac{1}{N} - p_n(\frac{1}{N}) = 1 - p_n(1) > 0$ and both $p_n(1)$ and $p_n(\frac{1}{N})$ are positive, $\operatorname{supp}(\alpha_n)$ is the set

$$\left\{(x,\theta) : (U\circ p)(x,\theta) \geqslant (U\circ p)(y,\theta)\ \forall y \in \left\{0,\frac{1}{N},1\right\}\right\}.$$

The pair $(p_n,\alpha_n)$ is thus a proto-equilibrium of the economy $(\mathfrak{T}_n,U,c)$, where

$$\mathfrak{T}_n = \left(\{0,1\}\sqcup\left\{0,\frac{1}{N},1\right\}, 2^{\{0,1\}\sqcup\{0,\frac{1}{N},1\}}, \mu+\eta_n\right).$$

Moreover, $\alpha_n \rightharpoonup \beta_N$ while $\max\limits_{x\in\{0,\frac{1}{N},1\}} |x-p_n(x)| \to 0$ as $n\to\infty$. ■

**Corollary 3.42.1.** *The sequence* $\{\beta_N\}_{N=1}^{\infty}$, *where*

$$\beta_N \in \Delta\left(\left\{0,\frac{1}{N},1\right\}\times\{0,1\}\right)$$

*is as in proposition 3.37; converges weakly to $\beta \in \Delta(\{0,1\}^2)$, with $\beta$ satisfying:*

$$\begin{aligned} \beta(x=0,\theta=0) &= \frac{1}{2}, \\ \beta(x=0,\theta=1) = \beta(x=1,\theta=1) &= \frac{1}{4}, \\ \beta(x=1,\theta=0) &= 0. \end{aligned}$$

*Remark* 3.43. Note that, by several abuses of notation, the measures in corollary 3.42.1 are all implicitly assumed to be Borel measures on the space

$$\left(\bigcup_{N=1}^{\infty} \left\{0, \frac{1}{N}, 1\right\} \times \{0,1\}\right) \cup \{0,1\}^2.$$

Weak convergence is therefore well-defined.

*Remark* 3.44. Consider now the sequence of AG-equilibria $\{(p_N, \beta_N)\}_{N=1}^{\infty}$ where $p_N = \mathrm{id}_{\{0,\frac{1}{N},1\}}$ and

$$\beta_N \in \Delta\left(\left\{0, \frac{1}{N}, 1\right\} \times \{0,1\}\right)$$

is as in proposition 3.41. Observe that given $x \in \{0,1\}$ and a sequence $\{x_N\}_{N=1}^{\infty}$ with $x_N \in \{0, \frac{1}{N}, 1\}$ satisfying $x_N \to x$, $p_N(x_N) \to x$ holds. Moreover, corollary 3.42.1 establishes that $\beta_N \rightharpoonup \beta$, where $\beta$ is the measure in proposition 3.37 (importantly, recall at this stage that proposition 3.37 implies $(\mathrm{id}_{\{0,1\}}, \beta)$ is not an AG-equilibrium). Since these two conditions were used to characterised the notion of equilibrium (see definition 3.26), it might be said by analogy that the sequence $\{(p_N, \beta_N)\}_{N=1}^{\infty}$ of AG-equilibria converges to the equilibrium $(\mathrm{id}_{\{0,1\}}, \beta)$. In this sense, proposition 3.37 implies that a sequence of AG-equilibria might not converge to a sequence of AG-equilibria, but it might do so to an equilibrium instead. This conclusion remains valid if the notion of convergence of $p_N$ is replaced by $\lim_{N\to\infty} \max_{x\in\{0,1\}} |\mathrm{id}_{\{0,1\}} - p_N(x)|$ instead.

## 3.6 Existence of $\varepsilon$-proto-equilibria

Under the assumption that $|X| < \infty$ it is shown that a given economy has $\varepsilon$-proto-equilibria for arbitrary values of $\varepsilon$. More formally,

**Theorem 3.45.** *Let $\varepsilon \geqslant 0$ and suppose that*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

*is a skeleton with $X$ finite and $\text{supp}(\eta) = X$. Then for any continuous functions $U : X \times [0, \infty) \times \Theta \longmapsto \mathbb{R}$ and $c : X \times \Theta \longmapsto [0, \infty)$ the economy $(\mathfrak{E}, \hat{U}, \hat{c})$ has an $\varepsilon$-proto-equilibrium.*

The proof of this result uses a standard fixed point argument. Recall that finding an $\varepsilon$-proto-equilibrium is equivalent to finding a pair $(p, \alpha)$ (to avoid overburden the notation reference to $\varepsilon$ is dropped in this section) with $p$ continuous and $\alpha \in \Delta(X \times \Theta)$ satisfying:

(i') $\mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot p) = \mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot c) \;\; \forall A \in \mathcal{B}(X)$

(ii') $\alpha(\{(x, \theta) \in X \times \Theta : (U \circ p)(x, \theta) + \varepsilon \geqslant (U \circ p)(y, \theta) \; \forall y \in X\}) = 1.$

(iii') $\alpha(X \times B) = \mu(B) \;\; \forall B \in \mathcal{B}(\Theta),$

In these relations $\overline{\eta}$ is as in equation (3.22).

Observe that conditions (i')-(iii') are given in terms of the functions $U$ and $c$ instead of their extensions to the full set. Indeed such extensions shall no longer play any significant role and it is thus convenient to denote by $U$ (resp. $c$) both the original function and its extension.

Conditions (i')-(iii') can be simplified further. Note that because $\text{supp}(\eta) = X$, which is finite, condition (i') yields for $x_0 \in X$,

$$p(x_0) = \frac{\iint_{\{x_0\}\times\Theta} c(x, \theta) \; \alpha(dx, d\theta)}{\eta(x_0) + \alpha(\{x_0\} \times \Theta)}, \tag{3.41}$$

which may be re-written somewhat less formally as the *conditional expectation*

$$p(x_0) = \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi, \cdot)|\xi = x_0). \tag{3.42}$$

Naturally, starting from equation (3.41) condition (i') in proposition 3.30 can be retrieved after algebraic manipulation.

*Remark* 3.46. Let $x \in X$, simple algebraic manipulations yield the usual factoring and tower properties:

$$\begin{aligned} \mathbb{E}_{\alpha+\overline{\eta}}(\psi \cdot c(\xi,\cdot)|\xi = x) &= \psi(x) \cdot \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi,\cdot)|\xi = x), \qquad (3.43) \\ \mathbb{E}_{\alpha+\overline{\eta}}(\,\mathbb{E}_{\alpha+\overline{\eta}}(c(\xi,\cdot)|\xi = \cdot)\,) &= \mathbb{E}_{\alpha+\overline{\eta}}(c). \qquad (3.44) \end{aligned}$$

Here $\psi : X \longmapsto \mathbb{R}$ is a continuous mapping.

Now, the finiteness of $X$ implies that any function $f : X \longmapsto \mathbb{R}$ is continuous and can be associated with an element $\mathbb{R}^{|X|}$. This association is used now. Given a pair $(p, \alpha) \in \mathbb{R}^{|X|} \times \Delta(X \times \Theta)$ and $\varepsilon \geqslant 0$ let

$$A_{\varepsilon,p} = \{(x, \theta) \in X \times \Theta : (U \circ p)(x, \theta) + \varepsilon \geqslant (U \circ p)(y, \theta)\ \forall y \in X\}. \qquad (3.45)$$

Note that this generalises the notation introduced in the proof of proposition 3.29 since it does not assume that $(p, \alpha)$ is an $\varepsilon$-proto-equilibrium. Condition (ii') can therefore be written as $\alpha(A_{\varepsilon,p}) = 1$. Moreover, observe that (iii') means that the marginal $(\pi_\Theta)_*(\alpha)$ coincides with $\mu$. Consequently, given $\varepsilon \geqslant 0$ a pair $(p, \alpha) \in \mathbb{R}^{|X|} \times \Delta(X \times \Theta)$ is an $\varepsilon$-proto-equilibrium if, and only if, it satisfies $(\pi_\Theta)_*(\alpha) = \mu$ and

$$\begin{aligned} p(x) &= \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi,\cdot)|\xi = x), \qquad (3.46) \\ \alpha(A_{\varepsilon,p}) &= 1. \qquad (3.47) \end{aligned}$$

Let $P = [0, K]^{|X|}$, where $\max_{(x,\theta)\in X\times\Theta} c(x, \theta) = K$. Observe that since $c$ is continuous, $P$ is well-defined. In addition, let

$$\Lambda = \{\alpha \in \Delta(X \times \Theta) : (\pi_\Theta)_*(\alpha) = \mu\}.$$

By taking equations (3.46) and (3.47) as reference point, define the multi-valued mapping $F : P \times \Lambda \longmapsto 2^{P\times\Lambda}$ by

$$F : (p, \alpha) \mapsto F_1(\alpha) \times F_2(p)$$

where

$$
\begin{aligned}
F_1(\alpha) &= \{p \in P : p(x) = \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi,\cdot)|\xi = x)\ \forall x \in X\}, && (3.48)\\
F_2(p) &= \{\alpha \in \Lambda : \alpha(A_{\varepsilon,p}) = 1\}. && (3.49)
\end{aligned}
$$

Note that in fact $F_1$ is single-valued and thus is a mapping in the usual sense.

Existence of a $\varepsilon$-proto-equilibrium amounts to finding a pair $(p,\alpha) \in P \times \Lambda$ such that $(p,\alpha) \in F(p,\alpha)$. Existence of this pair is a consequence of the well-known

**Theorem 3.47** (Kakutani-Glicksberg-Fan)**.** *Let $V$ be a locally convex Hausdorff topological vector space. Given $S \subset V$ non-empty, convex and compact and an upper hemi-continuous multi-valued mapping $\Phi : S \longmapsto 2^S$ with the property that for every $s \in S$, $\Phi(s)$ is a non-empty, convex and compact . Then there exists $s_0 \in S$ such that $s_0 \in \Phi(s_0)$.*

For a proof of this result see [2].

*Remark* 3.48. Here $V$ in 3.47 corresponds to $\mathbb{R}^{|X|} \times \mathfrak{F}(X \times (\Theta \sqcup X))$ which by proposition 3.2 satisfies the assumptions of the Kakutani-Glicksberg-Fan theorem.

For our purposes, a multi-valued mapping $\Phi : 2^S \to 2^T$ is *upper hemi-continuous* if its *Graph*

$$\mathrm{Gr}(\Phi) = \{(s,t) \in S \times T : t \in \Phi(s)\}$$

is closed. This is motivated by the following

**Proposition 3.49.** *The multi-valued mapping $\Phi : 2^S \to 2^T$, where $T$ is a compact Hausdorff space, is upper hemi-continuous if, and only if, Gr is closed.*

A proof is given in [30] propositions 5.2 and 5.3. A more precise definition of upper hemi-continuity is omitted. The interested reader is referred to [23] for a detailed description of this and other matters of interest.

*Remark* 3.50. Observe that if $\Phi : S \longmapsto T$ in 3.49 is single-valued then hemi-continuity is replaced by continuity.

The proof that $F$ satisfies the assumptions of theorem 3.47 uses the same strategy as in [5] (see lemma 1 located in the appendix). The properties of each mapping $F_1$ and $F_2$ are studied separately.

**Lemma 3.51.** *Let $F_1 : \Lambda \longmapsto P$ be the mapping* (3.48)*,*

$$F_1 : \alpha \mapsto \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi, \cdot)|\xi = \cdot)$$

*Then $Gr(F_1)$ is a closed set. Moreover, for every $\alpha \in \Lambda$, $F_1(\alpha)$ is a non-empty convex set.*

The proof of this result is straightforward and is given in [5] (see step 1 in the proof of lemma 1). A slightly more formal version of their argument is provided below:

*Proof of lemma 3.51.* The proof that $F_1(\alpha)$ is non-empty and convex follows directly from definition.

To show that $\mathrm{Gr}(F_1)$ is closed let $\{(\alpha_n, p_n)\}_{n=1}^{\infty} \subset \mathrm{Gr}(F_1)$ converge to $(\alpha, p)$. It is to be shown that for every $A \subset X$

$$\mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot p) = \mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot c). \tag{3.50}$$

In order to do so, fix such a set $A$ and let $\delta > 0$ be given. Now,

$$\begin{aligned}
|\mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot p) - \mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot c)| \leqslant& \\
|\mathbb{E}_{\alpha}(\chi_{A\times\Theta} \cdot p) - \mathbb{E}_{\alpha_n}(\chi_{A\times\Theta} \cdot p_n)|+& \\
|\mathbb{E}_{\alpha_n+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot p_n) - \mathbb{E}_{\alpha_n+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)} \cdot c)|+& \\
|\mathbb{E}_{\alpha_n}(\chi_{A\times\Theta} \cdot c) - \mathbb{E}_{\alpha}(\chi_{A\times\Theta} \cdot c)|+& \\
\|\eta\| \cdot \|p - p_n\|_{\infty}.&
\end{aligned} \tag{3.51}$$

Note that in this equation we have made use of $\mathrm{supp}(\alpha_n) \subset X \times \Theta$ for every $n \in \mathbb{N}$ and the fact that $\mathbb{E}_{\alpha+\overline{\eta}} = \mathbb{E}_{\alpha} + \mathbb{E}_{\overline{\eta}}$. In addition, the following facts are employed:

$$
\begin{aligned}
\mathbb{E}_{\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot c) &= 0, && (3.52)\\
\mathbb{E}_{\overline{\eta}}\Big(\chi_{A\times(\Theta\sqcup X)}(p-p_n)\Big) &= \sum_{x\in X}\Big(p(x)-p_n(x)\Big)\cdot\eta(x). && (3.53)
\end{aligned}
$$

Recall $\chi_{A\times(\Theta\sqcup X)}$ and $\chi_{A\times\Theta}$ are both continuous since $X$ is finite. Hence given $\delta>0$, $\alpha_n \rightharpoonup \alpha$ implies there exists $n_1\in\mathbb{N}$ such that

$$
\begin{aligned}
|\mathbb{E}_{\alpha_n}(\chi_{A\times(\Theta\sqcup X)}\cdot p_n)-\mathbb{E}_{\alpha}(\chi_{A\times(\Theta\sqcup X)}\cdot p)| &< \frac{\delta}{3}\ \forall n>n_1, && (3.54)\\
|\mathbb{E}_{\alpha_n}(\chi_{A\times(\Theta\sqcup X)}\cdot c)-\mathbb{E}_{\alpha}(\chi_{A\times(\Theta\sqcup X)}\cdot c)| &< \frac{\delta}{3}\ \forall n>n_1. && (3.55)
\end{aligned}
$$

The fact that $\{(\alpha_n,p_n)\}_{n=1}^{\infty}\subset \mathrm{Gr}(F_1)$ implies that for all $n\in\mathbb{N}$

$$
\mathbb{E}_{\alpha_n+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot p_n)=\mathbb{E}_{\alpha_n+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot c). \qquad (3.56)
$$

Finally,

$$
|\mathbb{E}_{\alpha_n}(\chi_{A\times\Theta}\cdot p)-\mathbb{E}_{\alpha_n}(\chi_{A\times\Theta}\cdot p_n)|\leqslant \|p-p_n\|_\infty. \qquad (3.57)
$$

Since $p_n\to p$ there exists $n_2\in\mathbb{N}$ such that $\|p-p_n\|_\infty<\frac{\delta}{3(1+\|\eta\|)}$ whenever $n>n_2$.

Let $n=\max\{n_1,n_2\}+1$ the above observations imply that the left-hand side of equation (3.51) satisfies the bound

$$
|\mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot p)-\mathbb{E}_{\alpha+\overline{\eta}}(\chi_{A\times(\Theta\sqcup X)}\cdot c)|<\delta. \qquad (3.58)
$$

The fact that $\delta$ was arbitrary implies the result. ■

The analogue of lemma 3.51 for $F_2$ is slightly more complex. Moreover, the proof differs from the analogous one (see step 2 in the proof of lemma 1) in [5].

**Lemma 3.52.** *The mapping*

$$
F_2: p\mapsto\{\alpha\in\Lambda:\alpha(A_{\varepsilon,p})=1\},
$$

*has a closed graph.*

*Proof.* Let $\{(p_n, \alpha_n)\}_{n=1}^{\infty} \subset \mathrm{Gr}(F_2)$ converge to $(p, \alpha)$. Note that by corollary 3.10.1, $\alpha \in \Lambda$. To show that $\alpha(A_{\varepsilon,p}) = 1$ the following chain of inclusions is established:

$$\mathrm{supp}(\alpha) \subset \mathrm{L.i.}(\mathrm{supp}(\alpha_n)) \subset \mathrm{L.i.}(A_{\varepsilon,p_n}) \subset A_{\varepsilon,p}. \tag{3.59}$$

Recall that L.i. is the Kuratowski lower limit. The leftmost inclusion follows from proposition 3.9. Now, note that $A_{\varepsilon,p_n}$ is a closed set, (ii) in theorem 3.5 implies that $\mathrm{supp}(\alpha_n) \subset A_{\varepsilon,p_n}$. This yields the middle inclusion (see remark 3.8). All that remains to prove is the rightmost inclusion. To that end, let $(x, \theta) \in \mathrm{L.i.}(A_{\varepsilon,p_n})$, there exists $N \in \mathbb{N}$ and a sequence $\{y_n, \phi_n\}_{n \geqslant N}$ with $(y_n, \phi_n) \in A_{\varepsilon,p_n}$ converging to $(x, \theta)$. Fix $y \in X$, since $(y_n, \phi_n) \in A_{\varepsilon,p_n}$,

$$U(y_n, p_n(y_n), \phi_n) + \varepsilon \geqslant U(y, p_n(y), \phi_n). \tag{3.60}$$

Taking the limit $n \to \infty$ on both sides of the inequality yields

$$U(x, p(x)), \theta) + \varepsilon \geqslant U(y, p(y), \theta), \tag{3.61}$$

since $p_n \to p$ and $U$ is continuous. Hence $(x, \theta) \in A_{\varepsilon,p}$ as required.

Now, given $\mathrm{supp}(\alpha) \subset A_{\varepsilon,p}$, (i) in theorem 3.5 implies $\alpha(A_{\varepsilon,p}) = 1$. ∎

It is now proven that $\mathrm{Gr}(F_2)$ is a non-empty convex set. Convexity being elementary. The non-emptyness is proven constructively.

**Lemma 3.53.** *Let $\Theta$ be a compact Polish spaces, $X$ a finite set and $f : X \times \Theta \longmapsto \mathbb{R}$ be a continuous mapping. Given $\varepsilon \geqslant 0$ there exists $\psi : \theta \longmapsto X$ Borel measurable with the property that given $\theta \in \Theta$, $f(\psi(\theta), \theta) + \varepsilon \geqslant f(y, \theta)$ for every $y \in X$.*

*Proof.* Write $X = \{x_1, \ldots, x_n\}$ the argument proceeds by induction on $n$. The case $n = 1$ is immediate. Suppose that the lemma held for some $n = k \geqslant 1$ and consider the case $k+1$. The induction hypothesis implies the existence of a Borel mapping $\varphi : \theta \longmapsto \{x_1, \ldots, x_k\}$ such that $f(\varphi(\theta), \theta) + \varepsilon \geqslant f(y, \theta)$ for any $y \in \{x_1, \ldots, x_k\}$.

Let

$$K = \{\theta \in \Theta : f(\varphi(\theta), \theta) + \varepsilon < f(x_{k+1}, \theta)\}.$$

Since both $f$ and $\varphi$ are Borel, $K \in \mathcal{B}(\Theta)$. Define

$$\psi : \theta \mapsto \begin{cases} \varphi(\theta) & \text{if } \theta \notin K, \\ x_{k+1} & \text{if } \theta \in K. \end{cases} \tag{3.62}$$

$\psi$ is desired Borel mapping. ■

*Remark* 3.54. Lemma 3.53 in fact follows from more general results (see e.g. theorem 18.19, the "measurable maximum theorem", in [2]). Seeing that the above proof is fairly elementary we have decided to include it.

**Lemma 3.55.** *Let $F_2$ be the mapping defined in equation* (3.49)*. For any $p \in P$ the set $F_2(p)$ is non-empty and convex.*

*Proof.* As mentioned, convexity follows directly from the definition of $F_2$. To prove non-emptiness, fix $p \in P$ and let $\psi$ be a mapping as in lemma 3.53 (i.e. for every $\theta \in \Theta$ $(U \circ p)(\psi(\theta), \theta) + \varepsilon \geqslant (U \circ p)(y, \theta)$ for arbitrary $y \in X$). By construction, $\mathrm{Gr}(\psi) \subset A_{\varepsilon,p}$.

Now, let $\beta \in \Delta(X \times \Theta)$ be defined by extending

$$\beta : A \times B \mapsto \mu(B \cap \psi^{-1}(A)), \tag{3.63}$$

where $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(\Theta)$, via the Hahn-Kolmogorov theorem. Observe that

$$\beta(X \times B) = \mu(B \cap \Theta) = \mu(B). \tag{3.64}$$

Now, let $(y, \phi) \notin A_{\varepsilon,p}$. There exists $m \in \mathbb{N}$ such that

$$\left(B_{\frac{1}{m}}(y) \times B_{\frac{1}{m}}(\phi)\right) \cap A_{\varepsilon,p} = \emptyset.$$

If there existed $\vartheta \in B_{\frac{1}{m}}(\phi) \cap \psi^{-1}\left(B_{\frac{1}{m}}(y)\right)$; then $\psi(\vartheta) \in B_{\frac{1}{m}}(y)$ and, as a result,

$$\emptyset \neq \mathrm{Gr}(\psi) \cap \left(B_{\frac{1}{m}}(y) \times B_{\frac{1}{m}}(\phi)\right) \subset \left(B_{\frac{1}{m}}(y) \times B_{\frac{1}{m}}(\phi)\right) \cap A_{\varepsilon,p},$$

a contradiction. It follows that if $(y,\phi) \notin A_{\varepsilon,p}$ then $(y,\phi) \notin \mathrm{supp}(\beta)$. Hence, $\mathrm{supp}(\beta) \subset A_{\varepsilon,p}$. Since $A_{\varepsilon,p}$ is closed, theorem 3.5 concludes the proof. ■

*Remark* 3.56. Lemma 3.55 and its proof are omitted in Azevedo and Gottlieb. Seeing they contain some subtleties, and in the interest of completeness, they are included.

An equivalent version of theorem 3.45 is now proven.

**Theorem 3.57.** *The multivalued mapping $F : P \times \Lambda \longmapsto 2^{P\times\Lambda}$ is the multivalued mapping defined by*

$$F : (p,\alpha) \mapsto F_1(\alpha) \times F_2(p)$$

*where*

$$\begin{aligned} F_1(\alpha) &= \{p \in P : p(x) = \mathbb{E}_{\alpha+\overline{\eta}}(c(\xi,\cdot)|\xi = x)\ \forall x \in X\}, &(3.65)\\ F_2(p) &= \{\alpha \in \Lambda : \alpha(A_{\varepsilon,p}) = 1\}. &(3.66)\end{aligned}$$

*has a fixed point.*

*Proof.* Observe that by applying a permutation of the coordinates it follows that $\mathrm{Gr}(F)$ is homeomorphic to $\mathrm{Gr}(F_1) \times \mathrm{Gr}(F_2)$. Lemmas 3.51, 3.52 and 3.55 imply at once that $\mathrm{Gr}(F)$ is closed and $F(p,\alpha)$ is non-empty for every pair $(p,\alpha) \in P \times \Lambda$. Moreover, the same algebraic manipulations that yield the convexity of $F_1(\alpha)$ and $F_2(p)$ yield the convexity of $F(p,\alpha)$.

Since $P \times \Lambda$ is a non-empty, convex and compact set theorem 3.47 implies that there exists $(p,\alpha) \in P \times \Lambda$ such that $(p,\alpha) \in F(p,\alpha)$. ■

## 3.7 Equilibrium existence for regular economies

### 3.7.1 Regular functions

Hitherto no assumptions beyond continuity have been made about the function $U : X \times [0,\infty) \times \Theta \longmapsto \mathbb{R}$. We now impose some natural constraints on the behaviour of this function. Under these constraints it is proven that any economy $(\mathfrak{E}, U, c)$ where

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

has an equilibrium. Incidentally, this further implies that the set of proto-equilibria of the economy is non-empty (see theorem 3.74 below).

Before stating the main result in this section the additional assumptions on $U$ are introduced. Some direct consequences of these assumptions are also discussed.

**Definition 3.58** (Regular function)**.** A continuous function $U : X \times [0, \infty) \times \Theta \longmapsto \mathbb{R}$ is said to be *regular* if there exists constants $C_1, C_2, C_3 > 0$ such that:

(i) $U(x, \cdot, \theta)$ is decreasing for every pair $(x, \theta) \in X \times \Theta$;

(ii) $\sup\limits_{p,\theta} |U(x, p, \theta) - U(y, p, \theta)| \leqslant C_1 \cdot d_X(x, y)$;

(iii) $C_2 \leqslant \inf\limits_{x,\theta} \left| \frac{U(x,p,\theta) - U(x,q,\theta)}{p-q} \right| \leqslant \sup\limits_{x,\theta} \left| \frac{U(x,p,\theta) - U(x,q,\theta)}{p-q} \right| \leqslant C_3$.

Let

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

be a skeleton. An economy $(\mathfrak{E}, U, c)$ with $U$ regular is said to be *regular* itself. If in addition, $X =$supp$(\eta)$ (so in particular $|X| < \infty$) then the economy is *tame.*

*Remark* 3.59. Requirement (ii) in definition 3.58 is in fact stronger than needed. Indeed, all that it is required is that given $c : X \times \Theta \longmapsto [0, \infty)$ continuous

$$\sup_{p \in c(X \times \Theta)} \sup_{\theta \in \Theta} |U(x, p, \theta) - U(y, p, \theta)| \leqslant C_1 \cdot d_X(x, y). \tag{3.67}$$

Proofs in this section remain unchanged if this assumption is made instead.

*Example(s)* 3.2. Given $u : X \times \Theta \longmapsto \mathbb{R}$ satisfying

$$\sup_{\theta \in \Theta} |u(x, \theta) - u(y, \theta)| \leqslant C \cdot d_X(x, y),$$

the function $U : (x, p, \theta) \mapsto u(x, \theta) - p$ is regular. More generally, given $\Theta \subset \mathbb{R}^{d_1}$ and $X \subset \mathbb{R}^{d_2}$ (both $X$ and $\Theta$ are assumed to have a non-empty interiors); a differentiable function $U$ is regular if, and only if, it satisfies

$$\begin{aligned} \sup_{\theta\in\Theta}\sup_{p\geqslant 0}\left|\frac{\partial U}{\partial x}\right| &< \infty, \\ C_1 \leqslant \inf_{x\in X}\inf_{\theta\in\Theta}\frac{\partial U}{\partial p} \leqslant \sup_{\theta\in\Theta}\sup_{x\in X}\frac{\partial U}{\partial p} &\leqslant C_2 < 0. \end{aligned}$$

**Proposition 3.60** (Azevedo-Gottlieb condition)**.** *Let $U : X \times [0, \infty) \times \Theta \longmapsto \mathbb{R}$ be a given regular function. The function $U$ has the following property: let $0 \leqslant p < q$ be such that $U(x, p, \theta) \leqslant U(y, q, \theta)$ for some $\theta \in \Theta$; then*

$$|p - q| \leqslant C \cdot d_X(x, y).$$

*Here the constant $C > 0$ is independent of the choice of $p$, $q$, $\theta$, $x$ and $y$.*

*Proof.* By assumption

$$0 \leqslant U(y, q, \theta) - U(x, p, \theta) = U(y, q, \theta) - U(y, p, \theta) + U(y, p, \theta) - U(x, p, \theta). \tag{3.68}$$

But since $U$ is regular $U(y, q, \theta) \leqslant U(y, p, \theta)$ therefore

$$0 \leqslant U(y, p, \theta) - U(y, q, \theta) \leqslant U(y, p, \theta) - U(x, p, \theta). \tag{3.69}$$

Requirement (ii) and (iii) in definition 3.58 now imply

$$C_2 \cdot (q - p) \leqslant C_1 \cdot d_X(x, y). \tag{3.70}$$

The result follows from this inequality. ■

*Remark* 3.61. In the original work of Azevedo and Gottlieb, proposition 3.60 is adopted instead of requirement (ii) in definition 3.58 as the main regularity assumption (see assumption 2 in [5]). While 3.60 indeed imposes less restrictions on $U$ than definition 3.58 does; in practice, requirements such as (ii) are often needed to verify conditions such as the one in proposition 3.60.

**Lemma 3.62.** *Consider an economy* $(\mathfrak{E}, U, c)$ *where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta).$$

*Given* $(p_\varepsilon, \alpha_\varepsilon)$ *a* $\varepsilon$*-proto-equilibrium of the economy. If* $x \in$ *supp*$(\eta)$ *satisfies* $p_\varepsilon(x) > 0$*, then* $\alpha_\varepsilon(\{x\} \times \Theta) > 0$.

*Proof.* We prove the counter-positive. Let $x \in X$ be such that $\eta(x) > 0$ and $\alpha_\varepsilon(\{x\} \times \Theta) = 0$. Then

$$\mathbb{E}_{\alpha_\varepsilon + \overline{\eta}}(\chi_{\{x\}\times(\Theta\sqcup X)} p_\varepsilon) = \mathbb{E}_{\overline{\eta}}(\chi_{\{x\}\times X} \cdot p_\varepsilon) = p_\varepsilon(x)\eta(x). \tag{3.71}$$

On the other hand,

$$\mathbb{E}_{\alpha_\varepsilon + \overline{\eta}}(\chi_{\{x\}\times(\Theta\sqcup X)} c) = \mathbb{E}_{\alpha_\varepsilon}(\chi_{\{x\}\times\Theta} \cdot c) = 0. \tag{3.72}$$

Hence since $(p_\varepsilon, \alpha_\varepsilon)$ is a $\varepsilon$-proto-equilibrium, $p_\varepsilon(x) = 0$ ■

**Corollary 3.62.1.** *Given* $(\mathfrak{E}, U, c)$ *and* $(p_\varepsilon, \alpha_\varepsilon)$ *as in lemma 3.62. For any* $x \in$ *supp*$(\eta)$ *satisfying* $p_\varepsilon(x) > 0$*, there exists* $\theta \in \Theta$ *such that* $(x, \theta) \in$ *supp*$(\alpha_\varepsilon) \subset A_{\varepsilon, p_\varepsilon}$.

**Corollary 3.62.2.** *Given* $(\mathfrak{E}, U, c)$ *and* $(p_\varepsilon, \alpha_\varepsilon)$ *as in lemma 3.62. Given* $x \in$ *supp*$(\eta)$ *satisfying* $p_\varepsilon(x) > p_\varepsilon(y) \geqslant 0$ *for some* $y \in X$*. There exists* $\theta \in \Theta$ *such that* $(U \circ p_\varepsilon)(x, \theta) + \varepsilon \geqslant (U \circ p_\varepsilon)(y, \theta)$.

**Proposition 3.63.** *Let* $(\mathfrak{E}, U, c)$ *with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta)$$

*be a regular economy. Constants* $K_1, K_2 > 0$*, depending solely on the constants* $C_1, C_2, C_3$*, can be found such that for every* $\varepsilon$*-proto-equilibrium of the economy* $(p_\varepsilon, \alpha_\varepsilon)$*, where* $\alpha_\varepsilon \in \Delta(X \times \Theta)$*, the following inequality holds*

$$|p_\varepsilon(x) - p_\varepsilon(y)| \leqslant K_1 \cdot d_X(x, y) + \frac{\varepsilon}{K_2}. \tag{3.73}$$

*whenever* $x \in$ *supp*$(\eta)$ *and* $y \in X$ *is such that* $p_\varepsilon(x) > p_\varepsilon(y) \geqslant 0$.

*Proof.* Fix a $\varepsilon$-proto-equilibrium and let $x \in \text{supp}(\eta)$ and $y \in X$ be such that $p_\varepsilon(x) > p_\varepsilon(y) \geqslant 0$. Corollary 3.62.2 implies that there exists $\theta \in \Theta$ satisfying

$$U(x, p_\varepsilon(x), \theta) + \varepsilon \geqslant U(y, p_\varepsilon(y), \theta). \tag{3.74}$$

Hence,

$$|U(x, p_\varepsilon(x), \theta) - U(y, p_\varepsilon(x), \theta)| + \varepsilon \geqslant U(y, p_\varepsilon(y), \theta) - U(y, p_\varepsilon(x), \theta) \geqslant 0, \tag{3.75}$$

where the last inequality follows since $U$ is regular and thus decreasing in $p$. Moreover, the regularity of $U$ implies the following bounds

$$|U(y, p_\varepsilon(y), \theta) - U(y, p_\varepsilon(x), \theta)| \quad \geqslant \quad C_2 \cdot |p_\varepsilon(x) - p_\varepsilon(y)|, \tag{3.76}$$

$$|U(x, p_\varepsilon(x), \theta) - U(y, p_\varepsilon(x), \theta)| \quad \leqslant \quad C_1 \cdot d_X(x, y). \tag{3.77}$$

These bounds yield the lemma. ■

**Corollary 3.63.1.** *Suppose that the economy in proposition 3.63 were tame instead. There exists constants $K_1, K_2 > 0$ such that for every $\varepsilon$-proto-equilibrium of the economy $(p_\varepsilon, \alpha_\varepsilon)$ the function $p_\varepsilon$ satisfies*

$$|p_\varepsilon(x) - p_\varepsilon(y)| \leqslant K_1 \cdot d_X(x, y) + \frac{\varepsilon}{K_2} \tag{3.78}$$

*for all $x, y \in X$.*

*Remark* 3.64. Recall that if an economy is tame the set $X$ is finite and, in addition, $\mathrm{supp}(\eta) = X$.

*Remark* 3.65. Recall that a mapping $f : M \longmapsto N$ ($M$ and $N$ are metric spaces) is *$C$-Lipschitz continuous* (or simply *$C$-Lipschitz*) if there exists $C > 0$ such that for every pair $s, t \in M$ $d_N(f(s), f(t)) \leqslant C \cdot d_M(s, t)$. Corollary 3.63.1 thus implies that, for a tame economy, $p_\varepsilon$ is $K_1$-Lipschitz continuous whenever $\varepsilon = 0$. This observation encompasses lemma 2 in the appendix of [5].

### 3.7.2 Equilibrium existence

Proposition 3.63 and corollary 3.63.1 show that $U$ being regular places some restrictions on the growth of any function $p_\varepsilon : X \longmapsto [0, \infty)$ that is part of a $\varepsilon$-proto-equilibrium. This fact is used in the proof of the main result:

**Theorem 3.66.** *Let* $(\mathfrak{E}, U, c)$ *with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

*be a regular economy. Then the economy has an equilibrium.*

Theorem 3.66 is the analogue of theorem 1 in [5]. The proof of theorem 3.66 follows almost immediately from the following

**Lemma 3.67.** *Fix a regular economy* $(\mathfrak{E}, U, c)$ *with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu).$$

*Suppose that*

$$\{\mathfrak{E}_n = (\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty}$$

*with supp*$(\eta_n) = X_n$ *is a collection of bones of* $\mathfrak{E}$ *such that* $\mathfrak{E}_n \to \mathfrak{E}$. *In addition, let* $\{\varepsilon_n\}_{n=1}^{\infty}$ *be a sequence converging to 0.*

*For any given a sequence* $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ *where each individual* $(p_n, \alpha_n)$ *is a* $\varepsilon_n$*-proto-equilibrum of the tame economy*

$$(\mathfrak{E}_n, U|_{X_n \times [0,\infty) \times (\Theta \sqcup X_n)}, c|_{X_n \times (\Theta \sqcup X_n)})$$

*there exists* $(p, \alpha)$ *with* $p : X \longmapsto \mathbb{R}$ *continuous and* $\alpha \in \Delta(X \times \Theta)$ *and a subsequence* $\{(p_{\varepsilon_{n_k}}, \alpha_{\varepsilon_{n_k}})\}_{k=1}^{\infty}$ *such that:*

*(i) Given* $x \in X$ *for any sequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *converging to* $x$ *such that* $x_{n_k} \in X_{n_k}$ *for every* $k$ $p_{n_k}(x_{n_k}) \to p(x)$ *as* $k \to \infty$.

*(ii) The sequence* $\alpha_{n_k}$ *converges weakly to* $\alpha$.

*Remark* 3.68. Lemma 3.67 simply states that if for a given regular economy $(\mathfrak{E}, U, c)$ a collection of bones

$$\{(\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty}$$

of $\mathfrak{E}$ can be found such that $\mathfrak{E}_n \to \mathfrak{E}$ then the economy has an equilibrium. Indeed, (i) and (ii) in lemma 3.67 correspond to conditions (ii) and (iii) in definition 3.26.

*Proof of theorem 3.66.* Fix a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ converging to 0 and observe that since $X$ is Polish it contains a countable dense subset

$$Q = \{q_1, q_2, q_3, \dots\}.$$

For every $n \in \mathbb{N}$ set

$$X_n = \{q_1, \dots, q_n\}.$$

By construction,

$$X_n \to \overline{\bigcup_{n=1}^{\infty} X_n} = X \tag{3.79}$$

in the sense of Kuratowski (see section B chapter 4 in [37]) and, by theorem 3.16, in the sense of Hausdorff.

Moreover, for every $n \in \mathbb{N}$ let $\eta_n \in \mathfrak{F}(X)$ be defined as follows

$$\eta_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in X_n, \\ 0 & \text{if } x \notin X_n. \end{cases} \tag{3.80}$$

Then $\eta_n \Rightarrow \mathbf{0}$.

The above constructions show that the collection

$$\{\mathfrak{E}_n = (\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty}$$

is a collection of bones of $\mathfrak{E}$, and $\mathfrak{E}_n \to \mathfrak{E}$. In addition, for each $n$ the economy $(\mathfrak{E}_n, U|_{X_n \times [0,\infty \times (\Theta \sqcup X_n)}, c|_{X_n \times (\Theta \sqcup X_n)})$ is tame and therefore it has an $\varepsilon_n$-proto-equilibrium $(p_n, \alpha_n)$ by theorem 3.45. Consider now the sequence $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ lemma 3.67 implies that there exists $(p, \alpha)$, with $p : X \longmapsto \mathbb{R}$ continuous and $\alpha \in \Delta(X \times \Theta)$; and a subsequence $\{(p_{n_k}, \alpha_{n_k})\}_{k=1}^{\infty}$ satisfying

(i) Given $x \in X$ for any sequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to $x$ such that $x_{n_k} \in X_{n_k}$ for every $k$, $p_{n_k}(x_{n_k}) \to p(x)$ as $k \to \infty$.

(ii) The sequence $\alpha_{n_k}$ converges weakly to $\alpha$.

But these are conditions (ii) and (iii) in definition 3.26 specialised to the subsequence. Hence considering the collection of bones given by

$$\{\mathfrak{E}_k = (\Theta \sqcup X_{n_k}, \mathcal{B}(\Theta \sqcup X_{n_k}), \mu + \eta_{n_k})\}_{k=1}^{\infty}$$

the result follows. ■

The proof of lemma 3.67 is considered now. The strategy follows Azevedo and Gottlieb's closely (see proof of lemma 3 in the appendix of [5]). A notable difference is that when considering an arbitrary $\varepsilon_n$-proto-equilibrium of the tame economy $(\mathfrak{E}_n, U|_{X_n \times [0,\infty) \times (\Theta \sqcup X_n)}, c|_{X_n \times (\Theta \sqcup X_n)})$, it might not be possible to extend $p_n$ to a Lipschitz function $\tilde{p}_n : X \longmapsto \mathbb{R}$. Nonetheless, it possible to find a Lipschitz function $\tilde{p}_n$ over $X$ such that $\tilde{p}_n|_{X_n}$ is close to $p_n$. This fact follows from the next

**Proposition 3.69.** *Let $A \subset M$ be non-empty and let $f : A \longmapsto \mathbb{R}$ have the following property: for every $s$, $t \in A$*

$$|f(s) - f(t)| \leqslant C \cdot d_M(s, t) + \delta, \tag{3.81}$$

*where $C > 0$ and $\delta \geqslant 0$. There exists $g : M \longmapsto \mathbb{R}$ $C$-Lipschitz satisfying*

$$\sup_{s \in A} |f(s) - g(s)| \leqslant \delta.$$

*Remark* 3.70. Proposition 3.69 is a slight generalisation of the well-known McShane-Whitney extension theorem (see theorem 1.1 in [21]). Indeed, the proof follows the exact same lines. As in the original result, the extension is not unique. A detailed discussion can be found in [21].

*Proof of proposition 3.69.* Let $g : M \longmapsto \mathbb{R}$ be defined as follows

$$g(t) = \sup_{s \in A} \{f(s) - C \cdot d_M(s, t) - \delta\}. \tag{3.82}$$

Let $t \in M$ and fix $a \in A$, for arbitrary $s \in A$,

$$\begin{aligned} f(s) - C \cdot d_M(s, t) - \delta &\leqslant f(a) + |f(s) - f(a)| - C \cdot d_M(s, t) - \delta \\ &\leqslant f(a) + C \cdot d_M(a, t). \end{aligned} \tag{3.83}$$

Hence $g(t) < \infty$ and since $A \neq \emptyset$ $g(t) > -\infty$. $g$ is then a well-defined mapping.

Now, to prove that $g$ is $C$-Lipschitz let $s, t \in M$ be such that $g(s) \leqslant g(t)$ and let $\xi > 0$. There exists $a \in A$ satisfying

$$g(t) - \xi \leqslant f(a) - C \cdot d_M(a, t) - \delta. \tag{3.84}$$

Moreover, by definition

$$g(s) \geqslant f(a) - C \cdot d_M(a, s) - \delta. \tag{3.85}$$

Hence,

$$g(t) - g(s) \leqslant C \cdot d_M(a, s) - C \cdot d_M(a, t) + \xi \leqslant C \cdot d_M(s, t) + \xi. \tag{3.86}$$

The arbitrariness of $\xi$ implies the $g(t) - g(s) \leqslant C \cdot d_M(s, t)$.

Finally, let $a \in A$ then

$$f(a) - \delta \leqslant \sup_{s \in A}\{f(s) - C \cdot d_M(s, a) - \delta\} = g(a). \tag{3.87}$$

On the other hand, given $b \in A$ equation (3.83) implies

$$f(b) - C \cdot d_M(b, a) - \delta \leqslant f(a). \tag{3.88}$$

Therefore, $f(a) - \delta \leqslant g(a) \leqslant f(a)$ or, equivalently, $|f(a) - g(a)| \leqslant \delta$ for every $a \in A$. ■

Recall the following

**Theorem 3.71** (Arzelà-Ascoli)**.** *Let $M$ be a compact metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous mappings from $M$ to $\mathbb{R}$. If for arbitrary $t \in M$*

*(i)* $\sup_{n \in \mathbb{N}} |f_n(t)| < \infty$*;*

*(ii) for every $\xi > 0$ there exists $\delta > 0$ such that* $\sup_{n \in \mathbb{N}} |f_n(s) - f_n(t)| < \xi$ *whenever* $d_M(s, t) < \delta$

*then there exists* $f : M \longmapsto \mathbb{R}$ *continuous and subsequence* $\{f_{n_k}\}_{k=1}^{\infty}$ *satisfying* $\lim_{k\to\infty} \|f - f_{n_k}\|_\infty = 0$.

A proof of a particular form of this result can be found in e.g., theorem 7.25 in [38].

*Remark* 3.72. Recall that property (ii) in theorem 3.71 is referred to as *equicontinuity* whilst property (i) is *uniform boundedness*.

*Proof of lemma 3.67.* Let $(\mathfrak{E}, U, c)$, with

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu),$$

be a regular economy and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence converging to 0. Suppose that the collection of bones of $\mathfrak{E}$

$$\{\mathfrak{E}_n = (\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty},$$

where $\operatorname{supp}(\eta_n) = X_n$, converges to $\mathfrak{E}$.

Denote by $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ a sequence whose individual terms are $\varepsilon_n$-proto-equilibria of each separate tame economy

$$(\mathfrak{E}_n, U|_{X_n\times[0,\infty)\times(\Theta\sqcup X_n)}, c|_{X_n\times(\Theta\sqcup X_n)})$$

respectively. Recall that this sequence exists by theorem 3.45.

Fix $n \in \mathbb{N}$, since $(\mathfrak{E}_n, U|_{X_n\times[0,\infty)\times(\Theta\sqcup X_n)}, c|_{X_n\times(\Theta\sqcup X_n)})$ is tame, corollary 3.63.1 implies that $p_n$ satisfies

$$|p_n(x) - p_n(y)| \leqslant K_1 \cdot d_X(x, y) + \frac{\varepsilon_n}{K_2} \tag{3.89}$$

for some $K_1, K_2 > 0$ independent of $n$. Proposition 3.69 implies that there exists $\tilde{p}_n : X \longmapsto \mathbb{R}$ $K_1$-Lipschitz such that

$$\sup_{x\in X_n} |\tilde{p}_n(x) - p_n(x)| \leqslant \frac{\varepsilon_n}{K_2}. \tag{3.90}$$

Observe that $\{\tilde{p}_n\}_{n=1}^{\infty}$ is equicontinuous. It is moreover uniformly bounded. To see this, fix $x \in X$ and choose $n \in \mathbb{N}$. Now, observe that for some

$x_n \in X_n$,

$$\begin{aligned} |\tilde{p}_n(x)| &\leqslant |\tilde{p}_n(x) - \tilde{p}_n(x_n)| + |\tilde{p}_n(x_n) - p_n(x_n)| + |p_n(x_n)| \\ &\leqslant K_1 \cdot d_X(x, x_n) + \frac{\varepsilon_n}{K_2} + |p_n(x_n)|. \end{aligned} \tag{3.91}$$

Since $(p_n, \alpha_n)$ is a $\varepsilon_n$-proto-equilibrium,

$$|p_n(x_n)| \leqslant \mathbb{E}_{\alpha_n + \overline{\eta}_n}(|c(\xi, \cdot)||\xi = x_n) \leqslant \sup_{(x,\theta) \in X \times \Theta} |c(x, \theta)| < \infty. \tag{3.92}$$

Hence, equation (3.91) yields the bound

$$|\tilde{p}_n(x)| \leqslant K_1 \cdot \mathrm{diam}(X) + \sup_n \frac{\varepsilon_n}{K_2} + \sup_{(x,\theta) \in X \times \Theta} |c(x, \theta)| < \infty. \tag{3.93}$$

The theorem of Arzelà and Ascoli implies that there exists a continuous mapping $p : X \longmapsto \mathbb{R}$ and subsequence $\{\tilde{p}_{n_k}\}_{k=1}^{\infty}$ converging to $p$ uniformly. Consider now the sequence $\{(\tilde{p}_{n_k}, \alpha_{n_k})\}_{k=1}^{\infty}$. Theorem 3.4 implies that there exists a sub-sequence $\{\alpha_{n_{k_l}}\}_{l=1}^{\infty}$ converging weakly to $\alpha \in \Delta(X \times \Theta)$.

It is claimed now that the sub-sequence $\{(p_{n_{k_l}}, \alpha_{n_{k_l}})\}_{l=1}^{\infty}$ is the desired sub-sequence. To that end, observe that condition (ii) in lemma 3.67 is satisfied by construction. To prove condition (i) fix $x \in X$ and let $\{x_{n_{k_l}}\}_{l=1}^{\infty}$ be a sequence converging to $x$ satisfying $x_{n_{k_l}} \in X_{n_{k_l}}$ then

$$\begin{aligned} \limsup_{l \to \infty} |p(x) - p_{n_{k_l}}(x_{n_{k_l}})| &\leqslant \limsup_{l \to \infty} |\tilde{p}_{n_{k_l}}(x_{n_{k_l}}) - p_{n_{k_l}}(x_{n_{k_l}})| \\ &\quad + \limsup_{l \to \infty} |\tilde{p}_{n_{k_l}}(x_{n_{k_l}}) - p(x)| \leqslant \lim_{l \to \infty} \frac{\varepsilon_{n_{k_l}}}{K_2} \\ &\quad + \lim_{l \to \infty} |\tilde{p}_{n_{k_l}}(x_{n_{k_l}}) - p(x)| = 0. \end{aligned} \tag{3.94}$$

The fact that

$$\lim_{l \to \infty} |\tilde{p}_{n_{k_l}}(x_{n_{k_l}}) - p(x)| = 0$$

follows from the fact that the sub-sub-sequence $\{p_{n_{k_l}}\}_{l=1}^{\infty}$ converges to $p$ uniformly since the sub-sequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to $p$ uniformly. ■

## 3.8 Equilibrium properties

In this section some consequences of $(p, \alpha)$ being an equilibrium of the regular economy $(\mathfrak{E}, U, c)$ are discussed. In the interest of brevity, explicit references to the skeleton $\mathfrak{E}$ of the chosen economy are often omitted. Moreover, a sequence $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ satisfying for each $n \in \mathbb{N}$ $(p_n, \alpha_n)$ is a $\varepsilon_n$-proto-equilibria of the tame economy $(\mathfrak{E}_n, U|_{X_n \times (\Theta \sqcup X_n)}, c|_{X_n \times (\Theta \sqcup X_n)})$ (here $\mathfrak{E}_n$ is a bone of $\mathfrak{E}$ and $\mathfrak{E}_n \to \mathfrak{E}$) is said to *converge to* $(p, \alpha)$ if it satisfies assumptions (i) and (ii) in lemma 3.67 or, equivalently, conditions (ii) and (iii) in definition 3.26. In addition, the collection $\{\mathfrak{E}_n\}_{n=1}^{\infty}$ is then said to be *a collection of bones associated with* $(p, \alpha)$.

The results in this section are analogous to those in proposition 1 in [5]. In particular, corollary 3.73.1 in subsection 1.7.1 corresponds to statement 3 in Azevedo and Gottlieb's proposition 1. Meanwhile, 3.74 in subsection 1.7.2 corresponds to statement 1 and proposition 3.83 in subsection 1.7.3 refers to statement 2. The permutation in the order of properties with respect to [5] is due Lipschitz-ness being needed in the proof of theorem 3.66. Moreover, proposition 3.83 requires theorem 3.74.

### 3.8.1 Lipschitz continuity

Observe that the the mappings $\{\tilde{p}_n\}_{n=1}^{\infty}$ in the proof of lemma 3.67 are all $K_1$-Lipschitz. This implies that $p$ is itself $K_1$-Lipschitz continuous. For a general equilibrium $(p, \alpha)$ of the regular economy $(\mathfrak{E}, U, c)$ this holds as well. This is a direct consequence of the definition equilibrium and the assumption of regularity.

**Proposition 3.73.** *Let* $(\mathfrak{E}, U, c)$*, where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu),$$

*be a regular economy and* $(p, \alpha)$ *an equilibrium of the economy. Let*

$$\{\mathfrak{E}_n\}_{n=1}^{\infty} = \{(\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty}$$

*be a collection of bones associated with* $(p, \alpha)$*, then* $p|_{\cup_{n=1}^{\infty} X_n}$ *is Lipschitz* *continuous.*

*Proof.* Let $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ be the sequence of $\varepsilon_n$-proto-equilibria converging to $(p, \alpha)$. Recall that by corollary 3.63.1 for any $x, y \in X_n$

$$|p_n(x) - p_n(y)| \leqslant K_1 \cdot d_X(x, y) + \frac{\varepsilon_n}{K_2}; \tag{3.95}$$

where $K_1, K_2 > 0$ do not depend on $\mathfrak{E}_n$.

Fix $x, y \in \cup_{n=1}^{\infty} X_n$ and let $\{x_n\}_{n=1}^{\infty}$ (resp. $\{y_n\}_{n=1}^{\infty}$) with $x_n \in X_n$ (resp. $y_n \in X_n$) be such that $x_n \to x$ (resp. $y_n \to y$). Then

$$\begin{aligned} |p(x) - p(y)| &\leqslant |p(x) - p_n(x_n)| + |p_n(x_n) - p_n(y_n)| && (3.96)\\ &+ |p_n(y_n) - p(y)| \leqslant K_1 \cdot d_X(x_n, y_n) + \frac{\varepsilon_n}{K_2} && (3.97)\\ &+ |p_n(x_n) - p_n(y_n)| + |p_n(y_n) - p(y)|. && (3.98) \end{aligned}$$

Taking the limit $n \to \infty$,

$$|p(x) - p(y)| \leqslant K_1 \cdot d_X(x, y). \tag{3.99}$$

■

**Corollary 3.73.1.** *Let* $(p, \alpha)$ *be an equilibrium of the regular economy* $(\mathfrak{E}, U, c)$*, where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu);$$

*then* $p$ *is Lipschitz continuous.*

*Proof.* Since $p|_{\cup_{n=1}^{\infty} X_n}$ is uniformly continuous over $\cup_{n=1}^{\infty} X_n$ it can be extended to a uniformly continuous $\tilde{p} : X \longmapsto [0, \infty)$. This extension is unique. Moreover, since $p$ is Lipschitz, $\tilde{p}$ is also Lipschitz.

Observe that $p$ is uniformly continuous and extends $p|_{\cup_{n=1}^{\infty} X_n}$. Hence $p = \tilde{p}$. ■

**Corollary 3.73.2.** *If* $X = \overline{U}$ *where* $U \subset \mathbb{R}^d$ *is an open bounded set then* $p$ *is differentiable Lebesgue a.e. on* $U$*.*

This follows immediately from Rademacher's theorem (see theorem 3.2.5 in [4]). This corollary corresponds to statement 4 in proposition 1 of [5]. Note however that Azevedo and Gottlieb's statement 4 is slightly inaccurate, Rademacher's theorem only guarantees differentiability a.e. on open sets.

### 3.8.2 Existence of $\varepsilon$-proto-equilibria in regular economies

**Theorem 3.74.** *Let* $(\mathfrak{E}, U, c)$*, where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu),$$

*be a regular economy and* $(p, \alpha)$ *one of its equilibria. Then* $(p, \alpha)$ *is a proto-equilibrium.*

*Remark* 3.75. Theorem 3.74 implies that given $\varepsilon \geqslant 0$ a given regular economy has a $\varepsilon$-proto-equilibrium. Furthermore, theorem 3.74 motivates the referring to the pair $(p, \alpha)$ as an "equilibrium". Note however that not all proto-equilibria of a given regular economy are equilibria. Indeed, Azevedo and Gottlieb's motivation behind their notion of equilibrium is to select proto-equilibria with desirable properties.

A proof of theorem 3.74 is presented below. For convenience it is split into two propositions. Notably, the proof presented here differs from the equivalent proof in [5] (see lemma 4 and its proof). There are two reasons for this: first, the original proof by Azevedo and Gottlieb is partially incorrect and, second, some of their arguments do not generalise when considering limits of general $\varepsilon$-proto-equilibrium.

Recall that a pair $(p, \alpha)$ is a proto-equilbrium of the economy $(\mathfrak{E}, U, c)$ with

$$(\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

if, and only if, $(\pi_\Theta)_*(\alpha) = \mu$ and

$$\mathbb{E}_\alpha(\chi_{A\times\Theta} \cdot p) = \mathbb{E}_\alpha(\chi_{A\times\Theta} \cdot c) \;\; \forall A \in \mathcal{B}(X), \tag{3.100}$$

$$\alpha(A_{0,p}) = 1. \tag{3.101}$$

As before,

$$A_{0,p} = \{(x,\theta) \in X \times \Theta : (U \circ p)(x,\theta) \geqslant (U \circ p)(y,\theta)\ \forall y \in X\}.$$

Recall that since $\alpha$ is the weak limit of functions whose marginal with respect to $\Theta$ is $\mu$, $(\pi_\Theta)_*(\alpha) = \mu$ holds. To obtain equation (3.100) two lemmas are needed:

**Lemma 3.76.** *Let $M$ be a metric space. Given a closed set $C \subset M$ and $\delta > 0$ there exists a bounded Lipschitz function $\chi_{C,\delta} \longmapsto [0,1]$ satisfying $\chi_{C,\delta}(t) = 1$ whenever $t \in C$ and $\chi_{C,\delta}(t) = 0$ whenever $d(t,C) \geqslant \delta$.*

This lemma appears in chapter 7 of [43]. It follows by mollifying $\chi_C$. Details of the proof are omitted.

**Lemma 3.77.** *Let $(\mathfrak{E}, U, c)$ be a regular economy, where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu),$$

*and one of its equilibria $(p,\alpha)$. Fix a sequence $\{(p_n,\alpha_n)\}_{n=1}^{\infty}$ of $\varepsilon_n$-proto-equilibria converging to $(p,\alpha)$. If the elements of a sequence of $K_1$-Lipschitz mappings $\tilde{p}_n : X \longmapsto \mathbb{R}$ each satisfy*

$$\max_{x \in X_n} |\tilde{p}_n(x) - p_n(x)| \leqslant \frac{\varepsilon_n}{K_2};$$

*then $\tilde{p}_n \to p$ uniformly.*

*Remark* 3.78. Corollary 3.73.1 also follows from this lemma. The original proof being more elementary. Moreover, lemma 3.77 implies that the choice of $\{\tilde{p}_n\}_{n=1}^{\infty}$ in the proof of theorem 3.66 is irrelevant.

*Proof of lemma 3.77.* Fix $\xi > 0$ and observe that since $X$ is compact, the continuity of $p$ implies that there exists $\delta > 0$ such that $|p(x) - p(y)| < \frac{\xi}{K_1+2}$ whenever $d_X(x,y) < \delta$. In addition, the compactness of $X$ also implies the existence of a set $\{y_1, \ldots, y_l\}$ with the property $X \subset \cup_{k=1}^{l} B_\delta(y_k)$. Similarly, $y_{l+1}, \ldots, y_m$ can be found such that $X \subset \cup_{k=l+1}^{m} B_{\frac{\xi}{K_1+2}}(y_k)$. Consequently, the collection $\{B_r(y_k)\}_{k=1}^{\infty}$ where $r = \min\left\{\delta, \frac{\xi}{K_1+2}\right\}$ covers $X$.

Given $k \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$,

$$\begin{aligned} |\tilde{p}_n(y_k) - p(y_k)| &\leqslant |\tilde{p}_n(y_k) - \tilde{p}(x_n)| + |\tilde{p}_n(x_n) - p_n(x_n)| \\ &+ |p_n(x_n) - p(y_k)| \leqslant K_1 \cdot d_X(y_k, x_n) + \frac{\varepsilon_n}{K_2} \\ &+ |p_n(x_n) - p(y_k)|, \end{aligned} \tag{3.102}$$

where $x_n \in X_n \subset X$ is the $n^{\text{th}}$ term of a sequence converging to $y_k$. Now, since $p_n(x_n) \to p(y_k)$ (which follows since $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ converges to $(p, \alpha)$) the right-hand side of equation (3.102) approaches 0. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\max_{1 \leqslant k \leqslant m} |p(y_k) - \tilde{p}_n(y_k)| < \frac{\xi}{K_1 + 2} \;\; \forall\, n > n_0. \tag{3.103}$$

Let $x \in X$ and $k \in \{1, \ldots, m\}$ satisfying $x \in B_r(y_k)$, for $n > n_0$

$$\begin{aligned} |p(x) - \tilde{p}_n(x)| &\leqslant |\tilde{p}_n(y_k) - \tilde{p}_n(x)| + |p(y_k) - \tilde{p}_n(y_k)| \\ &+ |p(x) - p(y_k)| < K_1 \cdot r + \frac{2\xi}{K_1 + 2} \leqslant \xi. \end{aligned} \tag{3.104}$$

From where it follows that $\tilde{p}_n \to p$ uniformly. ■

**Proposition 3.79.** *Let $(\mathfrak{E}, U, c)$ be a regular economy with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu).$$

*Given an equilibrium $(p, \alpha)$ and $A \in \mathcal{B}(X)$,*

$$\mathbb{E}_\alpha(\chi_{A \times \Theta} \cdot p) = \mathbb{E}_\alpha(\chi_{A \times \Theta} \cdot c). \tag{3.105}$$

*Proof.* Let $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ be a sequence of $\varepsilon_n$-proto-equilibria converging to the equilibrium $(p, \alpha)$. Notice that if $c = 0$, then $p_n = 0$ for every $n \in \mathbb{N}$ and there is nothing to prove. Assume therefore that $c(x, \theta) > 0$ for some $(x, \theta) \in X \times \Theta$.

Now, for every $n \in \mathbb{N}$ denote by $\tilde{p}_n$ a $K_1$-Lipschitz function satisfying $\max_{x \in X_n} |\tilde{p}_n(x) - p_n(x)| \leqslant \frac{\varepsilon_n}{K_2}$. Recall that such function exists by corollary

3.63.1 and proposition 3.69. Lemma 3.77 implies $\tilde{p}_n \to p$ uniformly; hence,

$$\lim_{n\to\infty} \mathbb{E}_{\alpha_n}(\tilde{p}_n) = \mathbb{E}_{\alpha}(p).$$

Let $\psi : X \longmapsto \mathbb{R}$ be continuous, it is proven that $\mathbb{E}_\alpha(\psi \cdot p) = \mathbb{E}_\alpha(\psi \cdot c)$. To that end observe that given $n \in \mathbb{N}$:

$$\begin{aligned} |\mathbb{E}_\alpha(\psi \cdot p) - \mathbb{E}_\alpha(\psi \cdot c)| \leqslant\ & |\mathbb{E}_\alpha(\psi \cdot p) - \mathbb{E}_{\alpha_n}(\psi \cdot \tilde{p}_n)| \\ & + |\mathbb{E}_{\alpha_n}(\psi \cdot \tilde{p}_n) - \mathbb{E}_{\alpha_n}(\psi \cdot p_n)| \\ & + |\mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot p_n) - \mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot c)| \\ & + |\mathbb{E}_{\overline{\eta}_n}(\psi \cdot p_n)| + |\mathbb{E}_{\overline{\eta}_n}(\psi \cdot c)| \\ & + |\mathbb{E}_{\alpha_n}(\psi \cdot c) - \mathbb{E}_\alpha(\psi \cdot c)|. \end{aligned} \tag{3.106}$$

Recall that $\mathrm{supp}(\alpha_n) \subset X_n \times \Theta \subset X \times \Theta$, which implies

$$|\mathbb{E}_{\alpha_n}(\psi \cdot (\tilde{p}_n - p_n))| \leqslant \|\psi\|_\infty \max_{x\in X_n} |\tilde{p}_n(x) - p_n(x)| \leqslant \frac{\varepsilon_n}{K_2}\|\psi\|_\infty. \tag{3.107}$$

Moreover,

$$|\mathbb{E}_{\overline{\eta}_n}(\psi \cdot c)| \leqslant \|\psi\|_\infty \|c\|_\infty \|\eta_n\|. \tag{3.108}$$

Since $p_n(x) = \mathbb{E}_{\alpha_n+\overline{\eta}_n}(c(\xi,\cdot)|\xi = x)$ the bound

$$|\mathbb{E}_{\overline{\eta}_n}(\psi \cdot p)| \leqslant \|\psi\|_\infty \|c\|_\infty \|\eta_n\| \tag{3.109}$$

holds as well. Finally,

$$\begin{aligned} \mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot p) &= \mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot \mathbb{E}_{\alpha_n+\overline{\eta}_n}(c(\xi,\cdot)|\xi = \cdot)) \\ &= \mathbb{E}_{\alpha_n+\overline{\eta}_n}(\mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot c(\xi,\cdot)|\xi = \cdot)) = \mathbb{E}_{\alpha_n+\overline{\eta}_n}(\psi \cdot c). \end{aligned} \tag{3.110}$$

Equations (3.107) - (3.110) along with equation (3.106) yield the bound

$$\begin{aligned} |\mathbb{E}_\alpha(\psi \cdot p) - \mathbb{E}_\alpha(\psi \cdot c)| \leqslant\ & \|\psi\|_\infty \left(2\|c\|_\infty \|\eta_n\| + \frac{\varepsilon_n}{K_2}\right) \\ & + |\mathbb{E}_\alpha(\psi \cdot p) - \mathbb{E}_{\alpha_n}(\psi \cdot \tilde{p}_n)| + |\mathbb{E}_{\alpha_n}(\psi \cdot c) - \mathbb{E}_\alpha(\psi \cdot c)|. \end{aligned} \tag{3.111}$$

But letting $n \to \infty$ the right hand side of equation (3.111) approaches 0. Hence, $\mathbb{E}_\alpha(\psi \cdot p) = \mathbb{E}_\alpha(\psi \cdot c)$ as required. Following a standard approximation argument, let $A \in \mathcal{B}(X)$,

$$\mathbb{E}_\alpha(\chi_{(A\times\Theta),\frac{1}{n}} \cdot p) = \mathbb{E}_\alpha(\chi_{(A\times\Theta),\frac{1}{n}} \cdot c).$$

Here $\chi_{(A\times\Theta),\frac{1}{n}}$ is the function in lemma 3.76 with $\frac{1}{n}$ in place of $\delta$.

Let

$$A_n = \left\{ x \in X : d(x, A) < \frac{1}{n} \right\}$$

and observe that $A_{n+1} \subset A_n$ and $\bigcap_{n=1}^{\infty} A_n = A$. Hence,

$$\lim_{n\to\infty} \alpha(A_n \times \Theta) = \alpha(A \times \Theta). \tag{3.112}$$

But since

$$|\mathbb{E}_\alpha(p \cdot (\chi_{A\times\Theta} - \chi_{(A\times\Theta),\frac{1}{n}})| \leqslant \|p\|_\infty \alpha((A_n \setminus A) \times \Theta) \tag{3.113}$$

equation (3.112) implies that $\lim_{n\to\infty} \mathbb{E}_\alpha(p \cdot \chi_{(A\times\Theta),\frac{1}{n}}) = \mathbb{E}_\alpha(p \cdot \chi_{A\times\Theta})$. An analogous relation is satisfied with $c$ replacing $p$. Define now the finite $\mathcal{B}(X)$ measures

$$\begin{aligned} \nu_p &= \mathbb{E}_\alpha(p \cdot \chi_{\cdot\times\Theta}) \quad \text{and} && (3.114)\\ \nu_c &= \mathbb{E}_\alpha(c \cdot \chi_{\cdot\times\Theta}). && (3.115) \end{aligned}$$

Observe that both measures are positive and $\sigma-$additive. Moreover, for $A \in \mathcal{B}(X)$ closed,

$$\begin{aligned} \nu_p(A) = \mathbb{E}_\alpha(p \cdot \chi_{A\times\Theta}) &= \lim_{n\to\infty} \mathbb{E}_\alpha(p \cdot \chi_{(A\times\Theta),\frac{1}{n}}) \\ &= \lim_{n\to\infty} \mathbb{E}_\alpha(c \cdot \chi_{(A\times\Theta),\frac{1}{n}}) = \mathbb{E}_\alpha(c \cdot \chi_{A\times\Theta}) = \nu_c(A). \end{aligned} \tag{3.116}$$

Since $\nu_p(A) = \nu_c(A)$ for arbitrary $A \in \mathcal{B}(X)$ closed $\nu_p = \nu_c$ (see lemma 1.9.4 in [8]), which is equivalent to equation (3.100). ■

*Remark* 3.80. Proposition 3.79 and its proof corresponds to the first part of the proof lemma 4 in [5]. Note that the proof presented here is considerably longer since to show that

$$\mathbb{E}_\alpha(\chi_{A\times\Theta} \cdot p) = \mathbb{E}_\alpha(\chi_{A\times\Theta} \cdot c)$$

holds for an arbitrary $A \in \mathcal{B}(X)$ it is not enough to just apply $\alpha_n \rightharpoonup \alpha$ as in the original argument by Azevedo and Gottlieb.

**Proposition 3.81.** *Let* $(\mathfrak{E}, U, c)$ *be a regular economy with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

*and an equilibrium* $(p, \alpha)$*. Then*

$$\alpha(\{(x,\theta) \in X \times \Theta : (U \circ p)(x,\theta) \geqslant (U \circ p)(y,\theta)\ \forall y \in X\}) = 1. \tag{3.117}$$

*Proof.* The proof follows the same argument as in the proof of lemma 3.52.

Let $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ be a sequence of $\varepsilon_n$-proto-equilibria converging to the equilibrium $(p, \alpha)$. Recall that

$$A_{\varepsilon_n, p_n} = \{(x,\theta) \in X_n \times \Theta : (U \circ p)(x,\theta) + \varepsilon_n \geqslant (U \circ p)(y,\theta)\ \forall y \in X_n\}$$

while

$$A_{0,p} = \{(x,\theta) \in X \times \Theta : (U \circ p)(x,\theta) \geqslant (U \circ p)(y,\theta)\ \forall y \in X\}.$$

As in lemma 3.52, the proof amounts to proving the following chain of inclusions:

$$\operatorname{supp}(\alpha) \subset \text{L.i.}(\operatorname{supp}(\alpha_n)) \subset \text{L.i.}(A_{\varepsilon_n, p_n}) \subset A_{0,p}. \tag{3.118}$$

Recall that, as in the proof of proposition 3.52, it suffices to prove the rightmost inclusion. Fix $(x,\theta) \in \text{L.i}(A_{\varepsilon_n,p_n})$, $z \in X$ and $\{z_n\}_{n=1}^{\infty}$ where $z_n \in X_n$ and $z_n \to z$.

There exist $N \in \mathbb{N}$ and a sequence $\{(y_n, \phi_n)\}_{n \geqslant N}$ converging to $(x,\theta)$ with $(y_n, \phi_n) \in A_{\varepsilon_n, p_n}$. But this implies

$$U(y_n, p_n(y_n), \phi_n) + \varepsilon_n \geqslant U(z_n, p_n(z_n), \phi_n). \tag{3.119}$$

Since $U$ is continuous and both $p_n(y_n) \to p(x)$ and $p_n(z_n) \to p(z)$ letting $n \to \infty$ yields

$$U(x, p(x), \theta) \geqslant U(z, p(z), \theta). \tag{3.120}$$

Thus $(x, \theta) \in X \times \Theta$. ■

*Remark* 3.82. Proposition 3.81 and its proof are analogous to the latter part of lemma 4 and its proof in [5]. The argument presented here being different since the original technique from Azevedo and Gottlieb does not generalise to the general $\varepsilon$-proto-equilibrium setting.

Propositions 3.79 and 3.81 together with $(\pi_\Theta)_* = \mu$ yield theorem 3.74.

### 3.8.3 Non-zero price characterization

The next result provides conditions under which an element $x \in X$ satisfies $p(x) > 0$, for $p$ an element of the equilibrium $(p, \alpha)$ of a regular economy:

**Proposition 3.83.** *Let* $(\mathfrak{E}, U, c)$ *with*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu)$$

*be a regular economy and* $(p, \alpha)$ *one of its equilibria. Suppose that* $x_0 \in X$ *is such that for every* $\theta \in \Theta$*, it holds that either* $p(x_0) < c(x_0, \theta)$*, or that whenever* $x \in X$ *is such that* $(x, \theta) \in supp(\alpha), (U \circ p)(x, \theta) \neq (U \circ p)(x_0, \theta)$*. Then* $p(x_0) = 0$*.*

*Remark* 3.84. Recall that proposition 3.83 corresponds to the second statement in proposition 1. The statement is also re-stated in lemma 5 in the appendix of [5]. Once again, the statement and method of proof provided here differs from the original. In this case this difference is justified by the fact that if $(p, \alpha)$ is an equilibrium of a given economy then

$$\operatorname{supp}(\alpha) \subset \{(x, \theta) \in X \times \Theta : (U \circ p)(x, \theta) \geqslant (U \circ p)(y, \theta) \; \forall y \in X\},$$

this inclusion may however be strict. In their argument Azevedo and Gottlieb make the implicit assumption that the two sets coincide.

The proof of proposition 3.83 proceeds by contradiction. A preliminary result is needed:

**Lemma 3.85.** *Consider an economy* $(\mathfrak{E}, U, c)$ *where*

$$\mathfrak{E} = (\Theta \sqcup X, \mathcal{B}(\Theta \sqcup X), \mu + \eta).$$

*Given* $(p_\varepsilon, \alpha_\varepsilon)$*, a* $\varepsilon$*-proto-equilibrium of the economy. If* $x \in supp(\eta)$ *satisfies* $p_\varepsilon(x) > 0$ *then there exists* $\theta \in \Theta$ *satisfying* $c(x, \theta) \leqslant p(x)$ *and* $(x, \theta) \in supp(\alpha_\varepsilon)$*.*

This lemma is yet another corollary of lemma 3.62. For clarity, it is stated at this stage.

*Proof of 3.85.* By way of contradiction, suppose that $p(x) < c(x, \theta)$ for any $\theta$ satisfying $(x, \theta) \in \operatorname{supp}(\alpha)$. By assumption $p(x) > 0$, lemma 3.62 implies that $\alpha_\varepsilon(\{x\} \times \Theta) > 0$. Hence,

$$p(x) = \frac{\mathbb{E}_{\alpha_\varepsilon}(\chi_{\{x\}\times\Theta}\, c)}{\eta(x) + \alpha_\varepsilon(\{x\} \times \Theta)} > p(x), \tag{3.121}$$

a contradiction. ■

*Proof of proposition 3.83.* Denote by $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ the relevant sequence of $\varepsilon_n$-proto-equilibria converging to $(p, \alpha)$ and let

$$\{\mathfrak{E}_n\}_{n=1}^{\infty} = \{(\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu + \eta_n)\}_{n=1}^{\infty}$$

be the associated collection of bones.

By way of contradiction, suppose $p(x_0) > 0$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to $x$ and such that, for each $n$, $x_n \in X_n$. Since $p$ is an equilibrium, there exists $N \in \mathbb{N}$ satisfying $p_n(x_n) > 0$ for all $n \geqslant N$. Lemma 3.85 implies that for every $n \geqslant N$ there exists $\theta_n$ with the property that $p_n(x_n) \geqslant c(x_n, \theta_n)$ and $(x_n, \theta_n) \in \operatorname{supp}(\alpha_n)$. The compactness of $\Theta$ implies there exists a subsequence $\{\theta_{n_k}\}_{k=1}^{\infty}$ converging to some $\theta_0 \in \Theta$.

Observe first that since both $c$ and $p$ are continuous the fact that $x_n \to x$ yields

$$c(x_0, \theta_0) = \lim_{k\to\infty} c(x_{n_k}, \theta_{n_k}) \geqslant \lim_{k\to\infty} p_{n_k}(x_{n_k}) = p(x_0). \tag{3.122}$$

The assumption in the rubric restricted to $\theta = \theta_0$ now implies that whenever $(x, \theta_0) \in \operatorname{supp}(\alpha)$, either $(U \circ p)(x, \theta_0) < (U \circ p)(x_0, \theta_0)$ or $(U \circ p)(x, \theta_0) > (U \circ p)(x_0, \theta_0)$ must hold. Note however that if $(x, \theta_0) \in \operatorname{supp}(\alpha)$ then $(U \circ p)(x, \theta_0) \geqslant (U \circ p)(y, \theta_0)$ for every $y \in X$. This means that the first of these possibilities cannot happen.

Now, fix $y \in X$ with the property that $(y, \theta_0) \in \operatorname{supp}(\alpha)$ and let $\{y_n\}_{n=1}^{\infty}$ be a sequence converging to $y$ such that for every $n \in \mathbb{N}$, $y_n \in X_n$. Consider the sub-sequences $\{x_{n_k}\}_{k=1}^{\infty}$, $\{\theta_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ since $(x_{n_k}, \theta_{n_k}) \in \operatorname{supp}(\alpha_{n_k})$ for every $k \in \mathbb{N}$ by construction,

$$(U \circ p_{n_k})(x_{n_k}, \theta_{n_k}) + \varepsilon_{n_k} \geqslant (U \circ p_{n_k})(y_{n_k}, \theta_{n_k}). \tag{3.123}$$

Letting $k \to \infty$,

$$(U \circ p)(y, \theta_0) \leqslant (U \circ p)(x, \theta_0) < (U \circ p)(y, \theta_0), \tag{3.124}$$

resulting in the desired contradiction. ■

## 3.9 Comparison with Azevedo and Gottlieb

This section discusses the main differences between the work of Azevedo and Gottlieb and the previous presentation. Some of these differences have already been referred to previously.

Regarding the formalism, unlike Azevedo and Gottlieb no formal distinction is made here between what they termed "economies" (in our setting this corresponds to an economy where $\eta = \mathbf{0}$) and "perturbations" (an economy where $\operatorname{supp}(\eta) = X$ and, consequently, $|X| < \infty$). Broadly speaking, this removes some ambiguity when referring to "equilibrium" notions, as these can be defined without having to specify to which type of economy. Moreover, the definition of equilibrium (definition 3.26) and $\varepsilon$-proto-equilibrium (definition 3.22) treat "true" (described by the probability measure $\mu$) and "behavioral" (described by the probability measure $\eta$) agents on equal footing. While proceeding in this way simplifies the presentation by avoiding unnecessary repetition, it does have the notable disadvantage that the resulting definition of equilibrium is not very tractable. One then needs to show that this definition reduces to a more manageable one (see section 3). Indeed, this latter, more convenient, definition is the one Azevedo and Gottlieb consider.

In terms of equilibrium notions, Azevedo and Gottlieb's analysis is restricted to the case $\varepsilon = 0$. More precisely, their "weak-equilibrium" concept corresponds to the notion of proto-equilibrium. Similarly, their "equilibrium" is what we have termed AG-equilibrium. The advantage of considering the more general case $\varepsilon > 0$ being the potential generalisation to situations where $U$ or $c$ may be discontinuous. Additionally, as the next section shows, our equilibrium notion enjoys better continuity properties.

Note, however, that in our definitions of equilibrium and $\varepsilon$-proto-equilibrium we have imposed the requirement that the mapping $p$ is non-negative and continuous. It is possible to drop these requirements to simply $p : X \longmapsto \mathbb{R}$ Borel measurable. Indeed, this is the original definition by Azevedo and Gottlieb. Nonetheless, there is no significant loss of generality in requiring that in equilibrium $p$ be continuous and non-negative. This is because requirement (i) in definition 3.22 (and the analogous requirement in [5]) implies $p(x) \geqslant 0$ $\alpha$-a.s. Note that, as it is done here, Azevedo and Gottlieb also require that $c$ to be non-negative. Similarly, imposing continuity ex ante does not rule out interesting solutions. In fact, the assumptions of regularity on $U$ needed to prove existence of equilibria along with the fact that the sequence of $\{X_n\}_{n=1}^{\infty}$ converging to $X$ contains only finite sets, imply the continuity of $p$.

As already mentioned, the proof of the existence of $\varepsilon$-proto-equilibria discussed here (theorem 3.45) follows closely the Azevedo and Gottlieb's strategy (see lemma 1 in the appendix of [5]). In fact, their result implies theorem 3.45 (see remark 3.24). The rationale behind providing an alternative proof is that our argument relies on more general tools than Azevedo and Gottlieb's. Notably, the proof that $\mathrm{Gr}(F_2)$ (see lemma 3.52) is more general than the equivalent argument in [5] (see step 2 in the proof of lemma 1 in their appendix). It might be further argued that it is considerably more elementary, as it relies only in notions of set convergence only. By contrast, in their proof (step 2) Azevedo and Gottlieb rephrase condition $\alpha(A_{0,p}) = 1$

as $\alpha$ is a solution $\max_{\alpha \in \Lambda} \mathbb{E}(U \circ p)$. Proceeding in this way adds additional steps (some made implicit in their proof of step 2) and does not generalise when $\varepsilon > 0$.

As already discussed, we place more stringent regularity assumptions on $U$ than Azevedo and Gottlieb. In particular, Azevedo and Gottlieb impose that $U$ must satisfy the conclusion in proposition 3.60 (besides continuity and $U$ being decreasing on $p$). Heuristically, proposition 3.60 simply states that if contract $x$ is preferred to the more expensive contract $y$ by some $\theta$, then the price difference cannot exceed the distance between the two contracts. By contrast, while we retain the assumptions of continuity of $U$ and monotonicity over $p$, we impose stricter conditions. Notably, we require that $U(\cdot, p, \theta)$ be a $C_1$-Lipschitz function for any choice of $p$ and $\theta$ (see condition (ii) in definition 3.58). Note however that, in practice, replacing proposition 3.60 by condition (ii) in definition does not make our result significantly less general. Indeed, when considering applications, condition (ii) is more easily verifiable and thus may be more appealing when considering particular examples. Note further that in addition to requiring $U(\cdot, p, \theta)$ being uniformly Lipschitz, we also require that the differential quotient

$$\left| \frac{U(x, p, \theta) - U(x, q, \theta)}{p - q} \right|$$

be both bounded and bounded away from 0 (see requirement (iii) in definition 3.58). This additional assumption is mostly technical as it is needed to ensure existence of equilibria when $\varepsilon > 0$. In particular, it guarantees the existence of Lipschitz extensions (see proposition 3.69 and corollary 3.63.1). Although formally more limiting, requirement (iii) in definition 3.58 does not seem to be too restrictive when considering applications.

Regarding existence of equilibrium, the arguments presented here follow the original work of Azevedo and Gottlieb closely. Our discussion is however more detailed than theirs. In particular, the construction of a convergent sequence of skeleta is absent in Azevedo and Gottlieb (see proof of theorem 1 in the appendix). Note furthermore that if $\varepsilon > 0$ then $p_\varepsilon$ is not Lipschitz, unlike the case $\varepsilon = 0$ (see lemma 2 in the appendix of [5]). Interestingly, this does not result in a significant obstacle as the main tool in Azevedo and Gottlieb's proof, the McShane-Whitney extension theorem, remains roughly valid in our setting (see proposition 3.69).

The section on equilibrium properties corresponds to proposition 1 in [5]. The results in that section are virtually identical to those of Azevedo and Gottlieb. As mentioned earlier, some of the statements in their paper do however contain inaccuracies, which we fix here. Importantly, their proof of proposition 3.79 (see the first part of the proof of lemma 4 in the appendix of [5]) incorrectly computes weak limits. In addition, Rademacher theorem (see corollary 3.73.2) only holds for open subsets and not for arbitrary ones as implied by [5]. Similarly, the proposition 3.83 is actually a sort of counter-positive; the choice is not accidental, doing so avoids assuming that the equilibrium $(p, \alpha)$ under consideration satisfies

$$\operatorname{supp}(\alpha) = \{(x, \theta) \in X \times \Theta : (U \circ p)(x, \theta) \geqslant (U \circ p)(y, \theta) \ \forall y \in X\}.$$

Note that, as a consequence of theorem 3.74, any equilibrium is a proto-equilibrium and thus only

$$\alpha(\{(x, \theta) \in X \times \Theta : (U \circ p)(x, \theta) \geqslant (U \circ p)(y, \theta) \ \forall y \in X\}) = 1$$

is known however.

Further to the above, as it was the case with the proof of lemma 3.52, the proof of proposition 3.81 presented is more elementary and should generalise better than the argument by Azevedo and Gottlieb (see second part of the proof of lemma 4 in their appendix). Moreover, proposition 3.73 is provided to indicate that the assumptions of continuity and regularity immediately imply Lipschitz-ness of any equilibria, without requiring equilibrium existence. If one is willing to trade measurability (as it is done here) by continuity, then the proof can be regarded as being more general than the one given in lemma 3 (see appendix in [5]).

## 3.10 Convergence of equilibria

In this final section, it is proven that if a sequence of equilibria converges to a limit, then the limit is an equilibrium of a certain economy. More formally,

**Theorem 3.86.** *Let $X$ and $\Theta$ be compact Polish spaces. Moreover, consider $\mu \in \Delta(\Theta \sqcup X)$ with supp$(\mu) \subset \Theta$. In addition, let $c : X \times \Theta \longmapsto [0, \infty)$ be continuous and $U : X \times [0, \infty) \times \Theta \longmapsto \mathbb{R}$ be a regular function in the sense of definition 3.58. Fix now a collection $\{X_n\}_{n=1}^{\infty} \subset 2^X$ of compact sets converging to the compact set $X_0 \subset X$ in the sense of Haussdorf. Suppose in addition that the sequence $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$, where $p_n : X_n \longmapsto [0, \infty)$ is continuous and $\alpha_n \in \Delta(X_n \times \Theta)$, satisfies the following:*

*(i) for each $n$, the pair $(p_n, \alpha_n)$ is an equilibrium of the economy*

$$(\mathfrak{E}_n, U|_{X_n \times [0,\infty) \times \Theta}, c|_{X_n \times \Theta}),$$

*where*

$$\mathfrak{E}_n = (\Theta \sqcup X_n, \mathcal{B}(\Theta \sqcup X_n), \mu).$$

*(ii) $\alpha_n \rightharpoonup \alpha$, where $\alpha \in \Delta(X_0 \times \Theta)$.*

*(iii) There exists a continuous mapping $p_0 : X_0 \longmapsto [0, \infty)$ such that whenever $\{x_n\}_{n=1}^{\infty}$ satisfies $x_n \in X_n$ and $x_n \to x$, then $p_n(x_n) \to p_0(x)$ holds.*

*Then $(p_0, \alpha_0)$ is an equilibrium of the economy*

$$(\mathfrak{E}_0, U|_{X_0 \times [0,\infty) \times \Theta}, c|_{X_0 \times \Theta}),$$

*where*

$$\mathfrak{E}_0 = (\Theta \sqcup X_0, \mathcal{B}(\Theta \sqcup X_0), \mu).$$

*Remark* 3.87. The proof of theorem 3.86 amounts to constructing a sequence $\{\varepsilon_m\}_{m=1}^{\infty}$ converging to zero and a sequence of proto-equilibria converging to $(p_0, \alpha_0)$. An element of the sequence of proto-equilibria is constructed by modifying an $\varepsilon_n$-proto-equilibria of the sequence that converges to a $(p_n, \alpha_n)$. This modification involves constructing a finite $Y_n \subset X_0$ and "relocating" consumers from their choice of contract to their nearest contract in $Y_n$. $\varepsilon_n$ is then increased to $\varepsilon_n'$, ensuring that these choices are $\varepsilon_n'$-optimal for consumers.

The proof of theorem 3.86 relies on a number of technical lemmas. Before introducing them it is necessary to fix some notation. First, let $d$ be a metric over $\Delta(X \times \Theta)$ such that $\alpha_n \rightharpoonup \alpha$ if, and only if, $d(\alpha, \alpha_n) \to 0$. Next, given $n \in \mathbb{N}$, denote by $\{(p_{n,k}, \alpha_{n,k})_{k=1}^{\infty}$ a sequence satisfying:

(i) for each $k \in \mathbb{N}$ there exists $\varepsilon_{n,k} \geqslant 0$, with $\varepsilon_{n,k} \to 0$ as $k \to \infty$, such that $(p_{n,k}, \alpha_{n,k})$ is a $\varepsilon_{n,k}$-proto-equilibria of the economy with skeleton

$$(X_{n,k} \sqcup \Theta, \mathcal{B}(X_{n,k} \sqcup \Theta), \mu + \eta_{n,k});$$

(ii) The sequence converges to the equilibrium $(p_n, \alpha_n)$, where $(p_n, \alpha_n)$ is as in theorem 3.86. Recall that this implies in particular $X_{n,k} \to X_n$ in the sense of Hausdorff.

*Remark* 3.88. Recall that for each $k$, $\eta_{n,k}$ is a finitely supported positive Borel measure over $\Theta \sqcup X_{n,k}$ satisfying $\mathrm{supp}(\eta_{n,k}) = X_{n,k}$ (so that in particular, $|X_{n,k}| < \infty$). Heuristically, this measure accounts for the added behavioural agents. Hence, it is assumed $\eta_{n,k} \to \mathbf{0}$ as $k \to \infty$. Recall further that $\alpha_{n,k}$ is a probability measure supported on $X_{n,k} \times \Theta$ which represents the choices of "true" consumers.

*Remark* 3.89. To avoid cluttering the notation unnecessarily, in what follows whenever we refer to $(p_n, \alpha_n)$ it is implicitly understood that it is a particular element of the sequence $\{(p_n, \alpha_n)\}_{n=1}^{\infty}$ in the statement of theorem 3.86. Similarly, when referring to $p_0$ it is to be understood as the limit (in the sense of theorem 3.86) of the sequence $\{p_n\}_{n=1}^{\infty}$.

**Lemma 3.90.** *Let $n \in \mathbb{N}$. There exists $k_n \in \mathbb{N}$ such that $(p_{n,k_n}, \alpha_{n,k_n})$, the $\frac{1}{n}$-proto-equilibrium of the economy with skeleton*

$$(X_{n,k_n} \sqcup \Theta, \mathcal{B}(X_{n,k_n} \sqcup \Theta), \mu + \eta_{n,k_n}),$$

*satisfies $p_{n,k_n}$ can be extended to a $K_1$-Lipschitz function $\tilde{p}_{n,k_n} : X \longmapsto \mathbb{R}$. Meanwhile, $p_n$ can be extended to $K_1$-Lipschitz function $\tilde{p}_n : X \longmapsto \mathbb{R}$, so that the following holds true:*

*(i)* $\max\limits_{x \in X_{n,k_n}} |\tilde{p}_{n,k_n}(x) - p_{n,k_n}(x)| < \frac{1}{n}$*;*

*(ii)* $\max\limits_{x \in X} |\tilde{p}_{n,k_n}(x) - \tilde{p}_n(x)| < \frac{1}{n}$*;*

*(iii)* $d(\alpha_n, \alpha_{n,k_n}) < \frac{1}{n}$*;*

*(iv)* $H(X_n, X_{n,k_n}) < \frac{1}{n}$;

*(v)* $\|\eta_{n,k_n}\| < \frac{1}{n}$.

*Remark* 3.91. Note that quantities with a tilde are defined over the whole space $X$. This convention is maintained throughout the remainder of this section. This convention is analogous to similar notation used in sections 6 and 7.

*Proof of lemma 3.90.* Let $\{(p_{n,k}, \alpha_{n,k})_{k=1}^{\infty}$ be a sequence such that for each $k \in \mathbb{N}$, $(p_{n,k}, \alpha_{n,k})$ is a $\varepsilon_{n,k}$-proto-equilibria of the economy with skeleton

$$(X_{n,k} \sqcup \Theta, \mathcal{B}(X_{n,k} \sqcup \Theta), \mu + \eta_{n,k}),$$

converging to $(p_n, \alpha_n)$.

Fix some $k \in \mathbb{N}$. Observe that since $U$ is a regular function, corollary 3.63.1 implies that

$$|p_{n,k}(x) - p_{n,k}(y)| \leqslant K_1 \cdot d_X(x, y) + \frac{\varepsilon_{n,k}}{K_2},$$

for arbitrary $x, y \in X_{n,k}$. Proposition 3.69 implies there exists $\tilde{p}_{n,k} : X \longmapsto \mathbb{R}$, which is $K_1$-Lipschitz, such that

$$\max_{x \in X_{n,k}} |p_{n,k}(x) - \tilde{p}_{n,k}(x)| \leqslant \frac{\varepsilon_{n,k}}{K_2}.$$

Now, by the same arguments as in the proof of lemma 3.67, the sequence $\{\tilde{p}_{n,k}\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous. Hence, there exists a subsequence of $\{\tilde{p}_{n,k}\}_{k=1}^{\infty}$ converging uniformly over $X$ to $\tilde{p}_n$. Assume without loss of generality that we have passed to any such subsequence; since each $\tilde{p}_{n,k}$ is $K_1$-Lipschitz by construction, $\tilde{p}_n$ is also $K_1$-Lipschitz. It is proven now that $\tilde{p}_n(x) = p_n(x)$ whenever $x \in X_n$. To that end, fix any such $x$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence converging to $x$ and satisfying $x_k \in X_{n,k}$.

Then

$$
\begin{aligned}
|p_n(x) - \tilde{p}_n(x)| \leqslant |p_n(x) - p_{n,k}(x_k)| + |p_{n,k}(x_k) - \tilde{p}_{n,k}(x_k)| \\
+|\tilde{p}_{n,k}(x_k) - \tilde{p}_n(x_k)| + |\tilde{p}_n(x) - \tilde{p}_n(x_k)| \leqslant \\
|p_n(x) - p_{n,k}(x_k)| + \frac{\varepsilon_{n,k}}{K_2} + \|\tilde{p}_{n,k} - \tilde{p}_n\|_\infty \\
+K_1 \cdot d_X(x, x_{n,k}),
\end{aligned}
\tag{3.125}
$$

where $\|\cdot\|_\infty$ refers to the $L^\infty$ norm over $X$. Since the left-hand side of equation (3.125) approaches 0 as $k \to \infty$, $\tilde{p}_n(x) = p_n(x)$ holds.

Now, there exists $k' \in \mathbb{N}$ such that

$$
\max\left\{H(X_n, X_{n,k}), d(\alpha_n, \alpha_{n,k}), \varepsilon_{n,k}, \frac{\varepsilon_{n,k}}{K_2}, \|\eta_{n,k}\|\right\} < \frac{1}{n},
$$

whenever $k \geqslant k'$. Moreover, it follows from the previous construction that there exists $k_2 \in \mathbb{N}$ such that $\|\tilde{p}_{n,k} - \tilde{p}_n\|_\infty < \frac{1}{n}$ whenever $k \geqslant k''$. Letting $k_n = \max\{k', k''\}$, properties (ii)-(v) in the statement of lemma 3.90 follow from this observation. In addition, since $\varepsilon_{n,k_n} < \frac{1}{n}$, remark 3.24 implies $(p_{n,k_n}, \alpha_{n,k_n})$ is a $\frac{1}{n}$-proto-equilibrium. All that remains it show that property (i) holds, but this follows since, by proposition 3.69,

$$
|\tilde{p}_{n,k_n}(x) - p_{n,k_n}(x)| \leqslant \frac{\varepsilon_{n,k_n}}{K_2} < \frac{1}{n} \tag{3.126}
$$

for any $x \in X_{n,k_n}$. ■

**Lemma 3.92.** *Let $\{\tilde{p}_{n,k_n}\}_{n=1}^\infty$ be such that for each $n$, $\tilde{p}_{n,k_n}$ satisfies the conclusions of lemma 3.90. There exists $\tilde{p}_0 : X \longmapsto \mathbb{R}$ $K_1$-Lipschitz which extends $p_0$, with the following property: there exists $\{\tilde{p}_{n_m,k_{n_m}}\}_{m=1}^\infty$ satisfying $\tilde{p}_{n_m,k_{n_m}} \to \tilde{p}_0$ uniformly.*

*Proof.* The argument follows the similar lines as the proof of lemma 3.90. The sequence $\{\tilde{p}_{n,k_n}\}_{n=1}^\infty$ is equicontinuous since it consists of $K_1$-Lipschitz functions. It is moreover uniformly bounded. To see this, fix $x \in X$ and for each $n$, fix some $x_n \in X_{n,k_n}$. Then

$$\begin{aligned}|\tilde{p}_{n,k_n}(x)| &\leqslant |\tilde{p}_{n,k_n}(x) - \tilde{p}_{n,k_n}(x_n)| + |p_{n,k_n}(x_n) - \tilde{p}_{n,k_n}(x_n)| \\ &\quad + |p_{n,k_n}(x_n)| \leqslant K_1 \cdot d_X(x, x_n) + \frac{1}{n} + \|c\|_\infty.\end{aligned} \tag{3.127}$$

The last inequality follows since $\tilde{p}_{n,k_n}$ is $K_1$-Lipschitz and it satisfies (i) in lemma 3.90. Furthermore, since $p_{n,k_n}$ is a proto equilibrium:

$$|p_{n,k_n}(x_n)| \leqslant \mathbb{E}_{\alpha_{n,k_n}+\eta_{n,k_n}}(|c(\xi,\cdot)| \, |\xi = x_n) \leqslant \|c\|_\infty,$$

where $\|c\|_\infty = \sup_{(x,\theta)\in X\times\Theta} |c(x,\theta)|$.

The Arzelà-Ascoli theorem now implies that $\{\tilde{p}_{n,k_n}\}_{n=1}^\infty$ has a subsequence converging uniformly to a $K_1$-Lipschitz $\tilde{p}_0$. All that remains to show is that $\tilde{p}_0(x) = p_0(x)$ for all $x \in X_0$. Fix accordingly any such $x$ and let $\{x_n\}_{n=1}^\infty$, with $x_n \in X_n$, be such that $x_n \to x_0$. Then

$$\begin{aligned}|p_0(x) - \tilde{p}_0(x)| &\leqslant |p_0(x) - p_{n_m}(x_{n_m})| + |\tilde{p}_{n_m}(x_{n_m}) - \tilde{p}_{n_m,k_{n_m}}(x_{n_m})| \\ &\quad + |\tilde{p}_0(x_{n_m}) - \tilde{p}_{n_m,k_{n_m}}(x_{n_m})| + |\tilde{p}_0(x_{n_m}) - \tilde{p}_0(x)|,\end{aligned} \tag{3.128}$$

where $\{\tilde{p}_{n_m,k_{n_m}}\}_{m=1}^\infty$ is the subsequence converging to $\tilde{p}_0$. Lemma 3.90 (ii) now implies $|\tilde{p}_{n_m}(x_{n_m}) - \tilde{p}_{n_m,k_{n_m}}(x_{n_m})| < \frac{1}{n_m}$. In addition, since $\tilde{p}_0$ is $K_1$-Lipschitz equation (3.128) becomes

$$\begin{aligned}|p_0(x) - \tilde{p}_0(x)| &\leqslant |p_0(x) - p_{n_m}(x_{n_m})| + \frac{1}{n_m} \\ &\quad + \|\tilde{p}_0 - \tilde{p}_{n_m,k_{n_m}}\|_\infty + K_1 \cdot d_X(x, x_{n_m}).\end{aligned} \tag{3.129}$$

Since, by assumption, $p(x_n) \to p_0(x)$, the left-hand side of equation (3.129) approaches 0 as $m \to \infty$. As a result, $p_0(x) = \tilde{p}_0(x)$ as required. ■

**Lemma 3.93.** *Let $n \in \mathbb{N}$. There exists $Y_n \subset X_0$ finite, and $\hat{m} > n$ with the following properties:*

*(i)* $H(Y_n, X_0) < \frac{1}{n}$,

*(ii)* $|X_{\hat{m},k_{\hat{m}}}| \geqslant |Y_n|$ *for some finite* $X_{\hat{m},k_{\hat{m}}} \subset X_{\hat{m}}$.

(iii) *There exists a surjection* $\phi_n : X_{\hat{m},k_{\hat{m}}} \longmapsto Y_n$ *satisfying* $d_X(x, \phi(x)) < \frac{1}{n}$, *for all* $x \in X_{\hat{m},k_{\hat{m}}}$.

*Remark* 3.94. Heuristically, lemma 3.93 serves two functions. First, it determines the set of contracts (namely, $Y_n$) available to agents in the approximate economy. Second, it determines the contracts agents purchase in $Y_n$ through $\phi_n$.

*Proof of lemma 3.93.* Observe first that $X_0$ is separable since it is a closed subset of the Polish space $X$. Let $\{q_m\}_{m=1}^{\infty}$ be a countably dense subset of $X_0$. For $n \in \mathbb{N}$, let $A_m = \{q_1, \ldots, q_m\}$. Since

$$\overline{\bigcup_{n=1}^{\infty} A_m} = X_0,$$

$A_m \to X_0$ in the sense of Hausdorff. It follows that given $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $H(A_m, X_0) < \frac{1}{2n}$ for any $m \geqslant m_n$. Setting now $Y_n = A_{m_n}$ yields (i).

On the other hand, it follows from lemma 3.90 (iv) that there exists a subsequence $\{X_{m,k_m}\}_{m=1}^{\infty}$ converging to $X_0$ in the sense of Hausdorff. Hence, there exists $m_2 \in \mathbb{N}$ such that $H(X_{m,k_m}, X_0) < \frac{1}{2n}$ whenever $m \geqslant m_2$. Moreover, since $\{X_{m,k_m}\}_{m=1}^{\infty}$ contains only finite sets, there exists $\hat{m} \geqslant m_2$ such that $|X_{\hat{m},k_{\hat{m}}}| \geqslant |Y_n|$. Note that if no such set existed, the fact that $X_{m,k_m} \to Y_0$ in the sense of Hausdorff would imply $|X_0| < |Y_n|$ since the sets $X_{m,k_m}$ are all finite. This conclusion would contradict the fact that $Y_n \subset X_0$. Observe further that this reasoning implies that the number of subsets $X_{m,k_m}$ with $|X_{m,k_m}| \geqslant |Y_n|$ must be infinite. Consequently, it may be assumed that $\hat{m} > n$. This last conclusion establishes (ii).

To conclude the proof, let $x \in X_{\hat{m},k_{\hat{m}}}$ and define $\phi_n : x \mapsto y_x$, where $y_x \in Y_n$ satisfies $d_X(x, y_x) \leqslant d_X(x, y)$ for any $y \in Y_n$. Observe that since both $X_{\hat{m},k_{\hat{m}}}$ and $Y_n$ are finite, such $y_x$ exists. As a consequence of this definition

$$d_X(x, y_x) = d_X(x, Y_n) \leqslant H(X_{\hat{m},k_{\hat{m}}}, Y_n) \leqslant H(X_{\hat{m},k_{\hat{m}}}, X_0) + H(X_0, Y_n). \tag{3.130}$$

Hence $d(x, \phi_n(x)) < \frac{1}{n}$ as required. ∎

The proof of theorem 3.86 proceeds by modifying the $\frac{1}{\hat{m}}$-proto-equilibrium $(p_{\hat{m},k_{\hat{m}}}, \alpha_{\hat{m},k_{\hat{m}}})$ by shifting agents' purchases from $X_{\hat{m},k_{\hat{m}}}$ to $Y_n$. Details of the formal construction of the resulting candidate $\varepsilon_n$-proto-equilibrium, $(q_n, \beta_n)$, are provided below in remark 3.95.

*Remark* 3.95. Choose $n \in \mathbb{N}$. Fix some $Y_n \subset X_0$ finite and $\phi_n : X_{\hat{m},k_{\hat{m}}} \longmapsto Y_n$, where $X_{\hat{m},k_{\hat{m}}}$ is a set satisfying (ii) in lemma 3.93, both satisfying properties (i)-(iii) in lemma 3.93. Denote by $\overline{\phi}_n : X_{\hat{m},k_{\hat{m}}} \times \Theta \longmapsto Y_n \times \Theta$ the mapping $\overline{\phi}_n : (x, \theta) \mapsto (\phi_n(x), \theta)$. Using $\overline{\phi}_n$, define the measure $\beta_n \in \Delta(Y_n \times \Theta)$ as the pushforward of $\alpha_{\hat{m},k_{\hat{m}}}$ by $\overline{\phi}_n$. To construct $q_n : Y_n \longmapsto [0, \infty)$, proceed similarly by letting $\kappa_n$ be the pushforward of $\eta_{\hat{m},k_{\hat{m}}}$ by $\phi_n$. Recall now that $\overline{\eta}_{\hat{m},k_{\hat{m}}}$ is the finite measure with $\operatorname{supp}(\overline{\eta}_{\hat{m},k_{\hat{m}}}) \subset X \times X$ with $\overline{\eta}_{\hat{m},k_{\hat{m}}}(x, \theta) = \delta_x(\theta)\eta_{\hat{m},k_{\hat{m}}}(x)$ and define $\gamma_n \in \Delta(Y_n \times (\Theta \sqcup Y_n))$ as

$$\gamma_n = (\overline{\phi}_n)_*(\alpha_{\hat{m},k_{\hat{m}}} + \overline{\eta}_{\hat{m},k_{\hat{m}}}) = \beta_n + \overline{\kappa}_n,$$

where $\overline{\kappa}_n = (\overline{\phi}_n)_*(\overline{\eta}_{\hat{m},k_{\hat{m}}})$. Fix $x \in Y_n$ (understood as a contract, not a behavioural consumer) and observe that we have

$$\gamma_n(S|\xi = x) = \frac{\beta_n(\{x\} \times (S \cap \Theta)) + \overline{\kappa}_n(\{x\} \times (S \cap Y_n))}{\beta_n(\{x\} \times \Theta) + \kappa_n(x)}, \tag{3.131}$$

where $S \in \mathcal{B}(\Theta \sqcup Y_n)$. Observe that, using the definition of $\gamma$ as a pushforward, equation (3.131) may be rewritten as

$$\begin{aligned} \gamma_n(S|\xi = x) &= \frac{\sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(\{y\} \times S)}{\sigma_{\hat{m}}(\phi_n^{-1}(\{x\}) \times (\Theta \sqcup X_{\hat{m},k_{\hat{m}}}))} \\ &= \sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(S|\xi = y) \cdot \frac{\sigma_{\hat{m}}(\{y\} \times (\Theta \sqcup X_{\hat{m},k_{\hat{m}}}))}{\sigma_{\hat{m}}(\phi_n^{-1}(\{x\}) \times (\Theta \sqcup X_{\hat{m},k_{\hat{m}}}))}, \end{aligned} \tag{3.132}$$

where $\sigma_{\hat{m}} = \alpha_{\hat{m},k_{\hat{m}}} + \overline{\eta}_{\hat{m},k_{\hat{m}}}$. Equation (3.132) can be simplified further by writing

$$\sigma_{\hat{m}}(y|\phi_n^{-1}(\{x\})) = \frac{\sigma_{\hat{m}}(\{y\} \times (\Theta \sqcup X_{\hat{m},k_{\hat{m}}}))}{\sigma_{\hat{m}}(\phi_n^{-1}(\{x\}) \times (\Theta \sqcup X_{\hat{m},k_{\hat{m}}}))},$$

then

$$\gamma_n(S|\xi = x) = \sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(\{x\}))\sigma_{\hat{m}}(S|\xi = y). \tag{3.133}$$

From equation (3.133) define $q_n$ as

$$\begin{aligned} q_n(x) = \mathbb{E}_{\gamma_n}(c(\xi,\cdot)|\xi = x) &= \sum_{y\in\phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(x))\mathbb{E}_{\sigma_{\hat{m}}}(c(\xi,\cdot)|\xi = y) \\ &= \sum_{y\in\phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(x))p_{\hat{m},k_{\hat{m}}}(y). \end{aligned} \tag{3.134}$$

The proof of theorem 3.86 can now be provided. Formally, the argument involves two steps. First, one needs to construct a sequence of proto-equilibria of certain economies, which approximate the economy

$$(\mathfrak{T}, U|_{X_0\times[0,\infty)\times\Theta}, c|_{X_0\times\Theta}),$$

where

$$\mathfrak{T} = (\Theta \sqcup X_0, \mathcal{B}(\Theta \sqcup X_0), \mu).$$

Second, it is necessary to demonstrate that the sequence constructed satisfies the conditions in definition 3.26. In the interest of clarity, the proof is thus split into two propositions.

**Proposition 3.96.** *Choose some $n \in \mathbb{N}$. Set*

$$\varepsilon_n = \frac{2C_1 + 4C_3(K_1 + 1) + 1}{n},$$

*where $C_1$ and $C_3$ are the positive constants in definition 3.58, and let $Y_n \subset X_0$ finite, and $\hat{m} > n$ both be as in lemma 3.93. Define $q_n : Y_n \longmapsto [0,\infty)$ as*

$$q_n : x \mapsto \mathbb{E}_{\gamma_n}(c(\xi,\cdot)|\xi = x),$$

*where $\gamma_n = (\overline{\phi}_n)_*(\alpha_{\hat{m},k_{\hat{m}}} + \overline{\eta}_{\hat{m},k_{\hat{m}}})$, with $\overline{\phi}_n : (x,\theta) \mapsto (\phi_n(x),\theta)$. Here, $\phi_n$ is the mapping satisfying (iii) in lemma 3.93. Moreover, let $\beta_n = (\overline{\phi}_n)_*(\alpha_{\hat{m},k_{\hat{m}}})$. The pair $(q_n,\beta_n)$ is an $\varepsilon_n$-proto-equilibrium of the economy*

$$(\mathfrak{T}_n, U|_{Y_n\times[0,\infty)\times\Theta}, c|_{Y_n\times\Theta}),$$

*where*

$$\mathfrak{T}_n = (\Theta \sqcup Y_n, \mathcal{B}(\Theta \sqcup Y_n), \mu + (\phi_n)_*(\eta_{\hat{m},k_{\hat{m}}})).$$

*Proof.* The argument amounts to checking properties (i')-(iii') in proposition 3.30. Observe that the definition of $q_n$ as a conditional expectation yields (i'). Moreover,

$$\begin{aligned}\beta_n(Y_n \times B) &= \sum_{x \in Y_n} \sum_{y \in \phi_n^{-1}(\{x\})} \alpha_{\hat{m},k_{\hat{m}}}(\{y\} \times B) \\ &= \sum_{x \in X_{\hat{m},k_{\hat{m}}}} \alpha_{\hat{m},k_{\hat{m}}}(\{x\} \times B) = \mu(B),\end{aligned}$$

for any $B \in \mathcal{B}(\Theta)$. Condition (iii') in proposition 3.30 is thus fulfilled. To establish the remaining condition, namely that

$$\beta_n(\{(x,\theta) \in Y_n \times \Theta : (U \circ q_n)(x,\theta) + \varepsilon_n \geqslant U(y,\theta)\ \forall y \in Y_n\}) = 1$$

holds for $\varepsilon_n > 0$ as in the rubric; we prove that for any $(x,\theta) \in X_{\hat{m},k_{\hat{m}}}$ the bound

$$|(U \circ p_{\hat{m},k_{\hat{m}}})(x,\theta) - U(\phi_n(x),(q_n \circ \phi_n)(x),\theta)| \leqslant \frac{C_1 + 2C_3(K_1+1)}{n}, \tag{3.135}$$

where $C_1$ and $C_3 > 0$ are the constants in definition 3.58, holds. To that end, recall first that for $x \in Y_n$,

$$q_n(x) = \sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(x)) p_{\hat{m},k_{\hat{m}}}(y).$$

Hence, applying (i) in lemma 3.90

$$\begin{aligned}|q_n(x) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| \leqslant& \sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(x))|p_{\hat{m},k_{\hat{m}}}(y) - \tilde{p}_{\hat{m},k_{\hat{m}}}(y)| \\ &+ \sum_{y \in \phi_n^{-1}(\{x\})} \sigma_{\hat{m}}(y|\phi_n^{-1}(x))|\tilde{p}_{\hat{m},k_{\hat{m}}}(y) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| \\ &\leqslant \frac{1}{\hat{m}} + K_1\, d_X(y,\phi_n(y)) < \frac{K_1+1}{n}.\end{aligned} \tag{3.136}$$

Now, given $(x,\theta) \in X_{\hat{m},k_{\hat{m}}} \times \Theta$, the fact that $U$ is a regular function implies

$$\begin{aligned}|(U \circ p_{\hat{m},k_{\hat{m}}})(x,\theta) - U(\phi_n(x),(p \circ \phi_n)(x,\theta)| \leqslant& C_3|(p_{\hat{m},k_{\hat{m}}}(x) - (q_n \circ \phi_n)(x)| \\ &+ C_1 \cdot d_X(x,\phi_n(x)).\end{aligned} \tag{3.137}$$

Meanwhile, if $x \in X_{\hat{m},k_{\hat{m}}}$,

$$\begin{aligned}|p_{\hat{m},k_{\hat{m}}}(x) - (q_n \circ \phi_n)(x)| &\leqslant |p_{\hat{m},k_{\hat{m}}}(x) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| \\ +|(\tilde{p}_{\hat{m},k_{\hat{m}}} \circ \phi_n)(x) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| &+ |(\tilde{p}_{\hat{m},k_{\hat{m}}} \circ \phi_n)(x) - (q_n \circ \phi_n)(x)| \\ &< \frac{1}{n} + K_1 \cdot d_X(x, \phi_n(x)) + \frac{K_1+1}{n} < (K_1+1)\cdot\frac{2}{n}.\end{aligned} \tag{3.138}$$

The second to last inequality is a consequence of lemma 3.90 (i); the fact that $\tilde{p}_{\hat{m},k_{\hat{m}}}$ is $K_1$-Lipschitz; and relation (3.136). Inequality (3.137) now yields

$$|(U \circ p_{\hat{m},k_{\hat{m}}})(x,\theta) - U(\phi_n(x), (q_n \circ \phi_n)(x), \theta)| \leqslant \frac{C_1 + 2C_3(K_1+1)}{n}$$

as required.

Now, since $(p_{\hat{m},k_{\hat{m}}}, \alpha_{\hat{m},k_{\hat{m}}})$ is a $\frac{1}{n}$-proto-equilibrium, it follows that

$$\alpha_{\hat{m},k_{\hat{m}}}\left(\left\{(U \circ p_{\hat{m},k_{\hat{m}}})(x,\theta) + \frac{1}{n} \geqslant (U \circ p_{\hat{m},k_{\hat{m}}})(y,\theta)\ \forall y \in X_{\hat{m},k_{\hat{m}}}\right\}\right) = 1; \tag{3.139}$$

(note that the above set is a subset of $X_{\hat{m},k_{\hat{m}}} \times \Theta$. To avoid excessive clutter, this fact has been omitted).

Now, the bound in (3.135) implies that

$$A_{\varepsilon_n,q_n} = \{(U \circ q_n)(\phi_n(x),\theta) + \varepsilon_n \geqslant (U \circ q_n)(\phi_n(y),\theta)\ \forall y \in X_{\hat{m},k_{\hat{m}}}\}$$

satisfies $\alpha_{\hat{m},k_{\hat{m}}}(A_{\varepsilon_n,q_n}) = 1$. Moreover, since $\phi_n$ is surjective,

$$A_{\varepsilon_n,q_n} = \{(U \circ q_n)(\phi_n(x,\theta)) + \varepsilon_n \geqslant (U \circ q_n)(y,\theta)\ \forall y \in Y_n\}.$$

It follows from this last equality along with the fact that $\alpha_{\hat{m},k_{\hat{m}}}(A_{\varepsilon_n,q_n}) = 1$ and the definition of $\beta_n$ as $(\overline{\phi}_n)_*(\alpha_{\hat{m},k_{\hat{m}}})$, that

$$\begin{aligned}\beta_n(\{(x,\theta) \in Y_n \times \Theta : (U \circ q_n)(x,\theta) + \varepsilon_n \geqslant (U \circ q_n)(y,\theta)\ \forall y \in Y_n\})& \\ = \alpha_{\hat{m},k_{\hat{m}}}(A_{\varepsilon_n,q_n}) = 1.&\end{aligned} \tag{3.140}$$

Hence condition (ii') in proposition 3.30 holds. ■

**Proposition 3.97.** *Let $\{(q_n, \beta_n)\}_{n=1}^{\infty}$, where for each $n$, the pair $(q_n, \beta_n)$ is the $\varepsilon_n$-proto-equilibrium in proposition 3.96. The sequence satisfies:*

*(i) Given $x \in X_0$ and an arbitrary sequence $\{y_n\}_{n=1}^{\infty}$ converging to $x$, satisfying for each $n$, $y_n \in Y_n$; $q_n(y_n) \to p_0(x)$.*

*(ii) $\beta_n \rightharpoonup \alpha_0$.*

The proof of proposition 3.97 makes use of the following technical

**Lemma 3.98.** *Let $M$ be a compact metric space and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous mappings, where for each $n$, $T_n : M_n \longmapsto M$ (with $M_n \subset M$). Suppose further that $\sup_{t\in M_n} d_M(t, T_n(t)) \to 0$ as $n \to \infty$. Consider now a sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Delta(M)$ such that supp$(\nu_n) = M_n$. If $\nu_n \rightharpoonup \nu$, then $(T_n)_*(\nu_n) \rightharpoonup \nu$.*

*Proof.* Fix $f : M \longmapsto \mathbb{R}$ continuous. Observe that for a given $n$ the change of variables formula implies

$$|\mathbb{E}_{\nu}(f) - \mathbb{E}_{(T_n)_*(\nu_n)}(f)| \leqslant \int_{M_n} |(f \circ T_n)(t) - f(t)|\nu_n(dt) + |\mathbb{E}_{\nu}(f) - \mathbb{E}_{\nu_n}(f)|. \tag{3.141}$$

Since $M$ compact, $f$ is uniformly continuous. Hence, given $\delta > 0$ there exists $\xi > 0$ such that $d_M(t, s) < \xi$ implies $|f(t) - f(s)| < \delta$. In particular, since $\sup_{t\in M_n} d_M(t, T_n(t)) \to 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, the bound $\sup_{t\in M_n} |f(t) - f(T_n(t))| \leqslant \delta$ holds. Equation (3.141) now yields,

$$\limsup_{n\to\infty} |\mathbb{E}_{\nu}(f) - \mathbb{E}_{(T_n)_*(\nu_n)}(f)| \leqslant \delta. \tag{3.142}$$

Since $\delta > 0$ was arbitrary the lemma follows. ■

*Proof of proposition 3.97.* To prove (i), fix $n \in \mathbb{N}$. Recall now that given $x \in Y_n$,

$$|q_n(x) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| \leqslant \frac{K_1 + 1}{n}.$$

Moreover, by passing to a subsequence if necessary, it can be assumed that

$$\|\tilde{p}_{\hat{m},k_{\hat{m}}} - \tilde{p}_0\|_{\infty} < \frac{1}{\hat{m}} \leqslant \frac{1}{n}.$$

This last inequality implies for any $x \in Y_n$ the bound

$$|q_n(x) - \tilde{p}_0(x)| \leqslant |q_n(x) - \tilde{p}_{\hat{m},k_{\hat{m}}}(x)| + \|\tilde{p}_{\hat{m},k_{\hat{m}}} - \tilde{p}_0\|_\infty < \frac{K_1 + 2}{n}. \quad (3.143)$$

As before $\| \cdot \|_\infty$ is taken over the whole space $X$.

Now, let $x \in X_0$ and an fix an arbitrary sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in Y_n$ converging to $x$. Inequality (3.143) implies

$$\limsup_{n\to\infty} |q_n(x_n) - \tilde{p}_0(x_n)| \leqslant \lim_{n\to\infty} \frac{K_1 + 2}{n} = 0. \quad (3.144)$$

Hence, since $\tilde{p}_0|_{X_0} = p_0$, $q_n(x_n) \to p_0(x)$. Consequently, (i) follows.

To prove (ii) observe that lemma 3.90 (iii) implies

$$d(\alpha_{\hat{m}}, \alpha_{\hat{m},k_{\hat{m}}}) < \frac{1}{\hat{m}} \leqslant \frac{1}{n}.$$

Since $\alpha_n \rightharpoonup \alpha_0$, it follows that $\alpha_{\hat{m},k_{\hat{m}}} \rightharpoonup \alpha_0$. Lemma 3.98 now yields $\beta_n \rightharpoonup \alpha_0$.

■

*Proof of theorem 3.86.* Let $\{\varepsilon_n\}_{n=1}^\infty$ with

$$\varepsilon_n = \frac{2C_1 + 4C_3(K_1 + 1) + 1}{n}$$

and let $\{(q_n, \beta_n)\}_{n=1}^\infty$ where each pair $(q_n, \beta_n)$ is as in proposition 3.96. In particular, proposition 3.96 implies that for each $n$, $(q_n, \beta_n)$ is a $\varepsilon_n$-proto-equilibrium of the economy with skeleton

$$\mathfrak{T}_n = (Y_n \sqcup \Theta, \mathcal{B}(Y_n \sqcup \Theta), \mu + \kappa_n).$$

Now, $\varepsilon_n \to 0$ by construction and $Y_n \to X_0$ in the sense of Hausdorff by (i) in lemma 3.93. Moreover, given $x \in Y_n$

$$\kappa_n(x) = \sum_{y \in \phi_n^{-1}(\{x\})} \eta_{\hat{m},k_{\hat{m}}}(y) \leqslant \eta_{\hat{m},k_{\hat{m}}}(X_{\hat{m},k_{\hat{m}}} \times X_{\hat{m},k_{\hat{m}}}) \to 0. \quad (3.145)$$

It follows that the sequence of bones $\{\mathfrak{T}_n\}_{n=1}^{\infty}$ converges to

$$(\Theta \sqcup X_0, \mathcal{B}(\Theta \sqcup X_0), \mu).$$

All that remains, is to verify that conditions (i)-(iii) in definition 3.26 hold. As mentioned at the onset, proposition 3.96 implies (i) holds. On the other hand, proposition 3.97 implies that (ii) and (iii) are satisfied. ■

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