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b-functions for rational Cherednik algebras

by

Rhys Davies

A thesis submitted in fulfilment of the requirements
for the degree of

Doctor of Philosophy

at the

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College of Science & Engineering
University of Glasgow



15 October 2026

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Abstract

In this thesis, we study Bernstein-Sato b-functions for rational Cherednik algebras. We prove (unconditionally in the parameter c) their existence in the case that the W-invariant polynomial f is homogeneous with isolated singularities. Our main theorem is an explicit (factored) formula for the b-function of a W-invariant polynomial homogeneous f with isolated singularities. The theorem shows that the degree of the b-function varies depending on the parameter c. Moreover, examples show that the b-function can have repeated roots when $c \neq 0$. Using the Koszul complex, we give an explicit formula for the graded W-character of the Milnor algebra M_f associated to f. This allows us to compute the eigenvalues of the Euler operator, which appear in our formula for the roots of the b-function. As a consequence, we give MAGMA code that will compute the b-function in examples. Finally, we classify those irreducible complex reflection groups that admit a set of fundamental homogeneous invariants, all of which have isolated singularities.

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Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Chapter 1

Introduction

Pavel Etingof and Victor Ginzburg first introduced symplectic reflection algebras in the seminal paper [EG02], and defined rational Cherednik algebras $\mathtt{H}_c(W)$ as a particular family of symplectic reflection algebras—those associated to a complex reflection group W acting on its reflection representation $\mathfrak{h}$. These algebras are parametrized by conjugation invariant functions on the set of complex reflections in W. They can be realised as a two-step degeneration of the double affine Hecke algebra $\mathtt{H}$:

$$\mathtt{H} = \mathtt{H}^{\text{elliptic}} \rightsquigarrow \mathtt{H}^{\text{trigonometric}} \rightsquigarrow \mathtt{H}^{\text{rational}} = \mathtt{H}_c.$$

Double affine Hecke algebras, also called Cherednik algebras, were introduced by Ivan Cherednik in the paper [Che92] in order to prove Macdonald's constant term conjectures about orthogonal polynomials for root systems. They also arise as deformation quantizations in the theory of Poisson algebras [Eti07].

The rational Cherednik algebra has a very rich representation theory. For instance, its representation theory is closely related to that of Schur algebras [Rou08] and Hecke algebras (via the Knizhnik–Zamolodchikov functor [GGOR03]) and it provides a categorification of the higher level Fock spaces [Sha11].

As a result of these various guises, much of the theory of rational Cherednik algebras sits at the intersection of areas such as Lie theory, algebraic geometry (Hilbert schemes, resolutions of symplectic quotient singularities), algebraic analysis (that is, $\mathcal{D}$-module theory) combinatorics (Macdonald theory, $n!$ conjecture), integrable systems, quiver varieties and representation theory; see for example the lecture notes [Bel16] or the survey article [Gor08].

§ 1.1. What this thesis is about

In this thesis, we study the representation theory of rational Cherednik algebras $\mathtt{H}_c$ using b-functions. These b-functions (also known as Bernstein—Sato polynomials) were originally introduced for $\mathcal{D}$-modules independently by Bernstein and Sato, in the papers

[Ber71] and [SS72] respectively, to prove that the push-forward of a holonomic $\mathcal{D}$-module under an affine open embedding is again holonomic. This is the key step in the general proof that the (derived) push-forward and pull-back functors preserve holonomicity.

Classically, for a polynomial $f \in k[x_1, \ldots, x_n]$, the b-function $b_f(s)$ is described by the following lemma, known as "The b-function Lemma":

Lemma 1.1. *Let k be an algebraically closed field of characteristic 0, let $f \in k[x_1, \ldots, x_n]$ be a polynomial, and $\mathcal{D}_n$ the ring of differential operators acting on $k[x_1, \ldots, x_n]$. Then there exists a differential operator $P \in \mathcal{D}_n[s]$ and monic non-zero polynomial $b_f \in k[s]$ of minimal degree (called the b-**function** of f) such that*

$$P(s)f(x)^{s+1} = b_f(s)f(x)^s.$$

We note that one can show that, for $f \in k[x_1, \ldots, x_n]$, the b-function generates an ideal $(b_f(s))$ of $k[s]$, which is a PID. The b-function is therefore unique. The operator $P(s)$ satisfying the b-function equation however is not unique since the annihilator of $f(x)^{s+1}$ in $\mathcal{D}_n[s]$ is non-zero in general.

Since their introduction, b-functions have had many important applications in the theory of $\mathcal{D}$-modules. For instance, they are essential in defining and studying the vanishing and nearby cycle functors for $\mathcal{D}$-modules [Kas83]. They also play a significant role in algebraic geometry, particularly in the study of jumping numbers and multiplier ideals; see for instance the survey [Sai09]. Another important application of the theory of b-functions lies in the computation of the eigenvalues of the monodromy operator on the Milnor fibre of a function with isolated singularities [Mal74].

Despite their seemingly simple definition and the fact that there is an abstract existence theorem for them, b-functions are notoriously difficult to compute explicitly, and relatively few non-trivial examples are known.

In the paper [BET26], the authors extended the definition of the b-function to the setting of rational Cherednik algebras. They proved the existence of the b-function under certain genericity assumptions on the parameter c, and used this result to show that the push-forward of a holonomic module for the rational Cherednik algebra remains holonomic. This result was the main motivation for introducing the notion of the b-function in that context.

One important class of examples where the roots of the b-function are explicitly known for $\mathcal{D}$-modules arises when the function f is (quasi-)homogeneous and has isolated singularities, a result due to Yano [Yan78, Proposition 3.6; Theorem 3.8] and Briançon, Granger, Maisonobe and Miniconi [BGMM89, Théorème C.2.2.3]. Recall that a point $p \in X$ in the vanishing locus $X = V(f)$ of f is a *singularity* (or *singular point*) of f if $\dim_p X > \dim X$. We say that f has *isolated singularities* if, for each $p \in X_{sing}$

there exists some open set U such that $U \cap X_{sing} = \{p\}$, where X_{sing} denotes the set of singularities of f.

A key property of the b-function associated with a polynomial having an isolated singularity is that it is reduced [Yan78, Corollary 3.9 ii)], meaning it has no repeated roots. In this thesis, we prove a general formula (Theorem 5.22) for the b-function associated with any homogeneous polynomial having isolated singularities which is invariant under the action of a complex reflection group W. This establishes that an analogue of the above result holds for rational Cherednik algebras. The first step in doing so is to establish (for all parameters c) the existence of the b-function. This strengthens the general existence result obtained in [BET26, Corollary 1.2] for this class of polynomials.

If f is a W-invariant homogeneous polynomial then $f \in J(f)$, the Jacobian ideal generated by all partial derivatives $f_i = \frac{\partial f}{\partial x_i}$. To understand the b-function in this case we first introduce the module

$$\mathcal{M} = \mathtt{H}_c f^s / \mathtt{H}_c J(f) f^s.$$

As shown in (4.2.2), the b-function b_c^f can be characterised as the generator of the annihilator ideal

$$(b_c^f(s)) = \mathrm{Ann}_{\mathbb{C}[s]}(\mathtt{H}_c[s] f^s / \mathtt{H}_c[s] f^{s+1}).$$

As in the classical case, it is easily seen that $s+1$ divides b_c^f. The *augmented b-function* $\widetilde{b}_c^f$ is defined to be $(s+1)^{-1} b_c^f$. We show that the augmented b-function of f is the minimal polynomial of s acting on $\mathcal{M}$. Therefore, in the case that f also has isolated singularities, we must show that this minimal polynomial exists. We do so by first proving that $\mathcal{M}$ is in category $\mathcal{O}_c^*$ (as defined in Section 3.7), from which we deduce that it is a holonomic $\mathtt{H}_c$-module and $\mathrm{End}_{\mathtt{H}_c}(\mathcal{M})$ is finite dimensional over $\mathbb{C}$; see Lemma 5.15 and Proposition 5.16. This implies the existence of the b-function. Moreover, this shows that $\mathcal{M}$ is a finite length module and is semi-simple for generic c.

In deriving the formula for the b-function of a W-invariant homogeneous polynomial with isolated singularities, we need two key objects. The first is the Milnor algebra

$$M_f = \mathbb{C}[\mathfrak{h}]/J(f),$$

which is a finite dimensional graded algebra when f is homogeneous with isolated singularities. Let N be the positive integer such that $(M_f)_i = 0$ for all $i > N$ and $(M_f)_N \neq 0$. The second object is the Euler operator

$$\mathbf{eu}_c = \sum_{i=1}^{n} y_i x_i + z_c \in \mathtt{H}_c$$

where z_c is a certain element of the centre $Z(\mathbb{C}W)$ of the group W; see Lemma 3.7 for the definition. It has the property that $[\mathbf{eu}_c, x] = x, [\mathbf{eu}_c, y] = -y$ and $[\mathbf{eu}_c, w] = 0$ for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$. The action of z_c on M_f is diagonalisable and we let E_i denote the set of eigenvalues of z_c acting on the i-th graded piece $(M_f)_i$. Then the main theorem of the thesis is:

Theorem 5.22. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be homogeneous of degree d with an isolated singularity. Then the augmented b-function $\tilde{b}_c^f(s)$ divides*

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha). \tag{5.3.1}$$

Moreover, the roots of $\tilde{b}_c^f(s)$ and (5.3.1) *agree. In particular, when* (5.3.1) *is reduced, it equals $\tilde{b}_c^f(s)$.*

Colloquially, we say that the b-function *behaves flatly in c* if the degree of the b-function does not change as we vary c; equivalently, we say that b-function is a flat deformation of the classical b-function at $c = 0$ (the case of usual $\mathcal{D}$-modules). In [BET26], only one (small) example of a b-function was provided—since, at the time, this was the only known example. It was simply not known how to compute the b-function in any other cases. Although no non-trivial examples were available, it was conjectured that the b-function for the rational Cherednik algebra should be a flat deformation in the parameter c.

As a consequence of our main theorem, we observe two striking features of b-functions in this setting:

- The b-function is **not** a flat deformation of the classical b-function at $c = 0$, contrary to the earlier expectation.
- In the case of isolated singularities, the b-function for rational Cherednik algebras may have repeated roots, unlike in the classical case, where the b-function is always reduced.

For example, in Section 4.4.3 we show that at $c = 0$, the b-function for $x_1^4 + x_2^4$ has degree 6, but for generic parameters for the rational Cherednik algebra $\mathtt{H}_c(B_2)$, the b-function has degree 7; see Remark 4.21.

An important step in the proof of the main theorem is Lemma 5.28, which is an analogue of [Yan78, Theorem 2.19]: if $f \in \mathbb{C}[\mathfrak{h}]^W$ is homogeneous with isolated singularities, then

$$\mathrm{Ann}_{\mathtt{H}_c}(f^s) = \langle\, 1 - e,\ (f_i y_j - f_j y_i)e \,|\, 1 \leq i < j \leq n \rangle,$$

where $e = \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W \subset \mathtt{H}_c$ is the trivial idempotent.

§ 1.2. A rationality conjecture

A celebrated result of Kashiwara [Kas77] states that, in the case of $\mathcal{D}$-modules, the roots of the b-function—*a priori* arbitrary complex numbers—are in fact always negative rational numbers. For rational Cherednik alegrbas, one can always choose the parameter c such that this is no longer the case. However, it is apparent from our main result that an analogue of this rationality result should still hold in the case of rational Cherednik algerbras.

In order to state this conjecture we first note that the rational Cherednik algebra can also be parameterised by tuples $\{\kappa_{H,i} \in \mathbb{C} \,|\, H \in \mathcal{A}, 0 \leq i \leq \ell_H\}$, where $\mathcal{A}$ is the set of reflecting hyperplanes in $\mathfrak{h}$, ℓ_H is the order of the cyclic group

$$W_H = \{w \in W \,|\, w(y) = y, \forall y \in H\},$$

and $\kappa_{H,i} = \kappa_{H',i}$ if H, H' are in the same W-orbit; see Section 3.4.

Conjecture 1.2. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$. If α is a root of the b-function $b_c^f(s)$ then there exists $a \in \mathbb{Q}_{<0}$ and $t_{H,i} \in \mathbb{Q}_{\leq 0}$ such that*

$$\alpha = a + \sum_{H \in \mathcal{A}} \sum_{i=0}^{\ell_H - 1} t_{H,i} \kappa_{H,i}.$$

In particular, if $\kappa_{H,i} \in \mathbb{Q}_{\geq 0}$ for all H and i then $\alpha \in \mathbb{Q}_{<0}$.

It is a consequence of our main theorem that the conjecture is true when f is homogeneous with isolated singularities, see Corollary 5.36. By Kashiwara's result, it is also true for any f when $c = 0$.

In Section 4.3, we prove a result (Theorem 4.14) analogous to the multiplicative Thom-Sebastini formula proved in [Lee25, Theorem 1.1]. This says that if $f \in \mathbb{C}[\mathfrak{h}_1]^{W_1}$ and $g \in \mathbb{C}[\mathfrak{h}_2]^{W_2}$ then the b-function of their product $f \otimes g \in \mathbb{C}[\mathfrak{h}_1 \times \mathfrak{h}_2]^{W_1 \times W_2}$ is the product

$$b_c^{f \otimes g}(s) = b_{c_1}^f(s) \cdot b_{c_2}^g(s).$$

At least for reducible complex reflection groups, this theorem produces infinitely many examples of polynomials f without isolated singularities for which we can check that the above conjecture holds.

Finally, in Section 4.4.3, we give an explicit example when $W = B_2$ of a polynomial f that does not have isolated singularities where we can explicitly compute the b-function. Of course, the conjecture holds in this case too.

§ 1.3. Computations

Our main result raises the obvious question of whether one can actually compute the b-function in explicit examples. The computation of the roots of the b-function in the isolated singularity case is highly combinatorial in nature. As noted above, a key object in the proof of the main theorem is the Milnor algebra $M_f = \mathbb{C}[\mathfrak{h}]/J(f)$. In order to compute the eigenvalues of z_c acting on M_f, we must first compute the graded W-module (or character) structure of M_f. Let $\chi_{M_f}(q) = \sum_{\lambda \in \operatorname{Irr} W} h_\lambda(q)\chi_\lambda$ denote the graded W-character of M_f, where $h_\lambda(q) = \sum_{i \geq 0}[(M_f)_i : \lambda]q^i$ and χ_λ is the character of λ. We show:

Theorem 6.7.

$$\chi_{M_f}(q) = \sum_{\lambda \in \operatorname{Irr} W} \langle \chi_{M_f}, \chi_\lambda \rangle \chi_\lambda = \sum_{\lambda \in \operatorname{Irr} W} \frac{1}{|W|} \sum_{g \in W} \frac{\det(1 - gq^\delta)}{\det(1 - g^{-1}q)} \chi_\lambda(g^{-1})\chi_\lambda. \tag{1.3.1}$$

The proof of Theorem 6.7 is a generalisation of the usual proof of Molien's formula and equation (1.3.1) can be viewed (up to a sign change) as a generalisation of Solomon's formula, an important result in the theory of complex reflection groups.

Our formula implies that that $\dim M_f = (d-1)^n$ and $N = n(d-2)$. Using this formula we have written code for the Magma computer algebra system [BCP97] that will explicitly compute the b-function associated to any given W-invariant homogeneous polynomial with isolated singularities. For this to be useful, we need a ready source of such polynomials. The main examples of homogeneous W-invariant polynomials are sets of homogeneous fundamental invariants. By definition, this means that any other invariant is a polynomial in these fundamental invariants. Remarkably, fundamental invariants very often also have isolated singularities. For instance, the power polynomials $p_i = x_1^i + \cdots + x_n^i$, for i a multiple of m, form a set of homogeneous fundamental invariants for the group $G(m, 1, n)$ and all have isolated singularities. On the other hand, the elementary symmetric polynomials

$$e_i = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

(invariant under $G(1, 1, n) = \mathfrak{S}_n$) never have isolated singularities, except for the degenerate case $i = 1$. Therefore, we say that the complex reflection group W has *isolated fundamental invariants* (IFI) if there exists a set of homogeneous fundamental invariants $g_1, \ldots, g_n$ such that every g_i has isolated singularities. We show that $G(m, p, n)$ has IFI if and only if p divides n, see Lemma 6.17. For the exceptional complex reflection groups, we are able to use MAGMA to show that of the 23 groups $G_4, \ldots, G_{26}$, all except G_{13}, G_{15} and G_{25} admit IFI. The computation becomes too hard for the remaining 11

groups $G_{27}, \ldots, G_{37}$.

§ 1.4. Applications

Classically, b-functions play a crucial role in the classification of irreducible holonomic $\mathcal{D}$-modules. Such modules are described as minimal extensions of integrable connections [HTT07, Definition 3.4.1], but these minimal extensions are usually very difficult to describe explicitly, despite their abstract existence being well understood. We expect that our results can similarly be used to explicitly describe $\mathtt{H}_c$-modules constructed as minimal extensions $j_{*!}(\mathcal{O}(U)f^\lambda)$ of rank one local systems $\mathcal{O}(U)f^\lambda$, where $U = (f \neq 0) \overset{j}{\hookrightarrow} \mathfrak{h}$ and $\lambda \in \mathbb{C}$. See [BET26, Section 6.3] for the definition of minimal extensions for modules over rational Cherednik algebras and the final Chapter 7 for a precise expectation.

As a more concrete application, we explain in Lemma 4.6 how to use the b-function to give an elementary proof of (the highly non-obvious) fact that $\mathbb{C}[\mathfrak{h}][f^{-1}]$ is a cyclic $\mathtt{H}_c$-module. By analogy with the theory of $\mathcal{D}$-modules, we expect there are many other applications of the theory of b-functions for rational Cherednik algebras that we have not yet thought of.

§ 1.5. Structure

In this final section, we describe the structure of the thesis.

In Chapters 2 and 3 we introduce preliminary material, with a view to defining holonomic modules for rational Cherednik algebras (Section 3.8). Chapter 2 discusses $\mathcal{D}$-modules, holonomic ones in particular. Then, in Chapter 3 we define complex reflection groups and the rational Cherednik algebras associated to them. We also recall some important results relating to rational Cherednik algebras, such as the PBW theorem (Section 3.3) and the Dunkl embedding (Section 3.5).

In Chapter 4 we introduce b-functions. First, we discuss classical b-functions over the Weyl algebra $\mathcal{D}_n$, and their generalisation to holonomic $\mathcal{D}_U$-modules ($U \subset X$). In Section 4.1 we define an analogous b-function for rational Cherednik algebras following [BET26]. In Section 4.2, we discuss some basic properties of the b-function for rational Cherednik algebras, including the fact that $s = -1$ is always a root by Proposition 4.8. In particular, we define the module $\mathcal{M}$, and note that the augmented b-function of $f \in \mathbb{C}[\mathfrak{h}]^W$ is precisely the minimal polynomial of s acting on $\mathcal{M}$ by Lemma 4.11. In Section 4.4, we compute some examples by hand: In Section 4.4.1 we compute the b-function for x^ℓ over the rational Cherednik algebra associated to the cyclic group $\mathbb{Z}_\ell$. In Section 4.4.2, we then apply our analogue of the multiplicative Thom-Sebastini formula to compute the b-function for $f = \prod_{i=1}^n x_i^{\ell_i}$ over the rational Cherednik algebra associated to the product $W = \mathbb{Z}/\ell_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_n\mathbb{Z}$ of cyclic groups. In Section 4.4.3 we consider

b-functions over our favourite example $\mathtt{H}_c(B_2)$, the rational Cherednik algebra associated to the dihedral group $B_2 = I_2(4)$. Finally, in Section 4.4.4, we compute the b-function for the W-invariant $f = x_1^2 + \cdots + x_n^2$, when W is a Coxeter group.

In Chapter 5 we restrict our attention to homogeneous polynomials $f \in \mathbb{C}[\mathfrak{h}]^W$ with isolated singularities. We begin by recalling the properties of polynomials with isolated singularities that we need. We also introduce the Milnor algebra $M_f := \mathbb{C}[\mathfrak{h}]/J(f)$ and show that it is a finite dimensional local graded algebra if f is homogeneous with isolated singularities. Then, in Section 5.2 we prove some of the more deeper properties of the module $\mathcal{M}$ in the case f has isolated singularities. In particular, we prove the existence of the b-function in this case. Then, in sections 5.3 and 5.4, we state and prove, respectively, our main theorem. In the final section of Chapter 5 we use the main theorem to illustrate the fact that the b-function is not flat in the parameter c. We also show that for a polynomial with isolated singularities, the b-function is reduced for generic c. Finally, we use the main result Theorem 5.22 to confirm Conjecture 1.2 in the case f has isolated singularities.

Chapter 6 is focused on deriving the results needed to actually compute the roots of the b-function in concrete examples. We first derive the formula for the graded W-character of the Milnor algebra, based on the fact that the derivatives $f_1, \ldots, f_n$ of f form a regular sequence when f has isolated singularities. In particular, this allows us to compute the dimension of the Milnor algebra M_f. Then we consider the question of when an irreducible complex reflection group admits a fundamental set of invariants that all have isolated singularities. In addition to the general result, Lemma 6.17, that considers the infinite family $G(m, p, n)$, we explain the MAGMA computations necessary to answer this question for the exceptional complex reflection groups. We end this chapter by giving a large number of explicit examples of b-functions associated to fundamental sets of invariants that have isolated singularities.

In the short final Chapter 7, we list a number of open questions that arise naturally from our work.

The thesis has three appendices. In Chapter A, we concisely recall the proof in the case of $\mathcal{D}$-modules, of the formula for the b-function of a homogeneous polynomial with isolated singularities, since our main result is an adaptation of this proof. It is based on the exposition given in [Pop18]. In Chapter B, we include the MAGMA code that (a) decides if a given polynomial has isolated singularities, (b) computes the graded character of the Milnor algebra when f is a W-invariant homogeneous polynomial with isolated singularities, and (c) computes the b-function of such a polynomial. In the final Chapter C, we include a table listing those exceptional complex reflection groups which are known not to have any set of fundamental invariants with only isolated singularities.

More specifically, if there exists a fundamental degree d_i for the exceptional complex reflection group G_k for which there does not exist a fundamental invariant of that degree with isolated singularities, then we give a representative fundamental invariant of that degree.

Chapter 2

Preliminaries on $\mathcal{D}$-modules

As explained in the introduction, $\mathcal{D}$-modules are important because they are related to many other areas of mathematics. For instance, "algebraic analysis" is the study of analysis using algebraic methods, such as ring theory and homological algebra. This is the application of $\mathcal{D}$-modules to analysis; see the introduction in [HTT07] for more details. Another example is the Riemann–Hilbert correspondence, whose applications include: providing a positive answer to Hilbert's 21st problem, a proof of the Kazhdan–Lusztig conjectures, and playing a central role in the geometric Langlands programme. In later chapters we consider the representation theory of rational Cherednik algebras. This will be developed by analogy with the theory of $\mathcal{D}$-modules. In this chapter we recall the basic theory of $\mathcal{D}$-modules.

§ 2.1. The Weyl algebra

Let $n \in \mathbb{Z}_{\geq 1}$ be a positive integer. We begin by considering the n-th complex Weyl algebra $\mathcal{D}_n$, which is defined formally as the noncommutative algebra generated by symbols $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ with relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}$$

for $i, j \in \{1, \ldots, n\}$, where the commutation relation $[\partial_i, x_j]$ is given by the Kronecker delta

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

The reason for the choice of notation is that from these relations one can recover rules for differentiating polynomials, for example the Leibniz rule

$$\partial_i x_i = 1 + x_i \partial_i. \tag{2.1.1}$$

The ordered monomials

$$x^\alpha \partial^\beta := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n},$$

as α, β range over $\mathbb{Z}_{\geq 0}^n$, form a basis of $\mathcal{D}_n$ over $\mathbb{C}$.

§ 2.2. Rings of differential operators

Let R be a commutative $\mathbb{C}$-algebra, say for example $\mathbb{C}[x_1, \ldots, x_n]$. A **derivation** on R is a $\mathbb{C}$-linear map $\delta \in \mathrm{End}_{\mathbb{C}}(R)$ satisfying the Leibniz rule

$$\delta(ab) = \delta(a)b + a\delta(b),$$

for all $a, b \in R$. We denote by $\mathrm{Der}(R)$ the set of all derivations on R, and define the **derivation ring** of R to be the subalgebra $\Delta(R)$ of $\mathrm{End}_{\mathbb{C}}(R)$ generated by R and $\mathrm{Der}(R)$.

Let A be a $\mathbb{C}$-algebra. A **filtration** $\mathcal{F}_\bullet$ on A is a nested sequence of subspaces

$$0 = \mathcal{F}_{-1}A \subset \mathcal{F}_0 A \subset \mathcal{F}_1 A \subset \cdots$$

such that $(\mathcal{F}_i A)(\mathcal{F}_j A) \subseteq \mathcal{F}_{i+j} A$ for all i, j, and $A = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{F}_i A$.

Example 2.1. *We consider, as examples, two filtrations of the Weyl algebra $\mathcal{D}_n$. Let $P = \sum_{\alpha,\beta \in \mathbb{Z}_{\geq 0}^n} a_{\alpha\beta} x^\alpha \partial^\beta$ be a general element ($a_{\alpha\beta} \in \mathbb{C}$) of $\mathcal{D}_n$.*

*i) The **Bernstein filtration** is $\mathcal{B}_i \mathcal{D}_n = \{P \in \mathcal{D}_n : \deg(P) \leq i\}$, $i \in \mathbb{Z}_{\geq 0}$, where the **degree** of P is defined to be*

$$\deg(P) = \max_{\alpha,\beta \in \mathbb{Z}^n} \{|\alpha| + |\beta| : a_{\alpha\beta} \neq 0\}.$$

*ii) The **order filtration** is $\mathcal{F}_i \mathcal{D}_n = \{P \in \mathcal{D}_n : \mathrm{ord}(P) \leq i\}$, $i \in \mathbb{Z}_{\geq 0}$, where the **order** of P is defined to be*

$$\mathrm{ord}(P) = \max_{\alpha,\beta \in \mathbb{Z}^n} \{|\beta| : a_{\alpha\beta} \neq 0\}.$$

Note that $\mathbb{C}[x_1, \ldots, x_n] \subseteq \mathcal{F}_0 \mathcal{D}_n$ and $\partial_i \in \mathcal{F}_1 \mathcal{D}_n$ for $i = 1, \ldots, n$.

The **associated graded** algebra of A with respect to the filtration $\mathcal{F}_\bullet$ is

$$\mathrm{gr}_{\mathcal{F}} A = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{F}_i A / \mathcal{F}_{i-1} A.$$

We claim that this is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra. First, it is a graded vector space with grading $(\mathrm{gr}_{\mathcal{F}} A)_i = \mathcal{F}_i A / \mathcal{F}_{i-1} A$ for $i \geq 0$ and $(\mathrm{gr}_{\mathcal{F}} A)_i = \{0\}$ for $i < 0$. Next, we define a multiplication on homogeneous elements in $\mathrm{gr}_{\mathcal{F}} A$ by setting

$$(a \bmod \mathcal{F}_{i-1}) \cdot (b \bmod \mathcal{F}_{j-1}) = ab \bmod \mathcal{F}_{i+j-1},$$

for $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$. To see that this only depends on the class of a in $\mathcal{F}_i/\mathcal{F}_{i-1}$ and b in $\mathcal{F}_j/\mathcal{F}_{j-1}$, assume that $a' \in \mathcal{F}_i, b' \in \mathcal{F}_j$ such that $a \bmod \mathcal{F}_{i-1} = a' \bmod \mathcal{F}_{i-1}$ and $b \bmod \mathcal{F}_{j-1} = b' \bmod \mathcal{F}_{j-1}$. Then $a - a' \in \mathcal{F}_{i-1}$ and $b - b' \in \mathcal{F}_{j-1}$ so

$$\begin{aligned} ab \bmod \mathcal{F}_{i+j-1} &= [a + (a' - a)]b \bmod \mathcal{F}_{i+j-1} \\ &= a'b \bmod \mathcal{F}_{i+j-1} \\ &= a'[b + (b' - b)] \bmod \mathcal{F}_{i+j-1} \\ &= a'b' \bmod \mathcal{F}_{i+j-1} \end{aligned}$$

since $(a' - a)b, a'(b - b') \in \mathcal{F}_{i+j-1}$. We extend the multiplication from homogeneous elements to all elements using bilinearity. We check that this is an associative product. By bilinearity, it suffices to do check this for homogeneous elements. Then

$$\begin{aligned} \Big((a \bmod \mathcal{F}_{i-1}) \cdot (b \bmod \mathcal{F}_{j-1})\Big) \cdot (c \bmod \mathcal{F}_{k-1}) &= (ab \bmod \mathcal{F}_{i+j-1}) \cdot (c \bmod \mathcal{F}_{k-1}) \\ &= (ab \bmod \mathcal{F}_{i+j-1}) \cdot (c \bmod \mathcal{F}_{k-1}) \\ &= (abc \bmod \mathcal{F}_{i+j+k-1}) \\ &= (a \bmod \mathcal{F}_{i-1}) \cdot (bc \bmod \mathcal{F}_{j+k-1}) \\ &= (a \bmod \mathcal{F}_{i-1}) \cdot \Big((b \bmod \mathcal{F}_{j-1}) \cdot (c \bmod \mathcal{F}_{k-1})\Big). \end{aligned}$$

Now, for a commutative ring R we inductively define $\mathcal{D}(R)$, **the ring of differential operators** on R, via filtrations, in the following way:

$$\begin{aligned} \mathcal{F}_{-1}\mathcal{D}(R) &:= \{0\}; \\ \mathcal{F}_0\mathcal{D}(R) &:= R; \\ \mathcal{F}_i\mathcal{D}(R) &:= \{P \in \mathrm{End}_{\mathbb{C}}(R) \,|\, [P, f] \in \mathcal{F}_{i-1}\mathcal{D}(R) \text{ for all } f \in R\}; \\ \mathcal{D}(R) &:= \bigcup_{i \geq 0} \mathcal{F}_i\mathcal{D}(R). \end{aligned}$$

Note that, by definition, $\mathcal{D}(R)$ comes equipped with an order filtration $\mathcal{F}_i\mathcal{D}(R)$, where $i \in \mathbb{Z}$ and $\mathcal{F}_{-i}\mathcal{D}(R) := \{0\}$ for $i \in \mathbb{Z}_{\geq 1}$.

Recall that a Noetherian ring is regular if it has finite global dimension; see for instance Theorem 19.12 of [Eis95]. In particular, we note that $\mathbb{C}[x_1, \ldots, x_n]$ is regular.

Lemma 2.2 ([MR01, Corollary 15.5.6])**.** *If R is a regular commutative algebra then $\mathcal{D}(R) = \Delta(R)$.*

The above lemma allows us to explicitly identify the ring of differential operators for $R = \mathbb{C}[x_1, \ldots, x_n]$.

Proposition 2.3. *The ring of differential operators on $R = \mathbb{C}[x_1, \ldots, x_n]$ is isomorphic to the Weyl algebra $\mathcal{D}_n$.*

Proof. Since the generators $\partial_1, \ldots, \partial_n$ of

$$\mathcal{D}_n = R[\partial_1, \ldots, \partial_n]$$

satisfy the Leibniz rule (2.1.1), they are derivations on R. Therefore, we have an inclusion $\mathcal{D}_n \subseteq \mathcal{D}(R)$ of algebras. It thus remains to show the reverse inclusion.

Let $\delta \in \mathrm{End}_{\mathbb{C}}(R)$ be a derivation on R. We show that

$$\delta = f_1(x)\partial_1 + \cdots + f_n(x)\partial_n$$

for some $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$, and then conclude that $\mathrm{Der}(R) \subseteq \mathcal{D}_n$. This will imply that $\Delta(R) = \mathcal{D}(R) \subseteq \mathcal{D}_n$ by Lemma 2.2. Let $f_i(x) := \delta(x_i)$ and let

$$\rho := \delta - f_1(x)\partial_1 + \ldots f_n(x)\partial_n.$$

Note that $\rho(x_i) = 0$ for all $i = 1, \ldots, n$. We conclude the proof by showing $\rho = 0$. A derivation is uniquely defined by its action on generators:

$$\rho\left(\sum_{\underline{i} \in I} a_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n}\right) = \sum_{\underline{i} \in \mathbb{Z}^n_{\geq 0}} a_{\underline{i}} \rho(x_1^{i_1} \cdots x_n^{i_n}),$$

where $a_{\underline{i}} \in \mathbb{C}$ are coefficients labelled by multi-indices $\underline{i} = (i_1, \ldots, i_n)$. By inductively using the Leibniz rule, one finds that

$$\rho(x_j^{i_j}) = i_j \rho(x_j) x_j^{i_j - 1}$$

for $j = 1, \ldots, n$. So

$$\rho(x_1^{i_1} \cdots x_n^{i_n}) = \sum_{j=1}^{n} i_j \rho(x_j) x_j^{i_j - 1} x_1^{i_1} \cdots \widehat{x_j^{i_j}} \cdots x_n^{i_n}, \qquad (\widehat{x_j^{i_j}} \text{ deleted})$$

which is equal to 0 since $\rho(x_i) = 0$ for all $i = 1, \ldots, n$ by assumption. □

We can now define $\mathcal{D}$-modules. Recall that for an affine variety X, defined as an irreducible closed subset of $\mathbb{A}^n$, its ring of functions is

$$\mathcal{O}(X) := \mathbb{C}[x_1, \ldots, x_n]/I(X),$$

where $I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \,|\, f(x) = 0 \text{ for all } x \in X\}$ is a prime ideal.

Definition 2.4. *A $\mathcal{D}$-**module** on a ring R is just a left $\mathcal{D}(R)$-module. A $\mathcal{D}$-module on an affine variety X is then a left $\mathcal{D}(X) := \mathcal{D}(\mathcal{O}(X))$-module.*

Remark 2.5. The Weyl algebra $\mathcal{D}_n$ does not have any finite-dimensional representations. To see this, suppose we do have a finite dimensional representation V and let $\mathrm{Tr}_V : \mathcal{D}_n \to \mathbb{C}$ be its trace. Then

$$n = \mathrm{Tr}_V(1) = \mathrm{Tr}(x_1\delta_1 - \delta_1 x_1) = \mathrm{Tr}_V(x_1\delta_1) - \mathrm{Tr}_V(\delta_1 x_1) = 0,$$

a contradiction.

§ 2.2.1. Good filtrations

Assume that X is a smooth affine variety. Then, geometrically, the **tangent bundle** is the vector bundle $TX = \{(x, v) : x \in X, v \in T_xX\}$, whose fiber over a point $x \in X$ is the tangent space T_xX, and whose sections are the vector fields $\theta_X = \mathrm{Der}(\mathcal{O}(X))$ on X. The **cotangent bundle** T^*X is then the dual vector bundle to TX, as in [Har77, p.180-181]. The sections of the cotangent bundle T^*X are **1-forms** $\omega_X \in \Omega^1_X$, where Ω^1_X is defined to be the $\mathcal{O}(X)$-module with generators $\{df | f \in \mathcal{O}(X)\}$ and relations

$$d(f+g) = df + dg, \qquad d(fg) = fdg + gdf, \qquad d(a) = 0 \text{ for all } a \in \mathbb{C}.$$

For a smooth variety X, $\dim X = \dim T_xX$ for all $x \in X$. Let $n = \dim X$ and let $\mathcal{F}$ be the order filtration on $\mathcal{D}(X)$, as defined in Example 2.1. Assuming Ω^1_X is a free module (which is always true locally by [Har77, II, Theorem 8.15]), we have $\Omega^1_X = \bigoplus_{i=1}^n \mathcal{O}(X)\omega_i$ for some $\omega_i \in \Omega^1_X, i = 1, \ldots, n$. Then,

$$\mathrm{Der}(X) = \mathrm{Hom}_{\mathcal{O}(X)}(\Omega^1_X, \mathcal{O}(X)) = \bigoplus_{i=1}^n \mathcal{O}(X)\partial_i,$$

where $\partial_i(\omega_i) := \delta_{ij}$.

Theorem 2.6 ([HTT07, p.17])**.** *If $\mathcal{F}$ is the order filtration on $\mathcal{D}(X)$ then*

$$\mathcal{O}(T^*X) \cong \mathrm{gr}_{\mathcal{F}}\mathcal{D}(X) = \mathcal{O}(X)[\xi_1, \ldots, \xi_n],$$

where $\xi_i \in \mathcal{F}_1\mathcal{D}(X)/\mathcal{F}_0\mathcal{D}(X)$ is given by $\xi_i := \partial_i \bmod \mathcal{F}_0\mathcal{D}(X)$.

Example 2.7. *Let $\mathfrak{h}$ be a vector space of dimension n. Then, for $X = \mathfrak{h}$, the tangent space at each point $x \in \mathfrak{h}$ is $T_xX = \mathfrak{h}$, so the corresponding cotangent space is $T^*_xX = \mathfrak{h}^*$, and the cotangent bundle is just $T^*X = \mathfrak{h} \times \mathfrak{h}^*$. Moreover, its ring of functions is*

$$\mathcal{O}(T^*X) = \mathcal{O}(X)[\xi_1, \ldots, \xi_n] = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n].$$

Let M be $\mathcal{D}$-module, that is, a left $\mathcal{D}(X)$-module. A filtration $\mathcal{F}_\bullet M$ on M (relative

to the order filtration on $\mathcal{D}(X)$) is a nested sequence of subspaces

$$\cdots \subset \mathcal{F}_{-1}M \subset \mathcal{F}_0 M \subset \mathcal{F}_1 M \subset \cdots$$

such that $\mathcal{F}_i M = \{0\}$ for $i \ll 0$, $(\mathcal{F}_i\mathcal{D}(X))(\mathcal{F}_j M) \subseteq \mathcal{F}_{i+j}M$ for all $i, j \in \mathbb{Z}$, and $M = \bigcup_{i\in\mathbb{Z}} \mathcal{F}_i M$. The associated graded of M with respect to the filtration $\mathcal{F}_\bullet$ is

$$\operatorname{gr}_{\mathcal{F}} M = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i M/\mathcal{F}_{i-1}M,$$

which is naturally a graded $\operatorname{gr}_{\mathcal{F}}\mathcal{D}(X)$-module.

Definition 2.8. *Let M be a finitely generated $\mathcal{D}$-module. A filtration $\mathcal{F}_\bullet M$ is **good** if $\operatorname{gr}_{\mathcal{F}} M$ is a finitely generated $\mathcal{O}(T^*X)$-module.*

Lemma 2.9 ([HTT07, Theorem 2.1.3])**.** *If M is a finitely generated $\mathcal{D}$-module then M has a good filtration.*

Remark 2.10. In fact, in the book [HTT07] the above result, and those that follow, are written in terms of *coherent sheaves* of $\mathcal{D}$-modules, rather than in terms of finitely generated $\mathcal{D}(X)$-modules. Since we only need to deal with affine varieties and the global sections functor defines an equivalence between coherent $\mathcal{D}$-modules on X and finitely generated $\mathcal{D}(X)$-modules when X is smooth and affine, we work throughout with finitely generated $\mathcal{D}(X)$-modules.

§ 2.3. Holonomic $\mathcal{D}$-modules

We recall some basic definitions from symplectic geometry that will be needed in understanding characteristic varieties. Recall that a symplectic vector space (V, ω) is a vector space V with a skew-symmetric non-degenerate bilinear form ω. We note that if (V, ω) is a symplectic vector space, then $\dim V$ is even.

Definition 2.11. *Let $U \subset V$ be a subspace of a symplectic vector space (V, ω).*

*(a) U is **isotropic** if $U \subset U^\perp$.*

*(b) U is **co-isotropic** if $U^\perp \subset U$.*

*(c) U is **Lagrangian** if $U = U^\perp$.*

If $U \subset V$ is Lagrangian then $\dim U = (1/2)\dim V$.

We now generalise Definition 2.11 to symplectic varieties. A symplectic variety Y is a (irreducible) variety with a closed two-form ω such that ω_y is a symplectic form on the tangent space T_yY for all $y \in Y$. A symplectic variety is necessarily smooth. We will consider mainly T^*X, which has a canonical symplectic form; see [HTT07, Section E.2].

Definition 2.12. *If Y is a symplectic variety, then an irreducible subvariety $\Lambda \subset Y$ is*

(a) ***isotropic*** *if $T_y\Lambda \subset (T_y\Lambda)^\perp$ for all $y \in \Lambda^{sm}$.*

(b) ***coisotropic*** *if $(T_y\Lambda)^\perp \subset T_y\Lambda$ for all $y \in \Lambda^{sm}$.*

(c) ***Lagrangian*** *if $(T_y\Lambda)^\perp = T_y\Lambda$ for all $y \in \Lambda^{sm}$.*

Here $\Lambda^{sm} \subset \Lambda$ denotes the smooth locus.

We note that coisotropic subvarieties Λ of a symplectic variety Y must satisfy $\dim \Lambda \geq (1/2)\dim Y$ and if Λ is Lagrangian then $\dim \Lambda = (1/2)\dim Y$.

Recall that if N is a finitely generated $\mathcal{O}(T^*X)$-module then

$$\mathrm{Supp}(N) := \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}(T^*X)) = T^*X | N_\mathfrak{p} \neq 0\}$$

is the **support** of N.

Example 2.13. *Let $X = \mathbb{A}^1$ so that $T^*\mathbb{A}^1 = \mathbb{A}^2$ and $\mathcal{O}(T^*X) = \mathbb{C}[x,y]$. Let N be the cyclic $\mathcal{O}(T^*X)$-module $\mathbb{C}[x,y]/(y^2 - x^2(x+1))$. Then,*

$$\mathrm{Supp}(N) = \{(x-a, y-b) \,|\, a, b \in \mathbb{C},\, b^2 = a^2(a+1)\} \cup \{(y^2 - x^2(x+1))\}$$

since $y^2 - x^2(x+1)$ is irreducible and hence $(y^2 - x^2(x+1))$ is a prime ideal.

Definition 2.14. *Let M be a finitely generated $\mathcal{D}$-module. The* ***characteristic variety*** *(or* singular support*) of M is defined to be*

$$\mathrm{Ch}(M) = \mathrm{Supp}(\mathrm{gr}_\mathcal{F} M)$$

for some good filtration $\mathcal{F}_\bullet M$ on M.

Proposition 2.15 ([HTT07, Theorem 2.2.1(i)])**.** *For a finitely generated $\mathcal{D}(X)$-module M, its characteristic variety $\mathrm{Ch}(M)$ is independent of the choice of good filtration.*

Proposition 2.16 ([HTT07, Corollary 2.3.2])**.** *Let M be a finitely generated $\mathcal{D}(X)$-module. Then for any irreducible component Λ of $\mathrm{Ch}(M)$ we have $\dim \Lambda \geq \dim X$. In particular, we have $\dim \mathrm{Ch}(M) \geq \dim X$ if $M \neq 0$.*

Definition 2.17. *A finitely generated $\mathcal{D}(X)$-module M is* ***holonomic*** *if $\mathrm{Ch}(M)$ is Lagrangian.*

Remark 2.18. The space T^*X is a symplectic variety. Gabber's Theorem [Gab81] says that the characteristic variety of a module is coistropic, which implies the statement of Proposition 2.16 above. Then holonomic modules can either be characterised geometrically as in Definition 2.17, or numerically by saying that a $\mathcal{D}$-module M is holonomic if and only if $\dim \mathrm{Ch}(M) = \dim X$.

We have recalled here the definition of holonomic modules since it motivates the definition of holonomic modules for rational Cherednik algebras given in the next chapter. But they are a very important category of modules in their own right, partly because of their remarkable properties when compared to the category of all $\mathcal{D}$-modules. For instance, they form an abelian category [HTT07, Proposition 3.1.2], where all objects are finite length [HTT07, Proposition 3.1.2], and all Ext-groups are finite dimensional [HTT07, Theorem 3.3.1]. Moreover, the category of holonomic modules is preserved by push-forward and pull-back functors [HTT07, Section 3.2.1] (which is not true of coherent or finitely generated modules). The category of holonomic modules includes the category of vector bundles with a flat (= integrable) connection

$$\begin{aligned}\mathrm{Conn(X)} &\simeq \{\text{coherent } \mathcal{D}_X\text{-modules that are locally free over } \mathcal{O}_X\} \\ &= \{\text{algebraic vector bundles on } X \text{ with integrable connection}\},\end{aligned}$$

which are traditionally important in differential geometry. Finally, we should mention that the Riemann-Hilbert correspondence is a derived equivalence

$$DR_X(\bullet) = R\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \bullet)[\dim X] \colon D^b_{rh}(\mathcal{D}_X) \to D^b_c(\mathbb{C}_X)$$

between the derived category $D^b_{rh}(\mathcal{D}_X)$ of bounded complexes of $\mathcal{D}_X$-modules whose cohomology sheaves are regular holonomic $\mathcal{D}_X$-modules and the bounded derived category of constructible sheaves. Then, as explained in [HTT07, Theorem 7.2.5], the de Rham functor induces an equivalence

$$DR_X : \mathrm{Mod}_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv(\mathbb{C}_X)}$$

between the category of holonomic $\mathcal{D}$-modules with regular singularities and the (abelian) category of perverse sheaves on X.

Chapter 3

Preliminaries on rational Cherednik algebras

§ 3.1. Complex reflection groups

Let $\mathfrak{h}$ be a finite dimensional complex vector space and let $W \subset GL(\mathfrak{h})$ be a finite group acting on $\mathfrak{h}$. We do not assume that $\mathfrak{h}$ is an irreducible W-module.

Definition 3.1. *An element $s \in W$ is said to be a **complex reflection** (or pseudo-reflection) if* $\mathrm{rk}(1-s) = 1$. *We say that W is a **complex reflection group** if it is generated by the set $\mathcal{S}$ of complex reflections in W.*

Remark 3.2. Note that for a complex reflection s, we have

$$\dim \ker(1-s) = \dim \mathfrak{h} - \mathrm{rk}(1-s) = \dim \mathfrak{h} - 1$$

by the rank-nullity theorem. That is, s fixes a subspace of $\mathfrak{h}$ of dimension $\dim \mathfrak{h} - 1$. Therefore it has eiegenvalue 1 with multiplicity $\dim \mathfrak{h} - 1$ and exactly one non-trivial eigenvalue λ_s with multiplicity one.

A complex reflection group W is said to be *irreducible* if the reflection representation $\mathfrak{h}$ is an irreducible representation for the group W.

The importance of complex reflection groups can be summed up in the following combined result of Chevalley and Shephard–Todd [Che55, ST54]. Let $\mathfrak{h}/W := \operatorname{Spec} \mathbb{C}[\mathfrak{h}]^W$ denote the GIT quotient, naïvely the space of W-orbits.

Theorem 3.3. *The space $\mathfrak{h}/W$ is smooth if and only if W is a complex reflection group.*

The complex reflection groups were classified in [ST54]. They are subgroups of the group $U(n)$ of $n \times n$ unitary matrices, and consist of one infinite family $G(m,p,n)$, for $m,p,n \in \mathbb{Z}_{>0}$ with p dividing m, and 34 exceptional groups labelled $G_4, \ldots, G_{37}$. We note some important subfamilies of $G(m,p,n)$:

- $G(m,1,n) = S_n \wr \mathbb{Z}_m$, the **wreath product** of the symmetric group S_n and cyclic group $\mathbb{Z}_m$, which can be taken to be the semidirect product $S_n \ltimes \mathbb{Z}_m^n$ generated by transpositions $(i, i+1)$, $1 \leq i \leq n-1$, and by the complex reflections τ_i given by

$$x\tau_i = (x_1, \ldots, \underbrace{\eta x_i}_{i\text{-th position}}, \ldots, x_n),$$

  where $\eta = e^{2\pi i/m}$. The group $G(m,p,n)$ is then an index-p normal subgroup of $G(m,1,n)$ of order $\frac{m^n n!}{p}$.

- $G(m,1,1) \cong \mathbb{Z}_m$ is cyclic of order m;

- $G(1,1,n) = S_n$ is the symmetric group on n letters, and is the Weyl group of type A_{n-1}.

- $G(2,1,n)$ is the hyperoctahedral group of order $2^n n!$ (the symmetries of the hypercube) and is the Weyl group of type B_n.

- $G(2,2,n)$ is a Weyl group of type D_n.

- $G(m,m,2)$ is the dihedral group of order $2m$ and is the Coxeter group of type $I_2(m)$.

We also note, as discussed in Chapter 2 of [LT09], that these groups, together with their faithful representation $\mathfrak{h} = \mathbb{C}^n$, are all irreducible except for the symmetric groups $G(1,1,n) = S_n$ for $n > 1$, and the dihedral group $G(2,2,2) \cong I_2(2)$; see [LT09, Proposition 2.10].

§ 3.2. Definition

Let W be a complex reflection group acting on its reflection representation $\mathfrak{h}$, a minimal faithful W-module upon which W acts by reflections. We consider the space $\mathfrak{h} \times \mathfrak{h}^*$ on which W acts diagonally, so that $w \cdot (y,x) = (w \cdot y, w \cdot x)$, where the action on $\mathfrak{h}^*$ is given by $(w \cdot x)(y) = x(w^{-1}y)$ for $w \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}$. It has a natural pairing $(\cdot,\cdot) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ defined by $(y,x) = x(y)$. Let $\mathcal{S}$ be the set of complex reflections in W, and for each $s \in \mathcal{S}$ fix α_s to be a basis element of the one dimensional space $\mathrm{im}(1-s)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee$ a basis of the one dimensional space $\mathrm{im}(1-s)|_{\mathfrak{h}}$, normalised so that $\alpha_s(\alpha_s^\vee) = 2$. Then, with $\lambda_s \in \mathbb{C}$ the (only) nontrivial eigenvalue of s defined by $s(\alpha_s) = \lambda_s \alpha_s$, the reflections are given by

$$s(x) = x - \frac{(1-\lambda_s)}{2}(x, \alpha_s^\vee)\alpha_s.$$

We note that $\det_{\mathfrak{h}}(s) = \lambda_s^{-1}$.

In defining the rational Cherednik algebra we must choose a pair of parameters t, c. We begin with the latter: Let $c : \mathcal{S} \to \mathbb{C}$ be a conjugate invariant function. That is,

$c(wsw^{-1}) = c(s)$ for all $s \in \mathcal{S}, w \in W$. The space of all such functions is $\mathcal{C} := (\mathcal{S}/W)^*$, which is dual to $\mathcal{S}$ modulo W, where W acts by conjugation on $\mathcal{S}$.

If V is a vector space over a field k then the **tensor algebra** of V is defined to be the k-algebra

$$TV = \bigoplus_{n=0}^{\infty} V^{\otimes n} = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

One interprets the elements of TV as non-commutative polynomials (precisely, linear combinations of tensor powers) on V, and will usually omit the tensor product in the notation.

We then define $TV \rtimes W$, the **skew group ring** of the tensor algebra, as a W-module equal to $TV \otimes_k kW$, with W-action given by

$$w \cdot (f \otimes u) = w(f) \otimes wu,$$

for all $u, w \in W$ and $f \in TV$. Multiplication in $TV \rtimes W$ is then given by

$$(v_1 \otimes \ldots \otimes v_n \otimes w) \cdot (u_1 \otimes \ldots \otimes u_m \otimes w') := v_1 \otimes \ldots \otimes v_n \otimes w(u_1) \otimes \ldots \otimes w(u_m) \otimes ww'$$

for all $v_1, \ldots, v_n, u_1, \ldots, u_m \in V, w, w' \in W$. Again, we usually omit the tensor products from the notation.

We now define the rational Cherednik algebra as in [BE09], which in turn is a particular form of the definition given in [EG02].

Definition 3.4. *Let $t \in \mathbb{C}$. The **rational Cherednik algebra** $\mathtt{H}_{t,c}(W)$ is the quotient of $T(\mathfrak{h} \times \mathfrak{h}^*) \rtimes W$ by the following relations:*

$$[x, x'] = [y, y'] = 0, \quad [y, x] = t(y, x) - \sum_{s \in \mathcal{S}} c(s)(y, \alpha_s)(\alpha_s^\vee, x)s, \tag{3.2.1}$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

Remark 3.5. The rational Cherednik algebra arises via various constructions, including the following:

- As a two-step degeneration (trigonometric, then rational, as explained on page 251 of [EG02]) of the double affine Hecke algebra (DAHA), also known as a Cherednik algebra, first defined by Cherednik in [Che92].

- As a symplectic reflection algebra with underlying symplectic vector space $V = \mathfrak{h} \oplus \mathfrak{h}^*$ and symplectic form $\omega((y, x), (y', x')) = (y, x') - (y', x)$; see [EG02].

As in Remark 2.10, one can also define the $\mathcal{D}_X$-modules of Section 2.2 as sheaves of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ with a flat connection. This is an operator $\nabla_\delta : \mathcal{F} \to \mathcal{F}$ for each vector field $\delta \in \theta_X$, which by definition must satisfy $[\nabla_\delta, \nabla_\rho] = \nabla_{[\delta,\rho]}$ (amongst other relations). The analogous construction for rational Cherednik algebras then leads to the notion of sheaves $\mathcal{H}_{t,c}(X, W)$ of Cherednik algebras on regular varieties X with the action of a finite group W, as in [Eti17], together with their quasi-coherent sheaves of modules. We do not consider Cherednik algebras in this greater generality in this thesis.

§ 3.3. PBW Theorem

Consider the natural filtration $\mathcal{B}$ on $\mathtt{H}_{t,c}$ given by putting $\mathfrak{h}$ and $\mathfrak{h}^*$ in degree one and W in degree zero, generalising the Bernstein filtration of Example 2.1. The following is an important result called the Poincaré–Birkhoff–Witt theorem after the authors who proved the analogous result for the universal enveloping algebra of a Lie algebra. Algebras satisfying theorems of the same form are said to have the PBW property.

Theorem 3.6 (PBW [EG02, Thm. 1.3])**.** *The map $x \mapsto \sigma(x), y \mapsto \sigma(y), g \mapsto \sigma(g)$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $g \in W$ defines an isomorphism of algebras*

$$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \xrightarrow{\sim} \mathrm{gr}_{\mathcal{B}}(\mathtt{H}_{t,c}(W)),$$

where $\sigma(D)$ denotes the symbol, or leading term, of $D \in \mathtt{H}_{t,c}(W)$ in $\mathrm{gr}_{\mathcal{B}}(\mathtt{H}_{t,c}(W))$.

The proof of this theorem, as in [EG02], relies on Theorem 0.5 of [BG96], a general result that states that if I is a two-sided ideal of $TV \rtimes W$ generated by a space U of elements of degree at most two then the quotient $TV \rtimes W/I$ has the PBW property if and only if U satisfies certain explicit conditions. This proof works for all symplctic reflection algebras.

In the particular case of rational Cherednik algebras, we can also consider the order filtration $\mathcal{F}$ on $\mathtt{H}_c(W)$, where $\deg x = \deg w = 0$ and $\deg y = 1$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$. Though not immediate, the PBW theorem will hold for the filtration $\mathcal{B}$ if and only if it does for the filtration $\mathcal{F}$. The Dunkl embedding $\phi_c : \mathtt{H}_{t,c}(W) \to \mathcal{D}_t(\mathfrak{h}_{reg}) \rtimes W$ is strictly filtered if we give both $\mathtt{H}_{t,c}(W)$ and $\mathcal{D}_t(\mathfrak{h}_{reg}) \rtimes W$ the order filtration (the Bernstein filtration doesn't make sense on $\mathcal{D}_t(\mathfrak{h}_{reg}) \rtimes W$). Then one can show that the Dunkl embedding $\phi_c : \mathtt{H}_{t,c}(W) \to \mathcal{D}_t(\mathfrak{h}_{reg}) \rtimes W$ given by (3.5.1) induces an embedding

$$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \to \mathrm{gr}_{\mathcal{F}} \mathtt{H}_{t,c}(W) \xrightarrow{\mathrm{gr}\,\phi_c} \mathrm{gr}_{\mathcal{F}}(\mathcal{D}_t(\mathfrak{h}_{reg}) \rtimes W) \to \mathbb{C}[\mathfrak{h}_{reg} \times \mathfrak{h}^*] \rtimes W. \qquad (3.3.1)$$

This embedding gives a direct proof of the PBW Theorem that does not rely on [BG96]. Specifically, the hard part of the theorem is to show that the map $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \to \mathrm{gr}_{\mathcal{F}} \mathtt{H}_{t,c}(W)$ is injective since the defining relations make it clear that it is surjective. But

the composite (3.3.1) is the inclusion $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \hookrightarrow \mathbb{C}[\mathfrak{h}_{reg} \times \mathfrak{h}^*] \rtimes W$. Thus, the first map in the sequence (3.3.1) must be injective.

A key consequence of the PBW Theorem is that it provides an explicit basis for the rational Cherednik algebra, namely, we get an isomorphism of vector spaces $\mathtt{H}_{t,c}(W) \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W$.

§ 3.4. The parameters

For a non-zero scalar $\lambda \in \mathbb{C}^\times$, we have an isomorphism $\mathtt{H}_{\lambda t,\lambda c}(W) \cong \mathtt{H}_{t,c}(W)$. Therefore, it is customary to consider only the (non-isomorphic) cases $t = 0, 1$. The case $t = 0$ corresponds to the "(quasi-)classical" case, and is closely related to Calogero–Moser spaces and Poisson geometry. The case $t = 1$ corresponds to the quantum case; t is sometimes referred to as the quantum parameter or Planck constant and often written $\hbar$. See [Eti07] for more details. This thesis focuses on the case $t = 1$, so to simplify notation, we write $\mathtt{H}_c(W) := \mathtt{H}_{1,c}(W)$ throughout, or simply $\mathtt{H}_c$ when W is clear. We note, however, some important cases:

- $\mathtt{H}_{0,0}(W) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$,
- $\mathtt{H}_{1,0}(W) = \mathcal{D}_n \rtimes W$.

As explained for instance in Section 1.3 of [Val07], there is an alternative parametrization of the rational Cherednik algebra, which will be very important for us when considering the roots of b-functions. For each reflecting hyperplane $H \in \mathcal{A}$, the group $W_H = \{w \in W \,|\, w(y) = y, \forall\, y \in H\}$ is a cyclic group and we set $\ell_H = |W_H|$. Every element of W_H, except the identity, is a reflection. For $0 \leq i \leq \ell_H$, define

$$e_{H,i} := \frac{1}{\ell_H} \sum_{w \in W_H} (\det_{\mathfrak{h}} w)^i w.$$

This is an idempotent in $\mathbb{C}W_H$ with $w \cdot e_{H,i} = (\det_{\mathfrak{h}} w)^{-i} e_{H,i}$. Choose $\kappa_{H,i} \in \mathbb{C}$ subject to $\kappa_{H,0} = \kappa_{H,\ell_H} = 0$ and the condition that $\kappa_{H,i} = \kappa_{H',i}$ if H, H' are in the same W-orbit in $\mathcal{A}$. Then we introduce, following [Val07, Definition 1.17], the alternative defining relations for $\mathtt{H}_c(W)$:

$$[y, x] = t(y, x) + \frac{1}{2} \sum_{H \in \mathcal{A}} (y, \alpha_H)(\alpha_H^\vee, x) \ell_H \sum_{i=0}^{\ell_H - 1} (\kappa_{H,i+1} - \kappa_{H,i}) e_{H,i}. \tag{3.4.1}$$

Comparing (3.2.1) with (3.4.1) gives

$$-2c(w) = \sum_{i=0}^{\ell_H - 1} (\det_{\mathfrak{h}} w)^i (\kappa_{H,i+1} - \kappa_{H,i}), \tag{3.4.2}$$

where $c(1) := 0$, since

$$-2 \sum_{w \in W_H} c(w) w = \ell_H \sum_{i=0}^{\ell_H - 1} (\kappa_{H,i+1} - \kappa_{H,i}) e_{H,i}.$$

Applying $e_{H,i}$ to both sides of the above equation and using $w \cdot e_{H,i} = (\det_{\mathfrak{h}} w)^{-i} e_{H,i}$ gives

$$\ell_H(\kappa_{H,i+1} - \kappa_{H,i}) = -2 \sum_{w \in W_H} c(w)(\det_{\mathfrak{h}} w)^{-i}. \tag{3.4.3}$$

For the remainder of the thesis, we consider only the case $t = 1$ and drop t from the notation.

§ 3.5. The Dunkl embedding

Let $\mathcal{A} = \{H_1, \ldots, H_r\}$ be the arrangement of hyperplanes $H_i \subset \mathfrak{h}$ occurring as the fixed points $H = \mathfrak{h}^s$ of the reflections $s \in \mathcal{S} \subset W$. Note that if $s(x) = x$ for all $x \in H_i$ then $H_i = \ker \alpha_s = \mathfrak{h}^s$. Let $\mathfrak{h}_{reg}$ denote the regular part of $\mathfrak{h}$, meaning the complement $\mathfrak{h} \setminus \bigcup_{H \in \mathcal{A}} H$ of the hyperplane arrangement $\mathcal{A}$. For all $y \in \mathfrak{h}$, let $\mathrm{Stab}_W(y) = \{w \in W \mid w(y) = y\}$ be its stabliser. A consequence of Steinberg's Theorem, stated as Theorem 1.1 in [Leh04], is that if $\mathcal{S} \subset W$ is the set of all reflections then for all $y \in \mathfrak{h}$, we have $\mathrm{Stab}_W(y) = \langle s \in \mathcal{S} \mid s(y) = y \rangle$. Therefore, $\mathfrak{h}_{reg}$ is also the set $\{y \in \mathfrak{h} \mid \mathrm{Stab}_W(y) = \{1\}\}$ of points in $\mathfrak{h}$ where W acts freely.

Let $n := \dim \mathfrak{h}$. Recall that by Proposition 2.3 the ring of differential operators $\mathcal{D}(\mathfrak{h}) := \mathcal{D}(\mathbb{C}[\mathfrak{h}])$ is given explicitly as the Weyl algebra $\mathcal{D}_n$. We consider the Dunkl embedding

$$\phi_c \colon \mathtt{H}_c(W) \hookrightarrow \mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$$

given by

$$y \mapsto \partial_y - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1 - \lambda_s} \frac{(y, \alpha_s)}{\alpha_s} (1 - s). \tag{3.5.1}$$

Here, we denote by ∂_y the differential operator given by

$$(\partial_y \cdot f)(v) = \lim_{\epsilon \to 0} \frac{f(v + \epsilon y) - f(v)}{\epsilon}$$

for $f \in \mathbb{C}[\mathfrak{h}], v \in \mathfrak{h}$. This is an element of $\mathcal{D}(\mathfrak{h}_{reg})$ because it is an endomorphism of $\mathbb{C}[\mathfrak{h}_{reg}]$ that satisfies the product rule.

To prove that ϕ_c is indeed an embedding of algebras $\mathtt{H}_c(W) \hookrightarrow \mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$, one needs to show that it is an injective homomorphism of rings. To check that ϕ_c is a homomorphism, one needs to verify that the images of $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $w \in W$ satisfy the defining relations Equation (3.2.1) of the rational Cherednik algebra. Injectivity is checked by showing, via the PBW theorem, that the map induced by passing to the

associated graded algebras is just localisation $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \hookrightarrow \mathbb{C}[\mathfrak{h}_{reg} \times \mathfrak{h}^*] \rtimes W$.

§ 3.6. The Euler element

We will now introduce the Euler element $\mathbf{eu}_c$, which will play a key role in the proof of the main result of the thesis. The usual Euler element in the Weyl algebra $\mathcal{D}_n$ is the vector field $\mathbf{eu} = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}$ and has the key property that $[\mathbf{eu}, x_i] = x_i$ for all i. We can think of $\mathbf{eu}$ as an element in $\mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$ and hence define the **Euler element**

$$\mathbf{eu}_c := \phi_c^{-1}\left(\sum_{i=1}^n x_i \frac{\partial}{\partial y_i}\right),$$

where ϕ_c is the Dunkl embedding (3.5.1). At first it is not even clear that $\mathbf{eu}$ is in the image of the embedding ϕ_c.

Lemma 3.7. *The element $\mathbf{eu}_c$ is a well-defined element in $\mathtt{H}_c(W)$ and can be written*

$$\mathbf{eu}_c = \sum_{i=1}^n x_i y_i + z_c' = \sum_{i=1}^n y_i x_i + z_c,$$

where

$$\begin{aligned} z_c' &= \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-s), \\ z_c &= \left(-n + \sum_{s \in \mathcal{S}} 2c(s)\right) - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s^{-1}}(1-s) \\ &= -n + 2\sum_{s \in \mathcal{S}} c(s)\left(1 - \frac{1}{1-\lambda_s^{-1}}(1-s)\right). \end{aligned}$$

Proof. Using the Dunkl embedding (3.5.1)

$$\phi_c : y \mapsto \partial_y - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s} \frac{(y, \alpha_s)}{\alpha_s}(1-s),$$

we see, substituting ∂_y into the defintion of $\mathbf{eu}_c$ from Section 3.6, that

$$\begin{aligned} \mathbf{eu}_c = \phi_c^{-1}\left(\sum_{i=1}^n x_i \partial_y\right) &= \sum_{i=1}^n x_i \left(y_i + \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s} \frac{(y_i, \alpha_s)}{\alpha_s}(1-s)\right) \\ &= \sum_{i=1}^n x_i y_i + \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s} \sum_{i=1}^n x_i \frac{(y_i, \alpha_s)}{\alpha_s}(1-s). \end{aligned}$$

Now, writing $\alpha_s = \sum_{j=1}^n a_j x_j$ for some $a_j \in \mathbb{C}$, we have

$$\sum_{i=1}^n x_i \frac{(y_i, \alpha_s)}{\alpha_s} = \sum_{i=1}^n x_i \sum_{j=1}^n \frac{(y_i, a_j x_j)}{\alpha_s} = \frac{1}{\alpha_s} \sum_{j=1}^n a_j x_j = 1.$$

Therefore,

$$\mathbf{eu}_c = \sum_{i=1}^{n} x_i y_i + \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-s) = \sum_{i=1}^{n} x_i y_i + z_c' = \sum_{i=1}^{n} y_i x_i + z_c,$$

where

$$z_c' = \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-s)$$

and

$$z_c = \sum_{i=1}^{n} x_i y_i + z_c' - \sum_{i=1}^{n} y_i x_i = z_c' - \sum_{i=1}^{n} [y_i, x_i].$$

Now, $[y_i, x_i] = 1 - \sum\limits_{s\in\mathcal{S}} c(s)(y_i, \alpha_s)(\alpha_s^\vee, x_i)s$ by (3.2.1), so

$$\begin{aligned} z_c &= \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-s) - \sum_{i=1}^{n} \left(1 - \sum_{s\in\mathcal{S}} c(s)(y_i,\alpha_s)(\alpha_s^\vee, x_i)s\right) \\ &= -n + \sum_{s\in\mathcal{S}} c(s) \left(\sum_{i=1}^{n} (y_i,\alpha_s)(\alpha_s^\vee, x_i)s + \frac{2}{1-\lambda_s}(1-s)\right). \end{aligned}$$

The same trick as above with $\alpha_s = \sum\limits_{j=1}^{n} a_j x_j$ gives

$$\sum_{i=1}^{n} (y_i,\alpha_s)(\alpha_s^\vee, x_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} (y_i, a_j x_j)(\alpha_s^\vee, x_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} (y_i, x_j)(\alpha_s^\vee, a_j x_i) = (\alpha_s^\vee, \alpha_s), \tag{3.6.1}$$

which by assumption is equal to 2; see Section 3.2. Therefore,

$$\begin{aligned} z_c &= -n + \sum_{s\in\mathcal{S}} c(s) \left((\alpha_s^\vee,\alpha_s)s + \frac{2}{1-\lambda_s}(1-s)\right) \\ &= -n + \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s} \left((1-\lambda_s)s + (1-s)\right) \\ &= -n + \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s} \left((1-\lambda_s) + \lambda_s(1-s)\right) \\ &= -n + \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-\lambda_s) + \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}\lambda_s(1-s) \\ &= \left(-n + \sum_{s\in\mathcal{S}} 2c(s)\right) - \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s^{-1}}(1-s) \\ &= -n + 2\sum_{s\in\mathcal{S}} c(s) \left(1 - \frac{1}{1-\lambda^{-1}}(1-s)\right), \end{aligned}$$

where we have used that

$$\frac{\lambda_s}{1-\lambda_s} = -\frac{1}{1-\lambda_s^{-1}}.$$

□

Using the relation (3.4.2), we may rewrite z_c' in terms of the parameters $\kappa_{H,i}$,

$$\begin{aligned}
z_c' &= \sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s}(1-s) \\
&= -\sum_{H\in\mathcal{A}} \sum_{w\in W_H\setminus\{1\}} \frac{1}{1-\det_{\mathfrak{h}}(w)^{-1}} \left(\sum_{i=0}^{\ell_H-1} \det_{\mathfrak{h}}(w)^i(\kappa_{H,i+1}-\kappa_{H,i})\right)(1-w) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \sum_{w\in W_H\setminus\{1\}} \frac{\det_{\mathfrak{h}}(w)^{i+1}}{\det_{\mathfrak{h}}(w)-1}(\kappa_{H,i+1}-\kappa_{H,i})(w-1) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \kappa_{H,i}\left(\sum_{w\in W_H\setminus\{1\}} \frac{\det_{\mathfrak{h}}(w)^i-\det_{\mathfrak{h}}(w)^{i+1}}{\det_{\mathfrak{h}}(w)-1}(w-1)\right) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \kappa_{H,i}\left(\sum_{w\in W_H\setminus\{1\}} \det_{\mathfrak{h}}(w)^i(w-1)\right) \\
&= -\sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \kappa_{H,i}\ell_H e_{H,i},
\end{aligned}$$

where we have used that $\kappa_{H,0}=0$ in the final equality. Then

$$\sum_{s\in\mathcal{S}} 2c(s) = -\sum_{H\in\mathcal{A}} \ell_H\kappa_{H,1}, \tag{3.6.2}$$

again using that $\kappa_{H,0}=0$ for all H. Similarly,

$$\begin{aligned}
\sum_{s\in\mathcal{S}} \frac{2c(s)}{1-\lambda_s^{-1}}(1-s) &= -\sum_{H\in\mathcal{A}} \sum_{w\in W_H\setminus\{1\}} \frac{1}{1-\det_{\mathfrak{h}}(w)} \left(\sum_{i=0}^{\ell_H-1} \det_{\mathfrak{h}}(w)^i(\kappa_{H,i+1}-\kappa_{H,i})\right)(1-w) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \sum_{w\in W_H\setminus\{1\}} \frac{\det_{\mathfrak{h}}(w)^i}{1-\det_{\mathfrak{h}}(w)}(\kappa_{H,i+1}-\kappa_{H,i})(w-1) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \kappa_{H,i}\left(\sum_{w\in W_H\setminus\{1\}} \frac{-\det_{\mathfrak{h}}(w)^i+\det_{\mathfrak{h}}(w)^{i-1}}{1-\det_{\mathfrak{h}}(w)}(w-1)\right) \\
&= \sum_{H\in\mathcal{A}} \sum_{i=0}^{\ell_H-1} \kappa_{H,i}\left(\sum_{w\in W_H} \det_{\mathfrak{h}}(w)^{i-1}(w-1)\right) \\
&= \sum_{H\in\mathcal{A}} \left(\sum_{i=0}^{\ell_H-1} \ell_H\kappa_{H,i+1}e_{H,i}\right) - \ell_H\kappa_{H,1}.
\end{aligned}$$

Hence

$$z_c = -n - \sum_{H\in\mathcal{A}} \ell_H \sum_{i=0}^{\ell_H-1} \kappa_{H,i+1}e_{H,i}. \tag{3.6.3}$$

Lemma 3.8. *The elements $z_c, z_c' \in \mathbb{C}W$ belong to the centre $Z(\mathbb{C}W)$ of the group algebra.*

Proof. An element of the group algebra $z \in \mathbb{C}W$ is a $\mathbb{C}$-linear combinations of group elements of W. For $g \in W$, denote by $\mathrm{coeff}(g,z)$ the coefficient of g in this combination.

A consequence of Proposition 12.22 of [JL01] is that $z \in Z(\mathbb{C}W)$ if and only if $\mathrm{coeff}(g, z) = \mathrm{coeff}(h, z)$ for all conjugate $g, h \in W$.

Start with

$$z_c' = \sum_{s \in \mathcal{S}} \frac{2c(s)}{1 - \lambda_s}(1 - s),$$

and assume that $g, h \in W$ are conjugate. The identity $1 \in W$ is the only element of its conjugacy class. Therefore, to show that $z_c' \in Z(\mathbb{C}W)$ it suffices to show that the coefficients of the $\mathbb{C}W$-basis elements g and h coincide. But $c : \mathcal{S} \to \mathbb{C}$ is conjugate invariant by definition, so it remains to show that $\lambda_g = \lambda_h$. Recall that, by Remark 3.2, the complex reflection g has eigenvalue 1 with multiplicity $n - 1$. Therefore, as the product of eigenvalues of g, we have $\det(g) = \lambda_g$. Since g and h are conjugate, $g = xhx^{-1}$ for some $x \in W$, and

$$\lambda_g = \det(g) = \det(xhx^{-1}) = \det(h) = \lambda_h.$$

Thus $z_c' \in Z(\mathbb{C}W)$.

Now,

$$z_c = \left(-n + \sum_{s \in \mathcal{S}} 2c(s)\right) - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1 - \lambda_s^{-1}}(1 - s) \in Z(\mathbb{C}W)$$

is central as the coefficients $-\frac{2c(s)}{1-\lambda_s^{-1}}$ of s are similarly conjugate invariant. □

The key property of the Euler element, that motivated its introduction, is that

$$[\mathbf{eu}_c, x] = x, \quad [\mathbf{eu}_c, y] = -y, \quad [\mathbf{eu}_c, w] = 0$$

for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$. This follows from Equation (3.1) of [BFH26], since their definition of the Euler element agrees with ours, up to a constant.

This shows that $\mathtt{H}_c(W)$ admits an *inner grading*, which is a $\mathbb{Z}$-grading given by $\deg x = 1, \deg y = -1$ and $\deg w = 0$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$ and a homogeneous element $h \in \mathtt{H}_c(W)$ has degree $d \in \mathbb{Z}$ if and only if $[\mathbf{eu}_c, h] = dh$. In particular:

Lemma 3.9. *Let $u \in \mathbb{C}[\mathfrak{h}]$ be homogeneous. Then $[\mathbf{eu}_c, u] = (\deg u)u$.*

Proof. It is sufficient to prove this for a monomial u, which we do by induction on the degree $\deg u = m_1 + \cdots + m_n$ of $u = x_1^{m_1} \cdots x_n^{m_n}$. We have $[\mathbf{eu}_c, x] = x$ as the base case. Up to relabelling, we may assume without loss of generality that $m_n > 0$. Then, using the commutator identity $[A, BC] = [A, B]C + B[A, C]$,

$$\begin{aligned}
[\mathbf{eu}_c, x_1^{m_1} \cdots x_n^{m_n}] &= [\mathbf{eu}_c, x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} x_n^{m_n - 1}] \cdot x_n + x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} x_n^{m_n - 1} \cdot [\mathbf{eu}_c, x_n] \\
&= (m_1 + \cdots + m_n - 1) x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} x_n^{m_n - 1} \cdot x_n + x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} x_n^{m_n - 1} \cdot x_n \\
&= (m_1 + \cdots + m_n) x_1^{m_1} \cdots x_n^{m_n} = (\deg u)u,
\end{aligned}$$

where we have used induction in the second equality since $m_1 + \dots m_n - 1 < \deg u$. $\square$

§ 3.7. Category $\mathcal{O}$

Let $\mathtt{H}_c(W)$−$\mathtt{mod}$ denote the category of all finitely generated left $\mathtt{H}_c(W)$-modules. A module M is said to be locally nilpotent for $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ if, for each $m \in M$ there exists some $N \gg 0$ such that $\mathfrak{h}^N \cdot m = 0$.

Definition 3.10. *Category $\mathcal{O}_c$ is defined to be the full subcategory of $\mathtt{H}_c(W)$−$\mathtt{mod}$ consisting of all modules M such that the action of $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ is locally nilpotent.*

Category $\mathcal{O}_c$ was introduced simultaneously by Berest–Etingof–Ginzburg in [BEG03], by Guay in [Gua03] and by Ginzburg–Guay–Opdam–Rouquier [GGOR03]. The definition given above is the category denoted $\mathcal{O}_{\mathtt{H}_c(W)}(0)$ in [BEG03] and [Gua03]. It can be shown to be equivalent to category $\mathcal{O}_c$ as defined in [GGOR03]. It is analogous to the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ for simple Lie algebras.

For each $\lambda \in \operatorname{Irr} W$, extend the action of W on λ to $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ by making $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ act by zero. Then

$$\Delta(\lambda) := \mathtt{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W} \lambda$$

belongs to $\mathcal{O}_c$ and is called the **standard module** labelled by λ. Then each standard module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$ and the set $\{L(\lambda) \mid \lambda \in \operatorname{Irr} W\}$ gives a complete set of pairwise non-isomorphic simple modules in $\mathcal{O}_c$ [Gua03, Introduction]. We note in passing that all objects in $\mathcal{O}_c$ have finite length and all simple modules admit projective covers, as shown in [Gua03].

Remark 3.11. Along with some extra conditions, these properties show that category $\mathcal{O}$ is a *highest weight category* in the sense of Cline–Parshall–Scott [CPS88]. This is shown in, for example, [Gua03, Theorem 3].

Definition 3.12. *A finitely generated $\mathtt{H}_c(W)$-module M is said to be $\mathbf{eu}_c$-locally finite if, for each $m \in M$, the subspace $\mathbb{C}[\mathbf{eu}_c] \cdot m \subset M$ is finite dimensional.*

Lemma 3.13. *Let M, N be $\mathbf{eu}_c$-locally finite $\mathtt{H}_c(W)$-modules.*

(i) $M = \bigoplus_{\alpha \in \mathbb{C}} \mathcal{W}_\alpha(M)$, *where* $\mathcal{W}_\alpha(M) = \{m \in M \mid (\mathbf{eu}_c - \alpha)^k \cdot m = 0,\ k \gg 0\}$.

(ii) If $\dim_{\mathbb{C}} \mathcal{W}_\alpha(M), \dim_{\mathbb{C}} \mathcal{W}_\alpha(N) < \infty$ *for all* $\alpha \in \mathbb{C}$ *then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathtt{H}_c(W)}(M, N) < \infty.$$

Proof. Part (i). If M is finite dimensional then, since we are working over an algebraically closed field, the primary decomposition theorem for the endomorphism $\mathbf{eu}_c$ is precisely

the decomposition $M = \bigoplus_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(M)$; see [Lan02, XIV, Section 2]. Therefore, the claim of part (i) is really that the same decomposition holds when M is infinite dimensional, if the action of $\mathbf{eu}_c$ is locally finite. First, we note that the sum $\sum_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(M)$ is still direct. Indeed, if there are $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ and $m_i \in \mathcal{W}_{\alpha_i}(M)$ such that $m_1 + \dots + m_k = 0$ then there exists a finite dimensional $V \subset M$ that is stable under $\mathbf{eu}_c$ and contains all m_i. But the sum

$$V = \sum_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(V) = \sum_{\alpha\in\mathbb{C}} (\mathcal{W}_\alpha(M) \cap V)$$

is direct by the primary decomposition theorem so $m_i = 0$ for all i. Thus, $\bigoplus_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(M) \subset M$. Finally, if $m \in M$ then $m \in U := \mathbb{C}[\mathbf{eu}_c] \cdot m \subset M$ and $\dim_{\mathbb{C}} U < \infty$ by assumption. Therefore, $U \subset \bigoplus_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(M)$ and hence $m \in \bigoplus_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha(M)$.

Part (ii). Since M is assumed to be finitely generated and the action of $\mathbf{eu}_c$ on M is locally finite, we can pick $I = \{\alpha_1, \dots, \alpha_k\} \subset \mathbb{C}$ such that the finite dimensional $\mathbf{eu}_c$-stable subspace $M_0 := \bigoplus_{\alpha\in I} \mathcal{W}_\alpha(M)$ generates M. Notice that if $\phi \in \mathrm{Hom}_{\mathtt{H}_c(W)}(M, N)$ then $\phi(\mathcal{W}_\alpha(M)) \subset \mathcal{W}_\alpha(N)$ for all $\alpha \in \mathbb{C}$. In particular, $\phi(M_0) \subset N_0 := \bigoplus_{\alpha\in I} \mathcal{W}_\alpha(N)$. Therefore, there is a restriction map

$$\mathrm{Hom}_{\mathtt{H}_c(W)}(M, N) \to \mathrm{Hom}_{\mathbb{C}}(M_0, N_0).$$

Since M_0 generates M, any homomorphism ϕ is uniquely defined by what it does on M_0. This means that the restriction map is injective and hence

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathtt{H}_c(W)}(M, N) \leq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(M_0, N_0) < \infty.$$

□

The following summarises the properties of category $\mathcal{O}$ that we will need in the thesis.

Theorem 3.14. *(i) If $M \in \mathcal{O}_c$ then M is finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module.*

(ii) For generic c, $\mathcal{O}_c$ is semi-simple.

(iii) If $M \in \mathcal{O}_c$ then M is $\mathbf{eu}_c$-locally finite with $\dim_{\mathbb{C}} \mathcal{W}_\alpha(M) < \infty$ for all $\alpha \in \mathbb{C}$.

(iv) If $M \in \mathcal{O}_c$ then $\dim_{\mathbb{C}} \mathrm{End}_{\mathtt{H}_c(W)}(M) < \infty$.

Proof. Part (i). This is [Gua03, Lemma 2.5(1)]; see also Exercise 2.11.1 of [Bel16].

Part (ii). This is explained on page 281 of [BEG03]. See Exercise 2.11.6 of [Bel16] for a simple argument.

Part (iii) is Lemma 2.4.2 from [Bel16].

Finally, since $M \in \mathcal{O}_c$ is $\mathbf{eu}_c$-locally finite by part (iii), Lemma 3.13(ii) with $M = N$ implies that $\dim_{\mathbb{C}} \mathrm{End}_{\mathtt{H}_c(W)}(M) < \infty$. □

We will actually require the alternative definition of category $\mathcal{O}_c$, where the $x_i \in \mathbb{C}[\mathfrak{h}]$ are assumed to act locally nilpotently. Therefore, to be precise, we define

$$\mathcal{O}_c^* = \{N \in H_c\text{-mod} \,|\, x_i \text{ acts locally nilpotently}\}. \tag{3.7.1}$$

Since there is a clear symmetry between $\mathfrak{h}$ and $\mathfrak{h}^*$ in $\mathtt{H}_c(W)$, all the properties described above for $\mathcal{O}_c$ also hold for $\mathcal{O}_c^*$ if appropriately rewritten.

§ 3.8. Holonomic modules for rational Cherednik algebras

Holonomic modules for rational Cherednik algebras are defined analogous to those for $\mathcal{D}$-modules. Namely, we choose good filtrations to define the characteristic variety of a finitely generated module, and holonomic modules are those of "smallest" possible "size". As mentioned in Remark 2.18, there are two ways to characterise holonomic $\mathcal{D}$-modules, either to say that the dimension of their characteristic variety is the smallest possible (equal to the dimension of X), or to say that their characteristic variety is Lagrangian. In the case of rational Cherednik algebras these two (numerical and geometric) characterisations differ and the correct notion is the geometric one. We describe this below.

Recall from Section 3.3 that $\mathtt{H}_c(W)$ has the order filtration $\mathcal{F}_\bullet$ such that $\mathrm{gr}_{\mathcal{F}} \mathtt{H}_c(W) \cong \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$. Let M be a finitely generated $\mathtt{H}_c(W)$-module. As in Section 2.2.1, a filtration $\mathcal{F}_\bullet M$ on M, compatible with the order filtration $\mathcal{F}_\bullet$ on $\mathtt{H}_c(W)$, is a nested sequence $\cdots \subset \mathcal{F}_{-1}M \subset \mathcal{F}_0 M \subset \mathcal{F}_1 M \subset \cdots$ of subspaces such that $\mathcal{F}_{-k}M = 0$ for $k \gg 0$, the union of the $\mathcal{F}_i M$ equals M, and

$$(\mathcal{F}_i \mathtt{H}_c(W)) \cdot (\mathcal{F}_j M) \subset \mathcal{F}_{i+j} M, \qquad \text{for all } i, j \in \mathbb{Z}.$$

Then $\mathrm{gr}_{\mathcal{F}} M$ is a $\mathrm{gr}_{\mathcal{F}} \mathtt{H}_c(W) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$-module. In particular, it is a $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]$-module by restriction and we can consider its support as a subset of $\mathfrak{h} \times \mathfrak{h}^* = T^*\mathfrak{h}$. As for $\mathcal{D}$-modules, we say that $\mathcal{F}_\bullet M$ is a good filtration if $\mathrm{gr}_{\mathcal{F}} M$ is finitely generated over $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$. Just as for $\mathcal{D}$-modules, one can argue that any finitely generated module can be given a good filtration.

Definition 3.15. *Let M be a finitely generated $\mathtt{H}_c(W)$-module. The* ***characteristic variety*** *(or* singular support*) of M is defined to be*

$$\mathrm{Ch}(M) = \mathrm{Supp}(\mathrm{gr}_{\mathcal{F}} M) \subset \mathfrak{h} \times \mathfrak{h}^*,$$

for some good filtration $\mathcal{F}_\bullet M$ on M.

Copying word-for-word the proof of Proposition 2.15 gives:

Proposition 3.16. *For a finitely generated* $\mathtt{H}_c(W)$*-module* M*, its characteristic variety* $\mathrm{Ch}(M)$ *is independent of the choice of good filtration.*

Since $\mathfrak{h} \times \mathfrak{h}^* = T^*\mathfrak{h}$ is a symplectic variety, we can talk about isotropic, coisotropic and Lagrangian subvarieties as in Definition 2.12.

Definition 3.17. *Let* M *be a finitely generated* $\mathtt{H}_c(W)$*-module. Then* M *is holonomic if* $\mathrm{Ch}(M)$ *is an isotropic subvariety of* $\mathfrak{h} \times \mathfrak{h}^* = T^*\mathfrak{h}$.

Gabber's Theorem [Gab81] cannot be applied directly to $\mathtt{H}_c(W)$ because its associated graded ring is not commutative. Therefore, it is not the case that the characteristic variety of M is always coistropic. In particular, it is possible that $\dim \mathrm{Ch}(M) < \dim \mathfrak{h}$. Indeed, for some parameters c it is possible to find $\mathtt{H}_c(W)$-modules that are finite dimensional over $\mathbb{C}$, in which case $\mathrm{Ch}(M) = \{0\}$ (these modules are all holonomic).

We note that the definition of holonomic modules for rational Cherednik algebras is due originally to Losev in [Los17]. In that paper, Losev considers $\mathrm{gr}_{\mathcal{F}} M$ as a $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$-module and hence defines the characteristic variety $\widetilde{\mathrm{Ch}}(M)$ of M as a subvariety of the quotient space $(\mathfrak{h} \times \mathfrak{h}^*)/W$. Note that this space is singular and therefore not a symplectic variety. However, it is a Poisson variety since $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ is a Poisson subalgebra of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]$. Moreover, it has a finite stratification by symplectic leaves, which are symplectic (so smooth) locally closed subsets. In this setting, a subvariety $\Lambda \subset (\mathfrak{h} \times \mathfrak{h}^*)/W$ is said to be Lagrangian if $\Lambda \cap \mathcal{L}$ is Lagrangian (in the usual symplectic sense) for every leaf $\mathcal{L}$. Then Losev defines M to holonomic if $\widetilde{\mathrm{Ch}}(M) \subset (\mathfrak{h} \times \mathfrak{h}^*)/W$ is Lagrangian.

To see that the two definitions are equivalent, we note first that Gabber's Theorem [Gab81] can be shown to hold with respect to $\widetilde{\mathrm{Ch}}(M) \subset (\mathfrak{h} \times \mathfrak{h}^*)/W$, in the sense that $\widetilde{\mathrm{Ch}}(M) \cap \mathcal{L}$ is a coisotropic subvariety of $\mathcal{L}$ for each leaf intersecting $\widetilde{\mathrm{Ch}}(M)$. See [Los17] for an explanation of this fact. Next, it was shown in [BET26, Lemma 2.8] that $\Lambda \subset (\mathfrak{h} \times \mathfrak{h}^*)/W$ is isotropic (in the sense that $\Lambda \cap \mathcal{L}$ is isotropic for every leaf) if and only if $\pi^{-1}(\Lambda) \subset \mathfrak{h} \times \mathfrak{h}^*$ is isotropic. Since $\mathrm{Ch}(M) = \pi^{-1}(\widetilde{\mathrm{Ch}}(M))$, it follows that $\widetilde{\mathrm{Ch}}(M)$ is Lagrangian if and only if it is isotropic if and only if $\pi^{-1}(\widetilde{\mathrm{Ch}}(M))$ is isotropic. Note that the example $\mathrm{Ch}(M) = \{0\}$ shows that it need not be the case that $\Lambda \subset (\mathfrak{h} \times \mathfrak{h}^*)/W$ Lagrangian implies that $\pi^{-1}(\Lambda)$ is Lagrangian.

§ 3.9. The Weyl group B_2

In this section, we work out the defining relations and the Dunkl embedding in detail for the Weyl group B_2. This will be a recurring example in the thesis so these additional details will be useful.

The Weyl group $B_2 = \langle \mathcal{S} \rangle$ is the dihedral group of order 8 (that is, the symmetries of

the square) acting on $\mathfrak{h} = \mathbb{C}^2$ by the reflections

$$\mathcal{S} = \left\{ s_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The reflecting hyperplanes $H_i := \ker(1 - s_i)$ form a hyperplane arrangement with positive roots $\alpha_1 = x_1,\ \alpha_2 = x_1 + x_2,\ \alpha_3 = x_2,\ \alpha_4 = x_1 - x_2$.

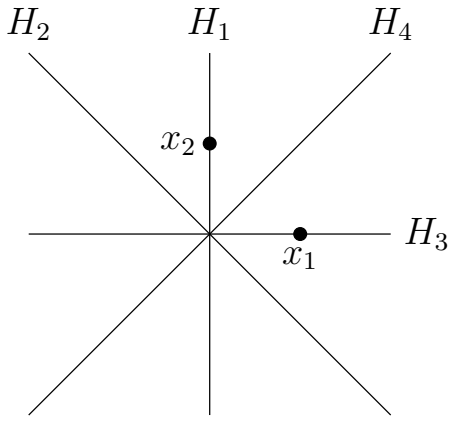


Figure 3.1: Hyperplane arrangement for B_2

We study the rational Cherednik algebra $\mathtt{H}_c(B_2)$. Note that $s_1 = s_2 s_3 s_2$ and $s_2 = s_1 s_4 s_1$, so we have two conjugacy classes of reflections, and the parameter c becomes the class function

$$c(s) = \begin{cases} c_1 & \text{if } s = s_1, s_3, \\ c_2 & \text{if } s = s_2, s_4, \end{cases}$$

for a choice of complex numbers $c_1, c_2 \in \mathbb{C}$. In terms of the parameters κ, there are two orbits of reflections and $W_H = \{1, s\}$ for each reflecting hyperplane. So the parameters are $\kappa_{H_1,0} = 0, \kappa_{H_1,1}$ and $\kappa_{H_2,0} = 0, \kappa_{H_2,1}$. Equation (3.4.2) gives

$$c_1 = -\kappa_{H_1,1}, \quad c_2 = -\kappa_{H_2,1}. \tag{3.9.1}$$

Recall from Definition 3.4 that the defining relations for $\mathtt{H}_c(B_2)$ are $[x_1, x_2] = [y_1, y_2] = 0$ and

$$[y_i, x_j] = (y_i, x_j) - \sum_{s \in \mathcal{S}} c(s)(y_i, \alpha_s)(\alpha_s^\vee, x_j)s.$$

To explicitly compute these relations in the case $W = B_2$, we summarise the values of the pairing $(-,-)\colon \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ on the various vectors involved. By assumption $(y_i, x_j) = \delta_{ij}$ (for δ_{ij} the Kronecker delta function). We also assume that $(\alpha^\vee, \alpha) = 2$, which makes the choices of α and $\alpha^\vee$ unique only up to simultaneous scaling. This implies a choice for $\alpha^\vee$

is

$$\alpha_1^\vee = 2y_1, \quad \alpha_2^\vee = y_1 + y_2, \quad \alpha_3^\vee = 2y_2, \quad \alpha_4^\vee = y_1 - y_2.$$

We can then compute:

$$\begin{aligned}
&(y_1, \alpha_1) = (y_1, x_1) = 1, && (y_2, \alpha_2) = (y_2, x_1) = 0,\\
&(y_1, \alpha_2) = (y_1, x_1 + x_2) = 1, && (y_2, \alpha_2) = (y_2, x_1 + x_2) = 1,\\
&(y_1, \alpha_3) = (y_1, x_2) = 0, && (y_2, \alpha_3) = (y_2, x_2) = 1,\\
&(y_1, \alpha_4) = (y_1, x_1 - x_2) = 1, && (y_2, \alpha_4) = (y_2, x_1 - x_2) = -1,
\end{aligned}$$

$$\begin{aligned}
&(\alpha_1^\vee, x_1) = (\alpha_1^\vee, \alpha_1) = 2, && (\alpha_1^\vee, x_2) = 0,\\
&(\alpha_2^\vee, x_1) = 1, && (\alpha_2^\vee, x_2) = 1,\\
&(\alpha_3^\vee, x_1) = 0, && (\alpha_3^\vee, x_2) = 2,\\
&(\alpha_4^\vee, x_1) = 1, && (\alpha_4^\vee, x_2) = -1.
\end{aligned}$$

Plugging these values into the formula for commutators gives:

$$\begin{aligned}
[y_1, x_1] &= 1 - 2c_1 s_1 - c_2 s_2 - c_2 s_4,\\
[y_1, x_2] &= -c_2 s_2 + c_2 s_4,\\
[y_2, x_1] &= -c_2 s_2 + c_2 s_4,\\
[y_2, x_2] &= 1 - c_2 s_2 - 2c_1 s_3 - c_2 s_4.
\end{aligned}$$

Computations of b-functions for $W = B_2$ will be made much easier by using the Dunkl operators obtained by embedding y_i into the ring of differential operators on $\mathbb{C}[x_1, x_2][\delta^{-1}]$, which specialising (3.5.1) to this case gives:

$$y_i = \frac{\partial}{\partial x_i} - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1 - \lambda_s} \frac{(y_i, \alpha_s)}{\alpha_s}(1 - s) = \frac{\partial}{\partial x_i} - \sum_{s \in \mathcal{S}} \frac{c(s)}{\alpha_s}(y, \alpha_s)(1 - s),$$

where $\lambda_s = -1$ is the eigenvalue given by $s(\alpha_s) = \lambda_s \alpha_s$. So we have

$$\begin{aligned}
y_1 &= \frac{\partial}{\partial x_1} - \frac{c_1}{x_1}(1 - s_1) - \frac{c_2}{x_1 + x_2}(1 - s_2) - \frac{c_2}{x_1 - x_2}(1 - s_4),\\
y_2 &= \frac{\partial}{\partial x_2} - \frac{c_2}{x_1 + x_2}(1 - s_2) - \frac{c_1}{x_2}(1 - s_3) + \frac{c_2}{x_1 - x_2}(1 - s_4).
\end{aligned}$$

Here $\delta = x_1^2 x_2^2 (x_1^2 - x_2^2)^2$.

Chapter 4

b-functions

Joseph Bernstein introduced the b-function (also known as the b-polynomial or Bernstein–Sato polynomial) in [Ber71], although it was also independently introduced by Mikio Sato and Takuro Shintani in [SS72]. Its definition and motivation for study can be gained from the following result by Bernstein, found for instance in Lemma 1 of [BG12].

Let X be a smooth affine algebraic variety over $\mathbb{C}$. Let f be a function on X and let $U \hookrightarrow X$ be the non-zero locus of f. Let $\mathcal{D}_X$ be the sheaf of differential operators on X and let $\mathcal{M}$ be a holonomic (left) $\mathcal{D}$-module on U.

Consider the sheaf $\mathcal{D}_X[s]$ given by tensoring $\mathcal{D}_X$ with the ring $k[s]$ of polynomials in one variable, and the $\mathcal{D}_U[s]$-module $\mathcal{O}_U[s]f^s$ defined to be the free $\mathcal{O}_U[s]$-module of rank one with generator denoted f^s, with action of a vector field ξ given by $\xi(f^s) = s{\cdot}\xi(f){\cdot}f^{s-1}$ where $f^{s-1} := f^{-1} \cdot f^s$.

Lemma 4.1 (The b-function Lemma)**.** *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module and $m_1, \ldots, m_r$ generators of $\mathcal{M}$ as a $\mathcal{D}_U$-module. Then there exist elements $P_{i,j} \in \mathcal{D}_X[s]$ and a non-zero element $b_f \in k[s]$ such that for every i,*

$$\sum_j P_{i,j}(m_j \otimes f^{s+1}) = b_f \cdot (m_i \otimes f^s).$$

In particular, Lemma 1.1 from the introduction is a simplified corollary of the b-function Lemma which gives us the b-function when $X = \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$, $\mathcal{D}_X = \mathcal{D}_n$ is the ring of differential operators acting on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$, and $\mathcal{M} = \mathcal{O}_U$:

$$P(s)f(x)^{s+1} = b_f(s)f(x)^s.$$

This is often referred to as the Bernstein equation; with $P(s)$ being referred to as the Bernstein operator.

§ 4.1. b-functions for rational Cherednik algebras

Let X be a smooth affine variety with an action of a finite group W and pick $f \in \mathcal{O}(X)^W$. Then a b-function for f, with respect to a Cherednik algebra $\mathtt{H}_c(X, W)$, was defined in [BET26, Corollary 1.2]. In this thesis, we consider the case where $X = \mathfrak{h}$ is a vector space and W acts linearly on $\mathfrak{h}$ as a complex reflection group.

Lemma 4.2. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be W-invariant. Then the vector space $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$ is a $\mathtt{H}_c(W)$-module, where*

$$x \cdot f^s = xf^s, \quad y \cdot f^s = s\frac{\partial f}{\partial y}f^{-1}f^s, \quad s \cdot f^s = sf^s, \quad w \cdot f^s = f^s, \tag{4.1.1}$$

for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$.

Before we give a proof of Lemma 4.2, we make some clarifying remarks.

- We note that f^s is a formal symbol and not the s-th power of the function f.

- When $f \in \mathbb{C}[\mathfrak{h}]^W$ and $s \in \mathbb{Z}_{>0}$ then the formulae given in Equation (4.1.1) describe the action of the generators $x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W$ of the rational Cherednik algebra on $f^s \in \mathbb{C}[\mathfrak{h}]$. This can be seen immediately from the Dunkl embedding (3.5.1).

- If $n \in \mathbb{Z}$ then we often write $f^{s+n} := f^n f^s$ in $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$. In particular,

$$f^{s+1} := ff^s.$$

Proof. This was explained in the first paragraph of [BET26, Section 6.2]. We give an alternative proof based on the Dunkl embedding (3.5.1). First, we note that $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$ is a $\mathbb{C}[\mathfrak{h}]$-submodule of $\mathbb{C}[\mathfrak{h}_{reg}][f^{-1}][s]f^s = \mathbb{C}[\mathfrak{h}][f^{-1}, \delta^{-1}][s]f^s$. The latter is a $\mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$-module, where $w \cdot f^s = f^s$ and $\partial_{x_i} \cdot f^s = sf^{-1}\frac{\partial f}{\partial x_i}f^s$. Via the Dunkl embedding ϕ_c, we think of $\mathtt{H}_c(W)$ as a subalgebra of $\mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$. Therefore, $\mathbb{C}[\mathfrak{h}][f^{-1}, \delta^{-1}][s]f^s$ is a $\mathtt{H}_c(W)$-module. We need to check that $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$ is a $\mathtt{H}_c(W)$-submodule with the action of the generators of $\mathtt{H}_c(W)$ given by (4.1.1). This is immediate for $x \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$ and $w \in W$. Let $y \in \mathfrak{h}$ and $r \in \mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$. Then there exist (unique) $q_i \in \mathbb{C}[\mathfrak{h}][f^{-1}]$ such

that $r = \sum_i q_i s^i f^s$. We compute that

$$\begin{aligned} y \cdot r &= \sum_i y \cdot q_i s^i f^s \\ &= \sum_i [y, q_i] s^i f^s + q_i s^i y \cdot f^s \\ &= \sum_i [y, q_i] s^i f^s + q_i s^i s f^{-1} \frac{\partial f}{\partial y} f^s \\ &= \sum_i [y, q_i] s^i f^s + q_i f^{-1} \frac{\partial f}{\partial y} s^{i+1} s f^s \in \mathbb{C}[\mathfrak{h}][f^{-1}, \delta^{-1}][s] f^s \end{aligned}$$

where we have used that

$$[y, q] = \frac{\partial q}{\partial y} - \sum_{\sigma \in \mathcal{S}} \frac{2c(\sigma)}{1 - \lambda_\sigma} \frac{(y, \alpha_\sigma)(q - \sigma(q))}{\alpha_\sigma} \sigma \in \mathbb{C}[\mathfrak{h}][f^{-1}] \otimes \mathbb{C}W$$

and that

$$\left(\partial_y - \sum_{\sigma \in \mathcal{S}} \frac{2c(\sigma)}{1 - \lambda_\sigma} \frac{(y, \alpha_\sigma)}{\alpha_\sigma} (1 - \sigma) \right) \cdot f^s = \partial_y \cdot f^s = s f^{-1} \frac{\partial f}{\partial y} f^s.$$

It follows that $y \cdot r \in \mathbb{C}[\mathfrak{h}][f^{-1}][s] f^s$ and the action of y on the generator f^s is given by (4.1.1). □

In the next couple of sections we develop the basic properties of the b-function for rational Cherednik algebras, by analogy with the classical theory for $\mathcal{D}$-modules.

For fixed W, we denote by $\mathtt{H}_c[s]$ the tensor product algebra $\mathtt{H}_c \otimes_{\mathbb{C}} \mathbb{C}[s]$. The action of $\mathtt{H}_c$ on $\mathbb{C}[\mathfrak{h}][f^{-1}][s] f^s$ extends to $\mathtt{H}_c[s]$. The following result is a key step in the proof of [BET26, Theorem 6.13], which says that the push-forward of a holonomic $\mathtt{H}_c$-module under an open embedding is again holonomic.

Theorem 4.3 ([BET26], Corollary 1.2)**.** *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be a non-constant function. Under a certain genericity condition on the parameter c, there exists an operator $P \in \mathtt{H}_c[s]$ and non-zero $b \in \mathbb{C}[s]$ such that*

$$P \cdot f^{s+1} = b(s) f^s \tag{4.1.2}$$

in $\mathbb{C}[\mathfrak{h}][f^{-1}][s] f^s$.

Colloquially, this theorem says that a b-function exists for any W-invariant f.

Remark 4.4. We do not make explicit the genericity condition on the parameter c in Theorem 4.3, since in the case of polynomials with isolated singularities we will prove directly in Corollary 5.17 that the b-function exists *for any parameter c*.

Remark 4.5. Let $e := \frac{1}{|W|} \sum_{w \in W} w \in \mathtt{H}_c(W)$ be the trivial idempotent. Since f^s is defined to be W-invariant, we can always assume that $P(s) \in \mathtt{H}_c e[s]$, or even that $P \in e\mathtt{H}_c e[s]$, in

the defining equation (4.1.2) of the b-function, since $ef^{s+1} = f^{s+1}$ implies that $Pf^{s+1} = Pef^{s+1}$ and $Pf^{s+1} = b(s)f^s$ implies that $ePf^{s+1} = eb(s)f^s = b(s)f^s$.

To appreciate why the existence of the b-function is not obvious and is important, we consider the $\mathtt{H}_c(W)$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]$. First we note that since $f \in \mathbb{C}[\mathfrak{h}]^W$, the operator $\mathrm{ad}(f)\colon \mathtt{H}_c \to \mathtt{H}_c$, $\mathrm{ad}(f)(h) = [f, h]$, is locally nilpotent on $\mathtt{H}_c(W)$ as can be seen from the Dunkl embedding (3.5.1). Hence the powers $\{f^k\}_{k>0}$ of f form a two-sided Ore set in $\mathtt{H}_c(W)$; see for instance [Mat00, Lemma 4.2]. Therefore, we can localise any $\mathtt{H}_c$-module with respect to f. In particular, $\mathbb{C}[\mathfrak{h}][f^{-1}]$ is a $\mathtt{H}_c[f^{-1}]$-module and thus a $\mathtt{H}_c$-module by restriction. It is highly non-obvious that $\mathbb{C}[\mathfrak{h}][f^{-1}]$ is finitely generated over $\mathtt{H}_c$. The existence of the b-function allows us to prove something even stronger.

Lemma 4.6. *Assume that Theorem 4.3 holds for f. Then $\mathbb{C}[\mathfrak{h}][f^{-1}]$ is a cyclic $\mathtt{H}_c$-module.*

Proof. Since $\mathbb{C}[\mathfrak{h}] \subset \mathtt{H}_c$, it suffices to show that there exists $r > 0$ such that $f^{-k} \in \mathtt{H}_c \cdot f^{-r}$ for all $k > 0$. Pick r such that $-k$ is not a root of b for all $k \geq r$. Then specialising $P(s) \cdot f^{s+1} = b(s)f^s$ to $s = -k - 1$ for $k \geq r$ implies that

$$P(-k-1)f^{-k} = b(-k-1)f^{-k-1}.$$

Since $b(-k-1) \neq 0$, this implies by induction that $f^{-k-1} \in \mathtt{H}_c \cdot f^{-r}$ for all $k \geq r$. □

In fact, Theorem 6.16 of [BET26] shows that $\mathbb{C}[\mathfrak{h}][f^{-1}]$ is even a holonomic $\mathtt{H}_c$-module, provided the b-function exists.

§ 4.2. Basic properties of the b-function

We begin by describing the basic properties of the b-function, as always by analogy with the case of $\mathcal{D}$-modules.

Proposition 4.7. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$. Then the set*

$$B = \{b \in \mathbb{C}[s] \mid \exists P \in \mathtt{H}_c[s] \text{ such that } P(s)f^{s+1} = b(s)f^s\}$$

is an ideal in $\mathbb{C}[s]$.

Proof. We note first that $0 \in B$. Let $b, b' \in B$ and $g \in \mathbb{C}[s]$. Then there exist $P, P' \in \mathtt{H}_c[s]$ satisfying $P(s)f(x)^{s+1} = b(s)f(x)^s$, and similar for P'. The equation

$$(P(s) + g(s)P'(s))f(x)^{s+1} = (b(s) + g(s)b'(s))f(x)^s.$$

implies that $b + gb' \in B$, and hence B is an ideal. □

If the ideal B is non-zero then the b-function b_c^f of f is defined to be the (unique) monic generator of B.

Consider the $\mathtt{H}_c$-module $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$, as in Lemma 4.2. For all $\lambda \in \mathbb{C}$ there exists a surjective algebra homomorphism $\mathtt{H}_c[s] \to \mathtt{H}_c$ which is the identity on $\mathtt{H}_c$ and sends s to λ. Similarly, we write $\mathbb{C}[\mathfrak{h}][f^{-1}]v_\lambda$ for the $\mathtt{H}_c[s]$-module which is free of rank one as a $\mathbb{C}[\mathfrak{h}][f^{-1}]$-module, and

$$y \cdot v_\lambda = \lambda f^{-1} v_\lambda, \qquad w \cdot v_\lambda = v_\lambda, \qquad s \cdot v_\lambda = \lambda \cdot v_\lambda,$$

for $y \in \mathfrak{h}$ and $w \in W$. Then there is a surjection of $\mathtt{H}_c$-modules

$$\mathbb{C}[\mathfrak{h}][f^{-1}[s]f^s \to \mathbb{C}[\mathfrak{h}][f^{-1}]v_\lambda, \quad f^s \mapsto v_\lambda.$$

Proposition 4.8. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be non-constant. Then $(s+1)$ divides $b_c^f(s)$.*

Proof. We take $\lambda = -1$ in the above construction. If $P(s)f^{s+1} = b_c^f(s)f^s$ in $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$ then $P(-1)fv_{-1} = b_c^f(-1)v_{-1}$ in $\mathbb{C}[\mathfrak{h}][f^{-1}]v_{-1}$. If $\mathbb{C}[\mathfrak{h}]$ denotes the polynomial representation of $\mathtt{H}_c$ then $\mathtt{H}_c$ acts on the localization $\mathbb{C}[\mathfrak{h}][f^{-1}]$ and the defining relations of $\mathbb{C}[\mathfrak{h}][f^{-1}]v_{-1}$ show that there is an isomorphism of $\mathtt{H}_c$-modules $\mathbb{C}[\mathfrak{h}][f^{-1}] \cong \mathbb{C}[\mathfrak{h}][f^{-1}]v_{-1}$ given by $1 \mapsto fv_{-1}$. Therefore, in the $\mathtt{H}_c$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]$, we have

$$P(-1) \cdot 1 = b_f(-1)f^{-1}. \tag{4.2.1}$$

If f is not invertible then the left hand side of this equation is regular but the right hand side has a pole along f, so $b_c^f(-1) = 0$ which implies that $(s+1)$ divides $b_c^f(s)$. □

In light of this fact, we will often consider the **augmented b-function**

$$\tilde{b}_c^f(s) := b_c^f(s)/(s+1).$$

Let $\mathcal{N}_f = \mathtt{H}_c[s] \cdot f^s$ be the $\mathtt{H}_c[s]$-submodule of $\mathbb{C}[\mathfrak{h}][s][f^{-1}]f^s$ generated by f^s and write $t(\mathcal{N}_f) = \mathtt{H}_c[s]ff^s$, which is a submodule of $\mathcal{N}_f$.

Lemma 4.9. *The polynomial $b_c^f(s)$ exists if and only if the minimal polynomial of s acting on $\mathcal{N}_f/t(\mathcal{N}_f)$ exists. In this case they are equal.*

Proof. Assume that b_c^f exists. Therefore, there exists $P(s) \in \mathtt{H}_c[s]$ such that $P(s)f^{s+1} = b_c^f(s)f^s$. Notice that for $D \in \mathtt{H}_c[s]$,

$$b_c^f(s)Df^s = Db_c^f(s)f^s = DP(s)f^{s+1} = 0$$

in $\mathcal{N}_f/t(\mathcal{N}_f)$. This implies that the minimal polynomial $m(s)$ exists and, by definition, $m(s)$ divides $b_c^f(s)$ since $b_c^f(s) \cdot (\mathcal{N}_f/t(\mathcal{N}_f)) = 0$.

Conversely, assume that the minimal polynomial $m(s)$ of s acting on $\mathcal{N}_f/t(\mathcal{N}_f)$ exists. Then $m(s)\cdot f^s = 0$ in $\mathcal{N}_f/t(\mathcal{N}_f)$ and thus $m(s)f^s \in t(\mathcal{N}_f)$. That is, there exists an operator $Q(s) \in \mathtt{H}_c[s]$ such that $m(s)f^s = Q(s)f^{s+1}$. Therefore, $b_c^f(s)$ exists and divides $m(s)$, and hence $b_c^f(s) = m(s)$ because they are monic. □

Remark 4.10. In examples, the existence of the b-function follows from Theorem 4.3; see also Remark 4.4.

Note that Lemma 4.9 is really saying that

$$(b_f(s)) = \mathrm{Ann}_{\mathbb{C}[s]}(\mathtt{H}_c[s]f^s/\mathtt{H}_c[s]f^{s+1}) \tag{4.2.2}$$

as ideals in $\mathbb{C}[s]$.

We recall that if $f_i = \frac{\partial f}{\partial x_i}$ is the i-th partial derivative of f then the **Jacobian ideal** $J(f)$ is the ideal $(f_1, \ldots, f_n)$ in $\mathbb{C}[\mathfrak{h}]$. Let

$$\mathcal{M} = \mathcal{N}_f \Big/ \mathtt{H}_c[s] \cdot (f + J(f))f^s, \tag{4.2.3}$$

and note that $\mathcal{N}_f/t(\mathcal{N}_f) \twoheadrightarrow \mathcal{M}$.

Lemma 4.11. *The polynomial $\tilde{b}_c^f(s)$ exists if and only if the minimal polynomial of s acting on $\mathcal{M}$ exists. In this case they are equal.*

Proof. Assume that $\tilde{b}_c^f$ exists. As in Proposition 4.8, we consider the $\mathtt{H}_c[s]$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]v_{-1}$. Recall that in the proof of Proposition 4.8 we showed that if $P(s)f^{s+1} = b_c^f(s)f^s$ then $b_c^f(-1) = 0$ and $P(-1)\cdot 1 = 0$ in the $\mathtt{H}_c$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]$. Since $\mathbb{C}[\mathfrak{h}]$ is a $\mathtt{H}_c$-submodule of $\mathbb{C}[\mathfrak{h}][f^{-1}]$, this shows that $P(-1)\cdot 1 = 0$ in $\mathbb{C}[\mathfrak{h}]$. Now, $\mathbb{C}[\mathfrak{h}] = \mathtt{H}_c/\sum_{i=1}^n \mathtt{H}_c y_i$ as a left $\mathtt{H}_c$-module, hence $P(-1)\cdot 1 = 0$ implies that

$$P(-1) = \sum_{i=1}^n A_i y_i$$

for some $A_i \in \mathtt{H}_c$. This implies that there exist $Q(s), A_i(s) \in \mathtt{H}_c[s]$ such that

$$P(s) = (s+1)Q(s) + \sum_{i=1}^n A_i(s)y_i. \tag{4.2.4}$$

Substituting this into $P(s)f^{s+1} = b_c^f(s)f^s$, and noting that

$$y_i f^{s+1} = (s+1)f^{-1}f_i f^{s+1} = (s+1)f_i f^s, \tag{4.2.5}$$

we obtain

$$P(s)f^{s+1} = (s+1)\left(Q(s)f + \sum_{i=1}^n A_i(s)f_i\right)f^s = (s+1)\tilde{b}_c^f(s)f^s.$$

Since $\mathbb{C}[\mathfrak{h}][f^{-1}][s]f^s$ is free over $\mathbb{C}[s]$, this implies that

$$\left(Q(s)f + \sum_{i=1}^{n} A_i(s)f_i\right) f^s = \tilde{b}_c^f(s)f^s.$$

Similarly to the proof of Lemma 4.9, for any $D \in \mathtt{H}_c[s]$,

$$\tilde{b}_c^f(s)Df^s = D\tilde{b}_c^f(s)f^s = D\left(Q(s)f + \sum_{i=1}^{n} A_i(s)f_i\right) f^s = DQ(s)f^{s+1} + \sum_{i=1}^{n} DA_i(s)f_i f^s = 0$$

in $\mathcal{M}$ since all summands on the right hand side are in $\mathtt{H}_c[s] \cdot (f + J(f))f^s$. This implies that the minimal polynomial $m(s)$ exists. By the definition, $m(s)$ must divide $\tilde{b}_c^f(s)$ since $\tilde{b}_c^f(s) \cdot \mathcal{M} = 0$.

Conversely, assume that the minimal polynomial $m(s)$ of s acting on $\mathcal{M}$ exists. Thus, $m(s) \cdot f^s = 0$ in $\mathcal{M}$. This means that $m(s)f^s \in \mathtt{H}_c[s] \cdot (f + J(f))f^s$. That is, there exist operators $Q'(s), A_i'(s) \in \mathtt{H}_c[s]$ such that $m(s)f^s = (Q'(s)f + \sum_{i=1}^{n} A_i'(s)f_i)\, f^s$. Therefore $\tilde{b}_c^f(s)$ exists and divides $m(s)$, and hence $\tilde{b}_c^f(s) = m(s)$ because they are monic. □

Remark 4.12. Note that it follows from the proof of Lemma 4.11 that $\tilde{b}_f(s)$ is the minimal polynomial such that

$$\tilde{b}_f(s) \cdot f^s = \left(Q(s)f + \sum_{i=1}^{n} A_i(s)f_i\right) f^s \in \mathtt{H}_c[s](f + J(f))f^s.$$

§ 4.3. Products of b-functions (Thom-Sebastini)

It was asked by Budur in [Bud12, § 2.11 Questions] and Popa in [Pop18, Remark 2.2.10 (5)], in the case of $\mathcal{D}$-modules, whether the b-polynomial of a product $f \otimes g$ of polynomials $f \in \mathcal{O}(X), g \in \mathcal{O}(Y)$ is the product of their b-polynomials. In his paper [Lee25], Lee proves the so-called multiplicative Thom-Sebastini formula, which essentially says that the the answer to this question is yes. In this section we will prove that the analogous result holds for rational Cherednik algebras too. The following result is [Lee25, Theorem 1.1].

Theorem 4.13 (Multiplicative Thom-Sebastini formula)**.** *Let X, Y be smooth affine varieties, $f \in \mathcal{O}(X), g \in \mathcal{O}(Y)$ non-zero regular functions, and consider the product function $fg = f \otimes g \in \mathcal{O}(X \times Y)$. Then $b_{fg}(s) = b_f(s)b_g(s)$.*

We prove the analogous result for rational Cherednik algebras. If $(W_1, \mathfrak{h}_1), (W_2, \mathfrak{h}_2)$ are complex reflection groups then so too is $(W, \mathfrak{h}) = (W_1 \times W_2, \mathfrak{h}_1 \times \mathfrak{h}_2)$. Moreover, if $f \in \mathbb{C}[\mathfrak{h}_1]^{W_1}$ and $g \in \mathbb{C}[\mathfrak{h}_2]^{W_2}$ then $f \otimes g \in \mathbb{C}[\mathfrak{h}]^W$. If $c_i \colon \mathcal{S}_i \to \mathbb{C}$ for $i = 1, 2$ are parameters then we define $c \colon \mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2 \to \mathbb{C}$ by $c(s) = c_i(s)$ if $s \in \mathcal{S}_i$.

Theorem 4.14. *Let $f \in \mathbb{C}[\mathfrak{h}_1]^{W_1}$ and $g \in \mathbb{C}[\mathfrak{h}_2]^{W_2}$. If the b-function for f, g exist then so too does the b-function for $f \otimes g \in \mathbb{C}[\mathfrak{h}]^W$, and it is given by*

$$b_c^{f\otimes g}(s) = b_{c_1}^f(s)\, b_{c_2}^g(s).$$

This result can be applied to products of groups to get b-functions for polynomials without isolated singularities; see Lemma 5.37.

The proof of Theorem 4.14 relies on Lemma 4.15 below on principal ideal domains, which is [Lee25, Lemma 3.1]. Fix $r \geq 1$ and a principal ideal domain k. Let $A_1, \ldots, A_r$ be principal ideal domains flat over k such that the tensor product $A = A_1 \otimes_k \cdots \otimes_k A_r$ is a unique factorisation domain. Let $Q_i \hookrightarrow P_i$ be an inclusion of flat A_i-modules for $i = 1, \ldots, r$. Let $\delta_i \in P_i$ be nonzero elements such that

(1) their images $\bar{\delta_i}$ in P_i/Q_i are A_i-torsion and

(2) the quotients $P_i/A_i\delta_i$ are A_i-flat.

Let $P = P_1 \otimes_k \cdots \otimes_k P_r,\ Q = Q_1 \otimes_k \cdots \otimes_k Q_r$, and $\delta = \prod_{i=1}^r \delta_i \in P$, whose image in P/Q we denote by $\bar{\delta}$.

Lemma 4.15. *If $\mathrm{Ann}_{A_i}(\bar{\delta_i}) = (b_i)$, for $1 \leq i \leq r$, and $b = b_1 \otimes \cdots b_r \in A$, then*

$$\mathrm{Ann}_A(\bar{\delta}) = (b).$$

Proof of Theorem 4.14. Write $f_1 := f,\ f_2 := g,\ \mathtt{H}_i := \mathtt{H}_{1,c_i}(W_i)$ and $\mathtt{H} := \mathtt{H}_1 \otimes \mathtt{H}_2$. As in the proof of the main theorem in [Lee25], we check that the conditions of Lemma 4.15 are met for $k = A_1 = A_2 = \mathbb{C}[s],\ P_i := \mathtt{H}_i[s]f_i^s,\ Q_i := \mathtt{H}_i[s]f_i^{s+1}$ and $\delta_i := f_i^s,\ (\delta := \delta_1\delta_2)$.

- $k = A_1 = A_2 = \mathbb{C}[s]$ is a principal ideal domain flat over k, and $A_1 \otimes_k A_2 = \mathbb{C}[s] \otimes_{\mathbb{C}[s]} \mathbb{C}[s] \cong \mathbb{C}[s]$ is a unique factorisation domain.
- $P_i = \mathtt{H}_i[s]f_i^s$ is a subspace of $\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]f_i^s \cong \mathbb{C}[s] \otimes_{\mathbb{C}} (\mathbb{C}[\mathfrak{h}][f_i^{-1}]f_i^s)$ by Lemma 4.2, which is $\mathbb{C}[s]$-free and therefore torsion free, and similar for Q_i. For a principal ideal domain, being torsion free is the same as being flat, and a submodule of a torsion free module is torsion free. So we get an inclusion of flat $\mathbb{C}[s]$-modules $Q_i = \mathtt{H}_i[s]f_i^{s+1} \hookrightarrow \mathtt{H}_i[s]f_i^s = P_i$.
- $\overline{\delta_i} \in \frac{P_i}{Q_i} = \frac{\mathtt{H}_i[s]f_i^s}{\mathtt{H}_i[s]f_i^{s+1}}$ has $\mathbb{C}[s]$-torsion as $b_i \cdot \overline{\delta_i} = \overline{Df_i^{s+1}} = 0$ for some $D \in \mathtt{H}_i[s]$.

The quotient $\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]/\mathbb{C}[s]$ is $\mathbb{C}[s]$-free because $1 \in \mathbb{C}[\mathfrak{h}][f_i^{-1}]$ is part of a $\mathbb{C}$-vector space basis for $\mathbb{C}[\mathfrak{h}][f_i^{-1}]$ and the image of this basis in $\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]$ is a $\mathbb{C}[s]$-basis. The space $\mathtt{H}_i[s]f_i^s/\mathbb{C}[s]f_i^s$ is $\mathbb{C}[s]$-torsion free because

$$\frac{\mathtt{H}_i[s]f_i^s}{\mathbb{C}[s]f_i^s} \subseteq \frac{\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]f_i^s}{\mathbb{C}[s]f_i^s} \simeq_{\mathbb{C}[s]} \frac{\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]}{\mathbb{C}[s]},$$

which is $\mathbb{C}[s]$-free. Thus,

$$\frac{P_i}{A_i\delta_i} = \frac{\mathtt{H}_i[s]f_i^s}{\mathbb{C}[s]f_i^s}$$

is $\mathbb{C}[s]$-flat as a submodule of the $\mathbb{C}[s]$-free module $\frac{\mathbb{C}[\mathfrak{h}][f_i^{-1}][s]f_i^s}{\mathbb{C}[s]f_i^s}$. We conclude the proof by noting that Lemma 4.9 says that $(b_i) = \mathrm{Ann}_{A_i}(P_i/Q_i)$. □

§ 4.4. Examples

In this section we consider a number of examples of polynomials f where the b-function can be computed by hand.

§ 4.4.1. b-function for x^ℓ over $\mathtt{H}_c(\mathbb{Z}_\ell)$

Let $\ell \geq 2$ be an integer, and $\mathtt{H}_c := \mathtt{H}_{1,c}(\mathbb{Z}_\ell)$ the rational Cherednik algebra corresponding to the complex reflection group $W = \mathbb{Z}_\ell = \langle\sigma\rangle$, whose action on $\mathbb{C}[x]$ is given by $\sigma \cdot x = \xi^{-1}x$, where ξ is a fixed primitive ℓ-th root of unity.

We can ask what is the b-function over $\mathtt{H}_c$ of some W-invariant $f \in \mathbb{C}[x]^W$. For example:

Question 4.16. *What is the b-function for $f(x) = x^\ell \in \mathbb{C}[x]$ over $\mathtt{H}_c$?*

Intuition tells us that the operator P corresponding to the b-function will look something like y^ℓ, so let's compute $y^\ell \cdot x^{\ell(s+1)}$.

The group algebra of $\mathbb{Z}_\ell$ can be written as $\mathbb{C}\mathbb{Z}_\ell = \mathbb{C}\{e_0, \ldots, e_{\ell-1}\}$, where the idempotents

$$e_i = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \xi^{ij}\sigma^j,$$

are such that $\sigma \cdot e_i = \xi^{-i}e_i$. As in Section 3.4, we can then rewrite the commutator

$$[y, x] = 1 + \ell\sum_{i=1}^{\ell-1} c_i\sigma^i = 1 + \ell\sum_{i=0}^{\ell-1}(\kappa_{i+1} - \kappa_i)e_i \tag{4.4.1}$$

in terms of the idempotents e_i and parameters κ_i. For clarity, and to keep track of parameters, we write $\mathtt{H}_\kappa := \mathtt{H}_c$. The Dunkl operator becomes

$$y = \frac{\partial}{\partial x} + \frac{\ell}{x}\sum_{i=0}^{\ell-1}\kappa_i e_i.$$

Note that, acting on $\mathbb{C}[x]$,

$$e_i \cdot 1 = \frac{1}{\ell}\sum_{j=0}^{\ell-1}\xi^{ij}\underbrace{\sigma^j \cdot 1}_{=1} = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

This implies that we must have $\kappa_0 = 0$ since $0 = y \cdot 1 = \kappa_0 \cdot 1$. Also,

$$e_i x = \sum_{j=0}^{\ell-1} \xi^{ij} \sigma^j x = \sum_{j=0}^{\ell-1} \xi^{ij} \xi^{-j} x \sigma^j = x \sum_{j=0}^{\ell-1} \xi^{(i-1)j} \sigma^j = x e_{i-1}.$$

Let $m, m' \in \mathbb{Z}$ and set $e_m := e_{m'}, \kappa_m := \kappa_{m'}$ if $m \equiv m' \bmod \ell$. Then $\kappa_{\ell m} = \kappa_0 = 0$ for all $m \in \mathbb{Z}$, and for positive m,

$$y \cdot x^m = m x^{m-1} + \underbrace{x^m \frac{\partial}{\partial x} \cdot 1}_{=0} + \ell x^{m-1} \left(\sum_{i=0}^{\ell-1} \kappa_i e_{i-m} \right) \cdot 1 = (m + \ell \kappa_m) x^{m-1},$$

so if $1 \leq n \leq m$ is an integer then one can show by induction that

$$y^n \cdot x^m = \left(\prod_{i=0}^{n-1} (m - i + \ell \kappa_{m+i}) \right) x^{m-n}.$$

Substituting $n = \ell, m = \ell(s+1)$, we get

$$\begin{aligned} y^\ell \cdot x^{\ell(s+1)} &= \left(\prod_{i=0}^{\ell-1} (\ell(s+1) - i + \ell \kappa_{\ell(s+1)+i}) \right) x^{\ell(s+1)-\ell} = \left(\prod_{i=0}^{\ell-1} (\ell(s+1) - i + \ell \kappa_i) \right) x^{\ell s} \\ &= \ell^\ell \left(\prod_{i=0}^{\ell-1} (s + 1 - i/\ell + \kappa_i) \right) x^{\ell s}. \end{aligned} \tag{4.4.2}$$

Thus, a candidate for the b-function for x^ℓ over $\mathtt{H}_\kappa$ is

$$B(s) := \prod_{i=0}^{\ell-1} (s + 1 - i/\ell + \kappa_i)$$

since $P_0 \cdot x^{\ell(s+1)} = B(s) x^{\ell s}$, where $P_0 = \frac{1}{\ell^\ell} y^\ell$.

Theorem 4.17. *The b-function for $f(x) = x^\ell \in \mathbb{C}[x]$ over $\mathtt{H}_\kappa$ is*

$$b_c^f(s) = \prod_{i=0}^{\ell-1} (s + 1 - i/\ell + \kappa_i).$$

Proof. It remains to show that B is the monic polynomial of minimal degree satisfying the b-function equation. Assume that

$$P \cdot x^{\ell(s+1)} = b_c^f(s) x^{\ell s}. \tag{4.4.3}$$

Recall that the Euler element defines a $\mathbb{Z}$-grading on $\mathtt{H}_\kappa(W)$ where $\deg y = -1, \deg x = 1$ and $\deg w = 0$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$. We set $\deg s = 0$. Then (4.4.3) is a homogeneous equation if we declare that $\deg f^s = 0$ (which makes $\mathbb{C}[\mathfrak{h}][x^{-1}][s] f^s$ into a graded $\mathtt{H}_\kappa(W)[s]$-module). This implies that we may assume $P \in \mathtt{H}_\kappa[s]$ is homogeneous

of degree $-\ell$. Moreover, recall from Remark 4.5 that we will also have $e_0 P e_0 \cdot x^{\ell(s+1)} = B(s)x^{\ell s}$. Now, write

$$P = \sum_{i=0}^{\infty} \sum_{k=0}^{\ell-1} a_{ik} x^i y^{\ell+i} \sigma^k$$

for some $a_{ik} \in \mathbb{C}[s]$, finitely many of which are non-zero. Then

$$\begin{aligned} Pe_0 &= \sum_{i=0}^{\infty} \left(\sum_{k=0}^{\ell-1} a_{ik} \right) x^i y^{\ell+i} e_0 \\ &= \sum_{i=0}^{\infty} d_i x^i y^{\ell+i} e_0, \end{aligned}$$

for $d_i = \sum_{k=0}^{\ell-1} a_{ik}$, and hence

$$e_0 P e_0 = e_0 \sum_{i=0}^{\infty} d_i x^i y^{\ell+i} = Q \frac{1}{\ell^\ell} y^\ell$$

for some $Q \in \mathtt{H}_\kappa[s]$, because $x^i y^{\ell+i}$ commutes with e_0. But then

$$b_c^f(s) x^{\ell s} = e_0 P e_0 x^{\ell(s+1)} = Q \frac{1}{\ell^\ell} y^\ell x^{\ell(s+1)} = B(s) Q x^{\ell s}.$$

Now, $\mathbb{C}[\mathfrak{h}][x^{-1}][s] f^s$ is free as a $\mathbb{C}[s]$-module, so the equality $b_c^f(s) x^{\ell s} = B(s) Q x^{\ell s}$ implies that $B(s)$ divides $b_c^f(s)$. By minimality, they must be equal. □

Remark 4.18. Notice that the roots of b_c^f are $-1 + i/\ell - \kappa_i$ and these roots belong to $\mathbb{Q}_{<0}$ if $\kappa_i \in \mathbb{Q}_{\geq 0}$ for all $0 \leq i \leq \ell - 1$. This confirms Conjecture 1.2 in this particular example.

§ 4.4.2. Corollary

We apply Theorem 4.14 to Theorem 4.17 to derive the following corollary. Let $W = \mathbb{Z}/\ell_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_n\mathbb{Z}$ be a product of cyclic groups acting diagonally on its reflections representation $\mathfrak{h} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$. As in Equation (4.4.1), we can write

$$[y_i, x_i] = 1 + \ell_i \sum_{j=0}^{\ell_i} (\kappa_{i,j+1} - \kappa_{i,j}) e_{ij},$$

with idempotents $e_{ij} = \frac{1}{\ell_i} \sum_{k=0}^{\ell_i} \xi_i^{jk} \sigma_i^k$ for primitive ℓ_i-th roots of unity ξ_i and generators σ_i of $\mathbb{Z}/\ell_i\mathbb{Z}$. As in Equation (4.4.2), we get

$$y_i^{\ell_i} \cdot x_i^{\ell_i(s+1)} = \ell_i^{\ell_i} \prod_{k=0}^{\ell_i - 1} \left(s + 1 - \frac{k}{\ell_i} + \kappa_i \right),$$

and can apply Theorem 4.14 to $f = \prod_{i=1}^n x_i^{\ell_i} \in \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathfrak{h}]$ to get $b_f = \prod_{i=1}^n b_{x^{\ell_i}}$.

Corollary 4.19. *If $f = \prod_{i=1}^{n} x_i^{\ell_i}$ then its b-function over $\mathtt{H}_\kappa(\mathbb{Z}/\ell_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_n\mathbb{Z})$ is*

$$b_f(s) = \prod_{i=1}^{n} \prod_{k=0}^{\ell_i - 1} \left(s + 1 - \frac{k}{\ell_i} + \kappa_{k,i} \right).$$

Corollary 4.19 gives an example of an invariant polynomial that does *not* have isolated singularities for which Conjecture 1.2 holds. It can be seen that it does not have isolated singularities because, for $\ell_i > 1$, we have $\partial_i(x_1^{\ell_1} \cdots x_n^{\ell_n}) = \ell_i x_1^{\ell_1} \cdots x_i^{\ell_i - 1} \cdots x_n^{\ell_n}$ and hence the singular set of $V(f)$ equals $V(f)$ i.e., every point is singular (and $\dim V(f) = n - 1$ so $V(f)$ not a finite set).

§ 4.4.3. *b*-functions over $\mathtt{H}_c(B_2)$

Recall from Section 3.9 that $B_2 = \langle \mathcal{S} \rangle$, where

$$\mathcal{S} = \left\{ s_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

We compute the *b*-functions of the polynomials $x_1^2 + x_2^2$ and $x_1^2 x_2^2$ in $\mathbb{C}[x_1, x_2]^{B_2}$.

The computations will be made easier by noting that y_i acts as $\frac{\partial}{\partial x_i}$ on f^s as in (4.2.5).

$\underline{f = x_1^2 + x_2^2}$**:**

For $f = x_1^2 + x_2^2$, we guess the candidate $y_1^2 + y_2^2$ for the operator. Let's compute!

$$\begin{aligned}
(y_1^2 + y_2^2) \cdot (x_1^2 + x_2^2)^{s+1} &= (s+1)(y_1 \cdot 2x_1 f^s + y_2 \cdot 2x_2 f^s) \\
&= 2(s+1)(x_1 y_1 + 1 - 2c_1 - c_2 - c_2 + x_2 y_2 + 1 - c_2 - 2c_1 - c_2) f^s \\
&= 4(s+1)(1 - 2c_1 - 2c_2 + s(x_1^2 + x_2^2) f^{-1}) f^s \\
&= 4(s+1)(s + 1 - 2c_1 - 2c_2) f^s,
\end{aligned}$$

so our candidate *b*-function is $b_c^f(s) = (s+1)(s+1-2c_1-2c_2)$ for $f = x_1^2 + x_2^2$ with operator $\frac{1}{4}(y_1^2 + y_2^2)$.

We can check our computations using the `dmod.lib` library in SINGULAR [DGPS24], which has a procedure `operatorBM(F)` which can give us the normalised *b*-function and the required operator, but only in the case where $c = 0$. That is, when we are working over $\mathtt{H}_0 = \mathcal{D}(\mathfrak{h}) \rtimes W$, and we can just compute the *b*-function over the ring $\mathcal{D}(\mathfrak{h})$ of differential operators on $\mathfrak{h}$. For example:

```
> LIB "dmod.lib";
> ring r = 0,(x,y),Dp;
> poly F = x^2+y^2;
> printlevel = 0;
> def A = operatorBM(F);
```

```
> setring A;
> bs;
s^2+2s+1
> PS;
1/4*Dx^2+1/4*Dy^2
```

We see that we get the b-function $(s+1)^2$ and operator $\frac{1}{4}(y_1^2+y_2^2)$, which agrees with our findings at $c=0$.

$\underline{f=x_1^2x_2^2}$**:**

Now we do a similar computation with $f=x_1^2x_2^2$ and operator candidate $y_1^2y_2^2$, noting that $\frac{\partial}{\partial x_i}(f)f^{-1}s=\frac{2}{x_i}s$:

$$\begin{aligned}
(y_1^2y_2^2)\cdot(x_1^2x_2^2)^{s+1} &= (s+1)y_1^2y_2\cdot 2x_1^2x_2f^s \\
&= 2(s+1)y_1^2\left(x_1^2+2x_1^2s-\frac{c_2}{x_1+x_2}(x_1^2x_2+x_1x_2^2s_2)\right. \\
&\qquad\left.-\frac{c_1}{x_2}(x_1^2x_2+x_1^2x_2s_3)+\frac{c_2}{x_1-x_2}(x_1^2x_2-x_1x_2^2s_4)\right)f^s \\
&= 2(s+1)y_1^2\left(x_1^2+2x_1^2s-c_2x_1x_2-2c_1x_1^2+c_2x_1x_2\right)f^s \\
&= 2(s+1)(2s+1-2c_1)y_1^2x_1^2f^s \\
&= 2(s+1)(2s+1-2c_1)y_1\left(2x_1+2x_1s-\frac{c_1}{x_1}(x_1^2-x_1^2s_1)\right. \\
&\qquad\left.-\frac{c_2}{x_1+x_2}(x_1^2-x_2^2s_2)-\frac{c_2}{x_1+x_2}(x_1^2-x_2^2s_4)\right)f^s \\
&= 2(s+1)(2s+1-2c_1)y_1\left(2x_1+2x_1s-c_2(x_1-x_2)-c_2(x_1+x_2)\right)f^s \\
&= 4(s+1)(2s+1-2c_1)(s+1-2c_2)y_1x_1f^s \\
&= 4(s+1)(2s+1-2c_1)(s+1-2c_2)\left(1+2s-\frac{c_1}{x_1}\cdot 2x_1\right. \\
&\qquad\left.-\frac{c_2}{x_1+x_2}(x_1+x_2)-\frac{c_2}{x_1-x_2}(x_1-x_2)\right)f^s \\
&= 4(s+1)(2s+1-2c_1)(s+1-2c_2)(1+2s-2c_1-2c_2)f^s,
\end{aligned}$$

so $b_c^f(s)$ divides $\beta_c(s):=(s+1)(s+1-2c_2)(s+1/2-c_1)(s+1/2-c_1-c_2)$. We expect they are equal. As a check we can use SINGULAR:

```
> poly F = x^2*y^2;
> def A = operatorBM(F);
> setring A;
> bs;
s^4+3*s^3+13/4*s^2+3/2*s+1/4
```

This shows that when $c = 0$, $b^f_{c=0}(s) = \beta_c(s)|_{c=0}$. If we knew that the degree of b^f_c did not depend on c then we could conclude that $b^f_c(s) = \beta_c(s)$ for all c. In fact, it would suffice to know that the minimum degree of b^f_c is obtained at $c = 0$.

Using (3.9.1) shows that $b^f_c(s)$ divides

$$(s+1)(s+1+2\kappa_{H_2,1})(s+1/2+\kappa_{H_1,1})(s+1/2+\kappa_{H_1,1}+\kappa_{H_2,1}).$$

In particular, this shows that Conjecture 1.2 holds for all parameters.

Further computations

We ought to continue with similar computations for other W-equivariants $f \in \mathbb{C}[\mathfrak{h}]^W$, such as for $f = x_1^4 + x_2^4$. However, for such cases, the number of terms in a given computation quickly gets out of hand. In fact, for many cases, it is not at all clear which candidate operator to try. For example, let's investigate the case of $f = x_1^4 + x_2^4$ in SINGULAR:

```
> poly F = x^4+y^4;
> def A = operatorBM(F);
> setring A;
> PS;
3/2048*y^2*Dx^4*Dy^2+3/2048*y^2*Dy^6+3/1024*y*Dx^4*Dy*s
+3/512*x*Dx*Dy^4*s-9/1024*y*Dy^5*s+21/2048*y*Dx^4*Dy+3/512*x*Dx*Dy^4
-3/2048*y*Dy^5+1/256*Dx^4*s^2+1/64*Dy^4*s^2+7/512*Dx^4*s
+13/512*Dy^4*s+15/1024*Dx^4+15/1024*Dy^4
```

There are a couple of things to note. Firstly, we would have expected the operator to be symmetric in `x` $= x_1$ and `y` $= x_2$, but the above clearly isn't. This is because the choice of operator is not unique. Secondly, in the grading in which we put x, y in degree 1 and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ in degree -1, the degree (of each of the summands) of the operator is -4, which we would expect given $\deg f = 4$. However, since it contains terms with a positive power of x_1 or x_2, we can expect that the degree of $b_f(s)$ will be greater than 4:

```
> bs;
s^6+6*s^5+235/16*s^4+75/4*s^3+841/64*s^2+153/32*s+45/64
```

Thus,

$$\begin{aligned} b_f(s) &= s^6 + 6s^5 + \frac{235}{16}s^4 + \frac{74}{4}s^3 + \frac{841}{64}s^2 + \frac{153}{32}s + \frac{45}{64} \\ &= (s+1)^2(s+1/2)(s+3/2)(s+3/4)(s+5/4), \end{aligned}$$

which is of degree 6. Though it is not possible to compute directly the b-function $b^f_c(s)$ for arbitrary parameters c for $x_1^4 + x_2^4$, our main result Theorem 5.22, together with the formulae we derived in Chapter 6 imply that:

Proposition 4.20. *If c is generic then the b-function for $f = x_1^4 + x_2^4$ is given by*

$$\begin{aligned} b_c^f(s) = \frac{1}{16384}&(s+1)(-4c_1-4c_2+4s+6)(-4c_1-4c_2+4s+4)(-4c_1-4c_2+4s+2) \\ &(-4c_2+4s+4)(-4c_1+4s+4)(-2c_1-2c_2+4s+5) \\ &(-2c_1-2c_2+4s+3) \\ = (s+1)&(-c_1-c_2+s+3/2)(-c_1-c_2+s+1/2)(-c_2+s+1)(-c_1+s+1) \\ &(-1/2c_1-1/2c_2+s+5/4)(-1/2c_1-1/2c_2+s+3/4). \end{aligned}$$

Remark 4.21. Notice that in this example the b-function at $c = 0$ has degree 6 but the b-function for generic c has degree 7.

Based on the motivating [BET26, Question 6.10], we would ideally like to compute the b-function for the discriminant

$$\delta = \prod_{\alpha \in R} \alpha = x_1^2 x_2^2 (x_1 + x_2)^2 (x_1 - x_2)^2$$

of our hyperplane arrangement. However, the kernel of SINGULAR, which is available at (https://www.singular.uni-kl.de:8003/) doesn't have enough memory to even define `operatorBM(F)` for `F` $= \delta$. We turn to another procedure, `bernsteinBM(F)`, of the `dmod.lib` library, which gives us the roots of the b-function of `F` along with their multiplicities:

```
> poly F = x^2*y^2*(x+y)^2*(x-y)^2;
> bernsteinBM(F);
[1]:
    _[1]=-1/4
    _[2]=-3/4
    _[3]=-5/4
    _[4]=-3/8
    _[5]=-5/8
    _[6]=-7/8
    _[7]=-9/8
    _[8]=-1
    _[9]=-1/2
[2]:
    1,1,1,1,1,1,1,2,2
```

Hence,

$$b_0^f(s) = (s+1)^2(s+1/2)^2(s+1/4)(s+3/4)(s+5/4)(s+3/8)(s+7/8)(s+9/8).$$

We do not have a candidate for b_c^f when $c \neq 0$.

§ 4.4.4. Coxeter groups

Let W be a finite Coxeter group (that is, a real reflection group). Then there exists a W-invariant non-degenerate symmetric bilinear form $(-,-)$ on $\mathfrak{h}$. If $x_1, \ldots, x_n$ is an orthonormal basis of $\mathfrak{h}^*$ with respect to this bilinear form then $f = x_1^2 + \cdots + x_n^2$ is W-invariant.

In this example, we guess that a suitable choice of operator P is given by $y_1^2 + \cdots + y_n^2$. We compute that

$$\begin{aligned}(y_1^2 + \cdots + y_n^2) \cdot (x_1^2 + \ldots x_n^2)^{s+1} &= (s+1) \sum_{i=1}^{n} y_i \cdot 2x_i f^s \\ &= 2(s+1) \sum_{i=1}^{n} (x_i y_i + [y_i, x_i]) f^s.\end{aligned}$$

By (3.6.1),

$$\sum_{i=1}^{n} [y_i, x_i] = n - \sum_{\sigma \in \mathcal{S}} c(\sigma) \alpha_\sigma(\alpha_\sigma^\vee) = n - \sum_{\sigma \in \mathcal{S}} 2c(\sigma).$$

Thus,

$$(y_1^2 + \cdots + y_n^2) \cdot (x_1^2 + \cdots + x_n^2)^{s+1} = (s+1) \left(2s + n - \sum_{\sigma \in S} 2c(\sigma) \right) f^s,$$

since $d = 2$ in this case. This shows that $b(s)$ divides $(s+1)(ds + n - \sum_{\sigma \in S} 2c(\sigma))$. But we know from Proposition 4.8 that $(s+1)$ divides $b(s)$ and $\tilde{b}(s) = (s+1)^{-1} b(s)$ is the minimal polynomial of s acting on $\mathcal{M}$. The minimal polynomial of any operator (even the zero operator) always has degree at least one if $\mathcal{M} \neq 0$. Since Proposition 5.30 implies that $\mathcal{M} \neq 0$, we deduce that $\deg \tilde{b}(s) \geq 1$. Therefore,

$$b_c^f(s) = (s+1) \left(s + n/2 - \sum_{\sigma \in S} c(\sigma) \right).$$

In terms of the parameters κ, equation (3.6.2) implies that

$$b_c^f(s) = (s+1) \left(s + n/2 + \sum_{H \in \mathcal{A}} \kappa_{H,1} \right),$$

since $\ell_H = 2$ for Coxeter groups. Notice that if $\kappa_{H,1} \in \mathbb{Q}_{\geq 0}$ for all H then the roots of b_c^f all belong to $\mathbb{Q}_{<0}$, confirming Conjecture 1.2 in this case.

Chapter 5

Isolated Singularities

§ 5.1. Homogeneous polynomials with isolated singularities

In this section, we introduce the notion of polynomials with isolated singularities and consider their properties in the case where they are also homogeneous. Recall that a point p in an affine variety X is singular if $\dim T_pX > \dim X$; see Section 10.3 of [Eis95]. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, a **singularity** of f is, by definition, a singular point in the variety $X = V(f)$.

Definition 5.1. *We say that a variety X has (only) isolated singularities if for each point p in its set of singularities X_{sing}, there exists an open neighbourhood $p \in U \subset X$ such that $X_{sing} \cap U = \{p\}$.*

We say that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ has isolated singularities if the variety $V(f) \subset \mathbb{A}^n$ has isolated singularties.

Lemma 5.2. *If the affine variety X has only isolated singularities, then $|X_{sing}| < \infty$.*

Proof. Affine varieties over $\mathbb{C}$ are **quasi-compact**. That is, every Zariski open cover admits a finite open subcover.

If X has isolated singularities then we get an open cover consisting of $U = X \setminus X_{sing}$ and $\{U_p\}_{p \in X_{sing}}$, where U_p is a neighbourhood of p not intersecting $X_{sing} \setminus \{p\}$. However, since X is quasi-compact, there must exist a *finite* open subcover $\{U, U_{p_1}, \ldots, U_{p_k}\}$, which implies that $X_{sing} = \{p_1, \ldots, p_k\}$ consists of finitely many points. □

Throughout this chapter, for a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, its partial derivatives will be denoted $f_i := \frac{\partial f}{\partial x_i}$, and $J(f) := (f_1, \ldots, f_n)$ is its Jacobian ideal.

Lemma 5.3. *If f is homogeneous then $f \in J(f)$.*

Proof. If f is homogeneous of degree d then

$$f = \frac{1}{d}\sum_{i=1}^{n} x_i f_i.$$

□

Lemma 5.4. *Let $X = V(f)$ for some homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, and let X_{sing} be its set of singularities. Then $X_{sing} = V(J(f))$.*

Proof. Note that $\mathbb{C}[x_1, \ldots, x_n]$ is a unique factorisation domain, so $f = \prod_i p_i$ factors uniquely as a product of irreducible polynomials p_i. Since (p_i) is a prime ideal of height one, $\dim V(p_i) = n - 1$. Therefore,

$$\dim V(f) = \dim \bigcup_i V(p_i) = n - 1.$$

That is, $V(f)$ has codimension 1.

By the Jacobian Criterion [Eis95, Theorem 16.19], a point $p \in V(f)$ is singular if and only if the rank of the Jacobian matrix $(\partial f/\partial x_i)_{i=1}^n$ evaluated at p is less than the codimension of $V(f)$, which is equal to 1 by the paragraph above. That is, the Jacobian matrix has rank zero. This means that $\partial f/\partial x_i|_p = 0$ for all $i = 1, \ldots, n$. In other words,

$$p \in V\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = V(J(f)).$$

Therefore, $X_{sing} = X \cap V(J(f))$. But f is assumed homogeneous so $f \in J(f)$ by Lemma 5.3, which implies that $V(J(f)) \subset X$ and hence $X_{sing} = V(J(f))$. □

Lemma 5.5. *Assume that f is homogeneous with isolated singularities. Then the set of singularities* $\mathrm{Sing}(f)$ *consists of a single point $0 \in \mathbb{C}^n$.*

Proof. Since f is homogeneous, Lemma 5.3 says that $f \in J(f) = (f_1, \ldots, f_n)$.

The set $\mathrm{Sing}(f)$ of singularities equals $V(J(f)) = \{x \in \mathbb{C}[x_1, \ldots, x_n] | f_i(x) = 0\}$ by Lemma 5.4, and is finite by Lemma 5.2. Since f is homogeneous, of degree d say, $f_i(\lambda x) = \lambda^{d-1} f_i(x) = 0$ for all $x \in \mathrm{Sing}(f)$ and all $\lambda \in \mathbb{C}^\times$, and therefore $\lambda x \in \mathrm{Sing}(f)$. That is, the set of singularities $\mathrm{Sing}(f) \subset \mathbb{C}^n$ of f is stable under the action of $\mathbb{C}^\times$ on $\mathbb{C}^n$. The only finite subset of $\mathbb{C}^n$ that is stable under rescaling is $\{0\}$. Therefore $\mathrm{Sing}(f) = \{0\}$. □

Lemma 5.6. *If $f = gh$ is reducible with* $\dim V(g, h) > 0$*, then f does not have isolated singularities.*

Proof. As in the proof of Lemma 5.4, a point $p \in V(f)$ is singular if and only if $\partial f/\partial x_i|_p = 0$ for all $i = 1, \ldots, n$. By the product rule for $f = gh$, this means

$$\left(g\frac{\partial h}{\partial x_i} + \frac{\partial g}{\partial x_i}h\right)\Big|_p = 0$$

for all $i = 1, \ldots, n$. Notice that this is the case if $p \in V(g,h)$ since then $g(p) = h(p) = 0$. So $V(g,h)$ is contained within the set of singularities X_{sing}. Thus, if $\dim V(g,h) > 0$ then $\dim X_{sing} > 0$. That is, f does not have isolated singularities. □

The above lemma will be applied in Chapter 5 in the following situation.

Proposition 5.7. *Assume $n \geq 3$ and f is a homogeneous polynomial. If f has isolated singularities then it must be irreducible.*

Proof. If f is reducible, $f = gh$, then g, h must also be homogeneous. This implies that $0 \in V(g,h)$. In particular, $V(g,h) \neq \emptyset$. Moreover, $\dim V(g,h) \geq n - 2 > 0$ since $n \geq 3$. Therefore, Lemma 5.6 says that f does not have isolated singularities in this case. □

Lemma 5.8. *Let R be an affine $\mathbb{C}$-algebra. Then R has zero Krull dimension if and only if $\dim_{\mathbb{C}} R < \infty$.*

Proof. Recall that being affine means that it is finitely generated as a $\mathbb{C}$-algebra. In particular, by Hilbert's basis theorem, R is noetherian. The result [AM69, Theorem 8.5] says that R is artinian if and only if it is noetherian and has Krull dimension zero. Thus, if R has Krull dimension zero then it is artinian. Then the structure theorem for artinian rings [AM69, Theorem 8.7] says that R is a finite direct product of local artinian rings. Therefore, to prove that $\dim_{\mathbb{C}} R < \infty$, we may assume that R is local artinian, with maximal ideal $\mathfrak{m}$ say. Then [AM69, Proposition 8.6] says that there exists some $k > 0$ such that $\mathfrak{m}^k = 0$. Each quotient $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finitely generated R-module, where the action factors through the quotient $R/\mathfrak{m} \cong \mathbb{C}$. That is, $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finite dimensional over $\mathbb{C}$. Therefore,

$$\dim_{\mathbb{C}} R = \sum_{i=0}^{k} \dim_{\mathbb{C}} \mathfrak{m}^i/\mathfrak{m}^{i+1} < \infty.$$

Conversely, if $\dim_{\mathbb{C}} R < \infty$ then R is artinian and hence by [AM69, Theorem 8.5] has Krull dimension zero. □

We call

$$M_f := \mathbb{C}[x_1, \ldots, x_n]/J(f)$$

the **Milnor algebra** associated to f, after John Willard Milnor who defined the algebra in his paper [Mil68].

Lemmata 5.5 and 5.8 have the following immediate, but crucial consequence.

Proposition 5.9. *If f is homogeneous with an isolated singularity then the Milnor algebra M_f is a finite dimensional graded $\mathbb{C}$-algebra.*

Proof. Note that $\dim V(J(f)) = 0$ since $V(J(f)) = \mathrm{Sing}(f) = \{0\}$ by Lemma 5.4 and Lemma 5.5. Therefore, the Krull dimension of M_f is zero since the Krull dimension of M_f equals the dimension of the variety $V(J(f))$ by [Har77, Proposition I.1.7]. Since M_f is a quotient of $\mathbb{C}[x_1, \ldots, x_n]$, it is an affine $\mathbb{C}$-algebra. Therefore, Lemma 5.8 implies that it is finite dimensional over $\mathbb{C}$.

The grading on M_f is inherited from $\mathbb{C}[x_1 \ldots, x_n]$ since $J(f)$ is a homogeneous ideal. □

Lemma 5.10. *(i) M_f is a local ring.*

(ii) M_f is indecomposable as a $\mathbb{C}[x_1, \ldots, x_n]$-module.

Proof. (i) We show that $\mathfrak{m} = (x_1, \ldots, x_n) \triangleleft M_f$ is the unique maximal ideal of M_f. [AM69, Proposition 1.6 i)] says that an ideal $\mathfrak{m} \triangleleft A$ of a ring A is the unique maximal ideal if every $a \in A \setminus \mathfrak{m}$ is invertible. Let $r \in M_f \setminus \mathfrak{m}$. Then $r \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial with non-zero constant term, so αr has constant term 1 for some $\alpha \in \mathbb{C} \setminus \{0\}$. Hence $(1 - \alpha r) \in \mathfrak{m}$. Therefore, by assumption $(1 - \alpha r) \in (M_f)_{\geq 1}$, and hence $(1 - \alpha r)^\ell \in (M_f)_{\geq \ell}$. However, M_f is finite dimensional by Proposition 5.9, so $(M_f)_{\geq \ell} = 0$ for large ℓ. Hence, $(1 - \alpha r)^\ell = 0$ in M_f for some $\ell \geq 0$. Thus

$$0 = (1 - \alpha r)^\ell = 1 + \sum_{k=1}^{\ell} \binom{\ell}{k} (-\alpha r)^{\ell - k},$$

and rearranging,

$$1 = r \left(- \sum_{k=1}^{\ell} \binom{\ell}{k} (-\alpha)^{\ell - k} r^{\ell - k - 1} \right).$$

That is, we have managed to invert arbitrary $r \in M_f \setminus \mathfrak{m}$. Therefore M_f is a local ring.

(ii) If we assume M_f is decomposable then we can write $M_f = A \oplus B$ as the direct sum of two non-trivial modules A and B. Then, since M_f is quotient of $\mathbb{C}[x_1, \ldots, x_n]$, this module direct sum decomposition would be a direct product decomposition as a ring. But this contradicts the fact that it is a local ring by part (i).

□

Consider the ring $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]_{\mathfrak{m}}$ which is a local ring with maximal ideal $\mathfrak{m} = (x_1, \ldots x_n)$. We state, with full proof, Lemma 2.20 from [Yan78] (where it is stated without proof), a result which is well-known in the theory of local rings but for which we were unable to find an explicit reference.

Lemma 5.11 (Local Version)**.** *Let $g_1, \ldots, g_n \in \mathfrak{m} \cdot \mathcal{O}$, let $\mathfrak{a} = \langle g_1, \ldots, g_n \rangle \triangleleft \mathcal{O}$ be the ideal generated by $g_1, \ldots, g_n$, and let $R = \mathcal{O}/\mathfrak{a}$.*

(i) The following three conditions on g_i $(i = 1, \ldots, n)$ are equivalent:

1. *g_i is not a zero-divisor of $\mathcal{O}/\sum_{j=0}^{i-1} g_i\mathcal{O}$, $i = 1, \ldots, n$, where we take $g_0 = 0$.*
2. *The morphism $R[\xi_1, \ldots, \xi_n] \to \bigoplus_\nu \mathfrak{a}^\nu/\mathfrak{a}^{\nu+1}$ given by $\xi_i \mapsto g_i \in \mathfrak{a}/\mathfrak{a}^2$ is an isomorphism.*
3. *$R = \mathcal{O}/\mathfrak{a}$ is finite dimensional over $\mathbb{C}$.*

If any of the equivalent conditions above hold then the following also holds:

(ii) The kernel of the morphism $\mathcal{O}[\xi_1, \ldots, \xi_n] \to \bigoplus_{\nu \geq 0} \mathfrak{a}^\nu$ given by $\xi_i \to g_i$ is generated by $g_i\xi_j - g_j\xi_i$ for $1 \leq i < j \leq n$.

A sequence $(g_1, \ldots, g_n)$ satisfying the equivalent conditions of Lemma 5.11 (i) is called a **regular sequence**.

Proof of Lemma 5.11. Let K be a field. We begin by recalling from [Sta18, Tag 00S8, Definition 10.135.1] that a finite type K-algebra S is a *(global) complete intersection* if there exists a presentation $S = K[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that the Krull dimension of S equals $n - c$.

(i) First, we show that conditions 1. and 3. are equivalent. To show that condition 1. implies 3. note that $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]_\mathfrak{m}$ has Krull dimension n by [Sta18, Tag 00OP]. Consider the projection $\mathcal{O} \to R = \mathcal{O}/\mathfrak{a}$, whose kernel is $\mathfrak{a} = \langle g_1, \ldots, g_n \rangle$, which is generated by a regular sequence $(g_1, \ldots, g_n)$ in the sense of condition 1. This is equivalent, by [Sta18, Tag 00S8, Lemma 10.135.7], to R being a complete intersection. In particular, this means that R has Krull dimension $\dim \mathcal{O} - n = n - n = 0$. Thus, since R has residue field $R/\mathfrak{m} = \mathbb{C}$, it remains to show that R is noetherian by Lemma 5.8. This is standard, but we recall the relevant results for completeness. By Corollary 7.6 of [AM69] of Hilbert's Basis Theorem, $\mathbb{C}[x_1, \ldots, x_n]$ is noetherian. By Proposition 7.3 of [AM69], $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]_\mathfrak{m}$ is noetherian since it is a localisation of a noetherian ring. Finally, $R = \mathcal{O}/\mathfrak{a}$ is noetherian, by Proposition 6.6 of [AM69], since it is the quotient of a noetherian ring.

To show that 3. implies 1. assume that R is finite dimensional over $\mathbb{C}$, which is equivalent, by Lemma 5.8, to R having Krull dimension 0. Recall that $\dim \mathcal{O} = n$ and note that $\mathcal{O}$ is a regular algebra; see for instance the Example on page 124 of [AM69]. Lemma 10.135.6 of [Sta18, Tag 00S8] says that if $A \to B \to C$ is

a sequence of surjective ring homomorphisms, where A and B are regular, then $\ker(B \to C)$ being generated by $\dim B - \dim C$ elements is equivalent to $\ker(B \to C)$ being generated by a regular sequence. We consider the sequence $\mathcal{O} \to \mathcal{O} \to R$ of surjective local ring homomorphisms, where the first morphism $\mathcal{O} \to \mathcal{O}$ is the identity map. Since $\ker(\mathcal{O} \to R) = \mathfrak{a} = \langle g_1, \ldots, g_n \rangle$ is generated by $\dim \mathcal{O} - \dim R = n$ elements, $\langle g_1, \ldots, g_n \rangle = \ker(\mathcal{O} \to R)$ is generated by a regular sequence in the sense of condition 1. by the above result from [Sta18].

Now we show that conditions 1. and 2. are equivalent. In [Sta18, Tag 061M], regular sequences in the sense of condition 1. are referred to as *regular sequences*, and regular sequences in the sense of condition 2. are referred to as *quasi-regular sequences*. Lemma 10.69.2 (1) of [Sta18, Tag 061M] says that a *regular sequence* is a *quasi-regular sequence*. That is, a regular sequence in the sense of condition 1. is a regular sequence in the sense of condition 2. Conversely, [Sta18, Tag 062D, Lemma 15.30.7] says, in the same terminology, that if $\mathcal{O}$ is noetherian then a *quasi-regular sequence* is a *regular sequence*. So in this case, 1. and 2. are equivalent as $\mathcal{O}$ was shown previously to be noetherian.

(ii) Finally, we show that the equivalent conditions of part (i) imply the condition of part (ii). Corollary 5.5.6 of [HS06] says that if $(g_1, \ldots, g_n)$ is a regular sequence in the sense of condition 1. then the "defining ideal of the Rees algebra" is generated by the 2×2 minors of a certain matrix. This ideal is exactly the kernel of part (ii) and the minors are the elements $g_i\xi_j - g_j\xi_i$, for $1 \leq i < j \leq n$.

□

Recall $f_i = \frac{\partial f}{\partial x_i}$. The following proposition shows that Lemma 5.11 applies to the local ring M_f. It will be important in the character computations in Chapter 6.

Proposition 5.12. *If f is homogeneous and has isolated singularities then $f_1, \ldots, f_n$ is a regular sequence in $\mathbb{C}[\mathfrak{h}]$.*

Proof. If f is has isolated singularities then $|X_{sing}| < \infty$ by Lemma 5.2. By Lemma 5.4, this is equivalent to $|V(J(f))| < \infty$, which in turn is equivalent to saying that the Krull dimension of $\mathbb{C}[\mathfrak{h}]/J(f)$ is zero. Furthermore, this is equivalent to $\dim_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]/J(f) < \infty$ by Lemma 5.8, which is equivalent to $f_1, \ldots, f_n$ being a regular sequence by part (i) 3. of Lemma 5.11. □

§ 5.2. Properties of $\mathcal{M}$ when f has isolated singularities

Let W be a complex reflection group, $\mathfrak{h}$ its reflection representation, and let $f \in \mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[x_1, \ldots, x_n]^W$ be a W-invariant homogeneous polynomial of degree d. We assume that f has an isolated singularity.

Recall from Lemma 3.7 that

$$\mathbf{eu}_c = \sum_{i=1}^{n} x_i y_i + z'_c, \quad z'_c = \sum_{\sigma \in \mathcal{S}} \frac{2c(\sigma)}{1-\lambda_\sigma}(1-\sigma).$$

This implies that

$$\mathbf{eu}_c \cdot f^s = s \sum x_i f_i f^{-1} f^s + z'_c f^s = ds f^s, \tag{5.2.1}$$

since f^s is W-invariant and hence $z'_c f^s = 0$. Recall from (4.2.3) that

$$\mathcal{M} = \mathcal{N}_f \Big/ \mathtt{H}_c[s] \cdot (f + J(f)) f^s.$$

Lemma 5.13. *If f is homogeneous then*

$$\mathtt{H}_c[s](f + J(f))f^s = \mathtt{H}_c[s]J(f)f^s = \mathtt{H}_c J(f) f^s. \tag{5.2.2}$$

Hence,

$$\mathcal{M} = \mathcal{N}_f / \mathtt{H}_c J(f) f^s = \frac{\mathtt{H}_c f^s}{\mathtt{H}_c J(f) f^s}.$$

Proof. Since $\mathbf{eu}_c \cdot f^s = ds f^s$ by Equation (5.2.1), we have

$$Ds^m f^s = D(d^{-1}(\mathbf{eu}_c))^m f^s$$

for any $D \in \mathtt{H}_c$. This gives us the second equality of (5.2.2), and the first equality follows from the fact that $f \in J(f)$ for homogeneous f by Lemma 5.3.

For the second claim we recall from Section 4.2 that $\mathcal{N}_f = \mathtt{H}_c[s] \cdot f^s$. Therefore, the claim follows directly from (5.2.2). □

If $f_i = \frac{\partial f}{\partial x_i}$ then $f_i \in J(f)$ implies that $f_i \cdot f^s = 0$ in $\mathcal{M}$. Therefore, since $V(f_1, \ldots, f_n) = \{0\}$, the module $\mathcal{M}$ is supported at zero.

Lemma 5.14. *Let R be a noetherian ring and M an R–module. If* $\mathrm{Supp}(M) = \{\mathfrak{m}\}$*, then for any $g \in \mathfrak{m}$, the action of g on M is locally nilpotent.*

Proof. Choose $m \in M$ non-zero and let $\widetilde{M} = R \cdot m \subset M$. Notice that because localisation is exact,

$$\widetilde{M}_{\mathfrak{p}} \subseteq M_{\mathfrak{p}} \quad \text{for all prime ideals } \mathfrak{p}.$$

Therefore, $\mathrm{Supp}(\widetilde{M}) \subset \mathrm{Supp}(M) = \{\mathfrak{m}\}$. By [Eis95, Corollary 2.7], the only way that $\mathrm{Supp}(\widetilde{M}) = \emptyset$ is if $\widetilde{M} = 0$. But, by assumption, $\widetilde{M} \neq 0$. Therefore, $\mathrm{Supp}(\widetilde{M}) = \{\mathfrak{m}\}$. Let $I = \mathrm{Ann}_R(\widetilde{M})$ and $J = \sqrt{I}$. We claim that $J = \mathfrak{m}$. Certainly, $I \subseteq \mathfrak{m}$ by [Eis95, Corollary 2.7], so $J = \sqrt{I} \subseteq \mathfrak{m}$.

Since J is a radical ideal, it is the intersection of all those prime ideals that contain J. Therefore, if $J \neq \mathfrak{m}$, then there exists another maximal ideal $\mathfrak{n}$ such that $J \subseteq \mathfrak{n}$. But then, again by [Eis95, Corollary 2.7],

$$\widetilde{M}_{\mathfrak{n}} \neq 0 \implies \mathfrak{n} \in \mathrm{Supp}(\widetilde{M}),$$

which is a contradiction. Hence, $J = \mathfrak{m}$.

This implies that some power of each element in J is contained in the annihilator of $\widetilde{M}$. In other words, if $g \in \mathfrak{m}$ then there exists k such that $g^k \in I$ and hence $g^k \cdot m = 0$. □

Recall from (3.7.1) that $\mathcal{O}_c^*$ is the category of finitely generated $\mathtt{H}_c$-modules on which the augmentation ideal $\mathbb{C}[\mathfrak{h}]_+$ acts locally nilpotently.

Lemma 5.15. $\mathcal{M} \in \mathcal{O}_c^*$.

Proof. We have already noted that $\mathrm{Supp}(\mathcal{M}) = \{0\} \subset \mathfrak{h}$. Therefore, by Lemma 5.14, the augmentation ideal $\mathbb{C}[\mathfrak{h}]_+$ acts locally nilpotently on $\mathcal{M}$. Thus, $\mathcal{M}$ belongs to $\mathcal{O}_c^*$. □

Proposition 5.16. *The* $\mathtt{H}_c$*-module* $\mathcal{M}$ *is holonomic and* $\mathrm{End}_{\mathtt{H}_c}(\mathcal{M})$ *is finite-dimensional over* $\mathbb{C}$.

Proof. Every module in $\mathcal{O}_c^*$ is holonomic by [BET26, Lemma 8.5]. Theorem 3.14(v) says that the endomorphism ring of any module in $\mathcal{O}_c$ is finite-dimensional over $\mathbb{C}$. □

Recall that the following is known for generic parameters c by [BET26, Corollary 1.2]. Our result holds for any c.

Corollary 5.17. *The b-function exists for* f *with isolated singularities.*

Proof. It suffices to show that the polynomial $\tilde{b}_c^f(s)$ exists. By Lemma 4.11, this in turn is equivalent to arguing that the minimal polynomial of

$$s \in \mathrm{End}_{\mathtt{H}_c}(\mathcal{M})$$

exists. This is the case because the endomorphism ring is finite-dimensional over $\mathbb{C}$ by Proposition 5.16. □

Corollary 5.18. *For generic parameters,* $\mathcal{M}$ *is a semisimple* $\mathtt{H}_c$*-module.*

Proof. This follows from Lemma 5.15 and Theorem 3.14(iii). □

The following corollary uses results from Chapter 6 but is not used elsewhere.

Corollary 5.19. *$\mathcal{M}$ is free of rank $(d-1)^n$ as a $\mathbb{C}[\mathfrak{h}^*]$-module.*

Proof. This follows from Proposition 5.30 and Corollary 6.12 since if V is a finite dimensional $\mathbb{C}[\mathfrak{h}] \rtimes W$-module then the rank of $\mathtt{H}_c \otimes_{\mathbb{C}[\mathfrak{h}]\rtimes W} V$ as a $\mathbb{C}[\mathfrak{h}^*]$-module equals the complex dimension of V. □

§ 5.3. The main result

The main result of this chapter is Theorem 5.22. The proof of the theorem is based loosely on that for $\mathcal{D}$-modules, as explained in Section 2.3 of the lecture notes [Pop18]. We give a concise summary of the relevant argument in the case of $\mathcal{D}$-modules in Appendix A.

Since the Milnor algebra M_f is a finite dimensional graded $\mathbb{C}[\mathfrak{h}]$-module, there exists some $N \geq 0$ such that $(M_f)_N \neq 0$ but $(M_f)_i = 0$ for all $i > N$.

Remark 5.20. We will show later in Lemma 6.11 that $N = (d-2)\dim \mathfrak{h}$ and Corollary 6.12 says that $\dim M_f = (d-1)^{\dim \mathfrak{h}}$. In fact, in Lemma 6.11 we show that as a graded vector space, M_f has Poincaré polynomial

$$H(q) = \frac{(1-q^{d-1})^n}{(1-q)^n}.$$

We note that $H(q)$ is (up to shift) palindromic. It is computable because M_f is a graded complete intersection. We do not use these facts in this chapter.

Let $u \in \mathbb{C}[\mathfrak{h}]$ be a homogeneous polynomial. Recall from Lemma 3.8 that the elements $z_c, z_c' \in \mathtt{H}_c$ belongs to the centre $Z(\mathbb{C}W)$ of the group algebra. If $\{w(u) | w \in W\}$ spans an irreducible representation of W in $\mathbb{C}[\mathfrak{h}]$ then central elements $z \in Z(\mathbb{C}W)$ act by scalars on this subspace. In particular, $z \cdot u = \alpha u$ for some $\alpha \in \mathbb{C}$. We write $z_c \cdot u = \beta(u)u$ for the scalar action of z_c on u.

For all $0 \leq i \leq N$, we can fix a complement $U_i \oplus J(f)_i = \mathbb{C}[\mathfrak{h}]_i$ as W-modules by Maschke's Theorem. Since $z_c \in Z(\mathbb{C}W)$, it acts semisimply on U_i. Therefore, U_i has a basis consisting of eigenvectors for z_c.

Definition 5.21. *Let E_i be the set of eigenvalues of z_c acting on U_i.*

We remark that each $\alpha \in E_i$ appears with multiplicity 1, but that roots $(i+\alpha)$ and $(j+\beta)$ of Equation (5.3.1) below may still coincide for $i \neq j \in \mathbb{Z}_{\geq 0}$ and $\beta \in E_j$. The fact that E_i is a set (rather than, say, a multiset) means that the degree of b_c^f depends on the parameter c.

Theorem 5.22. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be homogeneous of degree d with an isolated singularity. Then the augmented b-function $\tilde{b}_c^f(s)$ divides*

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha). \tag{5.3.1}$$

Moreover, the roots of $\tilde{b}_c^f(s)$ and (5.3.1) agree. In particular, when (5.3.1) is reduced, it equals $\tilde{b}_c^f(s)$.

Remark 5.23. We expect that $\tilde{b}_c^f(s)$ always equals the polynomial (5.3.1), even when it is not reduced. This is the case in all examples we can check. For an explicit example, see Theorem 4.17.

We will need a number of preparatory results before the proof of the theorem is given in Section 5.4.

Lemma 5.24. *If $u \in \mathbb{C}[\mathfrak{h}]$ is homogeneous and an eigenvector for z_c, with eigenvalue $\beta(u)$, then*

$$(ds + \deg u - \beta(u))(uf^s) = \sum_{i=1}^{n} y_i(x_i u) f^s.$$

Proof. Since $\mathbf{eu}_c = \sum_{i=1}^n y_i x_i + z_c$ and $z_c \cdot uf^s = \beta(u) uf^s$, we get

$$\begin{aligned} \sum_{i=1}^{n} y_i(x_i u) f^s + \beta(u) uf^s = \mathbf{eu}_c \cdot uf^s &= [\mathbf{eu}_c, u] \cdot f^s + u\mathbf{eu}_c f^s \\ &= (\deg u) uf^s + ds uf^s. \end{aligned}$$

□

Recall that $e = \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W$ is the trivial idempotent. The PBW property given in Theorem 3.6 implies that every element of $\mathtt{H}_c e$ can be written *uniquely* as a linear combination of monomials $\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j} e$ for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, where $\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j} = x_1^{i_1} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n}$. Recall the "order filtration" $\mathcal{F}_\bullet$ on $\mathtt{H}_c$ in which $\deg x_i = \deg w = 0$, $\deg y_i = 1$. Then $\operatorname{ord} P$ is defined to be the smallest integer i such that $P \in \mathcal{F}_i \mathtt{H}_c$. The left $\mathtt{H}_c$-submodule $\mathtt{H}_c e \subseteq \mathtt{H}_c$ inherits the order filtration by restriction. Then $P \in \mathtt{H}_c e$ has a well-defined order too.

Definition 5.25. *Given $P(s) = \sum s^i P_i(x, y) e \in (\mathtt{H}_c e)[s]$ we denote by*

- $\operatorname{ord}^T(P(s)) := \max_i(i + \operatorname{ord} P_i)$ *the* ***total order*** *of P;*
- $\sigma^T(P)(s, x, \xi) := \sum s^i \sigma_{l-i}(P_i)$ *the* ***total symbol*** *of P, where $l = \operatorname{ord}^T(P(s))$ and;*
- $\sigma_m(P_i)(x, df) := \sum_{|\alpha|=m} h_\alpha(x) f_\alpha$ *if* $P_i e = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} h_\alpha(x) y^\alpha e$ *for some $h_\alpha \in \mathbb{C}[x]$, where $df := (f_1, \ldots, f_n)$;*

- $(s)_m := \frac{s!}{(s-m)!} = s(s-1)\cdots(s-m+1)$.

Lemma 5.26. *If $\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j}e \in \mathtt{H}_c e$ has order $m = |\mathbf{j}|$ and $h \in \mathbb{C}[\mathfrak{h}]$ then $[\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j}, h]e \in \mathtt{H}_c e$ has order at most $m-1$.*

Proof. First we note that $[\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j}, h]e = \mathbf{x}^\mathbf{i}[\mathbf{y}^\mathbf{j}, h]e$ so it suffices to assume $\mathbf{i} = 0$. We prove by induction on m the slightly stronger statement that $[\mathbf{y}^\mathbf{j}, h]$ has order at most $m-1$ in $\mathtt{H}_c$. If $m = 1$ then $\mathbf{y}^\mathbf{j} = y_i$ for some i and it is immediate from the Dunkl embedding that $[y_i, h] \in \mathbb{C}[\mathfrak{h}] \otimes W$ has order zero.

If $m > 1$ then write $\mathbf{y}^\mathbf{j} = \mathbf{y}^{\mathbf{j}'} y_i$, where $|\mathbf{j}'| = m-1$. Then

$$[\mathbf{y}^\mathbf{j}, h] = \mathbf{y}^{\mathbf{j}'}[y_i, h] + [\mathbf{y}^{\mathbf{j}'}, h]y_i.$$

Now, $\operatorname{ord}[y_i, h] = 0$ and $\operatorname{ord}[\mathbf{y}^{\mathbf{j}'}, h] \leq m-2$ by the induction hypothesis. Therefore, it follows from the multiplicative property of filtrations (meaning $\operatorname{ord} ab \leq \operatorname{ord} a + \operatorname{ord} b$) that $\operatorname{ord}[\mathbf{y}^\mathbf{j}, h] \leq m-1$. □

Lemma 5.27. *Let $Pe \in \mathtt{H}_c e$ have order $\operatorname{ord}(Pe) = m$. Then*

$$P(x,y)ef^s = \sigma_m(P)(x, df)(s)_m f^{s-m} + (\textit{lower order terms in } s).$$

Proof. The proof is by induction on $\operatorname{ord}(Pe)$. The statement is clearly true for the base case $\operatorname{ord}(Pe) = 0$ since Pe is a scalar multiple of e in this case. Now assume the statement for $R \in \mathtt{H}_c e$ of order $\operatorname{ord} R < m$. As in Equation (4.2.4), we have

$$P(x,y)e = \sum_{i=1}^n P_i(x,y)y_i e + Qe, \quad \text{and therefore} \quad P(x,y)e \cdot f^s = \sum_{i=1}^n P_i(x,y)y_i f^s + Qf^s,$$

where $Q, P_i \in \mathtt{H}_c$ are of order $\operatorname{ord} Q, \operatorname{ord} P_i < m$. By induction, $Q \cdot f^s = Qef^s$ consists of powers of s less than m. So

$$\begin{aligned} P(x,y)ef^s &= \sum_{i=1}^n P_i(x,y)y_i f^s + (\text{lower order terms in } s) \\ &= \sum_{i=1}^n P_i(x,y)sf_i f^{s-1} + (\text{lower order terms in } s) \\ &= \sum_{i=1}^n [P_i, f_i]sf^{s-1} + sf_i P_i(x,y)f^{s-1} + (\text{lower order terms in } s). \end{aligned}$$

Now P_i is a linear combination of (ordered) monomials $\mathbf{x}^\mathbf{i}\mathbf{y}^\mathbf{j}$, where $|\mathbf{j}| < m$ for all $\mathbf{j}$. It follows therefore from Lemma 5.26 that $\operatorname{ord}([P_i, f_i]e) \leq m-2$.

By induction, $[P_i, f_i] \cdot f^{s-1} = [P_i, f_i]e \cdot f^{s-1}$ is a polynomial in $(s-1)$ of degree $\leq m-2$. Therefore, $s[P_i, f_i]f^{s-1}$ is a polynomial in s of degree at most $m-1$. Note that

- $s(s-1)_{m-1} = (s)_m$;

- $\sum_{i=1}^{n} f_i \sigma_{m-1}(P_i)(x, df) = \sigma_m(P)(x, df) \quad \text{since} \quad \sigma_1(y_i)(df) = f_i.$

Hence,

$$\begin{aligned} P(x,y)ef^s &= \sum_{i=1}^{n} s f_i P_i(x,y) \cdot f^{s-1} + (\text{lower order terms in } s) \\ &= \sum_{i=1}^{n} s f_i \sigma_{m-1}(P_i)(x, df)(s-1)_{m-1} f^{(s-1)-(m-1)} + (\text{lower order terms in } s) \\ &= \left(\sum_{i=1}^{n} f_i \sigma_{m-1}(P_i)(x, df) \right) (s)_m f^{s-m} + (\text{lower order terms in } s) \\ &= \sigma_m(P)(x, df)(s)_m f^{s-m} + (\text{lower order terms in } s), \end{aligned}$$

where the second equality is by induction. □

The following key result is an adaptation of a result for $\mathcal{D}$-modules (Lemma A.6 of Chapter A) to the situation of rational Cherednik algebras. It is based on [Yan78, Theorem 2.19].

Lemma 5.28. *For homogeneous $f \in \mathbb{C}[\mathfrak{h}]^W$ with an isolated singularity,*

$$\mathrm{Ann}_{\mathtt{H}_c}(f^s) = \langle 1 - e, (f_i y_j - f_j y_i)e \,|\, 1 \le i < j \le n \rangle.$$

Proof. If $f \in \mathbb{C}[\mathfrak{h}]^W$ then $e \cdot f^s = f^s$, so $(1-e) \cdot f^s = 0$, and $1 - e \in \mathrm{Ann}_{\mathtt{H}_c}(f^s)$. If $P \in \mathtt{H}_c$ then $P = Pe + P(1-e)$, and therefore $P \in \mathrm{Ann}_{\mathtt{H}_c}(f^s)$ if and only if $Pe \in \mathrm{Ann}_{\mathtt{H}_c}(f^s)$.

Now let $P(x,y)e \in \mathrm{Ann}_{\mathtt{H}_c}(f^s)$ with $\mathrm{ord}^T P = m$. We prove that Pe belongs to the left ideal $\langle 1 - e, (f_i y_j - f_j y_i)e | 1 \le i < j \le n \rangle$ by induction on $\mathrm{ord}^T P$. By Lemma 5.27, we have

$$0 = P(x,y)ef^s = \sigma_m(P)(x, df)(s)_m f^{s-m} + (\text{lower order terms in } s)$$

which gives $\sigma_m(P)(x, df) = 0$. When f has an isolated singularity, Proposition 5.12 says that $f_1, \ldots, f_n$ a regular sequence. Therefore,

$$\sigma_m(P)(x, \xi) = \sum q_{ij}(x, \xi)(f_i \xi_j - f_j \xi_i)$$

for $1 \le i < j \le n$ by part (ii) of Lemma 5.11. Choosing $Q_{ij} \in \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \subset \mathtt{H}_c$ such that $\sigma^T(Q_{ij}) = q_{ij}$,

$$\mathrm{ord}^T \left(Pe - \sum Q_{ij}(x,y)(f_i y_j - f_j y_i)e \right) < m.$$

Note also that

$$Pe - \sum Q_{ij}(x,y)(f_i y_j - f_j y_i)e \in \mathrm{Ann}_{\mathtt{H}_c}(f^s).$$

Thus, by induction, Lemma 5.28 is proved. □

Lemma 5.29. *If $f \in \mathbb{C}[\mathfrak{h}]^W$ and $f_i = \frac{\partial f}{\partial y_i}$ then $f_i y_j - f_j y_i = y_j f_i - y_i f_j$ in $\mathtt{H}_c$.*

Proof. Writing

$$y_i = \frac{\partial}{\partial y_i} - \sum_{s \in \mathcal{S}} \frac{2c(s)}{1-\lambda_s} \frac{(y, \alpha_s)}{\alpha_s}(1-s)$$

via the Dunkl embedding, we see that $[y_i, f] = [\frac{\partial}{\partial y_i}, f] = f_i$. Therefore,

$$[f_i, y_j] - [f_j, y_i] = [[y_i, f], y_j] - [[y_j, f], y_i] = -[[f, y_i], y_j] - [[y_j, f], y_i] = [[y_j, y_i], f] = 0,$$

where we use the Jacobi identity in the second equality. □

Recall the Milnor algebra $M_f = \mathbb{C}[\mathfrak{h}]/J(f)$ and the $\mathtt{H}_c[s]$-module

$$\mathcal{M} = \mathtt{H}_c f^s / \mathtt{H}_c J(f) f^s.$$

Proposition 5.30. *We have an isomorphism of $\mathtt{H}_c$-modules*

$$\phi : \mathtt{H}_c \underset{\mathbb{C}[\mathfrak{h}] \rtimes W}{\otimes} M_f \xrightarrow{\sim} \mathcal{M}.$$

Proof. Let $A = \mathrm{Ann}_{\mathtt{H}_c}(f^s)$ and $E = \mathbb{C}[\mathfrak{h}] \rtimes W$. Then

$$\mathcal{M} = \frac{\mathtt{H}_c}{A + \mathtt{H}_c J(f)},$$

as a left $\mathtt{H}_c$-module. But $f_i y_j - f_j y_i \in \mathtt{H}_c J(f)$ because $f_i y_j - f_j y_i = y_j f_i - y_i f_j$ by Lemma 5.29. Therefore, since $A = \langle 1 - e, (f_i y_j - f_j y_i) e | 1 \le i < j \le n \rangle$ by Lemma 5.28,

$$\mathcal{M} = \frac{\mathtt{H}_c}{\mathtt{H}_c(1-e) + \mathtt{H}_c J(f)}.$$

Notice that, as an E-module, $M_f = \mathbb{C}[\mathfrak{h}]/J(f)$ equals $E/E(1-e) + EJ(f)$. Therefore,

$$\mathcal{M} = \frac{\mathtt{H}_c}{\mathtt{H}_c(1-e) + \mathtt{H}_c J(f)} \cong \mathtt{H}_c \underset{E}{\otimes} \left(\frac{E}{E(1-e) + EJ(f)} \right) \cong \mathtt{H}_c \underset{E}{\otimes} M_f.$$

□

Example 5.31. *Let $W = \mathbb{Z}_2 = \{1, \sigma\}$ acting on $\mathfrak{h} = \mathbb{C} \cong \mathfrak{h}^*$, where σ acts by -1 on both spaces. Consider $f = x^4 \in \mathbb{C}[\mathfrak{h}]$, which has isolated singularities. Then Proposition 5.30 implies that $\mathcal{M} = H_c \otimes_{\mathbb{C}[x] \rtimes \mathbb{Z}_2} \mathbb{C}[x]/(x^3)$. Following Section 7 of [Rou05], the defining relation for $\mathtt{H}_c$ is*

$$[y, x] = 1 - 2cs.$$

Then, as noted in [Rou05, (7.2.3)], category $\mathcal{O}_c^*$ is semi-simple when $c \notin \frac{1}{2} + \mathbb{Z}$. When $c \notin \frac{1}{2} + \mathbb{Z}$,

$$\mathcal{M} = \Delta(\mathrm{triv})^{\oplus 2} \oplus \Delta(\mathrm{sign}) = L(\mathrm{triv})^{\oplus 2} \oplus L(\mathrm{sign}),$$

since $\Delta(\lambda) = L(\lambda)$ for all λ in this case. When $c \in -(\frac{1}{2} + \mathbb{Z}_{\geq 0})$, it is explained in [Rou05, (7.2.3)] that the projective cover $P(\mathrm{triv})$ of the simple $L(\mathrm{triv})$ is $H_c \otimes_{\mathbb{C}[x]\rtimes\mathbb{Z}_2} \mathbb{C}[x]/(x^2)$, giving rise to a short exact sequence

$$0 \to \Delta(\mathrm{triv}) \to \mathcal{M} \to P(\mathrm{triv}) \to 0.$$

Since $P(\mathrm{triv})$ is projective, this must split and hence $\mathcal{M} = \Delta(\mathrm{triv}) \oplus P(\mathrm{triv})$. When $c \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, it is explained in [Rou05, (7.2.3)] that the projective cover $P(\mathrm{triv})$ of the simple $L(\mathrm{triv})$ is $\Delta(\mathrm{triv})$ and $P(\mathrm{sign}) = H_c \otimes_{\mathbb{C}[x]\rtimes\mathbb{Z}_2} (\mathbb{C}[x]/(x^2) \otimes \mathrm{sign})$, giving rise to a short exact sequence

$$0 \to P(\mathrm{sign}) \to \mathcal{M} \to P(\mathrm{triv}) = \Delta(\mathrm{triv}) \to 0.$$

Since $P(\mathrm{triv})$ is projective, this must split and hence $\mathcal{M} = P(\mathrm{sign}) \oplus P(\mathrm{triv})$.

Notice that $\mathcal{M}$ is not always semi-simple and in this example is never indecomposable. In the special case $c \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, $\mathcal{M}$ is actually a projective module in $\mathcal{O}_c^*$. In fact, it is a projective generator for this category.

§ 5.4. The proof of Theorem 5.22

First we must show that $\tilde{b}_c^f$ divides $\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha)$. By Lemma 4.11, this means that we need to show that

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) \cdot \mathcal{M} = 0.$$

In other words, we need:

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) \cdot f^s \in \mathtt{H}_c J(f) f^s. \tag{5.4.1}$$

We show this by proving the following more general claim, and then setting $u = 1$.

Claim: If $u \in U_m$ is homogeneous of degree $0 \leq m \leq N + 1$ then

$$P_m := \prod_{i=m}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) u f^s \in \mathtt{H}_c J(f) f^s, \tag{5.4.2}$$

where we define $P_{N+1} 1 \cdot u f^s = u f^s$. We prove the claim by descending induction on m. Our induction statement is that $P_m \in \mathtt{H}_c J(f) f^s$. We begin with the base case $m = N+1$. In this case, $U_m = \{0\}$ and hence $u = 0$, so the statement is vacuously true.

By induction, it remains to show that $P_m \in \mathtt{H}_c J(f) f^s$ under this assumption. If we

decompose $u \in U_m$ into a sum $u_1 + \cdots + u_k$ of eigenvectors for z_c', and we know that

$$\prod_{i=m}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) u_j f^s \in \mathtt{H}_c J(f) f^s,$$

for $j = 1, \ldots, k$, then (5.4.2) holds for $u \in U_m$. Therefore, we may assume that u is an eigenvector for z_c. That is, there exists $\alpha' \in \mathbb{C}$ such that $z_c \cdot u = \alpha' u$. In particular $\alpha' \in E_m$. Note that

$$\left(\sum y_i x_i u f^s\right) + z_c u f^s = \mathbf{eu}_c \cdot (u f^s) = [\mathbf{eu}_c, u] f^s + u \mathbf{eu}_c f^s = m u f^s + u d s f^s = (m + ds) u f^s,$$

where $\mathbf{eu}_c \cdot f^s = ds f^s$ by Equation (5.2.1), and $[\mathbf{eu}_c, u] = (\deg u) u$ by Lemma 3.9. Hence

$$\sum_{i=1}^{n} y_i x_i u f^s = (ds + m - \alpha') u f^s. \tag{5.4.3}$$

Since $\deg x_i u = \deg u + 1 = m + 1 > m$, the inductive hypothesis implies that

$$\prod_{i=m+1}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) x_j u f^s \in \mathtt{H}_c J(f) f^s$$

and hence

$$\prod_{i=m+1}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) \cdot \sum_{j=1}^{n} y_j x_j u f^s = \sum_{j=1}^{n} y_j \left(\prod_{i=m+1}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) x_j u f^s \right) \in \mathtt{H}_c J(f) f^s.$$

Thus

$$(ds + m - \alpha') \prod_{i=m+1}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) \cdot u f^s \in \mathtt{H}_c J(f) f^s,$$

and the claim is proved. Setting $u = 1$ gives

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) f^s \in \mathtt{H}_c J(f) f^s$$

and hence $\tilde{b}_c^f$ divides $\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha)$.

It remains to show that if

$$\widetilde{B}_f(s) := \prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha)$$

then each linear factor $(ds + i - \alpha)$ of $\widetilde{B}_f(s)$ divides $\tilde{b}_c^f$. Suppose it does not. Then there exists some $0 \leq i_0 \leq N$ and $\alpha_0 \in E_{i_0}$ such that $(ds + i_0 - \alpha_0)$ does not divide $\tilde{b}_c^f(s)$. By

the Euclidean Algorithm, there exist $p, q \in \mathbb{C}[s]$ such that

$$1 = p(ds + i_0 - \alpha_0) + q\tilde{b}_c^f(s). \tag{5.4.4}$$

Choose $u \in U_{i_0}$ of degree i_0 an eigenvector for z_c with eigenvalue α_0. Then, by (5.4.3),

$$(ds + i_0 - \alpha_0)uf^s = \sum_{i=1}^{n} y_i x_i u f^s.$$

Now, as in Equation (4.2.4), there exist $Q(s), A_i(s) \in \mathtt{H}_c[s]$ such that

$$\tilde{b}_c^f(s)f^s = \left(Q(s)f + \sum_{i=1}^{n} A_i(s)f_i\right) f^s.$$

So

$$\tilde{b}_c^f(s)uf^s = \left(uQ(s)f + \sum_{i=1}^{n} uA_i(s)f_i\right) f^s = \left(Q'(s)f + \sum_{i=1}^{n} A_i'(s)f_i\right) f^s.$$

Therefore, multiplying uf^s by Equation (5.4.4), we have

$$\begin{aligned} uf^s &= p(s)(ds + i_0 - \alpha_0)uf^s + q(s)\tilde{b}_c^f(s)uf^s \\ &= \left(\sum_{i=1}^{n} p(s)y_i x_i u f^s\right) + \left(q(s)Q'(s)f + \sum_{i=1}^{n} q(s)A_i'(s)f_i\right) f^s \\ &= \left(\sum_{i=1}^{n} y_i x_i u p(s) + \sum_{i=1}^{n} Q'(s)q(s)\frac{1}{d}x_i f_i + \sum_{i=1}^{n} q(s)A_i'(s)f_i\right) f^s, \end{aligned}$$

noting that $f = \frac{1}{d}\sum x_i f_i$. Since $d^{-1}\mathbf{eu}_c f^s = s \cdot f^s$ implies $s^k \cdot f^s = (d^{-1}\mathbf{eu}_c)^k f^s$, the first sum on the right hand side of the above becomes

$$\left(\sum_{i=1}^{n} y_i x_i u p(s)\right) \cdot f^s = \sum_{i=1}^{n} y_i x_i u p(d^{-1}(\mathbf{eu}_c)) \cdot f^s = \sum_{i=1}^{n} y_i B_i f^s \quad \text{for some } B_i \in \mathtt{H}_c.$$

Since $\mathtt{H}_c[s]J(f)f^s = \mathtt{H}_c J(f)f^s$ by Lemma 5.13,

$$\left(\sum_{i=1}^{n} Q'(s)q(s)\frac{1}{d}x_i f_i + \sum_{i=1}^{n} q(s)A_i'(s)f_i\right) f^s = \sum_{i=1}^{n} B_i' f_i f^s \quad \text{for some } B_i' \in \mathtt{H}_c.$$

Thus

$$uf^s = \sum_{i=1}^{n} y_i B_i f^s + \sum_{i=1}^{n} B_i' f_i f^s, \quad \text{for some } B_i, B_i' \in \mathtt{H}_c.$$

Let $R = u - \sum y_i B_i - \sum B_i' f_i \in \mathtt{H}_c$, and note that $R \cdot f^s = 0$. This implies that $Re \cdot f^s = R \cdot f^s = 0$ too. Note also that

$$(f_i y_j - f_j y_i) \cdot f^s = f_j s f_i f^{-1} f^s - f_j s f_k f^{-1} f^s = 0$$

and that

$$f_i y_j - f_j y_i = y_j f_i - y_i f_j$$

in $\mathtt{H}_c$ by Lemma 5.29. With this, Lemma 5.28 implies that

$$\mathrm{Ann}_{\mathtt{H}_c}(f^s) = \langle 1-e, (y_j f_i - y_i f_j)e \,|\, 1 \le i < j \le n \rangle \subset \mathtt{H}_c J(f)e \oplus \mathtt{H}_c(1-e).$$

Hence $Re \in \mathtt{H}_c J(f)e$. That is,

$$\left(u - \sum y_i B_i - \sum B_i' f_i\right) e \in \mathtt{H}_c J(f)e = \bigoplus_{\mathbf{i} \in \mathbb{Z}^n_{\ge 0}} \mathbf{y}^{\mathbf{i}} J(f)e$$

Since $\sum B_i' f_i e \in \mathtt{H}_c J(f)e$ and

$$\sum y_i B_i \in \bigoplus_{\mathbf{i} \in \mathbb{Z}^n_{\ge 0} \setminus \{0\}} \mathbf{y}^{\mathbf{i}} J(f)e,$$

this implies that $ue \in J(f)e$. Hence, u belongs to the ideal $J(f)$, contradicting our choice of $u \in U_{i_0} \subset \mathbb{C}[\mathfrak{h}] \setminus J(f)$.

§ 5.5. Dependence on the parameter c

The authors of [BET26] expected that the b-function of an invariant polynomial f should behave "flatly" in c. As in the introduction, this means that the degree of the b-function should not depend on the parameter c. Since the sets E_i appearing in the statement of Theorem 5.22 are true sets rather than multisets, our main result shows that this is not the case. Recall from Section 3.2 that the space of parameters is $\mathcal{C} := \mathbb{C}[\mathcal{S}]^W$. If the number of conjugacy classes in $\mathcal{S}$ equals r then $\mathcal{C} \cong \mathbb{C}^r$. Let f be a homogeneous W-invariant polynomial with isolated singularities. We say that the parameter c is *generic* with respect to f if the b-function b_c^f of f is reduced.

Proposition 5.32. *Let f be a homogeneous W-invariant with isolated singularities. The set $\mathcal{C}_f \subset \mathcal{C}$ of parameter c such that b_c^f is reduced has complement contained in a finite union of affine hyperplanes in $\mathcal{C}$.*

Proof. By Theorem 5.22, it suffices to show that there are finitely many affine hyperplanes such that the polynomial

$$\prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha)$$

is reduced for any c not lying on one of these affine hyperplanes.

Clearly, $\alpha + i \neq \beta + i$ for $\alpha, \beta \in E_i$, $\alpha \neq \beta$. Therefore, we just need to show that $\alpha + i \neq \beta + j$ for $\alpha \in E_i$, $\beta \in E_j$, $i \neq j$. Note that if $\alpha + i = \beta + j$, then

$$\alpha - \beta = j - i \in \{-N, \dots, -1, 1, \dots, N\} = [-N, N] \setminus \{0\}$$

because $j \neq i$. Now write

$$z_c = \sum_{k=1}^{r} c_k z_k,$$

for $z_k \in Z(\mathbb{C}W)$ and $(c_1, \ldots, c_r) \in \mathcal{C} \cong \mathbb{C}^r$. Let $A_{k,\lambda}$ be the scalar by which z_k acts on $\lambda \in \operatorname{Irr}(W)$. Then

$$\alpha = \sum_{k=1}^{r} c_k A_{k,\lambda}$$

for some λ, and hence

$$\alpha - \beta = \sum_{k=1}^{r} c_k \Big(A_{k,\lambda} - A_{k,\mu} \Big).$$

Thus, if $(c_1, \ldots, c_r)$ does not lie on any of the affine hyperplanes

$$H_{\lambda,\mu,i} = \left\{ (c_1, \ldots, c_r) \in \mathcal{C} \;\middle|\; \sum_{k=1}^{r} c_k \Big(A_{k,\lambda} - A_{k,\mu} \Big) = i \right\} \tag{5.5.1}$$

for $i \in \{-N, \ldots, -1, 1, \ldots, N\}$, then c is generic. Notice that it is never the case that the solution to Equation (5.5.1) is the whole space, because $i \neq 0$. □

In particular, Proposition 5.32 implies that the set of generic c is dense in $\mathcal{C} \cong \mathbb{C}^r$.

It turns out that the degree of b_f depends on the parameter c; this is very unexpected. Indeed, assuming c is chosen so that the polynomial (5.3.1) is reduced (which is possible by Proposition 5.32), the degree $\tilde{b}_c^f$ is given by

$$\deg \tilde{b}_c^f = \deg \prod_{i=0}^{N} \prod_{\alpha \in E_i} (ds + i - \alpha) = \sum_{i=0}^{N} |E_i| \geq N + 1.$$

This inequality is an equality when $c = 0$. That is, $\deg \tilde{b}_0^f = N + 1$ because (5.3.1) is reduced in this case. The inequality is strict for many examples when c is in the open set defined by the proof of Proposition 5.32. More precisely, if we set

$$D_i = |\{\lambda \in \operatorname{Irr} W \mid [(M_f)_i : \lambda] \neq 0\}|$$

then Proposition 5.32 shows that for generic c, $|E_i| = D_i$ and hence

$$\deg \tilde{b}_c^f = \sum_{i=0}^{N} D_i.$$

This makes it easy to find examples where the degree of $\tilde{b}_c^f$ is greater than $N + 1$ for generic c.

Remark 5.33. It is possible in some examples to choose c such that there exist $0 \leq i < j \leq N$ with $\alpha \in E_i, \beta \in E_j$ such that $\alpha - i = \beta - j$, in which case the polynomial (5.3.1) has a repeated root. In principle, this could lead to a situation where (5.3.1) has fewer

than $N+1$ distinct roots and hence $\deg \tilde{b}_c^f < N+1 = \deg \tilde{b}_0^f$. We do not know of any example where this actually happens.

Example 5.34. *If we take $W = G(3,1,2)$ and $f = x_1^6 + x_2^6$, then we use the code in Appendix B to compute that $N = 8$,*

$$D_0 = 1,\ D_1 = 1,\ D_2 = 2,\ D_3 = 3,\ D_4 = 2,\ D_5 = 3,\ D_6 = 2,\ D_7 = 1,\ D_8 = 1,$$

and hence $\deg \tilde{b}_c^f = 16$, but $\deg \tilde{b}_0^f = 9$.

Finally, we turn to the question of the rationality of the roots of $\tilde{b}_c^f$.

Lemma 5.35. *Let V be a W-module and $u \in V$ an eigenvector for z_c with eigenvalue β. Then there exist $n_{H,i} \in \mathbb{Z}_{\geq 0}$ for $H \in \mathcal{A}$ and $0 \leq i \leq \ell_H - 1$ and an integer $1 \leq m \leq \dim V$ such that*

$$m\beta = -n - \sum_{H \in \mathcal{A}} \ell_H \sum_{i=0}^{\ell_H - 1} \kappa_{H,i+1} n_{H,i+1}.$$

In particular, if $\kappa_{H,i} \in \mathbb{Q}_{\geq 0}$ for all H and i then $\beta \in \mathbb{Q}_{<0}$.

Proof. We may assume without loss of generality that there is an irreducible submodule $U \subset V$ such that $u \in U$. Then z_c will act by the scalar β on U. This implies that

$$\beta = \frac{1}{\dim U} \mathrm{Tr}(z_c|_U).$$

Since the $e_{H,i}$ are idempotents, they act as diagonalizable matrices on U with eigenvalues 0 and 1. Therefore, $\mathrm{Tr}(e_{H,i}|_U) = n_{H,i}$ for some $n_{H,i} \in \mathbb{Z}_{\geq 0}$. The claimed formula, with $m = \dim U \leq \dim V$, then follows from (3.6.3). □

Combining Lemma 5.35 with our main Theorem 5.22 gives the following corollary, which in particular confirms Conjecture 1.2 for homogeneous polynomials with isolated singularities.

Corollary 5.36. *Let $f \in \mathbb{C}[\mathfrak{h}]^W$ be homogeneous with isolated singularities. If α is a root of the b-function $b_c^f(s)$ then there exists $a \in \mathbb{Q}_{<0}$ and $t_{H,i} \in \mathbb{Q}_{\leq 0}$ such that*

$$\alpha = a + \sum_{H \in \mathcal{A}} \sum_{i=0}^{\ell_H - 1} t_{H,i} \kappa_{H,i}.$$

In particular, if $\kappa_{H,i} \in \mathbb{Q}_{\geq 0}$ for all H, i then $\alpha \in \mathbb{Q}_{<0}$.

Finally, we explain how the main result of this chapter, together with the Thom-Sebastini Theorem 4.14, can be used to produce infinitely many examples of b-functions b_c^f for f *without isolated singularities*, at least if we allow W to be reducible. As in

Section 4.3, let $(\mathfrak{h}_1, W_1), (\mathfrak{h}_2, W_2)$ be a pair of complex reflection groups and $c_i \in \mathbb{C}[\mathcal{S}_i]^{W_i}$. Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $W = W_1 \times W_2$.

Lemma 5.37. *Let $f \in \mathbb{C}[\mathfrak{h}_1]^{W_1}$ and $g \in \mathbb{C}[\mathfrak{h}_2]^{W_2}$ be non-constant. Then*

$$V(f \otimes g)_{sing} = (V(f)_{sing} \times \mathfrak{h}_2) \cup (\mathfrak{h}_1 \times V(g)_{sing}) \cup V(f \otimes 1, 1 \otimes g),$$

and hence $f \otimes g$ does not have isolated singularities.

In particular, $f \otimes g \in \mathbb{C}[\mathfrak{h}]^W$ never has isolated singularities even if f and g both have isolated singularities.

Proof. We drop the tensor product for brevity. If $\dim \mathfrak{h}_1 = n$ and $\dim h_2 = m$ then

$$V(fg)_{sing} = V(fg, f_1 g, \ldots, f_n g, f g_1, \ldots, f g_m).$$

Let $x \in V(fg)_{sing}$. Then $f(x) = 0$ or $g(x) = 0$. If $f(x) = 0$ but $g(x) \neq 0$ then $f_i(x) = 0$ for all i and hence $x \in V(f)_{sing} \times \mathfrak{h}_2$. Similarly, if $g(x) = 0$ but $f(x) \neq 0$ then $x \in \mathfrak{h}_1 \times V(g)_{sing}$. Finally, if $f(x) = 0$ and $g(x) = 0$ then all other equations are satisfied, so $V(f, g) \subset V(fg)_{sing}$. □

Combining the Thom-Sebastini Theorem 4.14 with Corollary 5.36 gives rise (see Chapter 6) to many examples where Conjecture 1.2 holds for polynomials that do not have isolated singularities.

Chapter 6

Examples

Our main result, Theorem 5.22, gives an explicit formula for the b-function of a homogeneous polynomial with isolated singularities. In this chapter we derive an explicit formula for the graded character of the Milnor algebra. This is done by generalising the proof of the Molien formula, which gives the Poincaré polynomial for the ring of invariants. This allows us to compute many highly non-trivial examples of the b-function.

§ 6.1. The graded character of the Milnor algebra

The goal of this section is to use the fact that the Milnor algebra M_f is a quotient of $\mathbb{C}[\mathfrak{h}]$ by a regular sequence in order to compute the graded W-module structure of M_f. The key homological construction we use to do so is the Koszul resolution.

We recall that a zero divisor on a $\mathbb{C}[\mathfrak{h}]$-module N is a non-zero element $z \in \mathbb{C}[\mathfrak{h}]$ such that there exists a non-zero element $n \in N$ with $z \cdot n = 0$. Let M be a module over $\mathbb{C}[\mathfrak{h}]$. We recall, for instance from [Eis95, Exercise 10.13], that an **M-sequence** is a collection of elements $z_1, \ldots, z_n$ such that z_i is a non-zerodivisor on $M/(z_1, \ldots, z_{i-1})M$ for each $i = 1, \ldots, n$. Note that an M-sequence is a regular sequence in the sense of Lemma 5.11 in the case $M = \mathbb{C}[\mathfrak{h}]$.

If $z_1, \ldots, z_n \in \mathbb{C}[\mathfrak{h}]$ and $V = \mathbb{C} \cdot \{z_1, \ldots, z_n\}$ is the abstract vector space with basis the z_i then we define the i-th term of the Koszul resolution to be $K(z_1, \ldots, z_n)^i = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \bigwedge^i V$ and the differential between terms is given by

$$d^i : K(z_1, \ldots, z_n)^i \to K(z_1, \ldots, z_n)^{i-1},$$

$$d^i(a \otimes z_{j_1} \wedge \cdots \wedge z_{j_i}) = \sum_{r=1}^{i} (-1)^{r+1} a z_{j_r} \otimes z_{j_1} \wedge \cdots \wedge \hat{z}_{j_r} \wedge \cdots \wedge z_{j_i}.$$

We note that this indexing convention is opposite to the one appearing in [Eis95, Section 17].

Theorem 6.1. *[Eis95, Corollary 17.5]. If $z_1, \ldots, z_n$ is an M-sequence, then $M \otimes_{\mathbb{C}[\mathfrak{h}]}$*

$K(z_1, \ldots, z_n)$ is exact except at the extreme right. Specifically,

$$H^j(M \otimes_{\mathbb{C}[\mathfrak{h}]} K(z_1, \ldots, z_n)) = 0 \quad \textit{for } j < n,$$

and

$$H^n(M \otimes_{\mathbb{C}[\mathfrak{h}]} K(z_1, \ldots, z_n)) = M/(z_1, \ldots, z_n)M.$$

Lemma 6.2. *If a polynomial $f \in \mathbb{C}[\mathfrak{h}]$ has isolated singularities then*

$$H^i(K(f_1, \ldots, f_n)) = 0$$

for all $i \neq n$ and $H^n(K(f_1, \ldots, f_n)) = M_f$.

Proof. If f has isolated singularities then $f_1, \ldots, f_n$ is a regular sequence by Proposition 5.12. Therefore, it is an M-sequence for $M = \mathbb{C}[\mathfrak{h}]$. Hence the claim of the lemma follows directly from Theorem 6.1 above. □

Assume $f \in \mathbb{C}[\mathfrak{h}]^W$ has isolated singularities and is homogeneous. Then M_f is a graded W-module, and we define its graded W-character to be

$$\chi_{M_f}(q) = \sum_{\lambda \in \operatorname{Irr} W} h_\lambda(q)\chi_\lambda,$$

where

$$h_\lambda(q) = \sum_{i \geq 0} [(M_f)_i : \lambda] q^i,$$

and χ_λ is the character of λ.

We recall from Theorem 3.3 that $\mathfrak{h}/W$ is smooth when $(W, \mathfrak{h})$ is a complex reflection group. In fact, in this case $\mathfrak{h}/W \cong \mathbb{A}^n$ and hence $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring in n variables. It is also $\mathbb{N}$-graded. A *fundamental set of invariants* for W is a choice of homogeneous elements $g_1, \ldots, g_n \in \mathbb{C}[\mathfrak{h}]^W$ such that $\mathbb{C}[g_1, \ldots, g_n] = \mathbb{C}[\mathfrak{h}]^W$. Then the g_i are necessarily algebraically independent. Recall that it is a famous result in the invariant theory of complex reflection groups that if $g_1, \ldots, g_n \in \mathbb{C}[\mathfrak{h}]^W$ is a fundamental set of invariants then their degrees $d_1, \ldots, d_n$ are independent of the choice of the g_i; see for instance [Bro10, Lemma 2.28]. We fix these degrees.

The quotient

$$\mathbb{C}[\mathfrak{h}]^{\mathrm{co}W} := \mathbb{C}[\mathfrak{h}]/(g_1, \ldots, g_n)$$

is the *coinvariant ring*. It is a finite dimensional graded W-module, which is isomorphic to the regular representation of W as an ungraded representation [Kan01, Theorem 24.1]. Then, for each irreducible representation λ, we define the "fake degree", following for

instance [Bro10, Definition 4.32], of λ as

$$f_\lambda(q) = \sum_{i \geq 0} [\mathbb{C}[\mathfrak{h}]_i^{\mathrm{co}W} : \lambda] q^i.$$

Finally, set

$$\chi\left(q, \bigwedge^\bullet \mathfrak{h}\right) = 1 - q\chi_{\mathfrak{h}} + q^2 \chi_{\bigwedge^2 \mathfrak{h}} - q^3 \chi_{\bigwedge^3 \mathfrak{h}} + \cdots + (-1)^n q^n \chi_{\bigwedge^n \mathfrak{h}}.$$

Proposition 6.3. *If $f \in \mathbb{C}[\mathfrak{h}]^W$ is homogeneous with isolated singularities then the graded character of M_f is*

$$\chi_{M_f}(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{d_i}} \left(\sum_{\lambda \in \mathrm{Irr}\, W} f_\lambda(q) \chi_\lambda \right) \left(\chi\left(q^\delta, \bigwedge^\bullet \mathfrak{h}\right) \right)$$

where $\delta = (\deg f) - 1$.

Proof. By Lemma 6.2,

$$\chi_{M_f}(q) = \chi(K(f_1, \ldots, f_n)) = \chi\Big(0 \to \mathbb{C}[\mathfrak{h}] \otimes \bigwedge^n \left(\mathfrak{h}[\delta]\right) \to \mathbb{C}[\mathfrak{h}] \otimes \bigwedge^{n-1} \left(\mathfrak{h}[\delta]\right) \\ \to \cdots \to \mathbb{C}[\mathfrak{h}] \otimes \bigwedge^1 \left(\mathfrak{h}[\delta]\right) \to \mathbb{C}[\mathfrak{h}] \otimes \bigwedge^0 \left(\mathfrak{h}[\delta]\right) \to 0\Big)$$

where $V[\delta]$ means "put V in degree δ". Therefore,

$$\chi_{M_f}(q) = \sum_{i \geq 0} (-1)^i \chi\Big(\mathbb{C}[\mathfrak{h}] \otimes \bigwedge^i \left(\mathfrak{h}[\delta]\right)\Big), \tag{6.1.1}$$

and

$$\chi\Big(\mathbb{C}[\mathfrak{h}] \otimes \bigwedge^i \mathfrak{h}[\delta]\Big) = \prod_{j=1}^{n} \frac{1}{1 - q^{d_j}} \left(\sum_{\lambda \in \mathrm{Irr}\, W} f_\lambda(q)\, \chi_\lambda \right) \chi_{\bigwedge^i \mathfrak{h}}\, q^{i\delta}$$

because

$$\chi(\mathbb{C}[\mathfrak{h}]) = \prod_{j=1}^{n} \frac{1}{1 - q^{d_j}} \left(\sum_{\lambda \in \mathrm{Irr}\, W} f_\lambda(q)\, \chi_\lambda \right).$$

□

If $\mathfrak{h}$ is a finite dimensional complex vector space and $g \in GL(\mathfrak{h})$ then g acts on the space $\mathbb{C}[\mathfrak{h}]_i$ of homogeneous polynomials of degree i. We write g_i for the induced automorphism of $\mathbb{C}[\mathfrak{h}]_i$. We have the following trace formula.

Lemma 6.4. *If $g : \mathfrak{h} \to \mathfrak{h}$ is a linear map inducing linear maps $g_i : \mathbb{C}[\mathfrak{h}]_i \to \mathbb{C}[\mathfrak{h}]_i$ then*

$$\sum_{i \geq 0} \mathrm{Tr}(g_i) q^i = \frac{1}{\det(1 - g^{-1} q)}.$$

Proof. Proposition 17-2A of [Kan01] says that given a linear endomorphism $\varphi : V \to V$ on a vector space V we get induced maps $\varphi_i : (\operatorname{Sym} V)_i \to (\operatorname{Sym} V)_i$ such that

$$\sum_{i \geq 0} \operatorname{Tr}(\varphi_i) q^i = \frac{1}{\det(1 - \varphi q)}.$$

The lemma then follows by setting $\varphi = g^*$ to be the dual map to $g : \mathfrak{h} \to \mathfrak{h}$, which is equal to $(g^{-1})^T : \mathfrak{h}^* \to \mathfrak{h}^*$ by Exercise 1.1 of [FH91] (where $(-)^T$ denotes the transpose), and noting that $\operatorname{Tr}(g_i|_{\mathfrak{h}^*}) = \operatorname{Tr}(g_i^{-1}|_{\mathfrak{h}})$. □

Example 6.5. *We recall that the example $W = B_2$, as described in Section 3.9, has generators*

$$\mathcal{S} = \left\{ s_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

One can compute that

$$\det(1 - s_j^{-1} q) = \det(1 - s_j q) = 1 - q^2$$

for all $j = 1, 2, 3, 4$. We also compute that $\operatorname{Tr}((s_j)_0) = 1$, since $(s_j)_0$ acts trivially on the basis $\{1\}$, and that $\operatorname{Tr}((s_j)_1) = 0$, since $(s_j)_1$ acts by s_j on the basis $\{x_1, x_2\}$. Indeed, for $j = 1$ say, we have

$$(s_1)_{2k} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad (s_1)_{2k+1} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad (k \in \mathbb{Z}_{>0})$$

acting on bases $\{x_1^{2k}, x_1^{2k-1} x_2, \ldots, x_2^{2k}\}$ and $\{x_1^{2k+1}, x_1^{2k} x_2, \ldots, x_2^{2k+1}\}$, respectively. So, $\operatorname{Tr}((s_1)_{2k}) = 1$ and $\operatorname{Tr}((s_1)_{2k+1}) = 0$. Therefore,

$$\sum_{i \geq 0} \operatorname{Tr}((s_1)_i) q^i = 1 + q^2 + q^4 + \cdots = \frac{1}{1 - q^2} = \frac{1}{\det(1 - s_1^{-1} q)},$$

as expected.

We need a similar result for the action of g on the exterior powers of $\mathfrak{h}$. Let $g_{\wedge i}$ denote the induced representation of g on $\bigwedge^i \mathfrak{h}$. We assume for simplicity that g has finite order.

Lemma 6.6. *If $g : \mathfrak{h} \to \mathfrak{h}$ is a linear map inducing linear maps $g_{\wedge i} : \bigwedge^i \mathfrak{h} \to \bigwedge^i \mathfrak{h}$ then*

$$\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(g_{\wedge i}) q^i = \det(1 - gq).$$

Proof. Since g has finite order, it is diagonalisable. Therefore we may choose a basis $v_1, \ldots, v_n$ of $\mathfrak{h}$ consisting of eigenvectors for g with eigenvalues α_i. Then

$$g \cdot v_{j_1} \wedge \cdots \wedge v_{j_i} = \alpha_{j_1} \cdots \alpha_{j_i} \cdot (v_{j_1} \wedge \cdots \wedge v_{j_i}),$$

so

$$\operatorname{Tr}(g_{\wedge i}) = \sum_{1 \le j_1 < \cdots < j_i \le n} \alpha_{j_1} \cdots \alpha_{j_i} = e_i(\alpha_{j_1}, \ldots, \alpha_{j_i}).$$

Thus, by the formula

$$\prod_{i=1}^{n}(t - a_i) = \sum_{i=0}^{n}(-1)^i t^{n-i} e_i(a_1, \ldots, a_n), \quad (t, a_1, \ldots, a_n \in \mathbb{C})$$

with $t = 1$, we have

$$\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(g_{\wedge i}) q^i = \sum_{i=0}^{n}(-1)^i e_i(\alpha_1, \ldots, \alpha_n) q^i = \prod_{j=1}^{n}(1 - \alpha_j q) = \det(1 - gq).$$

□

We denote by $\langle -, - \rangle$ the usual inner product on characters:

$$\langle \chi, \psi \rangle = \frac{1}{|W|} \sum_{g \in W} \chi(g)\psi(g^{-1}).$$

The proof of following result is a generalisation of the proof of the Molien formula; see for instance [Kan01, Section 17].

Theorem 6.7. *Let χ_λ denote the character of $\lambda \in \operatorname{Irr} W$. The character of M_f is given by*

$$\chi_{M_f}(q) = \sum_{\lambda \in \operatorname{Irr} W} \langle \chi_{M_f}, \chi_\lambda \rangle \chi_\lambda = \sum_{\lambda \in \operatorname{Irr} W} \frac{1}{|W|} \sum_{g \in W} \frac{\det(1 - gq^\delta)}{\det(1 - g^{-1}q)} \chi_\lambda(g^{-1}) \chi_\lambda.$$

Proof. By Equation (6.1.1), the Koszul resolution gives

$$\chi(M_f) = \sum_{i=0}^{n}(-1)^i \chi\left(\mathbb{C}[\mathfrak{h}] \otimes \bigwedge^i \left(\mathfrak{h}[\delta]\right)\right),$$

where $\delta = (\deg f) - 1$. Therefore, if $g \in W$ then

$$\begin{aligned}\sum_{j \ge 0} \operatorname{Tr}_{(M_f)_j}(g) q^j &= \sum_{j=0}^{n}(-1)^j \left(\sum_{k \ge 0} \operatorname{Tr}(g_k) q^k\right) \cdot \left(\operatorname{Tr}(g_{\wedge j}) q^{j\delta}\right) \\ &= \left(\sum_{k \ge 0} \operatorname{Tr}(g_k) q^k\right) \left(\sum_{j=0}^{n}(-1)^j \operatorname{Tr}(g_{\wedge j}) q^{j\delta}\right) = \frac{\det(1 - gq^\delta)}{\det(1 - g^{-1}q)}.\end{aligned}$$

Now we use the inner product $\langle -, - \rangle$ on characters to compute $\langle \chi_{M_f}, \psi \rangle$ for all irreducible characters ψ of W. Then

$$\langle \chi_{M_f}, \psi \rangle = \frac{1}{|W|} \sum_{g \in W} \sum_{j \geq 0} \mathrm{Tr}_{(M_f)_j}(g) q^j \cdot \psi(g^{-1}) = \frac{1}{|W|} \sum_{g \in W} \frac{\det(1 - gq^\delta)}{\det(1 - g^{-1}q)} \psi(g^{-1}).$$

□

Remark 6.8. i) The above formula implies that the character χ_{M_f} depends only on the degree of f rather than on f itself.

ii) If f has an isolated singularity then M_f is finite dimensional by Proposition 5.9 and the character χ_{M_f} is polynomial-valued by definition. The above formula then computes χ_{M_f}. Therefore, if we compute χ_{M_f} for a general polynomial f and the output turns out to be fractional rather than polynomial, we know f cannot have isolated singularities. In fact, if this computation does not return a polynomial then it shows that there cannot exist any W-invariant homogeneous polynomial with isolated singularities of that degree.

iii) Note that Solomon's formula for the Poincaré series of $\mathbb{C}[\mathfrak{h}] \otimes \bigwedge^\bullet \mathfrak{h}$, where now $\bigwedge^\bullet \mathfrak{h}$ does not have homological degree, and $\mathfrak{h}$ is in graded degree one is precisely the coefficient of χ_{triv} in the main formula of Theorem 6.7, but with $\delta = 1$ and $\det(1-gq)$ replaced by $\det(1+gq)$. See [Coh76, Equation 2] or [Kan01, Section 22-4].

We wish to compute the terms of χ_{M_f} explicitly using the Magma computer algebra system [BCP97] (MAGMA). In order to do this, we rewrite the terms in the main equation of Theorem 6.7.

For any polynomial $p(t) \in \mathbb{C}[t]$, denote by $p^*(t) := t^{\deg p} p(t^{-1})$ the **reciprocal polynomial** of p, and for $g \in W$ let $p_g(t) := \det(t - g) \in \mathbb{C}[t]$ denote the **characteristic polynomial** of g. Suppose that $p_g(t)$ has roots α_i so that

$$p_g(t) = \prod_{i=1}^{m} (t - \alpha_i).$$

Then

$$p_g(t^{-1}) = \prod_{i=1}^{m} (t^{-1} - \alpha_i) = t^{-m} \prod_{i=1}^{m} (1 - t\alpha_i),$$

and

$$\det(1 - tg) = \prod_{i=1}^{m} (1 - t\alpha_i) = t^m p_g(t^{-1}) = p_g^*(t).$$

Corollary 6.9. *The terms of the formula of Theorem 6.7 are given by*

$$\langle \chi_{M_f}, \chi_\lambda \rangle \chi_\lambda = \frac{1}{|W|} \sum_{g \in W} \frac{\det(1 - gq^\delta)}{\det(1 - g^{-1}q)} \chi_\lambda(g^{-1}) \chi_\lambda = \frac{1}{|W|} \sum_{g \in W} \frac{p_g^*(q^\delta)}{p_{g^{-1}}^*(q)} \chi_\lambda(g^{-1}) \chi_\lambda.$$

Now we can use MAGMA to compute each coefficient

$$\frac{1}{|W|}\sum_{g\in W}\frac{p_g^*(q^\delta)}{p_{g^{-1}}^*(q)}\chi_\lambda(g^{-1})$$

of χ_λ. The following function, whose code is included in Appendix B allows one to quickly compute the graded character of the Milnor algebra.

```
MMultiplicityPolynomial := function(W,d,i)
```

Example 6.10. *We let W be the exceptional complex reflection group G_4 and $f \in \mathbb{C}[\mathfrak{h}]^W$ a polynomial of degree 6 with an isolated singularity. For instance the (unique up to scalar) fundamental invariant of degree 6 has isolated singularities, see Table 6.1. Then,*

$$\chi_{M_f}(q) = (q^4+1)\chi_1 + (q^8+q^4)\chi_2 + q^5\chi_4 + (q^7+q^5+q^3+q)\chi_5 + q^3\chi_6 + (q^6+q^4+q^2)\chi_7,$$

where the irreducible characters are labelled as in the character table produced by MAGMA.

Lemma 6.11. *The grading on M_f is given by*

$$H(q) = \frac{(1-q^{d-1})^n}{(1-q)^n}.$$

In particular, if $N = n(d-2)$ then $(M_f)_N \neq 0$ but $(M_f)_i = 0$ for all $i > N$.

Proof. The main claim is the special case $W = \{1\}$ of the formula of Theorem 6.7: Note that $\delta = d-1$. Taking $g = \mathrm{Id}_\mathfrak{h}$ gives

$$\det(1-gq^\delta) = (1-q^{d-1})^n, \quad \det(1-gq) = (1-q)^n.$$

For the second claim, notice that $\frac{(1-q^{d-1})^n}{(1-q)^n} = 1+\cdots+q^{d-2}$ is a polynomial of degree $d-2$. Therefore, $H(q)$ is a polynomial of degree $n(d-2)$. □

Corollary 6.12. *The dimension of the Milnor algebra M_f is $(d-1)^n$.*

Proof. By Lemma 6.11, $H(q) = (1+q+\cdots+q^{d-2})^n$, so $H(1) = (1+\cdots+1)^n = (d-1)^n$. □

Now we can explicitly compute some b-functions using MAGMA. Recall that for $f \in \mathbb{C}[\mathfrak{h}]^W$ homogeneous of degree d with isolated singularities, its augmented b-function is given by

$$\tilde{b}_c^f(s) = \prod_{i=0}^{N}\prod_{\alpha\in E_i}(ds+i-\alpha),$$

where E_i is the set of eigenvalues of z_c acting on $(M_f)_i$ by Definition 5.21. Let β_λ denote the eigenvalue obtained by acting on λ by z_c, and note that therefore $z_c|_\lambda = \beta_\lambda \mathrm{id}_\lambda$. Then

$\beta_\lambda \in E_i$ for some $i \in \{0, \dots, N\}$ if $[(M_f)_i : \lambda] \neq 0$. Since

$$\chi_{M_f}(q) = \sum_{\lambda \in \operatorname{Irr} W} \sum_{j \geq 0} [(M_f)_j : \lambda] q^j \chi_\lambda,$$

we can determine the set $\{i \,|\, [(M_f)_i : \lambda] \neq 0\}$ from $\chi_{M_f}(q)$, which is given by our code as the function:

```
MMultiplicityPolynomial := function(W,d,i)
```

Thus, to compute $\tilde{b}_c^f(s)$, it remains to determine β_λ, for each $\lambda \in \operatorname{Irr} W$.

Lemma 6.13. *If λ is an irreducible W-module then z_c acts on λ by multiplication by*

$$\beta_\lambda := -n + 2 \sum_{s \in \mathcal{S}} c(s) \left(1 - \frac{1}{1 - \lambda_s^{-1}} \left(1 - \frac{\chi_\lambda(s)}{\dim \lambda}\right)\right). \tag{6.1.2}$$

Proof. Since z_c is central it acts by a scalar on λ. This scalar equals $(\dim \lambda)^{-1} \operatorname{Tr}_\lambda(z_c)$. Taking the trace of z_c, as given in Lemma 3.7 and dividing by $\dim \lambda$ gives

$$\beta_\lambda = \frac{1}{\dim \lambda} \left(-n \dim \lambda + 2 \sum_{s \in \mathcal{S}} c(s) \left(\dim \lambda - \frac{1}{1 - \lambda_s^{-1}} \big(\dim \lambda - \chi_\lambda(s)\big)\right)\right).$$

□

To make computation in MAGMA easier, we rewrite the summation in (6.1.2) to be over conjugacy classes $[s]$ of reflections $s \in \mathcal{S}$,

$$\beta_\lambda = -n + 2 \sum_{\substack{\text{conjugacy} \\ \text{classes } [s]}} \big|[s]\big| c(t) \left(1 - \frac{1}{1 - \lambda_t^{-1}} \big(1 - \frac{\chi_\lambda(t)}{\dim \lambda}\big)\right), \qquad (t \in [s]).$$

This, we package into the MAGMA function

```
coeff := function(W, lambda)
```

and use it to build

```
b_function := function(W,deg)
```

which computes our b-functions. See Chapter B for the code itself.

§ 6.2. Fundamental invariants

Note that if f has isolated singularities and is homogeneous then it is shown in Lemma 6.11 that as a graded vector space

$$\chi_{M_f}(q) = \frac{(1 - q^\delta)^n}{(1 - q)^n} = (1 + q + \cdots + q^{\delta - 1})^n,$$

where $\deg f = \delta + 1$; see Lemma 6.11. In particular, $\dim M_f = (\deg f - 1)^n$ by Corollary 6.12.

Definition 6.14. *We say that a complex reflection group $(W, \mathfrak{h})$ has* IFI *(isolated fundamental invariants) if there exists a set of homogeneous, fundamental invariants $g_1, \ldots, g_n$ such that each g_i has isolated singularities.*

We note that if f is homogeneous with isolated singularities and g is any other homogeneous polynomial of the same degree then $\lambda f + \mu g$ has isolated singularities for generic $\lambda, \mu \in \mathbb{C}$. This immediately implies:

Theorem 6.15. *If there exists a homogeneous invariant polynomial f of degree k with isolated singularities then it is possible to choose a set of homogeneous fundamental invariants $g_1, \ldots, g_n$ such that all g_i of degree k have isolated singularities.*

Example 6.16. *When $n \geq 3$, the elementary symmetric polynomials $e_1, \ldots, e_n$, where*

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i},$$

are not a set of IFI for $\mathfrak{h} = \mathbb{C}^n, W = S_n$. However, the power polynomials

$$p_i = x_1^i + \cdots + x_n^i$$

do form a set of IFI. To see that the $e_1, \ldots, e_n$ are not a set of IFI, consider $e_n = x_1 \cdots x_n$. Then $\frac{\partial e_n}{\partial x_i} = x_i^{-1} e_n$ will vanish on any point $(a_1, \ldots, a_n)$ where two of the coordinates a_j are zero. This shows that the singular locus of $V(e_n)$ is a union of codimension two subspaces and hence has dimension $n - 2 > 0$ for $n \geq 3$. When $n = 2$, e_1, e_2 is a set of IFI.

Lemma 6.17. *For $(W, \mathbb{C}^n)$, where $W = G(m, p, n)$ and $n \geq 2$, the pair admits IFI if and only if p divides n.*

Proof. A set of fundamental invariants for $G(m, p, n)$ is given by $x_1^{km} + \cdots + x_n^{km}$ for $k = 1, \ldots, m-1$ and $(x_1 \cdots x_n)^{m/p}$; see [Coh76, Remark (2.5)(ii)]. All of the polynomials $x_1^{km} + \cdots + x_n^{km}$ have isolated singularities. If p divides n, say $n = pr$, then

$$\deg(x_1^{rm} + \cdots + x_n^{rm}) = \deg(x_1 \cdots x_n)^{m/p}.$$

Therefore, $x_1^{rm} + \cdots + x_n^{rm} + \lambda(x_1 \cdots x_n)^{m/p}$ will have isolated singularities for $\lambda \in \mathbb{C}$ generic and

$$\{x_1^{km} + \cdots + x_n^{km} | k = 1, \ldots, m-1\} \cup \{x_1^{rm} + \cdots + x_n^{rm} + \lambda(x_1 \cdots x_n)^{m/p}\}$$

is a fundamental set of invariants.

If p does not divide n then $\frac{nm}{p} \notin \{m, \ldots, (m-1)n\}$ since if $\frac{nm}{p} = rm$ then $n = pr$. So $(x_1 \cdots x_n)^{m/p}$ is the unique invariant of degree $\frac{nm}{p}$, and it does not have isolated singularities by Lemma 5.6 if $n > 1$. □

Group	**degrees**	**IFI**
G_4	4,6	Yes (M)
G_5	6,12	Yes (M)
G_6	4,12	Yes (M)
G_7	12,12	Yes
G_8	8,12	Yes (M)
G_9	8,24	Yes (M)
G_{10}	12,24	Yes (M)
G_{11}	24,24	Yes
G_{12}	6,8	Yes (M)
G_{13}	8,12	No
G_{14}	6,24	Yes (M)
G_{15}	12,24	No
G_{16}	20,30	Yes (M)
G_{17}	20,60	Yes (M)
G_{18}	30,60	Yes (M)
G_{19}	60,60	Yes
G_{20}	12,30	Yes (M)
G_{21}	12,60	Yes (M)
G_{22}	12,20	Yes (M)
G_{23}	2,6,10	Yes (M)
G_{24}	4,6,14	Yes (M)
G_{25}	6,9,12	No
G_{26}	6,12,18	Yes (M)
G_{27}	6,12,30	??
⋮	⋮	??

Table 6.1: Exceptional groups with IFI

Recall from Section 3.1 that the irreducible complex reflection groups either belong to the infinite family $G(m, p, n)$ or are one of the 34 exceptional groups $G_4, \ldots, G_{37}$. Since we have classified those groups $G(m, p, n)$ that admit IFI in Lemma 6.17, we consider next the exceptional complex reflection groups. In the proof of Theorem 6.18, it is important to note that MAGMA produces one choice of a set of fundamental invariants. In fact, different versions of MAGMA give different sets of fundamental invariants.

Theorem 6.18. *Of the twenty three exceptional complex reflection groups G_4 to G_{26}, all have IFI, except G_{13}, G_{15} and G_{25}.*

Proof. For these exceptional complex reflection groups, we run the MAGMA code (for MAGMA version 2.29-8) which produces a set of fundamental invariants and checks if the associated Milnor algebra is finite dimensional. If the answer is yes, then we record "Yes (M)" in Table 6.1. We consider below the cases where the answer is no:

G_7 has two fundamental invariants of degree 12. Of the pair chosen by MAGMA one has an isolated singularity and the other does not. Taking a generic linear combination of both produces a second fundamental invariant that does have an isolated singularity.

G_{11} has two fundamental invariants of degree 24. Of the pair chosen by MAGMA one has an isolated singularity and the other does not. Taking a generic linear combination of both produces a second fundamental invariant that does have an isolated singularity.

G_{13} has fundamental invariants of degree 8 and 12. Since 12 is not a multiple of 8, these fundamental invariants are unique up to scaling. MAGMA shows that the one of degree 8 has isolated singularities but the one of degree 12 does not. Therefore it does not have IFI.

G_{15} has a fundamental invariants of degree 12 and 24. The one of degree 12 is unique up to scaling. MAGMA shows that it does not isolated singularities. Therefore, it does not have IFI.

G_{19} has two fundamental invariants of degree 60. Of the pair chosen by MAGMA one has an isolated singularity and the other does not. Taking a generic linear combination of both produces a second fundamental invariant that does have an isolated singularity.

G_{25} has three fundamental invariants, of degree $6, 9$ and 12 respectively, which means that the one of degree 9 is unique up to scaling. MAGMA shows that the one of degree 9 does not have isolated singularities (but the ones of degree 6 and 12 do). Therefore, it does not have a IFI. □

MAGMA is currently unable to compute if the Milnor fibre for a set of fundamental invariants for the groups $G_{27}, \ldots, G_{37}$ is finite dimensional.

Example 6.19. *We list below the b-function of fundamental invariants with isolated singularities for a number of different complex reflection groups. Note that, by definition, the b-function is monic. However, to make the output comprehensible (avoiding a large number of fractions) we do not always put the polynomial in monic form.*

- *G_4 is the smallest of the exceptional complex reflection groups and has fundamental invariants of degree 4 and 6. Their b-functions are*

$$b_c^{g_1}(s) = (s+1)(-8c_1 - 8c_2 + 4s + 2)(4s+6)(-4c_1 - 4c_2 + 4s + 5)$$
$$(-4c_1 - 4c_2 + 4s + 3)(1/3(8\zeta_3 - 8)c_1 + 1/3(-8\zeta_3 - 16)c_2 + 4s + 4),$$

$$b_c^{g_2}(s) = (s+1)(-8c_1-8c_2+6s+6)(-8c_1-8c_2+6s+2)(6s+10)(6s+6)$$
$$(4\zeta_3 c_1+(-4\zeta_3-4)c_2+6s+7)((4\zeta_3-4)c_1+(-4\zeta_3-8)c_2+6s+5)$$
$$(-4c_1-4c_2+6s+9)(-4c_1-4c_2+6s+7)(-4c_1-4c_2+6s+5)$$
$$(-4c_1-4c_2+6s+3)(1/3(8\zeta_3-8)c_1+1/3(-8\zeta_3-16)c_2+6s+8)$$
$$(1/3(8\zeta_3-8)c_1+1/3(-8\zeta_3-16)c_2+6s+6)$$
$$(1/3(8\zeta_3-8)c_1+1/3(-8\zeta_3-16)c_2+6s+4).$$

- *$G(1,1,n)$ is the symmetric group on n letters:*
  - *$G(1,1,1)$ has a fundamental invariant of degree 1. Its b-function is*

$$b_c^g(s) = s+1.$$

  - *$G(1,1,2)$ has fundamental invariants of degree 1 and 2. Its b-functions are*

$$b_c^{g_1}(s) = s+1,$$
$$b_c^{g_2}(s) = (s+1)(-2c_1+2s+2).$$

  - *$G(1,1,3)$ has fundamental invariants of degree 1, 2 and 3. Its b-functions are*

$$b_c^{g_1}(s) = s+1,$$
$$b_c^{g_2}(s) = (s+1)(-6c_1+2s+3),$$
$$b_c^{g_3}(s) = (s+1)(-6c_1+3s+6)(-6c_1+3s+5)(-6c_1+3s+4)(-6c_1+3s+3)$$
$$(-3c_1+3s+5)(-3c_1+3s+4).$$

- *$G(2,1,n)$ is the hyperoctahedral group, or Weyl group of type B_n.*
  - *$G(2,1,1)$ has a fundamental invariant of degree 2. Its b-functions is*

$$b_c^g(s) = (s+1)(-2c_1+2s+1).$$

  - *$G(2,1,2)$ has fundamental invariants of degree 2 and 4. Its b-functions are*

$$b_c^{g_1}(s) = (s+1)(-4c_1-4c_2+2s+2),$$

$$b_c^{g_2}(s) = (s+1)(-4c_1-4c_2+4s+6)(-4c_1-4c_2+4s+4)(-4c_1-4c_2+4s+2)$$
$$(-4c_2+4s+4)(-4c_1+4s+4)(-2c_1-2c_2+4s+5)(-2c_1-2c_2+4s+3).$$

– $G(2,1,3)$ *has fundamental invariants of degree 2, 4 and 6. Its b-functions are*

$$b_c^{g_1}(s) = (s+1)(-6c_1 - 12c_2 + 2s + 3),$$

$$\begin{aligned} b_c^{g_2}(s) = (s+1)&(-6c_1 - 12c_2 + 4s + 9)(-6c_1 - 12c_2 + 4s + 7)(-6c_1 - 12c_2 + 4s + 5) \\ &(-6c_1 - 12c_2 + 4s + 3)(-12c_2 + 4s + 6)(-6c_1 - 6c_2 + 4s + 7) \\ &(-6c_1 - 6c_2 + 4s + 5)(-2c_1 - 8c_2 + 4s + 7)(-2c_1 - 8c_2 + 4s + 5) \\ &(-4c_1 - 8c_2 + 4s + 8)(-4c_1 - 8c_2 + 4s + 6)(-4c_1 - 8c_2 + 4s + 4) \\ &(-4c_1 - 4c_2 + 4s + 6), \end{aligned}$$

$$\begin{aligned} b_c^{g_3}(s) = (s+1)&(-6c_1 - 12c_2 + 6s + 15)(-6c_1 - 12c_2 + 6s + 13)(-6c_1 - 12c_2 + 6s + 11) \\ &(-6c_1 - 12c_2 + 6s + 9)(-6c_1 - 12c_2 + 6s + 7)(-6c_1 - 12c_2 + 6s + 5) \\ &(-6c_1 - 12c_2 + 6s + 3)(-6c_1 + 6s + 9)(-12c_2 + 6s + 12)(-12c_2 + 6s + 10) \\ &(-12c_2 + 6s + 8)(-12c_2 + 6s + 6)(-6c_2 + 6s + 10)(-6c_2 + 6s + 8) \\ &(-6c_1 - 6c_2 + 6s + 13)(-6c_1 - 6c_2 + 6s + 11)(-6c_1 - 6c_2 + 6s + 9) \\ &(-6c_1 - 6c_2 + 6s + 7)(-6c_1 - 6c_2 + 6s + 5)(-2c_1 - 8c_2 + 6s + 13) \\ &(-2c_1 - 8c_2 + 6s + 11)(-2c_1 - 8c_2 + 6s + 9)(-2c_1 - 8c_2 + 6s + 7) \\ &(-2c_1 - 8c_2 + 6s + 5)(-4c_1 - 8c_2 + 6s + 14)(-4c_1 - 8c_2 + 6s + 12) \\ &(-4c_1 - 8c_2 + 6s + 10)(-4c_1 - 8c_2 + 6s + 8)(-4c_1 - 8c_2 + 6s + 6) \\ &(-4c_1 - 8c_2 + 6s + 4)(-4c_1 - 4c_2 + 6s + 12)(-4c_1 - 4c_2 + 6s + 10) \\ &(-4c_1 - 4c_2 + 6s + 8)(-4c_1 - 4c_2 + 6s + 6)(-2c_1 - 4c_2 + 6s + 11) \\ &(-2c_1 - 4c_2 + 6s + 9)(-2c_1 - 4c_2 + 6s + 7). \end{aligned}$$

- $G(m,m,2)$ *is the dihedral group of order* $2m$.

– $G(1,1,2)$ *has fundamental invariants of degree 1 and 2. Its b-functions are*

$$\begin{aligned} b_c^{g_1}(s) &= s+1, \\ b_c^{g_2}(s) &= (s+1)(-2c_1 + 2s + 2). \end{aligned}$$

– $G(2,2,2)$ *has fundamental invariants of degree 2 and 2. Its b-functions are*

$$\begin{aligned} b_c^{g_1}(s) &= (s+1)(-2c_1 - 2c_2 + 2s + 2), \\ b_c^{g_2}(s) &= (s+1)(-2c_1 - 2c_2 + 2s + 2). \end{aligned}$$

– $G(3,3,2)$ *has fundamental invariants of degree 2 and 3. Its b-functions are*

$$b_c^{g_1}(s) = (s+1)(-6c_1+2s+2),$$
$$b_c^{g_2}(s) = (s+1)(-6c_1+3s+4)(-6c_1+3s+2)(-3c_1+3s+3).$$

Chapter 7

Open questions

In this short final chapter, we list a number of open questions and future research directions that arise from our work.

1. Is it possible to compute non-trivial examples of b-functions associated to polynomials $f \in \mathbb{C}[\mathfrak{h}]^W$ that do not have isolated singularities? Already for $c = 0$, that is, for the ring of differential operators $\mathcal{D}(\mathfrak{h})$, computing b-functions is very difficult. However, in this case there do exist computer algorithms [Oak97] that do so effectively. Can one write similar algorithms for rational Cherednik algebras?

2. As one can see from Table 6.1, MAGMA is not able to decide if the set of fundamental invariants it constructs for the ten exceptional complex reflection groups $G_{27}, \ldots, G_{37}$ have isolated singularities. Is it possible to write more efficient code in order to decide if these groups have IFI?

3. If $\mathcal{H}_q(W)$ is the Hecke algebra associated to the complex reflection group W then there exists a Knizhnik-Zamolodchikov functor $\mathrm{KZ}\colon \mathcal{O}_c^* \to \mathcal{H}_q(W)$-mod where, roughly speaking, $q = \exp(2\pi\sqrt{-1}c)$; see [GGOR03] for details. Therefore, given $f \in \mathbb{C}[\mathfrak{h}]^W$ with isolated singularities, we can define $S_f := \mathrm{KZ}(\mathcal{M})$ because of Lemma 5.15. What are these representations S_f of the Hecke algebra $\mathcal{H}_q(W)$ constructed this way? It is a consequence of Proposition 5.30 that $\dim_{\mathbb{C}} S_f = \dim_{\mathbb{C}} M_f$ and S_f will be filtered by "standard modules" $S(\lambda) := \mathrm{KZ}(\Delta(\lambda))$ (of dimension $\dim_{\mathbb{C}} \lambda$) since Proposition 5.30 implies that $\mathcal{M}$ is filtered by the standard modules $\Delta(\lambda)$ and KZ is an exact functor.

4. Recall that the discriminant $\delta = \prod_{\alpha \in \mathcal{A}} \alpha_H^{\ell_H}$ is the W-invariant polynomial of smallest degree whose zeros is the complement of the regular locus $\mathfrak{h}^{reg}$ in $\mathfrak{h}$. As in [BET26, Question 6.10], it is intersting question to ask what the b-function of δ is. Even when $c = 0$ the answer is not currently known. However, rather amazingly, an

explicit formula for the b-function of δ is given by Opdam [Opd89, Theorem 7.1] but over the algebra $\mathcal{D}(\mathfrak{h}/W)$.

5. It is a consequence of results in [BET26] that when the b-function exists (it is expected *always* to exist) then the $\mathtt{H}_c(W)$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]$ has finite length. Therefore, one can ask what the composition factors are. In particular, it would be interesting to give a general answer to this question in the case (a) f has isolated singularities, or (b) when $f = \delta$.

6. One can define the rational Cherednik algebra $\mathtt{H}_c(W)$ associated to any finite subgroup $W \subset GL(\mathfrak{h})$, not only complex reflection groups, as in [Eti07] and [Eti17]. If W° denotes the normal subgroup $\langle \mathcal{S} \rangle \lhd W$ generated by all reflections in W then $\mathtt{H}_c(W^\circ) \subset \mathtt{H}_c(W)$. One can ask if the b-function of $f \in \mathbb{C}[\mathfrak{h}]^W$ with respect to $\mathtt{H}_c(W^\circ)$ agrees with the b-function with respect to $\mathtt{H}_c(W)$.

7. The formula for the graded character of the Milnor fibre M_f given in Theorem 6.7 depends only on the degree d of f (with $\delta = d - 1$). For each $\lambda \in \mathrm{Irr}(W)$, it gives a polynomial when there exists an invariant function of that degree with isolated singularities. However, for some d and λ it is a rational function that is not polynomial. Therefore, one can ask for which $d \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathrm{Irr}(W)$ is the rational function of Theorem 6.7 actually a polynomial?

8. As shown by Beilinson and Gaitsgory in the paper [BG12], the b-function can often be used to describe minimal extensions concretely. In particular, there are situations in which the minimal extension of an integrable connection on an open set coincides with its naïve push-forward, which is much easier to handle. Following Lusztig [Lus85, Definition 7.7], an integrable connection satisfying this property is called *clean*. In the proof of Lemma 4.6, we specialised s to an integer. One can also specialise s to any complex number λ. This gives rise to a $\mathtt{H}_c(W)$-module $\mathbb{C}[\mathfrak{h}][f^{-1}]f^\lambda$, which is free of rank one over $\mathbb{C}[\mathfrak{h}][f^{-1}]$; one should think of it as an integrable connection on $U = (f \neq 0)$. We expect that the arguments of [Kas03, Section 6.5] can be adapted to show that $\mathbb{C}[\mathfrak{h}][f^{-1}]f^\lambda$ is an irreducible $\mathtt{H}_c(W)$-module if none of the complex numbers in $\lambda + \mathbb{Z}$ is a root of $b_c^f(s)$. In particular, $\mathbb{C}[\mathfrak{h}][f^{-1}]f^\lambda$ is clean.

9. Nearby and vanishing cycle functors are very important in the context of holonomic $\mathcal{D}$-modules. They are constructed from the "specialization functor" $\mathrm{Sp}(-)$. Though it is not clear that the nearby and vanishing cycles functors make sense for rational Cherednik algebras, the specialization functor does make sense and was introduced in Section 8.6 of [BG15] for any W-stable closed subset Y of $\mathfrak{h}$. However, this functor

can only be applied to $\mathtt{H}_c(W)$-modules which are specialisable. In the case where $Y = V(f)$ for some invariant function f, all holonomic modules are specialisable provided the b-function exists. We expect that our main result can be used to explicitly compute the image of $\mathbb{C}[\mathfrak{h}]$ under the specialisation functor when $Y = V(f)$ and f is homogeneous with isolated singularities.

Appendix A

b-functions for homogeneous polynomials with isolated singularities

In Section 5.1, we developed the theory of b-functions for homogeneous polynomials with isolated singularities when our operators belong to the rational Cherednik algebra $\mathtt{H}_c(W)$.

In this appendix, we explain in detail the corresponding result in the classical case of $\mathcal{D}$-modules. The proof is based on the lecture notes [Pop18] and the paper [Yan78].

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous of degree d with an isolated singularity. The Weyl algebra $\mathcal{D}_n$ acts on $\mathbb{C}[x_1, \ldots, x_n, s][f^{-1}]f^s$ via $\partial_i \cdot f^s := sf^{-1}\frac{\partial f}{\partial x_i}f^s$. Let $\mathcal{N}_f = \mathcal{D}_n[s]\cdot f^s$ as a $\mathcal{D}_n$-submodule of $\mathbb{C}[x_1, \ldots, x_n, s][f^{-1}]f^s$ and write $t(\mathcal{N}_f) = \mathcal{D}_n[s]ff^s$ which is a submodule of $\mathcal{N}_f$.

Lemma A.1. *$b_f(s)$ is the minimal polynomial of s acting on $\mathcal{N}_f/t(\mathcal{N}_f)$.*

Proof. Follows verbatim from Lemma 4.9 by replacing $\mathtt{H}_c[s]$ with $\mathcal{D}_n[s]$. □

Recall that $J(f)$ is the Jacobian ideal. Let

$$\mathcal{M} = \mathcal{N}_f \Big/ \mathcal{D}_n[s] \cdot (f + J(f))f^s,$$

and note that $\mathcal{N}_f/t(\mathcal{N}_f) \twoheadrightarrow \mathcal{M}$. Recall the augmented b-function $\tilde{b}_f(s) := b_f(s)/(s+1)$.

Lemma A.2. *$\tilde{b}_f(s)$ is the minimal polynomial of s acting on $\mathcal{M}$.*

Proof. Substitute $\mathtt{H}_c$ for $\mathcal{D}_n$ and y_i for ∂_i in the proof of Lemma 4.11. □

Let's assume now that f is homogeneous of degree d. Write $\mathbf{eu} = \sum_{i=1}^n x_i\partial_i$. Then $\sum_{i=1}^n x_i f_i = \mathbf{eu}(f) = df$. So $f \in J(f)$ and

$$\mathcal{D}_n[s](f + J(f))f^s = \mathcal{D}_n[s]J(f)f^s.$$

Lemma A.3. *If f is homogeneous then*

$$\mathcal{D}_n[s](f + J(f))f^s = \mathcal{D}_n[s]J(f)f^s = \mathcal{D}_n J(f)f^s.$$

Proof. We have

$$\mathbf{eu} \cdot f^s = s\sum_{i=1}^{n} x_i f_i f^{-1} f^s = s(df)f^{-1}f^s = dsf^s,$$

so for $D \in \mathcal{D}_n$,

$$Ds^m f^s = D(d^{-1}\mathbf{eu})^m f^s.$$

This gives us the second equality, and the first equality follows from the fact that $f \in J(f)$ for homogeneous f, as above. □

Thus $\mathcal{M} = \mathcal{N}_f/\mathcal{D}_n J(f)f^s$.

Definition A.4. *Given $P(s) = \sum s^i P_i(x, \partial) \in \mathcal{D}_n[s]$ we denote by*

- $\operatorname{ord}^T(P(s)) := \max_i(i + \operatorname{ord} P_i)$ *the* ***total order*** *of P;*
- $\sigma^T(P)(s, x, \xi) := \sum s^i \sigma_{l-i}(P_i)$ *the* ***total symbol*** *of P, where $l = \operatorname{ord}^T(P(s))$ and;*
- $\sigma_m(P_i)(x, df) := \sum_{|\alpha|=m} h_\alpha(x) f_\alpha$ *if* $P_i = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} h_\alpha(x)\partial^\alpha$ *for some* $h_\alpha \in \mathbb{C}[x]$, *where* $df = (f_1, \ldots, f_n)$;

Lemma A.5. *Let $P \in \mathcal{D}_n$ have order* $\operatorname{ord} P = m$. *Then*

$$P(x, \partial)f^s = \sigma_m(P)(x, df)(s)_m f^{s-m} + (\textit{lower order terms in } s).$$

Proof. By induction on $\operatorname{ord} P$. The statement is clearly true for the base case $\operatorname{ord} P = 0$. Now assume the statement for $P \in \mathcal{D}_n$ of order $\operatorname{ord} P < m$. As in Equation (4.2.4), we have

$$P(x, \partial) = \sum_{i=1}^{n} P_i(x, \partial)\partial_i + Q, \quad \text{and therefore} \quad P(x, \partial) \cdot f^s = \sum_{i=1}^{n} P_i(x, \partial)\partial_i f^s + Qf^s,$$

where $Q \in \mathcal{D}_n$ is of order $\operatorname{ord} Q < m$. By induction, $Q \cdot f^s$ consists of powers of s less than m. So

$$\begin{aligned} P(x, \partial)f^s &= \sum_{i=1}^{n} P_i(x, \partial)\partial_i f^s + (\text{lower order terms in } s) \\ &= \sum_{i=1}^{n} P_i(x, \partial) s f_i f^{s-1} + (\text{lower order terms in } s) \\ &= \sum_{i=1}^{n} [P_i, f_i] s f^{s-1} + s f_i P_i(x, \partial) f^{s-1} + (\text{lower order terms in } s). \end{aligned}$$

Now $\operatorname{ord}([P_i, f_i]) \leq m-2$ since $\operatorname{ord}(P_i) = m-1$ and taking the commutator of a differential operator with a polynomial reduces the degree by at least 1. By induction, $[P_i, f_i] \cdot f^{s-1}$ is a polynomial in $(s-1)$ of degree $\leq m-2$. So $s[P_i, f_i]f^{s-1}$ is a polynomial in s of degree at most $m-1$. Note that

- $s(s-1)_{m-1} = (s)_m$;
- $\sum_{i=1}^{n} f_i \sigma_{m-1}(P_i)(x, df) = \sigma_m(P)(x, df) \quad \text{since} \quad \sigma_1(\partial_i)(df) = f_i.$

Hence,

$$\begin{aligned} P(x,\partial)f^s &= \sum_{i=1}^{n} s f_i P_i(x,\partial) \cdot f^{s-1} + (\text{lower order terms in } s) \\ &= \sum_{i=1}^{n} s f_i \sigma_{m-1}(P_i)(x, df)(s-1)_{m-1} f^{(s-1)-(m-1)} + (\text{lower order terms in } s) \\ &= \left(\sum_{i=1}^{n} f_i \sigma_{m-1}(P_i)(x, df) \right) (s)_m f^{s-m} + (\text{lower order terms in } s) \\ &= \sigma_m(P)(x, df)(s)_m f^{s-m} + (\text{lower order terms in } s), \end{aligned}$$

where the second equality is by induction. □

The following is [Yan78, Theorem 2.19]. The version stated here can be found (without proof) in [Pop18, 2.3.7].

Lemma A.6. *For homogeneous f with an isolated singularity,*

$$\operatorname{Ann}_{\mathcal{D}_n}(f^s) = \langle f_i\partial_j - f_j\partial_i \,|\, 1 \leq i < j \leq n \rangle.$$

Proof. When f has an isolated singularity, $\dim \mathbb{C}[x]/\mathfrak{a} \leq N < \infty$, which makes $(f_1, \ldots, f_n)$ a regular sequence.

Now let $P(x, \partial) \in \operatorname{Ann}_{\mathcal{D}_n}(f^s)$ with $\operatorname{ord} P = m$. We prove the rest of the theorem by induction on $\operatorname{ord} P$. By Lemma A.5, we have

$$0 = P(x,\partial)f^s = \sigma_m(P)(x, df)(s)_m f^{s-m} + (\text{lower order terms in } s)$$

which gives $\sigma_m(P)(x, df) = 0$. Since $(f_1, \ldots, f_n)$ is a regular sequence,

$$\sigma_m(P)(x, \xi) = \sum q_{ij}(x, \xi)(f_i\xi_j - f_j\xi_i)$$

for $1 \leq i < j \leq n$ by part (ii) of Lemma 5.11. Choosing $Q_{ij} \in \mathcal{D}_n$ such that $\sigma^T(Q_{ij}) = q_{ij}$,

$$\operatorname{ord}\left(P - \sum Q_{ij}(x, D)(f_i\partial_j - f_j\partial_i)\right) < m.$$

Thus, by induction, Lemma A.6 is proved. □

Assuming that f is homogeneous with an isolated singularity, recall from Section 5.1 that the Milnor algebra

$$M_f := \mathbb{C}[x_1, \ldots, x_n]/J(f)$$

is a finite dimensional graded $\mathbb{C}[x_1, \ldots, x_n]$-module which, by Lemma 5.10, is (i) a local ring and (ii) indecomposable as a $\mathbb{C}[x_1, \ldots, x_n]$-module.

Let $N = \max\{i \in \mathbb{Z}_{\geq 0} | (M_f)_i \neq 0\}$.

Theorem A.7. *For homogeneous f with an isolated singularity,*

$$\tilde{b}_f(s) = \prod_{i=0}^{N}(ds + n + i).$$

Proof. First we must show that $\tilde{b}_f$ divides $\prod_{i=0}^{N}(ds + n + i)$. In other words, $\prod_{i=0}^{N}(ds + n + i) \cdot \mathcal{M} = 0$. This means that

$$\prod_{i=0}^{N}(ds + n + i) \cdot f^s \in \mathcal{D}_n J(f) f^s.$$

We show this by proving the following more general claim, and then setting $u = 1$.

Claim: If $u \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree $m \leq N$ then

$$\prod_{i=m}^{N}(ds + n + i) u f^s \in \mathcal{D}_n J(f) f^s.$$

We prove the claim by descending induction on m. With $\mathbf{eu} = \sum x_i \partial_i = \sum \partial_i x_i - n$, we have

$$\mathbf{eu} \cdot u f^s = \mathbf{eu}(u) f^s + u\, \mathbf{eu} \cdot f^s = m u f^s + ds u f^s,$$

so

$$\sum_{i=1}^{n} \partial_i x_i u f^s - n u f^s = (m + ds) u f^s.$$

That is,

$$(ds + n + m) u f^s = \sum_{i=1}^{n} \partial_i x_i u f^s. \tag{A.0.1}$$

Since $\deg x_i u = \deg u + 1 = m + 1 > m$, by inductive assumption we have

$$\prod_{i=m+1}^{N}(ds + n + i) \cdot \sum_{j=1}^{n} \partial_j x_j u f^s = \sum_{j=1}^{n} \partial_j \left(\prod_{i=m+1}^{N}(ds + n + i) x_j u f^s \right) \in \mathcal{D}_n J(f) f^s.$$

Thus

$$(ds + n + m) \prod_{i=m+1}^{N}(ds + n + i) \cdot u f^s \in \mathcal{D}_n J(f) f^s,$$

and the claim is proved. Setting $u = 1$ gives

$$\prod_{i=0}^{N}(ds+n+i)f^s \in \mathcal{D}_n J(f) f^s,$$

and hence $\tilde{b}_f$ divides $\prod_{i=0}^{N}(ds+n+i)$.
It remains to show that

$$\prod_{i=0}^{N}(ds+n+i)$$

divides $\tilde{b}_f$. Suppose it does not. Then there exists some $0 \leq i_0 \leq N$ such that $(ds+n+i_0)$ does not devide $\tilde{b}_f(s)$. By the Euclidean Algorithm there exist $p, q \in \mathbb{C}[s]$ such that

$$1 = p(ds+n+i_0) + q\tilde{b}_f(s). \tag{A.0.2}$$

Choose $u \in \mathbb{C}[x_1, \ldots, x_n] \backslash J(f)$ of degree i_0. We can do this because M_f is indecomposable by Lemma 5.10, so $(M_f)_i \neq 0$ for all $0 \leq i \leq N$ and therefore there exists an eigenvector $u \in \mathbb{C}[x_1, \ldots, x_n] \setminus (J(f))_i$. Then, by Equation (A.0.1),

$$(ds+n+i_0)uf^s = \sum_{i=1}^{n} \partial_i x_i u f^s.$$

Now, as in Equation (4.2.4), there exist $Q, A_i \in \mathcal{D}_n[s]$ such that

$$\tilde{b}_f f^s = \left(Q(s)f + \sum_{i=1}^{n} A_i(s) f_i \right) f^s.$$

So

$$\tilde{b}_f(s) u f^s = \left(uQ(s)f + \sum_{i=1}^{n} uA_i(s) f_i \right) f^s = \left(Q'(s)f + \sum_{i=1}^{n} A_i'(s) f_i \right) f^s.$$

Therefore, multiplying uf^s by Equation (A.0.2), we have

$$\begin{aligned} uf^s &= p(s)(ds+n+i_0)uf^s + q(s)\tilde{b}_f(s)uf^s \\ &= \left(\sum_{i=1}^{n} p(s)\partial_i x_i u f^s \right) + \left(q(s)Q'(s)f + \sum_{i=1}^{n} q(s)A_i'(s)f_i \right) f^s \\ &= \left(\sum_{i=1}^{n} \partial_i x_i u p(s) + \sum_{i=1}^{n} Q'(s)q(s)\frac{1}{d} x_i f_i + \sum_{i=1}^{n} q(s)A_i'(s)f_i \right) f^s, \end{aligned}$$

noting that $f = \frac{1}{d}\sum x_i f_i$. Since $d^{-1}\mathbf{eu}f^s = s \cdot f^s$ implies $s^k \cdot f^s = (d^{-1}\mathbf{eu})^k f^s$, the first sum on the right hand side of the above becomes

$$\left(\sum_{i=1}^{n} \partial_i x_i u p(s) \right) \cdot f^s = \sum_{i=1}^{n} \partial_i x_i u p(d^{-1}\mathbf{eu}) \cdot f^s = \sum_{i=1}^{n} \partial_i C_i f^s \quad \text{for some } C_i \in \mathcal{D}_n.$$

Since $\mathcal{D}_n[s]J(f)f^s = \mathcal{D}_n J(f)f^s$ by Lemma A.3,

$$\left(\sum_{i=1}^{n} Q'(s)q(s)\frac{1}{d}x_i f_i + \sum_{i=1}^{n} q(s)A_i'(s)f_i\right) f^s = \sum_{i=1}^{n} B_i f_i f^s \quad \text{for some } B_i \in \mathcal{D}_n.$$

Thus

$$uf^s = \sum_{i=1}^{n} \partial_i C_i f^s + \sum_{i=1}^{n} B_i f_i f^s, \quad \text{for some } C_i, B_i \in \mathcal{D}_n.$$

Let $R = u - \sum \partial_i C_i - \sum B_i f_i \in \mathcal{D}_n$, and note that $R \cdot f^s = 0$. Note also that

$$(f_i\partial_j - f_j\partial_i) \cdot f^s = f_j s f_i f^{-1} f^s - f_j s f_k f^{-1} f^s = 0$$

and that

$$f_i\partial_j - f_j\partial_i = \partial_j f_i - \partial_i f_j - \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} = \partial_j f_i - \partial_i f_j$$

in $\mathcal{D}_n$. With this, Lemma A.6 implies that

$$\mathrm{Ann}_{\mathcal{D}_n}(f^s) = \langle \partial_j f_i - \partial_i f_j \,|\, 1 \le i < j \le n \rangle \subset \mathcal{D}_n J(f).$$

Hence $R \in \mathcal{D}_n J(f)$, that is,

$$u - \sum \partial_i C_i - \sum B_i f_i \in \mathcal{D}_n J(f) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\ge 0}} \partial^\alpha J(f)$$

since $\sum B_i f_i \in \mathcal{D}_n J(f)$ and $\sum \partial_i C_i \in \bigoplus_{\alpha \in \mathbb{Z}^n_{\ge 0} \setminus \{0\}} \partial^\alpha J(f)$. This then implies that $u \in J(f)$, which contradicts our choice of $u \in \mathbb{C}[x_1, \ldots, x_n] \setminus J(f)$. □

Appendix B

Code

§ B.1. For computing fundamental invariants that do not have isolated singularities

```
PolynomialToTex := function(f)
    // returns polynomial f (in x) written in tex
    tex := "$";
    ints := "1234567890";
    f := Eltseq(Sprint(f));   // turns f into sequence of characters
    len := #f;
    for i in [1..len] do;   // convert f to tex
        a := f[i];
        if a eq "x" then
            tex cat:= "x_";
        elif a eq "z" then
            tex cat:= "\\z";
        elif a eq "*" then
        elif a in ints then   // wraps integers in braces {}
            first := i eq 1;
            last := i eq len;
            if first and last then
                prev := false;  // previous character is not an integer
                next := false;      // next character is not an integer
            elif first then
                prev := true;
                next := not f[i+1] in ints;
            elif last then
                prev := not f[i-1] in ints;
                next := true;
            else
                prev := not f[i-1] in ints;
                next := not f[i+1] in ints;
```

```
            end if;

            if prev and next then
                tex cat:= "{" cat a cat "}";
            elif prev then
                tex cat:= "{" cat a;
            elif next then
                tex cat:= a cat "}";
            else
                tex cat:= a;
            end if;
        else
            tex cat:= a;
        end if;
    end for;
    tex cat:= "$";
    return tex;
end function;

// the following outputs fundamental invariants of exceptional CRGs
   which don't have only isolated singularities
NonIFI := function(list)
    for k in list do
        G := ShephardTodd(k);
        n := Degree(G);
        K := CoefficientRing(G);
        R := InvariantRing(G);
        Inv := PrimaryInvariants(R);
        P := PolynomialRing(R);
        i := 0;
            for f in Inv do
            i +:= 1;
            <k,i>;
            J := ideal<P | [Derivative(f,i) : i in [1..n]]>;
            if IsZeroDimensional(J) eq false then
                print &cat["$G_{", IntegerToString(k), "}$ & ",
   PolynomialToTex(f), " \\\\ "];
            end if;
        end for;
    end for;
    return "DONE";
end function;
```

§ B.2. For computing b-functions

```
eigen := function(s)
    // returns the nontrivial eigenvalue of the reflection s
    for zetas in [zeta : zeta in Eigenvalues(s)] do
        zeta := zetas[1];
        if zeta ne 1 then
            return zeta;
        end if;
    end for;
return 1;
end function;

ReflClasses := function(W)
    // checks which conjugacy classes of W contain reflections and
   returns their indices
    conjclasses := ConjugacyClasses(W);
    reflclasses := [];
    for i in [1..#conjclasses] do
        if IsPseudoReflection(conjclasses[i][3]) then // only need to
   check a representative
            Append(~reflclasses, i);
        end if;
    end for;
    return reflclasses;
end function;

coeff := function(W, lambda)
    // returns the coefficient beta_lambda, summing over conjugacy
   classes of reflections
    n := Degree(W);
    dim := Dimension(lambda);
    char := Character(lambda);
    conjclasses := ConjugacyClasses(W);
    reflclasses := ReflClasses(W);
    m := #reflclasses;
    K := CoefficientRing(W);
    P := PolynomialRing(K, m);
    AssignNames(~P, ["c" cat IntegerToString(i) : i in [1..m]]);
    summands := [];
    for i in reflclasses do
        C := conjclasses[i];
        s := C[3];  // representative of conjugacy class
        c := P.Position(reflclasses, i);    // this is c(s)
```

```
40          c_coeff := C[2]*(1 - (1-char(s)/dim)/(1-eigen(s^(-1))));    //
    coefficient of c(s)
41          Append(~summands, P ! c_coeff * c);
42      end for;
43      return 2*(&+summands) - n;
44  end function;
45
46  dthpower := function(d,f)       //Substitutes t -> t^d in any polynomial
47      Q<t> := Parent(f);
48      phi := hom<Q -> Q | t^d >;
49      return phi(f);
50  end function;
51
52  PolynomialInversion := function(f) // Sends the polynomial \sum_{i =
    0}^n a_i t^i of degree n to \sum_{i = 0}^n a_i t^{n-i}
53      deg := Degree(f);
54      coe := Coefficients(f);
55      P<t> := Parent(f);
56      return &+[coe[i]*t^(deg + 1 - i) : i in [1..#coe]];
57  end function;
58
59  MMultiplicityPolynomial := function(W,d,lambda)
60    Welmt := [w : w in W];
61    summands := [];
62    for w in Welmt do
63          P<t> := Parent(CharacteristicPolynomial(w));
64          F := FunctionField(P);
65          num := dthpower(d,PolynomialInversion(P!
      CharacteristicPolynomial(w)));
66          den := PolynomialInversion(P!CharacteristicPolynomial(w^(-1)));
67          p := (F!num)/(F!den);
68          q := P!Character(lambda)(w^(-1));
69          Append(~summands,p*q);
70      end for;
71      return (Order(W)^(-1)*(&+summands));    // P! throws an error if
    expression is fractional
72  end function;
73
74  bf_coeffs := function(W,deg)
75      // returns a list of pairs <i, beta_lambda>, where i is the degree
    of the graded piece of the Milnor algebra of a fundamental invariant
      of W of degree deg, and beta_lambda is ths scalar by which z_c acts
      on  the representation lambda occurring in that graded piece
76      coeffs := [];
77      K := CoefficientRing(W);
```

```
78      irr := IrreducibleModules(W,K);
79      for lambda in irr do
80          Molien := MMultiplicityPolynomial(W,deg-1,lambda);
81          if not Molien eq 0 then
82              terms := Split(Sprint(Molien), "+");
83              for term in terms do
84                  if not "t" in term then
85                      i := 0;
86                  elif not "^" in term then
87                      i := 1;
88                  else
89                      i := Split(term, "^");
90                      i := StringToInteger(i[#i]);
91                  end if;
92                  Append(~coeffs, <i,coeff(W,lambda)>);
93              end for;
94          end if;
95      end for;
96      return coeffs;
97  end function;
98
99  b_function := function(W,deg)
100     // returns the reduced (not necessarily monic) b-function of the
        fundamental invariant of W of degree deg
101     coeffs := bf_coeffs(W,deg);
102     P := Parent(coeffs[1][2]);
103     Q := VariableExtension(P, 1, true);   // extend to polynomials in s
104     AssignNames(~Q, Append(["c" cat IntegerToString(i) : i in [1..Rank(
        P)]], "s"));
105     m := Rank(P);
106     phi := hom<P -> Q | [Q.i : i in [1..m]]>;
107     s := m + 1;
108     return  [deg*Q.s + phi(coeff[1] - coeff[2]) : coeff in coeffs];
109 end function;
```

Appendix C

Fundamental invariants that do not have isolated singularities

The following polynomials were produced using MAGMA version 2.29-8.

G_k	**Fundamental invariant without only isolated singularities**
G_7	$x_1^{10}x_2^2 - 5x_1^9x_2^3 + 45/4x_1^8x_2^4 - 15x_1^7x_2^5 + 1/2(\zeta_{12}^3 - 2\zeta_{12} + 28)x_1^6x_2^6 + 1/2(-3\zeta_{12}^3 + 6\zeta_{12} - 21)x_1^5x_2^7 + 1/16(30\zeta_{12}^3 - 60\zeta_{12} + 105)x_1^4x_2^8 + 1/8(-10\zeta_{12}^3 + 20\zeta_{12} - 25)x_1^3x_2^9 + 1/16(11\zeta_{12}^3 - 22\zeta_{12} + 22)x_1^2x_2^{10} + 1/16(-5\zeta_{12}^3 + 10\zeta_{12} - 9)x_1x_2^{11} + 1/64(4\zeta_{12}^3 - 8\zeta_{12} + 7)x_2^{12}$
G_{11}	$x_1^{22}x_2^2 - 11x_1^{21}x_2^3 + 1/36(44\zeta_{24}^5 - 44\zeta_{24}^3 - 44\zeta_{24} + 2145)x_1^{20}x_2^4 + 1/18(-220\zeta_{24}^5 + 220\zeta_{24}^3 + 220\zeta_{24} - 3795)x_1^{19}x_2^5 + 1/324(22737\zeta_{24}^5 - 22737\zeta_{24}^3 - 22737\zeta_{24} + 181907)x_1^{18}x_2^6 + 1/36(-10197\zeta_{24}^5 + 10197\zeta_{24}^3 + 10197\zeta_{24} - 44594)x_1^{17}x_2^7 + 1/432(361702\zeta_{24}^5 - 361702\zeta_{24}^3 - 361702\zeta_{24} + 1029369)x_1^{16}x_2^8 + 1/54(-100276\zeta_{24}^5 + 100276\zeta_{24}^3 + 100276\zeta_{24} - 216667)x_1^{15}x_2^9 + 1/108(347589\zeta_{24}^5 - 347589\zeta_{24}^3 - 347589\zeta_{24} + 633941)x_1^{14}x_2^{10} + 1/108(-497629\zeta_{24}^5 + 497629\zeta_{24}^3 + 497629\zeta_{24} - 816398)x_1^{13}x_2^{11} + 1/648(3546184\zeta_{24}^5 - 3546184\zeta_{24}^3 - 3546184\zeta_{24} + 5463309)x_1^{12}x_2^{12} + 1/108(-513583\zeta_{24}^5 + 513583\zeta_{24}^3 + 513583\zeta_{24} - 768034)x_1^{11}x_2^{13} + 1/108(201960\zeta_{24}^5 - 201960\zeta_{24}^3 - 201960\zeta_{24} + 305261)x_1^{10}x_2^{14} + 1/162(278839\zeta_{24}^5 - 278839\zeta_{24}^3 - 278839\zeta_{24} + 382646)x_1^9x_2^{15} + 1/432(-1540022\zeta_{24}^5 + 1540022\zeta_{24}^3 + 1540022\zeta_{24} - 2167539)x_1^8x_2^{16} + 1/36(108053\zeta_{24}^5 - 108053\zeta_{24}^3 - 108053\zeta_{24} + 152570)x_1^7x_2^{17} + 1/324(-437646\zeta_{24}^5 + 437646\zeta_{24}^3 + 437646\zeta_{24} - 618431)x_1^6x_2^{18} + 1/27(1001\zeta_{24}^5 - 1001\zeta_{24}^3 - 1001\zeta_{24} + 1408)x_1^5x_2^{19} + 1/18(8151\zeta_{24}^5 - 8151\zeta_{24}^3 - 8151\zeta_{24} + 11528)x_1^4x_2^{20} + 1/162(-60467\zeta_{24}^5 + 60467\zeta_{24}^3 + 60467\zeta_{24} - 85514)x_1^3x_2^{21} + 1/54(9384\zeta_{24}^5 - 9384\zeta_{24}^3 - 9384\zeta_{24} + 13271)x_1^2x_2^{22} + 1/27(-1393\zeta_{24}^5 + 1393\zeta_{24}^3 + 1393\zeta_{24} - 1970)x_1x_2^{23} + 1/324(2378\zeta_{24}^5 - 2378\zeta_{24}^3 - 2378\zeta_{24} + 3363)x_2^{24}$

G_{13} | $x_1^{12} - 6x_1^{11}x_2 + (-5\zeta_8^3 + 5\zeta_8 + 9)x_1^{10}x_2^2 + (25\zeta_8^3 - 25\zeta_8 + 10)x_1^9x_2^3 + 1/2(-90\zeta_8^3 + 90\zeta_8 - 75)x_1^8x_2^4 + (30\zeta_8^3 - 30\zeta_8 + 24)x_1^7x_2^5 + (70\zeta_8^3 - 70\zeta_8 + 108)x_1^6x_2^6 + (-210\zeta_8^3 + 210\zeta_8 - 300)x_1^5x_2^7 + 1/2(540\zeta_8^3 - 540\zeta_8 + 765)x_1^4x_2^8 + (-205\zeta_8^3 + 205\zeta_8 - 290)x_1^3x_2^9 + (70\zeta_8^3 - 70\zeta_8 + 99)x_1^2x_2^{10}$

G_{15} | $x_1^{12} - 6x_1^{11}x_2 + (-5\zeta_{24}^5 + 5\zeta_{24}^3 + 5\zeta_{24} + 9)x_1^{10}x_2^2 + (25\zeta_{24}^5 - 25\zeta_{24}^3 - 25\zeta_{24} + 10)x_1^9x_2^3 + 1/2(-90\zeta_{24}^5 + 90\zeta_{24}^3 + 90\zeta_{24} - 75)x_1^8x_2^4 + (30\zeta_{24}^5 - 30\zeta_{24}^3 - 30\zeta_{24} + 24)x_1^7x_2^5 + (70\zeta_{24}^5 - 70\zeta_{24}^3 - 70\zeta_{24} + 108)x_1^6x_2^6 + (-210\zeta_{24}^5 + 210\zeta_{24}^3 + 210\zeta_{24} - 300)x_1^5x_2^7 + 1/2(540\zeta_{24}^5 - 540\zeta_{24}^3 - 540\zeta_{24} + 765)x_1^4x_2^8 + (-205\zeta_{24}^5 + 205\zeta_{24}^3 + 205\zeta_{24} - 290)x_1^3x_2^9 + (70\zeta_{24}^5 - 70\zeta_{24}^3 - 70\zeta_{24} + 99)x_1^2x_2^{10}$

G_{19} | $x_1^{57}x_2^3 + (-19\zeta_{60}^{10} - 19)x_1^{56}x_2^4 + 532\zeta_{60}^{10}x_1^{55}x_2^5 + 1/72(-7695\zeta_{60}^{14} - 6156\zeta_{60}^{12} - 461149\zeta_{60}^{10} + 4617\zeta_{60}^8 + 3078\zeta_{60}^6 + 7695\zeta_{60}^4 - 4617\zeta_{60}^2 + 225188)x_1^{54}x_2^6 + 1/4(6156\zeta_{60}^{14} + 12312\zeta_{60}^{12} + 167352\zeta_{60}^{10} - 7695\zeta_{60}^8 - 1539\zeta_{60}^6 - 10773\zeta_{60}^4 + 1539\zeta_{60}^2 - 161196)x_1^{53}x_2^7 + 1/8(81567\zeta_{60}^{14} - 326268\zeta_{60}^{12} - 1177183\zeta_{60}^{10} + 163134\zeta_{60}^8 - 81567\zeta_{60}^6 + 163134\zeta_{60}^4 + 163134\zeta_{60}^2 + 2191232)x_1^{52}x_2^8 + 1/64(-26054966\zeta_{60}^{14} + 14845289\zeta_{60}^{12} + 11166414\zeta_{60}^{10} - 3678875\zeta_{60}^8 + 14888552\zeta_{60}^6 + 3722138\zeta_{60}^4 - 26011703\zeta_{60}^2 - 81113033)x_1^{51}x_2^9 + 1/64(272976021\zeta_{60}^{14} + 37568453\zeta_{60}^{10} - 46354053\zeta_{60}^8 - 136120275\zeta_{60}^6 - 136120275\zeta_{60}^4 + 226621968\zeta_{60}^2 + 310544474)x_1^{50}x_2^{10} + 1/64(-1519458215\zeta_{60}^{14} - 540829585\zeta_{60}^{12} - 165771675\zeta_{60}^{10} + 559216360\zeta_{60}^8 + 680633675\zeta_{60}^6 + 1100045945\zeta_{60}^4 - 1100045945\zeta_{60}^2 - 1100045945)x_1^{49}x_2^{11} + 1/512(37441610045\zeta_{60}^{14} + 28835464665\zeta_{60}^{12} + 4531001440\zeta_{60}^{10} - 22624711795\zeta_{60}^8 - 14018566415\zeta_{60}^6 - 37441610045\zeta_{60}^4 + 22624711795\zeta_{60}^2 + 23464587510)x_1^{48}x_2^{12} + 1/32(-2753450455\zeta_{60}^{14} - 5506900910\zeta_{60}^{12} - 2037430325\zeta_{60}^{10} + 3590284460\zeta_{60}^8 + 243345445\zeta_{60}^6 + 5263555465\zeta_{60}^4 - 836834005\zeta_{60}^2 - 716020130)x_1^{47}x_2^{13} + 1/192(-27239857680\zeta_{60}^{14} + 32285633625\zeta_{60}^{12} + 50938430095\zeta_{60}^{10} - 10674113835\zeta_{60}^8 + 27239857680\zeta_{60}^6 - 21611519790\zeta_{60}^4 - 21611519790\zeta_{60}^2 - 80265340400)x_1^{46}x_2^{14} + 1/384(-2293437598\zeta_{60}^{14} + 164018105758\zeta_{60}^{12} + 1065048059\zeta_{60}^{10} - 162953057699\zeta_{60}^8 + 1228389539\zeta_{60}^6 + 163341480\zeta_{60}^4 - 165083153817\zeta_{60}^2 + 596145879138)x_1^{45}x_2^{15} + 1/128(683378646705\zeta_{60}^{14} - 655696160560\zeta_{60}^{10} + 14935996140\zeta_{60}^8 - 396590665290\zeta_{60}^6 - 396590665290\zeta_{60}^4 + 698314642845\zeta_{60}^2 + 27682486145)x_1^{44}x_2^{16} + 1/64(-1674549024610\zeta_{60}^{14} - 787139110370\zeta_{60}^{12} + 1666403453860\zeta_{60}^{10} + 506619725235\zeta_{60}^8 + 912968646600\zeta_{60}^6 + 1293758835605\zeta_{60}^4 - 1293758835605\zeta_{60}^2 - 1293758835605)x_1^{43}x_2^{17} + 1/256(15574591060240\zeta_{60}^{14} + 14646984669810\zeta_{60}^{12} - 17362307713280\zeta_{60}^{10} - 9018080915540\zeta_{60}^8 - 8090474525110\zeta_{60}^6 - 15574591060240\zeta_{60}^4 + 9018080915540\zeta_{60}^2 + 20513686649315)x_1^{42}x_2^{18} + 1/128(-7836029588100\zeta_{60}^{14} - 15672059176200\zeta_{60}^{12} + 11690856912070\zeta_{60}^{10} + 9121169068480\zeta_{60}^8 + 3872569628375\zeta_{60}^6 + 11799489547825\zeta_{60}^4 - 1285139480380\zeta_{60}^2 - 19526886500170)x_1^{41}x_2^{19} + 1/256(1817418864170\zeta_{60}^{14} + 1712243626197\zeta_{60}^{12} -$

$$7897993390084\zeta_{60}^{10} - 10121456370361\zeta_{60}^{8} - 1817418864170\zeta_{60}^{6} - 7000980255836\zeta_{60}^{4} - 7000980255836\zeta_{60}^{2} + 22796967036004)x_1^{40}x_2^{20} + 1/128(-60894757518375\zeta_{60}^{14} + 37859206822470\zeta_{60}^{12} + 25929043931505\zeta_{60}^{10} - 11930162890965\zeta_{60}^{8} + 34965713586870\zeta_{60}^{6} + 9036669655365\zeta_{60}^{4} - 63788250753975\zeta_{60}^{2} + 13412243665810)x_1^{39}x_2^{21} + 1/384(1633166838105240\zeta_{60}^{14} - 1067539816403645\zeta_{60}^{10} - 199320008692845\zeta_{60}^{8} - 850542950946105\zeta_{60}^{6} - 850542950946105\zeta_{60}^{4} + 1433846829412395\zeta_{60}^{2} + 565627021701595)x_1^{38}x_2^{22} + 1/384(-7015443757788625\zeta_{60}^{14} - 2853239405322620\zeta_{60}^{12} + 5587511020859255\zeta_{60}^{10} + 2383169915275105\zeta_{60}^{8} + 3457374883406825\zeta_{60}^{6} + 5236409320597725\zeta_{60}^{4} - 5236409320597725\zeta_{60}^{2} - 5236409320597725)x_1^{37}x_2^{23} + 1/1536(75184153696704610\zeta_{60}^{14} + 65369279427010605\zeta_{60}^{12} - 71783608870953780\zeta_{60}^{10} - 44460411378160565\zeta_{60}^{8} - 3464553710846656\zeta_{60}^{6} - 75184153696704610\zeta_{60}^{4} + 44460411378160565\zeta_{60}^{2} + 90806649838062475)x_1^{36}x_2^{24} + 1/128(-9163559084873733\zeta_{60}^{14} - 18327118169747466\zeta_{60}^{12} + 12259508066780977\zeta_{60}^{10} + 11083470974630128\zeta_{60}^{8} + 3492303817720754\zeta_{60}^{6} + 14834814352026712\zeta_{60}^{4} - 1919911889756395\zeta_{60}^{2} - 2142306715165471 0)x_1^{35}x_2^{25} + 1/256(-12447915015408495\zeta_{60}^{14} + 83722986483984225\zeta_{60}^{12} - 20085677449782210\zeta_{60}^{10} - 44397307116835890\zeta_{60}^{8} + 12447915015408495\zeta_{60}^{6} - 39325679367148335\zeta_{60}^{4} - 39325679367148335\zeta_{60}^{2} + 79497034266712755)x_1^{34}x_2^{26} + 1/128(84876519031557617\zeta_{60}^{14} - 58717224002445323\zeta_{60}^{12} - 36298084064979800\zeta_{60}^{10} + 22419139937465523\zeta_{60}^{8} - 4857843496657781 7\zeta_{60}^{6} - 12280350901598017\zeta_{60}^{4} + 95015308067425123\zeta_{60}^{2} - 2999718961735006 0)x_1^{33}x_2^{27} + 1/512(-1196750400902413268\zeta_{60}^{14} + 79190136205611386 0\zeta_{60}^{10} + 13697199043266555 1\zeta_{60}^{8} + 62636619820946724 8\zeta_{60}^{6} + 62636619820946724 8\zeta_{60}^{4} - 10597784104697477 17\zeta_{60}^{2} - 4048490388462994 08)x_1^{32}x_2^{28} + 1/32(174032193699148455\zeta_{60}^{14} + 71173162598706642\zeta_{60}^{12} - 13923698937088483 7\zeta_{60}^{10} - 58869715559228388\zeta_{60}^{8} - 8605356261672160 5\zeta_{60}^{6} - 13004287815793503 0\zeta_{60}^{4} + 13004287815793503 0\zeta_{60}^{2} + 13004287815793503 0)x_1^{31}x_2^{29} + 1/1152(-10299112117785801306\zeta_{60}^{14} - 89683477650498962 88\zeta_{60}^{12} + 98489220140375797 49\zeta_{60}^{10} + 60870346277929424 19\zeta_{60}^{8} + 47562702750570374 01\zeta_{60}^{6} + 10299112117785801306\zeta_{60}^{4} - 60870346277929424 19\zeta_{60}^{2} - 1245215220344020 9228)x_1^{30}x_2^{30} + 1/96(82347188099422759 7\zeta_{60}^{14} + 16469437619884551 94\zeta_{60}^{12} - 11020194063461249 26\zeta_{60}^{10} - 99562677596856650 0\zeta_{60}^{8} - 3145345600932332 38\zeta_{60}^{6} - 13324092018952219 56\zeta_{60}^{4} + 17215489497433890 3\zeta_{60}^{2} + 1925491287340352 523)x_1^{29}x_2^{31} + 1/384(15219807216984082 79\zeta_{60}^{14} - 10257551236576513023\zeta_{60}^{12} + 24625811090443532 97\zeta_{60}^{10} + 54399975930742841 63\zeta_{60}^{8} - 15219807216984082 79\zeta_{60}^{6} + 48175536435022288 60\zeta_{60}^{4} +$$

$$4817553643502228860\zeta_{60}^{2} - 9742715861590935454)x_1^{28}x_2^{32} + 1/288(-10866330400094639519\zeta_{60}^{14} + 7519291174260058739\zeta_{60}^{12} + 4647183313326377999\zeta_{60}^{10} - 2872107860933680740\zeta_{60}^{8} + 621914708676826152 0\zeta_{60}^{6} + 15719637734418835 21\zeta_{60}^{4} - 12166474487586436738\zeta_{60}^{2} + 3843627099706879688)x_1^{27}x_2^{33} + 1/128(11984754418326799284\zeta_{60}^{14} - 7930775070297443560\zeta_{60}^{10} - 1371305415388902378\zeta_{60}^{8} - 6272792574059501133\zeta_{60}^{6} - 6272792574059501133\zeta_{60}^{4} + 1061344900293789690 6\zeta_{60}^{2} + 4053979348029355724)x_1^{26}x_2^{3\ 4} + 1/64(-98309809467706971 48\zeta_{60}^{14} - 4020645106577366154\zeta_{60}^{12} + 7865569140039760163\zeta_{60}^{10} + 3325439960707464122\zeta_{60}^{8} + 4861189187798963404\zeta_{60}^{6} + 7346085067284830276\zeta_{60}^{4} - 7346085067284830276\zeta_{60}^{2} - 7346085067284830276)x_1^{25}x_2^{35} + 1/512(90572749272018619825\zeta_{60}^{14} + 7887035826746246028 0\zeta_{60}^{12} - 8661428236842734920 5\zeta_{60}^{10} - 5353057355896358103 5\zeta_{60}^{8} - 4182818255440742149 0\zeta_{60}^{6} - 9057274927201861982 5\zeta_{60}^{4} + 5353057355896358103 5\zeta_{60}^{2} + 10950760709742669526 0)x_1^{24}x_2^{36} + 1/32(-37410413098798213 15\zeta_{60}^{14} - 7482082619759642630\zeta_{60}^{12} + 5006490457571088790\zeta_{60}^{10} + 4523131974369743545\zeta_{60}^{8} + 1428950530600891580\zeta_{60}^{6} + 6053132089158751050\zeta_{60}^{4} - 7820906644899222 30\zeta_{60}^{2} - 8747531767450910105)x_1^{23}x_2^{37} + 1/384(-13872467821066613265\zeta_{60}^{14} + 9349600549090313772 0\zeta_{60}^{12} - 2244612237281784870 5\zeta_{60}^{10} - 4958476299286743616 5\zeta_{60}^{8} + 1387246782106661326 5\zeta_{60}^{6} - 4391124249803570155 5\zeta_{60}^{4} + 4391124249803570155 5\zeta_{60}^{2} + 8880348724367139896 5)x_1^{22}x_2^{38} + 1/384(84867153279926235190\zeta_{60}^{14} - 5872651708200061487 5\zeta_{60}^{12} - 3629498393911334693 0\zeta_{60}^{10} + 2243153314288726794 5\zeta_{60}^{8} - 4857216934081288826 0\zeta_{60}^{6} - 1227718540169954133 0\zeta_{60}^{4} + 9502150102111396180 5\zeta_{60}^{2} - 3001925055137327014 5)x_1^{21}x_2^{39} + 1/256(-86680346344683030163\zeta_{60}^{14} + 5735974715269423194 5\zeta_{60}^{10} + 9918021538578373360\zeta_{60}^{8} + 4536829477053506074 4\zeta_{60}^{6} + 4536829477053506074 4\zeta_{60}^{4} - 7676232480610465680 3\zeta_{60}^{2} - 2932059919198879821 8)x_1^{20}x_2^{\ 40} + 1/192(64531445661451031725\zeta_{60}^{14} + 2639188102069963944 5\zeta_{60}^{12} - 5163031023152338220 5\zeta_{60}^{10} - 2182848510067509900 0\zeta_{60}^{8} - 3190928658129844516 5\zeta_{60}^{6} - 4822036612137473844 5\zeta_{60}^{4} + 4822036612137473844 5\zeta_{60}^{2} + 4822036612137473844 5)x_1^{19}x_2^{41} + 1/768(-18381423913689288105 5\zeta_{60}^{14} - 1600646514729621029 55\zeta_{60}^{12} + 1757806747572733484 00\zeta_{60}^{10} + 1086384316569724329 85\zeta_{60}^{8} + 8488884399304165488 5\zeta_{60}^{6} + 1838142391368928810 55\zeta_{60}^{4} - 1086384316569724329 85\zeta_{60}^{2} - 2222418789436039421 70)x_1^{18}x_2^{42} + 1/128(13327413130770948110\zeta_{60}^{14} + 2665482626154189622 0\zeta_{60}^{12} -$$

$$17835560057522066310\zeta_{60}^{10} - 161136011625390856 45\zeta_{60}^{8} - 50906188280216578 10\zeta_{60}^{6} - 215642074335202384 10\zeta_{60}^{4} + 27861880317681375 35\zeta_{60}^{2} + 31162973188293014420)x_1^{17}x_2^{43} + 1/512(11614322147052131725\zeta_{60}^{14} - 78276829895265778695\zeta_{60}^{12} + 18792367958572270420\zeta_{60}^{10} + 4151341061836872542 5\zeta_{60}^{8} - 11614322147052131725\zeta_{60}^{6} + 3676341927689705327 0\zeta_{60}^{4} + 3676341927689705327 0\zeta_{60}^{2} - 74348155194041594110)x_1^{16}x_2^{44} + 1/128(-13062777432362022650\zeta_{60}^{14} + 90392030427411497 30\zeta_{60}^{12} + 55865347126187045 95\zeta_{60}^{10} - 34526683301224451 35\zeta_{60}^{8} + 74762427197433180 55\zeta_{60}^{6} + 18897080071246134 60\zeta_{60}^{4} - 14625737755359854325\zeta_{60}^{2} + 4620572106023649710)x_1^{15}x_2^{45} + 1/192(22452251853417755355\zeta_{60}^{14} - 14857525912654958120\zeta_{60}^{10} - 25690012074653469 90\zeta_{60}^{8} - 11751457209752507100\zeta_{60}^{6} - 11751457209752507100\zeta_{60}^{4} + 19883250645952408365\zeta_{60}^{2} + 7594725940762797235)x_1^{14}x_2^{4\ 6} + 1/192(-16917297308938677175\zeta_{60}^{14} - 69187865537941317 65\zeta_{60}^{12} + 13535188935327621275\zeta_{60}^{10} + 5722465503788045005\zeta_{60}^{8} + 83652068062256763 65\zeta_{60}^{6} + 12641252057582176770\zeta_{60}^{4} - 12641252057582176770\zeta_{60}^{2} - 12641252057582176770)x_1^{13}x_2^{47} + 1/1536(75187661934342444200\zeta_{60}^{14} + 65473093743633173010\zeta_{60}^{12} - 71901600316731742620\zeta_{60}^{10} - 44437632841906576300\zeta_{60}^{8} - 34723064651197305110\zeta_{60}^{6} - 75187661934342444200\zeta_{60}^{4} + 44437632841906576300\zeta_{60}^{2} + 90906163451135745965)x_1^{12}x_2^{48} + 1/384(-65173132521835608 80\zeta_{60}^{14} - 13034626504367121760\zeta_{60}^{12} + 87218675365187407 70\zeta_{60}^{10} + 78798027299836275 00\zeta_{60}^{8} + 24893921469593877 15\zeta_{60}^{6} + 10545234357407734045\zeta_{60}^{4} - 13624894778000666 20\zeta_{60}^{2} - 15239180788702301650)x_1^{11}x_2^{49} + 1/384(-11196171027411156 70\zeta_{60}^{14} + 75458624707670675 31\zeta_{60}^{12} - 18115785265786241 42\zeta_{60}^{10} - 40018800921191536 93\zeta_{60}^{8} + 11196171027411156 70\zeta_{60}^{6} - 35439823786479138 38\zeta_{60}^{4} - 35439823786479138 38\zeta_{60}^{2} + 7167139431805162122)x_1^{10}x_2^{50} + 1/192(19237764983093459 66\zeta_{60}^{14} - 13312181479490259 08\zeta_{60}^{12} - 82273806187494249 9\zeta_{60}^{10} + 50848008607408340 9\zeta_{60}^{8} - 11010384364344034 67\zeta_{60}^{6} - 27830037455946096 8\zeta_{60}^{4} + 21539562098239684 07\zeta_{60}^{2} - 680479176631309627)x_1^{9}x_2^{51} + 1/512(-43935871572131484 66\zeta_{60}^{14} + 29074070372320858 17\zeta_{60}^{10} + 50271708970271504 1\zeta_{60}^{8} + 22995934578418564 95\zeta_{60}^{6} + 22995934578418564 95\zeta_{60}^{4} - 3890870067510433425\zeta_{60}^{2} - 1486180119981062649)x_1^{8}x_2^{52} + 1/128(62612047177655114 7\zeta_{60}^{14} + 25606891113951985 2\zeta_{60}^{12} - 50094638210732560 5\zeta_{60}^{10} - 21179227009057164 3\zeta_{60}^{8} - 30960189068363184 3\zeta_{60}^{6} - 46786118123009149 5\zeta_{60}^{4} + 46786118123009149 5\zeta_{60}^{2} + 467861181230091495)x_1^{7}x_2^{53} + 1/576(-12260633758559222 85\zeta_{60}^{14}$$

	$-106765073256399584\zeta_{60}^{12} + 117247852302614152\zeta_{60}^{10} + 724631578309489653\zeta_{60}^{8} + 566218935017563212\zeta_{60}^{6} + 122606337585592285\zeta_{60}^{4} - 724631578309489653\zeta_{60}^{2} - 1482380416949813513)x_1^6x_2^{54} + 1/384(221612367066904525\zeta_{60}^{14} + 443224734133809050\zeta_{60}^{12} - 296575235104198259\zeta_{60}^{10} - 267941967408129382\zeta_{60}^{8}$
	$-84648391892234594\zeta_{60}^{6} - 358576342241574456\zeta_{60}^{4} + 46329600341224857\zeta_{60}^{2} + 518187602171102784)x_1^5x_2^{55} + 1/1536(1112320344835623 61\zeta_{60}^{14} - 749668464796488207\zeta_{60}^{12} + 179977212432195942\zeta_{60}^{10} + 397579907741862358\zeta_{60}^{8} - 111232034483562361\zeta_{60}^{6} + 352088557054625849\zeta_{60}^{4} + 352088557054625849\zeta_{60}^{2} - 71204298191901 7733)x_1^4x_2^{56} + 1/1152(-185116239406647061\zeta_{60}^{14} + 128097051604164151\zeta_{60}^{12} + 79168331750022880\zeta_{60}^{10} - 48928719854141271\zeta_{60}^{8} + 105947907656624181\zeta_{60}^{6} + 267795759066013 01\zeta_{60}^{4} - 207265383354187031\zeta_{60}^{2} + 65479407968728396)x_1^3x_2^{57} + 1/768(557881703186720 68\zeta_{60}^{14} - 36917196171375532\zeta_{60}^{10} - 6383318600229209\zeta_{60}^{8} - 29199400603486768\zeta_{60}^{6} - 29199400603486768\zeta_{60}^{4} + 49404851718442859\zeta_{60}^{2} + 18870974147296536)x_1^2x_2^{58} + 1/384(-57883983702939 97\zeta_{60}^{14} - 2367322160410547\zeta_{60}^{12} + 4631180982739092\zeta_{60}^{10} + 1957990652428901\zeta_{60}^{8} + 2862227255384899\zeta_{60}^{6} + 4325312812839448\zeta_{60}^{4} - 4325312812839448\zeta_{60}^{2} - 4325312812839448)x_1x_2^{59} + 1/4608(5295607925968654\zeta_{60}^{14} + 4611392683991964\zeta_{60}^{12} - 5064164448457673\zeta_{60}^{10} - 3129825753765707\zeta_{60}^{8} - 2445610511789017\zeta_{60}^{6} - 5295607925968654\zeta_{60}^{4} + 3129825753765707\zeta_{60}^{2} + 6402691443107672)x_2^{60}$
G_{25}	$x_1^9 + 1/2(-18\zeta_3 - 9)x_1^8x_2 + 9/2x_1^8x_3 - 27x_1^7x_2^2 + (-36\zeta_3 - 18)x_1^7x_2x_3 + 9x_1^7x_3^2 + 1/2(126\zeta_3 + 63)x_1^6x_2^3 - 189/2x_1^6x_2^2x_3 + 1/2(-126\zeta_3 - 63)x_1^6x_2x_3^2 + 21/2x_1^6x_3^3 + 135/2x_1^5x_2^4 + (180\zeta_3 + 90)x_1^5x_2^3x_3 - 135x_1^5x_2^2x_3^2 + (-54\zeta_3 - 27)x_1^5x_2x_3^3 + 9/2x_1^5x_3^4 + (-54\zeta_3 - 27)x_1^4x_2^5 + 135x_1^4x_2^4x_3 + (180\zeta_3 + 90)x_1^4x_2^3x_3^2 - 135/2x_1^4x_2^2x_3^3 - 9/2x_1^4x_3^5 - 27/2x_1^3x_2^6 + (-54\zeta_3 - 27)x_1^3x_2^5x_3 + 135/2x_1^3x_2^4x_3^2 + 135/2x_1^3x_2^2x_3^4 + (54\zeta_3 + 27)x_1^3x_2x_3^5 - 21/2x_1^3x_3^6 - 135/2x_1^2x_2^4x_3^3 + (-180\zeta_3 - 90)x_1^2x_2^3x_3^4 + 135x_1^2x_2^2x_3^5 + 1/2(126\zeta_3 + 63)x_1^2x_2x_3^6 - 9x_1^2x_3^7 + (54\zeta_3 + 27)x_1x_2^5x_3^3 - 135x_1x_2^4x_3^4 + (-180\zeta_3 - 90)x_1x_2^3x_3^5 + 189/2x_1x_2^2x_3^6 + (36\zeta_3 + 18)x_1x_2x_3^7 - 9/2x_1x_3^8 + 27/2x_2^6x_3^3 + (54\zeta_3 + 27)x_2^5x_3^4 - 135/2x_2^4x_3^5 + 1/2(-126\zeta_3 - 63)x_2^3x_3^6 + 27x_2^2x_3^7 + 1/2(18\zeta_3 + 9)x_2x_3^8 - x_3^9$
G_{27}	??
$\vdots$	$\vdots$

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