W*-Bundles

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Abstract

This thesis collates, extends and applies the abstract theory of W*-bundles. Highlights include the standard form for W*-bundles, a bicommutant theorem for W*-bundles, and an investigation of completions, ideals, and quotients of W*-bundles.

The Triviality Problem, whether all W*-bundles with fibres isomorphic to the hyperfinite II$_1$ factor $\mathcal{R}$ are trivial, is central to this thesis. Ozawa's Triviality Theorem is presented, and property Γ and the McDuff property for W*-bundles are investigated thoroughly. Ozawa's Triviality Theorem is applied to some new examples such as the strict closures of Villadsen algebras and non-trivial $C(X)$-algebras. The solution to the Triviality Problem in the locally trivial case, obtained by myself and Pennig, is included.

A theory of sub-W*-bundles is developed along the lines of Jones' subfactor theory. A sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ encapsulates a tracially continuous family of subfactors in a single object. The basic construction and the Jones tower are generalised to this new setting and the first examples of sub-W*-bundles are constructed.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Contents

Chapter 1

Introduction

Operator algebras are *-subalgebras of $B(H)$, the *-algebra of bounded linear operators on a Hilbert space H with involution given by adjoints. C*-algebras arise if one considers uniformly-closed *-subalgebras, von Neumann algebras if one considers pointwise-closed *-subalgebras. Via spectral theory, commutative C*-algebras with identity are isomorphic to the algebras $C(X)$ of continuous functions on some compact Hausdorff space X whereas von Neumann algebra correspond to $L^\infty(X)$ for some measure space X. Accordingly, the study of C*- and von Neumann algebras are often called *non-commutative topology* and *non-commutative measure theory* respectively.

The key objects of study in this thesis, W*-bundles, transcend the divide between von Neumann algebras and C*-algebras, behaving locally like the former and globally like the later. This thesis collates, extends and applies the abstract theory of W*-bundles. But first, we need to set the scene.

W*-bundles were introduced by Ozawa in [62], motivated by recent progress in the classification programme for C*-algebra. This background, although not logically necessary for understanding the thesis, has been included for completeness, for motivation, and to underline the importance of W*-bundles.

Towards W*-bundles

Questions of structure and of classification are central to the study of operator algebras. First considered in the von Neumann setting, this goes back to the original papers of Murray and von Neumann [59]. However, the full picture only emerged after the Fields Medal winning work of Connes in the 1970's. In a ground-breaking combination of deep insight and technical expertise, Connes was able to establish the uniqueness of the amenable II_1

factor $\mathcal{R}$ [11]. Subsequently, he was able to classify almost all amenable factors with the one remaining case resolved later by Haagerup [33].

The following theorem of 2015 is the culmination of a 40 year endeavour to develop a classification result for amenable, simple C*-algebras analogous to the Connes–Haagerup classification for amenable factors. This programme was spearheaded by the work and insight of Elliott [20], and the final result encompasses and extends both the celebrated classification theorem of Kirchberg–Philips [46] and the classification of the inductive limit algebras considered by Elliott–Gong–Li [21].

Theorem 1.1. *[22, 28, 84] The class of unital, simple, separable, infinite-dimensional C*-algebras of finite nuclear dimension which satisfy the universal coefficient theorem is classified by K-theory and traces.*

Of great interest here is that, in the C*-setting, mere amenability does not suffice, as counterexamples of Rørdam [72] and Toms [87] lay bare. The stronger hypothesis that the nuclear dimension, a non-commutative extension of covering dimension developed by Winter and Zacharias [97], is finite must be shown before the classification result can be applied.

This leads to the problem of establishing finite nuclear dimension in natural examples. Powerful but ad hoc methods exist in certain special cases. A more abstract approach to this problem is offered by the Toms–Winter Conjecture [89], which predicts that the dichotomy between finite and infinite nuclear dimension is also witnessed by other very different properties.

Conjecture 1.2 (The Toms–Winter Conjecture)**.** *Let A be a unital, simple, separable, amenable, infinite-dimensional C*-algebra. The following are equivalent:*

(i) A has finite nuclear dimension;

(ii) A absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially, i.e. $A \otimes \mathcal{Z} \cong A$;

(iii) A has strict comparison.

Tensorial absorption properties, like condition (ii) of the Toms–Winter Conjecture, have a long history in the theory of operator algebras. This goes back to the work of McDuff in the von Neumann setting, who considered II_1 factors that absorb the hyperfinite II_1 factor $\mathcal{R}$ tensorially [55,56]. Tensorial absorption of the Cuntz algebra $\mathcal{O}_\infty$, is fundamental in the proof of the Kirchberg–Phillips Theorem and in the study of non-simple purely infinite C*-algebras [46].

The Jiang–Su algebra $\mathcal{Z}$ arrived on the scene at the turn of the millennium, and can be viewed as a counterexample to early forms of the Elliott conjecture because it has the same K-theory and traces as the complex numbers yet is not isomorphic to them.[1] Tensorial absorption of the Jiang–Su algebra is intimately related to K-theoretic classification because, under weak hypotheses on the C*-algebra A, the K-theory and traces of A and $A \otimes \mathcal{Z}$ will be identical [37].

The third condition in the Toms–Winter Conjecture also has a genealogy dating back to the foundation of the field, more precisely to Murray and von Neumann's discovery that the order on equivalence classes of projections in a II_1 factor is completely determined by the trace [59]. Whilst for finite von Neumann algebras it is always true that traces determine the order of projections, the analogue in the C*-setting can fail, leading to the regularity property of strict comparison.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of the Toms–Winter Conjecture were proven by Winter and Rørdam respectively [73, 95]. It is the recent progress on the reverse implications, which has lead to the theory of W*-bundles.

The breakthrough came in 2012 when Matui and Sato managed to prove (iii) $\Rightarrow$ (ii) in the case where A has finitely many extreme traces [52]. The key idea was to consider the von Neumann algebras $\pi_\tau(A)''$ coming from the GNS representations of A with respect to extreme traces. The hypotheses on A suffice to show that $\pi_\tau(A)''$ is an injective II_1 factor with separable predual, so $\pi_\tau(A)'' \cong \mathcal{R}$ by Connes' Theorem [11]. Matui and Sato then developed powerful techniques to lift structural properties from $\mathcal{R}$, which is well understood, to the C*-algebra A.

The implication (ii) $\Rightarrow$ (i) was shown to hold when A has a unique trace by Sato, White and Winter in [78], building on the work of Matui and Sato [53]. Once again, von Neumann algebras and von Neumann algebraic ideas were fundamental, in particular the fact that the von Neumann algebra $\pi_\tau(A)''$ coming from the GNS representation of A with respect to the unique trace is isomorphic to $\mathcal{R}$ by Connes' Theorem.

The assumption of a unique trace, or even that of finitely many extreme traces, is a strong one. In general, the trace space $T(A)$ of a C*-algebra A, when non-empty, is a

[1] Note the assumption *infinite-dimensional* in Theorem 1.1: the Jiang-Su algebra is infinite-dimensional, the complex numbers are not.

Choquet simplex, a compact convex set where every point can be uniquely represented as the barycentre of a probability measure concentrated on the set of extreme points $\partial_e T(A)$. Moreover, all metrisable Choquet simplices occur as the the trace space of some simple, unital, approximately finite-dimensional (AF) C*-algebra [29].

The concept of a W*-bundle has its roots in the efforts to prove that (iii) $\Rightarrow$ (ii) in the Toms–Winter Conjecture for more general trace simplices. The basic idea was to view an element a of the C*-algebra A as a section of a bundle-like object over the space of extreme traces $\partial_e T(A)$ with $a(\tau) := \pi_\tau(a)$. This viewpoint allowed (iii) $\Rightarrow$ (ii) to be proven in the case that $\partial_e T(A)$ was compact and of finite covering dimension [47, 77, 88].

Ozawa then formalised this intuition in [62]. His crucial insight was to consider not the C*-algebra A but a certain tracial completion $\overline{A}^{\mathrm{st}}$. Now the fibres of the bundle were precisely the von Neumann algebras $\pi_\tau(A)''$ and so all isomorphic to $\mathcal{R}$ in the setting of the Toms–Winter Conjecture. Ozawa then noticed that the tracial completions $\overline{A}^{\mathrm{st}}$ could be studied from an axiomatic view point, the key axiom being a form of completeness. In short, he defined W*-bundles.

The conceptual framework of W*-bundles bore fruit in [5], where it was proven that (ii) $\Rightarrow$ (i) in the Toms–Winter Conjecture whenever $\partial_e T(A)$ is compact. This paper also contributed to the abstract theory of W*-bundles, introducing morphisms, tensor products and ultrapowers of W*-bundles for the first time.

In the language of W*-bundles, the role of finite covering dimension in [47, 77, 88] also becomes clearer. When the base space X of a W*-bundle with fibres $\mathcal{R}$ has finite covering dimension, the bundle is in fact a trivial bundle. This trivialisation can be viewed as a global version of Connes' Theorem, ensuring that the isomorphisms $\pi_\tau(A)'' \cong \mathcal{R}$ for each $\tau \in \partial_e T(A)$ can be chosen in a consistent manner. In [62], Ozawa asks whether all W*-bundles with fibres $\mathcal{R}$ are trivial. The locally trivial case was answered positively in [23] by myself and Pennig; the general case remains elusive.

Thesis Structure

This thesis is structured as follows. After recalling some necessary preliminary results in Chapter 2, we set out the abstract theory of W*-bundles in Chapter 3, building on the work of [62] and [5], but also filling in details not contained in these papers such as the standard form of a W*-bundle and the theory of completions, ideals and quotients. In Chapter 3, we also develop an alternative topological viewpoint for W*-bundles based on the notion of Banach bundles from [26], expanding on the viewpoint set out in my joint paper with Pennig [23].

Chapter 4 is devoted to the triviality problem for W*-bundles. We present Ozawa's Triviality Theorem [62, Theorem 15] and investigate property Γ and the McDuff property for W*-bundles. We discuss the known applications of Ozawa's Triviality Theorem to finite-dimensional base spaces and $\mathcal{Z}$-stable C*-algebras, before applying it to some new examples such as the strict closures of Villadsen algebras and non-trivial $C(X)$-algebras. Chapter 4 ends with a discussion of the locally trivial case, which me and Pennig solved in [23].

In Chapter 5, we leave the familiar territory of the classification programme for C*-algebras, in which W*-bundle theory was born, in favour of the exciting world of subfactor theory. After recalling the necessary background material in Section 5.1, we develop the theory of sub-W*-bundles following the path trodden by Jones for subfactors in his seminal paper [38].

Chapter 2

Preliminaries

The purpose of this chapter is to set notation and introduce some of the axillary concepts that will be used in the course of the thesis. We assume that the reader has a solid grasp of the fundamentals of general topology [44, 57] and functional analysis [75, 79, 98]. In particular, we assume the reader is has experience with topologies defined on vector spaces by a norm or a family of seminorms, and they have studied the theory of Banach and Hilbert spaces in some detail.

We also assume the reader has some familiarity with the theory of operator algebras (for example [58, Chapters 1-6]). The material is this chapter should be viewed as supplementary, not a self-contained introduction to the field.

2.1 Completeness in Topological Vector Spaces

Completeness is a fundamental concept in analysis. We assume the reader is familiar with the definitions of completeness for metric spaces and, in particular, normed vector spaces. However, the existence of a metric is not essential in order to define completeness. The correct notion is that of a uniform space, which lies somewhere between that of a metric space and a topological space. We shall suppress an in depth discussion of uniform spaces in general, referring the interested reader to [44, Chapter 6], and focus instead on the special case of subsets of a topological vector space.

Fix a topological vector space V. For any $x_0 \in V$, the translation $x \mapsto x + x_0$ is a homeomorphism mapping 0 to x_0. This allows one to use the open neighbourhoods of 0, to define a uniform notion of size in the topological vector space. This can be formalised using the language of uniform spaces [44, Chapter 6] or be taken as motivation for the following definition.

Definition 2.1.1. Let A be a subset of a topological vector space V. A net $(x_\lambda)_{\lambda\in\Lambda}$ in A is *Cauchy* if, for any open neighbourhood U of 0 in V, there exists $\lambda_0 \in \Lambda$ such that $x_{\lambda_1} - x_{\lambda_2} \in U$ whenever $\lambda_1, \lambda_2 \geq \lambda_0$.

This definition is easily seen to be equivalent to the requirement that $x_\lambda - x_\mu \to 0$ as $(\lambda, \mu) \to \infty$, where the net is indexed by the directed set $\Lambda \times \Lambda$. The definition of completeness can now be given.

Definition 2.1.2. The subset A of a topological vector space V is *complete* if all Cauchy nets in A converge in A.

If the topological vector space V is first countable, then there is no loss of generality if just Cauchy sequences are considered. The definition above, therefore, extends the standard definition of completeness for subsets of a normed vector space.

2.2 The Strong Operator Topology

Let V be a normed vector space and $B(V)$ denote the normed algebra of bounded operators on V. We recall that the *strong operator topology* on $B(V)$ is the topology induced by the family of seminorms $\{\|\cdot\|_v : v \in V\}$, where $\|T\|_v = \|Tv\|$. The strong operator topology is weaker than the topology induced by the operator norm.

It is well-known that when V is complete, then $B(V)$ is complete in operator norm. The following completeness result is less well-known, perhaps because it requires the more general notion of completeness discussed in Section 2.1.

Theorem 2.2.1. *Let V be a Banach space. The unit ball of $B(V)$ is complete with respect to the the strong operator topology.*

Proof. Let $(T_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in the unit ball of $B(V)$ with respect to the the strong operator topology. Then for all $v \in V$, $(T_\lambda v)_{\lambda\in\Lambda}$ is a Cauchy net in V, so has a limit. Define $T : V \to V$ by $v \mapsto \lim_\lambda T_\lambda v$.

The linearity of each T_λ ensures the linearity of T. Since $\|T_\lambda\| \leq 1$ for all $\lambda \in \Lambda$, we have $\|T_\lambda v\| \leq \|v\|$ for all $v \in V$ and $\lambda \in \Lambda$. Hence, $\|Tv\| \leq \|v\|$ for all $v \in V$, and so $\|T\| \leq 1$. Thus, T is in the unit ball of $B(V)$. By construction, $T_\lambda \to T$ in the strong operator topology. $\square$

From the completeness of the unit ball, the completeness of all norm-bounded sets which are closed in the strong operator topology follows. Since we shall often be working with the strong operator topology restricted to norm-bounded sets, we record the following well-known result.

Proposition 2.2.2. *Let V be a normed vector space and let A be a bounded subset of $B(V)$. Suppose the span of S is dense in V. Then the strong operator topology on A is induced by the family of seminorms $\{\|\cdot\|_v : v \in S\}$*

Proof. We have $\|T\|_{\lambda v+\mu w} = \|T(\lambda v + \mu w)\| = \|\lambda T v + \mu T w\| \leq |\lambda|\|T\|_v + |\mu|\|T\|_w$. So the topologies induces by the families of seminnorms $\{\|\cdot\|_v : v \in S\}$ and $\{\|\cdot\|_v : v \in \text{span}(S)\}$ are the same. Without loss of generality, we may, therefore, assume that S is a dense subspace of V.

Suppose $(T\lambda)_{\lambda\in\Lambda}$ is a net in A, $T \in A$ and $\|T_\lambda - T\|_v \to 0$ for all $v \in S$. Since A is bounded there is $M > 0$ such that $\|T_\lambda\| \leq M$ and $\|T\| \leq M$. Let $v \in V$ and $\epsilon > 0$. There is $u \in S$ such that $\|u - v\| < \frac{\epsilon}{3M}$ and $\lambda_0 \in \Lambda$ such that $\|T_\lambda - T\|_u < \frac{\epsilon}{3}$ whenever $\lambda \geq \lambda_0$. We then have

$$\begin{aligned}
\|T_\lambda - T\|_v &= \|T_\lambda v - Tv\| && (2.2.1)\\
&= \|T_\lambda v - T_\lambda u\| + \|T_\lambda u - Tu\| + \|Tu - Tv\| && (2.2.2)\\
&\leq \|T_\lambda\|\|v - u\| + \|T_\lambda - T\|_u + \|T\|\|u - v\| && (2.2.3)\\
&< M\frac{\epsilon}{3M} + \frac{\epsilon}{3} + M\frac{\epsilon}{3M} && (2.2.4)\\
&= \epsilon && (2.2.5)
\end{aligned}$$

whenever $\lambda \geq \lambda_0$. So $T_\lambda \to T$ in the strong operator topology. Since the convergence of nets determines the topology, this completes the proof. □

In fact more is true. The family of seminorms $\{\|\cdot\|_v : v \in S\}$ induces the same uniform structure on A as $\{\|\cdot\|_v : v \in V\}$. Since we are only interested in Cauchy sequences and completeness, we just record the following.

Corollary 2.2.3. *Let V be a normed space and let A be a bounded subset of $B(V)$. Suppose the span of S is dense in V. A net $(T\lambda)_{\lambda\in\Lambda}$ in A is Cauchy if and only if $\|T_\lambda - T_\mu\|_v \to 0$ as $(\lambda, \mu) \to \infty$ for all $v \in S$.*

Proof. The set of differences $A - A$ is also bounded. So we can apply the proposition above to the net $(T_\lambda - T_\mu)_{(\lambda,\mu)\in\Lambda\times\Lambda}$. □

2.3 Topologies on $B(H)$

In this thesis, all Hilbert space will be complex and the inner product will be taken to be linear in the first entry. We shall denote the C*-algebra of bounded operators on a Hilbert space H by $B(H)$.[1] In preparation for our discussion of von Neumann algebras, we record here the additional topologies on $B(H)$ and the preferred names used in this thesis.

Definition 2.3.1. Let H be a Hilbert space. Write $H^{(\infty)} = \bigoplus_{i\in\mathbb{N}} H$ and $T^{(\infty)} \in B(H^{(\infty)})$ for the infinite diagonal inflation of $T \in B(H)$. We make the following definitions:

- The *weak (operator) topology* on $B(H)$ is induced by the family of seminorms

$$\|T\|_{v,w} := |\langle Tv, w\rangle| \qquad (v, w \in H). \tag{2.3.1}$$

- The *strong (operator) topology* on $B(H)$ is induced by the family of seminorms

$$\|T\|_{v} := \|Tv\| \qquad (v \in H). \tag{2.3.2}$$

- The *strong* topology* on $B(H)$ is induced by the family of seminorms

$$\|T\|_{v} := \|Tv\| \qquad (v \in H), \tag{2.3.3}$$

$$\|T\|_{v,*} := \|T^*v\| \qquad (v \in H). \tag{2.3.4}$$

- The *ultraweak topology* on $B(H)$ (also called σ-weak in some references) is induced by the family of seminorms

$$\|T\|_{v,w,\infty} := |\langle T^{(\infty)}v, w\rangle| \qquad (v, w \in H^{(\infty)}). \tag{2.3.5}$$

- The *ultrastrong topology* on $B(H)$ (also called σ-strong in some references) is induced by the family of seminorms

$$\|T\|_{v,\infty} := \|T^{(\infty)}v\| \qquad (v \in H^{(\infty)}). \tag{2.3.6}$$

- The *ultrastrong* topology* on $B(H)$ (also called σ-strong* in some references) is induced by the family of seminorms

$$\|T\|_{v,\infty} := \|T^{(\infty)}v\| \qquad (v \in H^{(\infty)}), \tag{2.3.7}$$

$$\|T\|_{v,*,\infty} := \|{T^{(\infty)}}^*v\| \qquad (v \in H^{(\infty)}). \tag{2.3.8}$$

[1] On occasion, we will also write $\mathcal{L}(H)$ for the space of bounded operators for compatability with the notation for Hilbert modules (see Section 2.11).

All the topologies defined in Definition 2.3.1 are weaker than the norm topology. The relationships between the topologies are set out in the following diagram

$$\begin{array}{ccccc} \text{ultraweak} & < & \text{ultrastrong} & < & \text{ultrastrong}^* \\ \vee & & \vee & & \vee \\ \text{weak} & < & \text{strong} & < & \text{strong}^*, \end{array} \tag{2.3.9}$$

where $<$ means weaker than. When restricted to norm-bounded subsets of $B(H)$, the weak, strong and strong* topologies agree with the ultraweak, ultrastrong and ultrastrong* topologies respectively. For a detailed examination of the properties of the topologies defined in Definition 2.3.1, the reader is referred to [82, Section II.2] or [4, Section I.3]. We record only the following corollary of Theorem 2.2.1, which is of particular importance to this thesis.

Corollary 2.3.2. *Let H be a Hilbert space. The unit ball of $B(H)$ is complete with respect to the strong operator topology, the strong* topology, the ultrastrong topology, and the ultrastrong* topology.*

Proof. On bounded sets, the ultrastrong topology agrees with the strong operator topology and the ultrastrong* topology agrees with the strong* topology. Thus, we only need to deduce the result for the strong*-topology.

Let $(T_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in the unit ball of $B(H)$ with respect to the the strong* topology. Apply Theorem 2.2.1 to both $(T_\lambda)_{\lambda\in\Lambda}$ and $(T_\lambda^*)_{\lambda\in\Lambda}$, and denote the limits of the these nets in the strong operator topology T and S respectively. Since $\langle T_\lambda v, w\rangle = \langle v, T_\lambda^* w\rangle$ for all $v, w \in H$ and $\lambda \in \Lambda$, we have, after taking limits, $\langle Tv, w\rangle = \langle v, Sw\rangle$ for all $v, w \in H$. Hence $S = T^*$ and $(T_\lambda)_{\lambda\in\Lambda}$ converges to T in the strong* topology. □

Remark 2.3.3. From the completeness of the unit ball, the completeness of all norm-bounded, sets closed in the topology in question follows. The analogous results for the weak operator topology and ultraweak topology also hold [4, Section I.3.2.2].

2.4 C*-Algebras

We assume the reader is familiar the basics of C*-algebras including the theory of inductive limits and tensor products [58, Chapters 1-6]. All C*-algebras in this thesis are complex.

We shall denote the self-adjoint elements of a C*-algebra A by A_{sa} and the positive elements by A_+.

One basic fact about C*-algebras that will be used particularly often is that injective *-homomorphisms between C*-algebras are necessarily isometric [58, Theorem 3.14]. Also worthy of particular emphasis is the following lifting result, which can be found in [74, Section 2.2.10].

Proposition 2.4.1. *Let A, B be C*-algebras and $\phi : A \to B$ be a surjective *-homomorphism. Given $b \in B$, call any $a \in A$ with $\phi(a) = b$ a lift of b.*

(i) Any $b \in B$ has a lift $a \in A$ with $\|a\| = \|b\|$.

(ii) Any $b \in B_{\mathrm{sa}}$ has a lift $a \in A_{\mathrm{sa}}$ with $\|a\| = \|b\|$.

(iii) Any $b \in B_+$ has a lift $a \in A_+$ with $\|a\| = \|b\|$.

Although fundamental to the classification programme for C*-algebras, nuclearity will only play a minor role in this thesis. The following brief overview will suffice.

A C*-algebra A is said to be *nuclear* if there is a unique C*-norm on the algebraic tensor product $A \otimes_{\mathrm{alg}} B$, and so a unique C*–tensor product $A \otimes B$, for all C*-algebras B. Nuclearity can equivalently be viewed as a form of amenability [12, 32] or a finite dimensional approximation property [9, 45]. The class of nuclear C*-algebras includes the finite-dimensional C*-algebras and the commutative C*-algebras. Moreover, the class of nuclear C*-algebras is closed under tensor products, inductive limits, and extensions. For further details on nuclearity, the reader is referred to [6, Chapters 2-3].

2.5 Completely Positive Maps

The most fundamental class of maps between C*-algebras are the *-homomorphisms. However, for some applications, *-homomorphisms are too restrictive to be of use, and we turn to the larger class of *completely positive* maps. In this section, we recall the basic results about complete positive maps that will be used in this thesis. For further information on complete positive maps, the reader should consult [64].

Definition 2.5.1. A bounded linear map $\phi : A \to B$ between C*-algebras is said to be *positive* if $\phi(A_+) \subseteq B_+$. We say ϕ is *completely positive* if the map $\phi^{(n)} : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ given by $(a_{ij}) \mapsto (\phi(a_{ij}))$ is positive for all $n \in \mathbb{N}$.

The first examples of completely positive maps are the *-homomorphisms. Indeed, any *-homomorphism $\phi : A \to B$ is positive because $\phi(a^*a) = \phi(a)^*\phi(a)$. Moreover, if $\phi : A \to B$ is a *-homomorphism, then so is $\phi^{(n)} : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ for every $n \in \mathbb{N}$. Hence, ϕ is completely positive.

A second family of completely positive maps is the following: let A be a C*-algebra, $b \in A$ and define $\phi_b : A \to A$ by $a \mapsto b^*ab$. The map ϕ_b is positive because $\phi_b(a^*a) = (ab)^*(ab)$. Moreover, for all $n \in \mathbb{N}$, we have $\phi_b^{(n)}((a_{ij})) = (b^*a_{ij}b) = (b\delta_{ij})^*(a_{ij})(b\delta_{ij})$, where $(a_{ij}) \in \mathbb{M}_n(A)$ and δ_{ij} is the Kronecker delta. Thus, $\phi_b^{(n)}$ is a map of the same form as ϕ_b. Therefore, ϕ_b is completely positive.

Finally, the class of completely positive maps is closed under positive linear combinations and point norm limits because the set of positive elements in a C*-algebra is a closed cone.

Since checking positivity of all matrix inflations of a maps, can be quite time consuming and notationally messy, the following proposition is somewhat convenient.

Proposition 2.5.2. *Let A, B be C*-algebras with B commutative. Then a map $\phi : A \to B$ is completely positive if and only if it is positive.*

Proof. Let $\phi : A \to B$ be positive. Without loss of generality, $B = C_0(X)$ for some locally compact Hausdorff space X. Then $M_n(B) = C_0(X, \mathbb{M}_n(\mathbb{C}))$. Since $f \in C_0(X, \mathbb{M}_n(\mathbb{C}))$ is positive if and only if $f(x)$ is positive for all $x \in X$, we get that ϕ is completely positive if and only if $\mathrm{eval}_x \circ \phi$ is completely positive for all $x \in X$.

Hence, we have reduced the problem to the case $B = \mathbb{C}$. Let $\xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{C}^n$ and $a = (a_{ij}) \in \mathbb{M}_n(A)_+$. Then

$$\langle \phi^{(n)}(a)\xi, \xi\rangle = \sum_{1\le i,j\le n} \overline{\xi_i}\phi(a_{ij})\xi_j \tag{2.5.1}$$

$$= \phi(\sum_{1\le i,j\le n} \overline{\xi_i}a_{ij}\xi_j) \tag{2.5.2}$$

$$= \phi(\xi^*a\xi) \tag{2.5.3}$$

$$\geq 0. \tag{2.5.4}$$

Therefore $\phi^{(n)}(a) \in \mathbb{M}_n(\mathbb{C})_+$. Hence, ϕ is completely positive. □

The reader should be warned that Proposition 2.5.2, fails dramatically without the commutativity assumption. Indeed, the transpose map $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is an example of map that is positive but not completely positive (See [6, Proposition 3.5.1]).

We now come to Stinespring's Theorem, which describes the structure of a general completely positive map.

Theorem 2.5.3. *[6, Theorem 1.5.3] Let A be a unital C^*-algebra and $\phi : A \to B \subseteq B(H)$ be a completely positive map. Then there exists a Hilbert space $\widehat{H}$, a *-homomorphism $\pi : A \to B(\widehat{H})$ and an operator $V \in B(H, \widehat{H})$ such that*

$$\phi(a) = V^*\pi(a)V \qquad (a \in A). \tag{2.5.5}$$

In particular, $\|\phi\| = \|V^*V\| = \|\phi(1)\|$.

Stinespring's Theorem is proved by a generalisation of the GNS construction (see for example [6, Theorem 1.5.3] or [64, Theorem 4.1]). An important corollary of Stinespring's Theorem is the following inequality.

Corollary 2.5.4. *Let A be a unital C^*-algebra and $\phi : A \to B \subseteq B(H)$ be a completely positive map. Then*

$$\phi(a)^*\phi(a) \leq \phi(a^*a) \qquad (a \in A). \tag{2.5.6}$$

Proof. In the notation of Theorem 2.5.3, we have

$$\phi(a^*a) - \phi(a)^*\phi(a) = V^*\pi(a)^*(1_{\widehat{H}} - VV^*)\pi(a)V \geq 0 \tag{2.5.7}$$

for all $a \in A$. □

Notation and Terminology 2.5.5. Following common practice, we introduce the following abbreviations to describe maps between C*-algebras: cp for completely positive, cpc for completely positive and contractive, and ucp for unital and completely positive. It follows from Theorem 2.5.3 that ucp maps are automatically cpc.

An important result on cpc maps, which will be used on a number of occasions in this thesis, is the following lifting theorem of Choi and Effros [8, Theorem 3.10] (see also [6, Theorem B.3]).

Theorem 2.5.6. *Let A, B, C be C^*-algebras with A nuclear and let $q : B \to C$ be a surjective *-homomorphism. Given a cpc map $\phi : A \to C$, there exists a cpc map $\Phi : A \to B$ such that $\phi = q \circ \Phi$.*

We now consider a special case of completely positive maps: the conditional expectations.

Definition 2.5.7. Let $B \subseteq A$ be C*-algebras. A *conditional expectation*[2] of A onto B is a completely positive contractive map $E : A \to B$ such that

$$E(b) = b \qquad (b \in B), \tag{2.5.8}$$

$$E(bab') = bE(a)b' \qquad (a \in A; b, b' \in B). \tag{2.5.9}$$

The following theorem of Tomiyama is very convenient when determining if a map is a conditional expectation. This result goes back to [85, Theorem 1]. The version stated below is from [6, Theorem 1.5.9], where a proof can also be found.

Theorem 2.5.8. *Let $B \subseteq A$ be C*-algebras and $E : A \to B$ be a linear map satisfying $E(b) = b$ for all $b \in B$. Then the following are equivalent:*

(i) E is a conditional expectation.

(ii) E is completely positive and contractive.

(iii) E is contractive.

Another important class of completely positive maps are the completely positive order zero maps. These maps were first introduced by Winter [94, Definition 2.1] in the special case that the domain is finite dimensional. The general case of completely positive order zero maps was investigated in [96].

Definition 2.5.9. A completely positive map $\phi : A \to B$ between C*-algebras is said to be *order zero* if, for all positive elements $a, b \in A_+$, $\phi(ab) = 0$ whenever $ab = 0$.

In [96], Winter and Zacharias prove a structure theorem for completely positive order zero maps [96, Theorem 3.3] and deduce the following result as a corollary [96, Corollary 4.1].

Theorem 2.5.10. *Let A, B be C*-algebras and $\phi : A \to B$ be a cpc order zero map. Then the map given by $\rho_\phi(\mathrm{id}_{(0,1]} \otimes a) := \phi(a)$ induces a *-homomorphism $\rho_\phi : C_0(0,1] \otimes A \to B$. Conversely, any *-homomorphism $\rho : C_0(0,1] \otimes A \to B$ induces a cpc order zero map $\phi_\rho : A \to B$ via $\phi(a) := \rho(\mathrm{id}_{(0,1]} \otimes a)$. These mutual assignments yield a canonical bijection between the spaces of cpc order zero maps from A to B and *-homomorphisms from $C_0(0,1] \otimes A$ to B.*

[2]The name conditional expectation is motivated by the case where $A = L^\infty(\Omega, \mathcal{F}, \mu)$ and $B = L^\infty(\Omega, \mathcal{G}, \mu)$ for some probability triple $(\omega, \mathcal{F}, \mu)$ and $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$. The existence of an expectation preserving conditional expectation $A \to B$ is basic to advanced probability theory

The main use of order zero maps in this thesis is the following lifting theorem, which is taken from [88, Lemma 2.1]. It is proved by combining Theorem 2.5.10 with Loring's result on the projectivity of cones over finite dimensional C*-algebras [51].

Proposition 2.5.11. *Let A, B, F be C*-algebras with F finite dimensional, and let $q : A \to B$ be a surjective *-homomorphism. Suppose $\phi : F \to B$ is a cpc order zero map. Then there exists a cpc order zero map $\Phi : F \to A$ such that $\phi = q \circ \Phi$.*

2.6 Representations of C*-Algebras and the GNS construction

Every C*-algebra has a faithful representation on a Hilbert space [27, 80]. The proof of this fundamental result consists of two parts: firstly the Gelfand–Naimark–Segal (GNS) construction, which builds representations out of positive linear functionals; secondly, the existence of sufficiently many positive linear functionals, which is a Hahn–Banach argument.

Many aspects of the GNS construction will be used and generalised in the thesis. We therefore sketch the argument here to set up notation and record the constituent sub-results. To avoid some (minor) technicalities, we shall stick to unital C*-algebras and unital representations throughout. For a full treatment of the material in this section, see [14, Section I.9] or [58, Chapter 3].

2.6.1 States and the GNS construction

Fix a unital C*-algebra A.

Definition 2.6.1. A *representation* of A is a unital *-homomorphism $\pi : A \to B(H_\pi)$, where H_π is a Hilbert space. A representation is said to be *faithful* if it is injective. Two representations π_1, π_2 are *equivalent* if there is a unitary $U : H_{\pi_1} \to H_{\pi_2}$ such that $\pi_2(a) = U\pi_1(a)U^*$ for all $a \in A$.

The key to constructing representations of a C*-algebras is to consider positive linear functionals.

Definition 2.6.2. A *positive linear functional* on A is a bounded linear map $\phi : A \to \mathbb{C}$ such that $\phi(a) \geq 0$ for all $a \in A_+$ (c.f. Definition 2.5.1).

It follows from Proposition 2.5.2 that positive linear functions are, in fact, completely positive. We denote the set of all positive linear functionals on A by A^*_+. It is a weak*-closed cone in A^*.

Given a positive functional ϕ on A, we can define a positive sesquilinear form on A via $\langle a, b\rangle_\phi = \phi(b^*a)$. We set $\|a\|_{2,\phi} = \langle a, a\rangle^{1/2}$. We have the Cauchy–Schwarz inequality

$$|\langle a, b\rangle_\phi| \leq \|a\|_{2,\phi}\|b\|_{2,\phi} \qquad (a, b \in A), \tag{2.6.1}$$

from which it follows that $\|\cdot\|_{2,\phi}$ is a seminorm. A positive linear functional ϕ on A is said to be *faithful* if $\phi(a^*a) > 0$ for all non-zero $a \in A$, or equivalently when $\|\cdot\|_{2,\phi}$ is a norm.

Let $a \in A$ have $\|a\| \leq 1$. Taking $b = 1$ in (2.6.1), we get that

$$\begin{aligned} |\phi(a)| &\leq \phi(a^*a)^{1/2}\phi(1)^{1/2} && (2.6.2)\\ &\leq \|\phi\|^{1/2}\|a^*a\|\phi(1)^{1/2} && (2.6.3)\\ &\leq |\phi\|^{1/2}\phi(1)^{1/2}, && (2.6.4) \end{aligned}$$

from which we deduce that $\|\phi\| \leq \|\phi\|^{1/2}\phi(1)^{1/2}$. Thus, $\|\phi\| = \phi(1)$. This brings us to the definition of a state.

Definition 2.6.3. A *state* on A is a positive linear function ϕ with $\|\phi\| = \phi(1) = 1$.

The set of all states, denoted by $S(A)$, is a weak* closed, convex subset of the unit ball of A^*. By the Banach–Algolou Theorem, the unit ball of A^* is weak* compact. Hence, $S(A)$ is also weak* compact.

We now sketch the GNS construction which produces a representation π_ϕ from a positive linear functional ϕ. Let $L^2(A, \phi)$ denote the Hilbert space obtained from A by quotienting out by $N_\phi = \{b \in A : \|a\|_{2,\phi} = 0\}$ and completing in the resulting norm. We denote the natural map $A \to L^2(A, \phi)$ by $a \mapsto \widehat{a}$; we write ξ_ϕ for $\widehat{1}$. The key technical lemma is the following.

Lemma 2.6.4. *With the notation above* $\|ab\|_{2,\phi} \leq \|a\|\|b\|_{2,\phi}$.

Proof. Since $a^*a \leq \|a\|^2 1_A$ in A, we have

$$\begin{aligned} \|ab\|^2_{2,\phi} &= \phi(b^*a^*ab) && (2.6.5)\\ &\leq \phi(b^*\|a\|^2 b) && (2.6.6)\\ &= \|a\|^2\phi(b^*b) && (2.6.7)\\ &= \|a\|^2\|b\|^2_{2,\phi}. && (2.6.8) \end{aligned}$$

□

It follows that one can define bounded linear operators on $L^2(A, \phi)$ by $\phi(a)\widehat{b} = \widehat{ab}$, and the resulting map $\pi_\phi : A \to B(L^2(A, \phi))$ is a representation. The original positive linear functional is recovered via $\phi(a) = \langle \pi(a)\xi_\phi, \xi_\phi \rangle_{L^2(A,\phi)}$. Moreover, ξ_ϕ generates $L^2(A, \phi)$ as a (Banach)-A-module in the sense that $L^2(A, \phi) = \overline{\pi_\phi(A)\xi_\phi}$. This motivates the following definition.

Definition 2.6.5. A *cyclic representation* of A is a representation π together with a vector $\xi_\pi \in H_\pi$ such that $H_\pi = \overline{\pi(A)\xi_\pi}$. Two cylic representations π_1, π_2 are *equivalent* if there is a unitary $U : H_{\pi_1} \to H_{\pi_2}$ such that $U\xi_{\pi_1} = \xi_{\pi_2}$ and $\pi_2(a) = U\pi_1(a)U^*$ for all $a \in A$.

The GNS construction is now one direction of a 1-1 correspondence between equivalence classes of cyclic representations and positive linear functionals. The positive linear functional associated with a cyclic representation π of A is $a \mapsto \langle \pi(a)\xi_\pi, \xi_\pi \rangle_{H_\pi}$.

Theorem 2.6.6. *There is a 1-1 correspondence between equivalence classes of cyclic representations and positive linear functionals. If we require the cyclic vector ξ_π of a cyclic representation to be a unit vector, then we obtain a 1-1 correspondence with states.*

To construct a faithful representation of a unital C*-algebra A, one needs to prove the existence of sufficiently many states. The following definition and proposition make this precise.

Definition 2.6.7. A family of states $(\phi_i)_{i\in I}$ on a unital C*-algebra A is separating if, for all non-zero $a \in A$, there exists $i \in I$ such that $\phi_i(a^*a) > 0$.

Proposition 2.6.8. *Let A be unital C*-algebra and $(\phi_i)_{i\in I}$ a separating family of states. Then $\oplus_{i\in I}\pi_{\phi_i}$ is a faithful representation.*

Proof. Let $a \in A$. By hypothesis, there is $j \in I$ such that $\phi_j(a^*a) > 0$. Then

$$\|\pi_{\phi_j}(a)\xi_{\phi_j}\|^2 = \langle \pi_{\phi_j}(a)\xi_{\phi_j}, \pi_{\phi_j}(a)\xi_{\phi_j} \rangle \tag{2.6.9}$$

$$= \langle \pi_{\phi_j}(a)^*\pi_{\phi_j}(a)\xi_{\phi_j}, \xi_{\phi_j} \rangle \tag{2.6.10}$$

$$= \langle \pi_{\phi_j}(a^*a)\xi_{\phi_j}, \xi_{\phi_j} \rangle \tag{2.6.11}$$

$$= \phi_j(a^*a) > 0, \tag{2.6.12}$$

so $\pi_{\phi_j}(a) \neq 0$. Therefore, $\oplus_{i\in I}\pi_{\phi_i}(a) \neq 0$. □

When end this subsection with an existence theorem for states on a unital C*-algebra. For a proof, see for example [58, Theorem 3.3.6].

Theorem 2.6.9. *Let A be unital C*-algebra. For all $a \in A$, there exists a state ϕ with $\phi(a^*a) = \|a\|^2$. In particular, there exists a separating family of sates for A and A has a faithful representation on a Hilbert space.*

2.6.2 The GNS Construction for Traces

In this thesis, the traces on a C*-algebra will play a major role. Since terminology varies in the literature, we make the following definition. In particular, note that traces for us are normalised and defined everywhere, i.e. tracial states, unless otherwise stated.

Definition 2.6.10. A *trace* on a unital C*-algebra A is a positive linear functional $\tau : M \to \mathbb{C}$ such that

$$\tau(ab) = \tau(ba) \qquad (a, b \in M), \tag{2.6.13}$$

$$\tau(1_A) = 1. \tag{2.6.14}$$

We denote the set of all traces on a unital C*-algebra A by $T(A)$. We can have $T(A) = \emptyset$, but when this is not the case $T(A)$ is a weak*-closed, convex subset of $S(A)$, so is compact weak* compact. We now consider the GNS construction with respect to a trace τ. The corresponding seminorm $\|\cdot\|_{2,\tau}$ enjoys some additional properties compared with seminorms coming from a mere state.

Proposition 2.6.11. *The semi-norm $\|\cdot\|_{2,\tau}$ arising from a trace τ on a unital C*-algebra A satisfies the following for all $a, b \in A$:*

(i) $\|a\|_{2,\tau} \leq \|a\|$,

(ii) $|\tau(a)| \leq \|a\|_{2,\tau}$

(iii) $\|a^*\|_{2,\tau} = \|a\|_{2,\tau}$,

(iv) $\|ab\|_{2,\tau} \leq \|a\|\|b\|_{2,\tau}$,

(v) $\|ab\|_{2,\tau} \leq \|a\|_{2,\tau}\|b\|$.

Proof. (i) We have $\|a\|_{2,\tau}^2 = \tau(a^*a) \leq \|\tau\|\|a^*a\| = \|a\|^2$.

(ii) By Cauchy-Schwarz, $|\tau(a)| = |\tau(a.1_A)| \leq \|a\|_{2,\tau}\|1_A\|_{2,\tau} = \|a\|_{2,\tau}$.

(iii) We have $\|a^*\|_{2,\tau}^2 = \tau(aa^*) = \tau(a^*a) = \|a\|_{2,\tau}^2$.

(iv) This is Lemma 2.6.4.

(v) Using (iii) and (iv), we have $\|ab\|_{2,\tau} = \|b^*a^*\|_{2,\tau} \leq \|b^*\|\|a^*\|_{2,\tau} = \|a\|_{2,\tau}\|b\|$. □

Properties (ii) and (v) of Proposition 2.6.11 required that τ was a trace. They give rise to additional bounded operators on $L^2(A,\tau)$. Firstly, we can define a representation π_τ^{op} of A^{op} on $L^2(A,\tau)$ via the formula $\pi_\tau^{\mathrm{op}}(a)\widehat{b} = \widehat{ba}$ for $a,b \in A$. Secondly, the map $\widehat{a} \mapsto \widehat{a^*}$ extends to a conjugate-linear isometric involution J on $L^2(A,\tau)$. Simple calculations give

$$\pi_\tau(a)\pi_\tau^{\mathrm{op}}(b) = \pi_\tau^{\mathrm{op}}(b)\pi_\tau(a) \qquad (a,b \in A), \tag{2.6.15}$$

$$\pi_\tau^{\mathrm{op}}(a) = J\pi_\tau(a^*)J \qquad (a \in A). \tag{2.6.16}$$

We can also describe the kernel of the GNS representation with respect to a trace easily.

Proposition 2.6.12. *We have* $\mathrm{Ker}(\pi_\tau) = \mathrm{Ker}(\pi_\tau^{\mathrm{op}}) = \{a \in A : \|a\|_{2,\tau} = 0\}$. *In particular,* $I_\tau = \{a \in A : \|a\|_{2,\tau} = 0\}$ *is an ideal. The vector state on* $B(H_\tau)$ *defined by* $T \mapsto \langle T\xi_\tau, \xi_\tau\rangle$ *restricts to a faithful trace on* $\pi_\tau(A)$ *and* $\pi_\tau^{\mathrm{op}}(A)$.

Proof. It follows from Proposition 2.6.11(v) that $a \in I_\tau$ implies $\pi_\tau(a)\widehat{b} = 0$ for all $b \in A$. Hence, by density, $a \in \mathrm{Ker}(\pi_\tau)$. Conversely, if $a \in \mathrm{Ker}(\pi_\tau)$, then $\widehat{a} = \pi_\tau(a)\xi_\tau = 0$, so $a \in I_\tau$. Therefore, $\mathrm{Ker}(\pi_\tau) = I_\tau$. Similarly, this time using Proposition 2.6.11(iv), $\mathrm{Ker}(\pi_\tau^{\mathrm{op}}) = I_\tau$.

We have $\langle \pi_\tau(a)\xi_\tau, \xi_\tau\rangle = \tau(a)$, so the vector state $T \mapsto \langle T\xi_\tau, \xi_\tau\rangle$ restricts to a trace on $\pi_\tau(A)$. If $0 = \langle \pi_\tau(a)^*\pi_\tau(a)\xi_\tau, \xi_\tau\rangle$, then $\|a\|_{2,\tau}^2 = \tau(a^*a) = \langle \pi_\tau(a)^*\pi_\tau(a)\xi_\tau, \xi_\tau\rangle = 0$, so $a \in I_\tau$ and $\pi_\tau(a) = 0$. Hence, the trace on $\pi_\tau(A)$ is faithful. The equivalent result for $\pi_\tau^{\mathrm{op}}(A)$ is proved similarly. □

Corollary 2.6.13. *The map* $\overline{\tau} : A/I_\tau \to \mathbb{C}$ *given by* $a + I_\tau \mapsto \tau(a)$ *is a well-defined faithful trace on* A/I_τ

Proof. This result can be proved directly using 2.6.11(ii). Alternatively, one observes that $A/I_\tau \cong \pi_\tau(A)$ by the first isomorphism theorem and that $\overline{\tau}$ can be obtain by following this isomorphism with the vector state of Proposition 2.6.12. □

Corollary 2.6.14. *All traces on a simple* C^**-algebra are faithful.*

2.7 Von Neumann Algebras

We assume that the reader is familiar with the basics of the theory of von Neumann algebras [58, Chapter 4] and the construction of the hyperfinite II_1 factor $\mathcal{R}$ [4, Section III.3.1.4]. The goal of this section is to highlight the key results about von Neumann algebras that will be used in the thesis.

Central to the theory of von Neumann algebras are von Neumann's Bicommutant Theorem [92, Satz 8] and the Kaplansky Density Theorem [42, Theorem 1]. We record them in their strongest forms.[3]

Theorem 2.7.1 (The Bicommutant Theorem)**.** *Let H be a Hilbert space and M be a *-subalgebra of $B(H)$ containing the identity. The following are equivalent:*

(i) $M'' = M$,

(ii) *M is closed in all of the topologies of Definition 2.3.1,*

(iii) *M is closed in any of the topologies of Definition 2.3.1.*

Theorem 2.7.2 (The Kaplansky Density Theorem)**.** *Let H be a Hilbert space and M be a *-subalgebra of $B(H)$ containing the identity. Then the unit ball of M is a dense subset of the unit ball of M'' with respect to any of the topologies of Definition 2.3.1*

The ultraweak, ultrastrong and ultrastrong* topologies depend only on the *-algebraic structure of M [82, Corollary III.3.10]. The same can be said for weak, strong and strong* topologies, so long as one restricts to norm-bounded subsets, where they agree with the ultraweak, ultrastrong and ultrastrong* topologies respectively.

When working with maps between von Neumann algebras, one is particularly interested in maps that are continuous with respect to the additional topologies on von Neumann algebras. Fortunately, we have the following result.[4]

Theorem 2.7.3. *Let $\phi : M \to N$ be a completely positive map between von Neumann algebras. Let $\phi_1 : B_1(M) \to N$ denote the restriction of ϕ to the unit ball. The following are equivalent:*

---

[3] Some textbooks don't state theses theorems in their strongest forms. For a proof that $M'' = M$ whenever M is ultrastrong* closed (the weakest hypothesis), see [82, Theorem II.3.9]. For a proof that the unit ball of M is strong* dense in the unit ball of M'', see [82, Theorem II.408], and note that the ultrastrong* topology agrees with the strong* topology on bounded sets.

[4] This is essentially taken from [4, Section III.2.2]. Since cpc maps are *-preserving, the additional implications $(ii) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iii)$ are trivial. See also [82, Theorem II.2.6(iv)].

(i) ϕ *is continuous with respect to the ultraweak topologies on* M *and* N*,*

(i') ϕ_1 *continuous with respect to the weak topologies on* M *and* N*,*

(ii) ϕ *continuous with respect to the ultrastrong topologies on* M *and* N*,*

(ii') ϕ_1 *continuous with respect to the strong topologies on* M *and* N*,*

(iii) ϕ *is continuous with respect to the ultrastrong** *topologies on* M *and* N*,*

(iii') ϕ_1 *is continuous with respect to the strong** *topologies on* M *and* N*.*

A completely positive map between W*-algebras is said to be *normal* if any of the conditions of Theorem 2.7.3 hold. An important result about normal maps is that the image of a von Neumann algebra under a normal *-homomorphism is also von Neumann algebra. We record this well-known result as a theorem for ease of future reference, as we've been unable to find a precise reference.

Theorem 2.7.4. *Let* M *and* N *be von Neumann algebras and* $\phi : M \to N$ *a normal* **-homomorphism. Then* $\phi(M)$ *is a von Neumann algebra.*

Proof. It standard that the image of a C*-algebra under a *-homomrophism is a C*-algebra [4, Corollary II.5.1.2]. The only difficultly is in proving that the image is ultraweakly closed. Since unit ball $B_1(M)$ is ultraweakly compact [58, Theorem 4.2.4], $\phi(B_1(M))$ ultraweakly compact and thus ultraweakly closed. It now follows from the Kaplansky Density Theorem that $\phi(M)$ is ultraweakly closed and hence a von Neumann algebra. □

Although von Neumann algebras come with a defining representation, at times it is useful to consider alternative representations. The 1-1 correspondence of Theorem 2.6.6 is supplemented by the following. (See for example [4, Proposition III.2.2.3]).

Proposition 2.7.5. *A state on a von Neumann algebra is normal if and only if the corresponding GNS representation is normal.*

We also record the analogue of Theorem 2.6.9 for von Neumann algebra. (See for example [82, Theorem II.2.6(iii)]).

Theorem 2.7.6. *The normal states of a von Neumann algebra* M *are weak** *dense in the state space. In particular,* M *has a separating family of normal states.*

A separating family of faithful states $(\phi_i)_{i\in I}$ gives rise to a faithful normal representation $\oplus_{i\in I}\pi_{\phi_i}$. Moreover, it provides us with an alternative description of the strong topology on bounded subsets. The following result is taken from [4, Proposition III.2.2.19]. We include the proof for the benefit of the reader.

Proposition 2.7.7. *Let M be a von Neumann algebra and $(\phi_i)_{i\in I}$ a separating family of normal states. The strong topology on norm-bounded subsets of M is induced by the family of seminorms $\{\|\cdot\|_{2,\phi_i} : i \in I\}$.*

Proof. We may identify M with its image under the faithful normal representation $\oplus_{i\in I}\pi_{\phi_i}$ since the restriction of the strong topology to bounded subsets is a *-algebraic invariant.

The subset $\{\xi_{\phi_i} : i \in I\} \subseteq \oplus_{i\in I}H_{\phi_i}$ is separating for M. Therefore, the set $\{y\xi_{\phi_i} : y \in M', i \in I\}$ has dense span in H_{ϕ_i}. By Proposition 2.2.2, the strong operator topology on norm-bounded subsets of $B(\oplus_{i\in I}H_{\phi_i})$ is induced by the family of seminorms $\{\|\cdot\|_{y\xi_{\phi_i}} : y \in M', i \in I\}$, where $\|T\|_{y\xi_{\phi_i}} = \|T(y\xi_{\phi_i})\|$.

For $a \in M$,

$$\begin{aligned}
\|a\|_{y\xi_{\phi_i}} &= \|a(y\xi_{\phi_i})\| && (2.7.1)\\
&= \|ya\xi_{\phi_i})\| && (2.7.2)\\
&\leq \|y\|\|a\xi_{\phi_i})\| && (2.7.3)\\
&= \|y\|\|a\|_{\xi_{\phi_i}} && (2.7.4)\\
&= \|y\|\|a\|_{2,\phi_i}. && (2.7.5)
\end{aligned}$$

Therefore, on norm-bounded subsets of M the subfamily $\{\|\cdot\|_{2,\phi_i} : i \in I\}$ induces the same topology as $\{\|\cdot\|_{y\xi_{\phi_i}} : y \in M', i \in I\}$. □

2.8 Finite von Neumann Algebras

This purpose of this section is to recall the main results about finite von Neumann algebras and tracial von Neumann algebras. We begin with a brief overview of the theory of Murray–von Neumann equivalence for projections in order to define finiteness for von Neumann algebras and the finite part of a von Neumann algebra. The main reference for this section is [4, Chapter III]

2.8.1 Projections in von Neumann Algebras

In this subsection, we review the definitions of finite and properly infinite projections. The reader who is familiar with these definition can safely skip to the next subsection. The background material in this subsection is based on [4, Section III.1], where full proofs and further details can be found. The original source is [59, Chapters VI-VII].

Fix a von Neumann algebra M. By a *projection*, we mean a self-adjoint idempotent in M, i.e. $p \in M$ such that $p^2 = p = p^*$. Two projections p, q are *Murray–von Neumann equivalent*, denoted $p \sim q$, if there exists $v \in M$ such that $vv^* = p$ and $v^*v = q$. The projection p is *Murray–von Neumann subequivalent* to q, denoted $p \precsim q$, if there exists $v \in M$ such that $vv^* = p$ and $v^*v \leq q$, where $\leq$ denotes the standard partial order in a C*-algebra.

Theorem 2.8.1. *The relation $\sim$ is an equivalence relation and $\precsim$ induces a partial order on the equivalence classes.*

Projections p and q are said to be *orthogonal* if $pq = 0$. In this case, $p + q$ is also a projection. An indexed family $(p_i)_{i\in I}$ of projections is said to be *pairwise orthogonal* if $p_i p_j = 0$ whenever $i \neq j$. In this case, $\sum_{i\in I} p_i$ converges in the ultrastrong* topology and defines a projection. The relations $\sim$ and $\precsim$ interact well with addition of orthogonal projections.

Theorem 2.8.2. *Suppose $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ are two families of pairwise orthogonal projections with the same index set.*

(i) If $p_i \precsim q_i$ for all $i \in I$, then $\sum_{i\in I} p_i \precsim \sum_{i\in I} q_i$.

(ii) If $p_i \sim q_i$ for all $i \in I$, then $\sum_{i\in I} p_i \sim \sum_{i\in I} q_i$.

There is a notable analogy between the above and set theory. In this analogy, projections correspond to sets, Murray–von Neumann (sub)equivalence to (sub)equinumerosity of sets, and orthogonality to disjointness. Theorems 2.8.1 and 2.8.2 correspond to well-known results in set theory. In particular, Theorem 2.8.1 contains an analogue of the Cantor-Schröder-Bernstein Theorem: if $p \precsim q$ and $q \precsim p$, then $p \sim q$. The definition of finite and infinite projections can be viewed as analogous to that of Dedekind finite and infinite sets.

Definition 2.8.3. A projection p is *finite* if it is not Murray–von Neumann equivalent to any proper subprojection, i.e. if $q \leq p$ and $p \sim q$, then $p = q$.

The following result is fundamental but somewhat tricky to prove; see [4, Proposition III.1.3.9] or [59, Lemma 7.3.5].

Theorem 2.8.4. *If p and q are orthogonal finite projections, then $p+q$ is finite.*

We know turn our attention to infinite projections.

Definition 2.8.5. A projection p is *infinite* if it is Murray–von Neumann equivalent to a proper subprojection, i.e. there exists $q < p$ with $p \sim q$.

Definition 2.8.6. An projection p is *properly infinite* if there exist orthogonal projections $p_1, p_2 \leq p$ with $p_1 \sim p_2 \sim p$.

A non-zero, properly infinite projection is clearly infinite. According to the definitions given here, the zero projection is properly infinite despite being finite (and therefore not infinite)! This convention is common, but not universal, in the literature. The reason for this apparent disregard of the English language is the clean statement of Theorem 2.8.9 which it affords.

We end this subsection with a couple of examples.

Example 2.8.7. For $M = B(\ell^2)$, the finite projections are those with finite-dimensional range. The projections with infinite-dimensional range are both infinite and properly infinite.

Example 2.8.8. For $M = B(\ell^2) \oplus B(\ell^2)$, the projection (p, q) is infinite but not properly infinite when one of p and q has infinite-dimensional range and the other has finite-dimensional (but non-zero) range.

2.8.2 Finite von Neumann Algebras and Traces

A von Neumann algebra M is said to be *finite* if the unit 1_M is a finite projection. In fact, all projections in a finite von Neumann alegbra M are finite because, if $p \in M$ is infinite, so is $1_M = p + (1-p)$ by Theorem 2.8.2. Similarly, M is said to be *properly infinite* if 1_M is a properly infinite projection. A general von Neumann algebra has a canonical central decomposition into a finite part and a properly finite part (see for example [4, Section III.1.4.1]).

Theorem 2.8.9. *Let M be a von Neumann algebra. There exists a unique central projection $z_f \in Z(M)$ such that z_f is finite and $1 - z_f$ is properly infinite. This leads to a central decomposition $M = Mz_f \oplus M(1 - z_f)$.*

For the remainder of this subsection, we focus on finite von Neumann algebras. Finiteness of a von Neumann algebra is closely related to the existence of traces.

Theorem 2.8.10. *A von Neumann algebra M is finite if it has a separating family of traces.*

Proof. Suppose $(\tau_i)_{i\in I}$ is a separating family of traces and p is a projection in M with $1_M \sim p$. Say $vv^* = 1_M$ and $v^*v = p$. Then $\tau_i((1_M - p)^*(1_M - p)) = \tau_i(1_M - p) = \tau_i(vv^*) - \tau_i(v^*v) = 0$ for all $i \in I$. Hence, $p = 1_M$. □

The converse to Theorem 2.8.10 is also true, but is highly non elementary. It follows from the existence of the *centre-valued trace* [15]. The construction of the centre-valued trace is beyond the scope of this thesis. We collect the all properties of the centre valued trace that will be needed in the following theorem. The interested reader is referred to [4, Section III.2.5] for proofs and further details.

Theorem 2.8.11. *Let M be a finite von Neumann algebra. There exists a unique map* $\mathrm{ctr} : M \to Z(M)$ *with the following properties:*

(i) ctr *is a conditional expectation (in particular ucp).*

(ii) $\mathrm{ctr}(ab) = \mathrm{ctr}(ba)$ *for all $a, b \in M$.*

This map ctr *has the following additional properties:*

(iii) ctr *is normal.*

(iv) ctr *is faithful, i.e. for $a \in M$,* $\mathrm{ctr}(a^*a) = 0$ *only if $a = 0$.*

(v) Every trace on M has the form $\varphi \circ \mathrm{ctr}$ for a unique state φ on $Z(M)$.

(iv) The center-valued trace completely determines the Murray–von Neumann comparison theory of M, i.e. projections p, q are equivalent if and only if $\mathrm{ctr}(p) = \mathrm{ctr}(q)$ *and* $p \precsim q$ *if and only if* $\mathrm{ctr}(p) \leq \mathrm{ctr}(q)$.

Property (v) of the theorem, reduces the study of traces on a finite von Neumann algebra M to that of states of $Z(N)$. We therefore have the following corollary.

Corollary 2.8.12. *A finite von Neumann algebra has a separating family of normal traces. The normal traces are weak* dense in the space of all traces.*

Proof. This follows from Theorem 2.8.11(v) combined with Theorem 2.7.6. □

2.8.3 Tracial von Neumann Algebras

A *tracial von Neumann algebra* is a von Neumann algebra together with a faithful, normal trace. Tracial von Neumann algebras are necessarily finite by Theorem 2.8.10. In general, a finite von Neumann algebra need not have faithful, normal traces but any separably-acting, finite von Neumann algebra will (see for example [4, Corollary III.2.5.8]). Choosing one such trace, we can view a separably-acting, finite von Neumann algebra as a tracial von Neumann algebra.

In the case of II_1 factors, no arbitrary choices or separability assumptions need to be made because a II_1 factor has a unique trace, which is both faithful and normal [60]. Tracial von Neumann algebras also arise from C*-algebras by considering the GNS construction with respect to a trace. This is the subject of the following proposition, which uses the notation of Section 2.6.2.

Proposition 2.8.13. *Let τ be a trace on the unital C*-algebra A. The vector state on $B(H_\tau)$ defined by $T \mapsto \langle T\xi_\tau, \xi_\tau\rangle$ restricts to a faithful trace on $\pi_\tau(A)''$. Hence, the GNS closure $\pi_\tau(A)''$ is a tracial von Neumann algebra.*

Proof. Let ψ be the vector state on $B(H_\tau)$ given by $T \mapsto \langle T\xi_\tau, \xi_\tau\rangle$. By Proposition 2.6.12, the restriction of ψ to $\pi_\tau(A)$ is a faithful trace. By density, ψ is a trace on $\pi_\tau(A)''$. Normality is clear as ψ is a vector state. We need to show that it's faithful on $\pi_\tau(A)''$. Suppose $T \in \pi_\tau(A)''$ and $\psi(T^*T) = \langle T^*T\xi_\tau, \xi_\tau\rangle = 0$. Then $T\xi_\tau = 0$. Let $b \in A$. Then $T\widehat{b} = T\pi_\tau^{\mathrm{op}}(b)\xi_\tau = \pi_\tau^{\mathrm{op}}(b)T\xi_\tau = 0$, since $\pi_\tau^{\mathrm{op}}(A) \subseteq \pi_\tau(A)'$. Hence, by density, $T = 0$. □

Applying Proposition 2.8.13 to A^{op}, we see that $\pi^{\mathrm{op}}(A)''$ is also a tracial von Neumann algebra. Taking limits in (2.6.16), we see that these two tracial von Neumann algebras are related: $\pi^{\mathrm{op}}(A)'' = J\pi_\tau(A)''J$. Furthermore, taking limits in (2.6.15), we get that $\pi^{\mathrm{op}}(A)''$ and $\pi_\tau(A)''$ commute. In fact, they are commutants of one another. This important result can be proved in a number of ways. The slick proof given below is taken from [6, Section 6.1].

Lemma 2.8.14. *Let τ be a trace on the unital C*-algebra A. Using the notation of Section 2.6.2, we have*

(i) $\langle Jv, w\rangle = \langle Jw, v\rangle = \overline{\langle v, Jw\rangle}$ *for all* $v, w \in L^2(A, \tau)$,

(ii) $JT\xi_\tau = T^*\xi_\tau$ *for all* $T \in \pi_\tau(A)'$.

Proof. (i) Let $a, b \in A$. Then

$$\langle J\widehat{a}, \widehat{b}\rangle = \langle \widehat{a^*}, \widehat{b}\rangle \tag{2.8.1}$$
$$= \tau(b^*a^*) \tag{2.8.2}$$
$$= \tau(a^*b^*) \tag{2.8.3}$$
$$= \langle \widehat{b^*}, \widehat{a^*}\rangle \tag{2.8.4}$$
$$= \langle J\widehat{b}, \widehat{a}\rangle \tag{2.8.5}$$
$$= \overline{\langle \widehat{a}, J\widehat{b}\rangle}, \tag{2.8.6}$$

so the claim follows by density.

(ii) Let $T \in \pi_\tau(A)'$ and $a \in A$. Then

$$\langle JT\xi_\tau, \widehat{a}\rangle = \langle J\widehat{a}, T\xi_\tau\rangle \tag{2.8.7}$$
$$= \langle \widehat{a^*}, T\xi_\tau\rangle \tag{2.8.8}$$
$$= \langle \pi_\tau(a^*)\xi_\tau, T\xi_\tau\rangle \tag{2.8.9}$$
$$= \langle T^*\pi_\tau(a^*)\xi_\tau, \xi_\tau\rangle \tag{2.8.10}$$
$$= \langle \pi_\tau(a^*)T^*\xi_\tau, \xi_\tau\rangle \tag{2.8.11}$$
$$= \langle T^*\xi_\tau, \pi_\tau(a)\xi_\tau\rangle \tag{2.8.12}$$
$$= \langle T^*x\xi_\tau, \widehat{a}\rangle. \tag{2.8.13}$$

Hence, by density, $JT\xi_\tau = T^*\xi_\tau$.

□

Theorem 2.8.15. *Let τ be a trace on the unital C*-algebra A. Then $\pi_\tau(A)' = \pi_\tau^{\mathrm{op}}(A)''$ and $\pi_\tau^{\mathrm{op}}(A)' = \pi_\tau(A)''$.*

Proof. Since $\pi_\tau^{\mathrm{op}}(A)''$ and $\pi_\tau(A)''$ commute, we have that $\pi_\tau(A)' \supseteq \pi_\tau^{\mathrm{op}}(A)''$ and $\pi_\tau^{\mathrm{op}}(A)' \supseteq \pi_\tau(A)''$. Therefore, it suffices to show that $\pi_\tau(A)'$ and $\pi_\tau^{\mathrm{op}}(A)'$ commute.

Since J is a conjugate-linear isometric isomorphism of order 2, the map $\mathrm{Ad}(J) : B(L^2(A,\tau)) \to B(L^2(A,\tau))$ given by $T \mapsto JTJ$ is a conjugate-linear *-isomorphism of C*-algebras. Since $\mathrm{Ad}(J)(\pi_\tau(A)) = \pi_\tau^{\mathrm{op}}(A)$, we have that $\mathrm{Ad}(J)(\pi_\tau(A)') = \pi_\tau^{\mathrm{op}}(A)'$. Therefore, it suffice to show that $JSJT = TJSJ$, where $T, S \in \pi_\tau(A)'$. This is achieved by the following computation.

Let $T, S \in \pi_\tau(A)'$ and $a, b \in A$. Then

$$\begin{aligned}
\langle JSJT\widehat{a}, \widehat{b}\rangle &= \langle J\widehat{b}, SJT\widehat{a}\rangle && (2.8.14)\\
&= \langle S^*\pi_\tau(b^*)\xi_\tau, JT\pi_\tau(a)\xi_\tau\rangle && (2.8.15)\\
&= \langle \pi_\tau(b^*)S^*\xi_\tau, J\pi_\tau(a)T\xi_\tau\rangle && (2.8.16)\\
&= \langle \pi_\tau(b^*)S^*\xi_\tau, \pi_\tau^{\mathrm{op}}(a^*)JT\xi_\tau\rangle && (2.8.17)\\
&= \langle \pi_\tau(b^*)JS\xi_\tau, \pi_\tau^{\mathrm{op}}(a^*)T^*\xi_\tau\rangle && (2.8.18)\\
&= \langle \pi_\tau^{\mathrm{op}}(a)JS\xi_\tau, \pi_\tau(b)T^*\xi_\tau\rangle && (2.8.19)\\
&= \langle J\pi_\tau(a^*)S\xi_\tau, \pi_\tau(b)T^*\xi_\tau\rangle && (2.8.20)\\
&= \langle JS\pi_\tau(a^*)\xi_\tau, T^*\pi_\tau(b)\xi_\tau\rangle && (2.8.21)\\
&= \langle JSJ\widehat{a}, T^*\widehat{b}\rangle && (2.8.22)\\
&= \langle TJSJ\widehat{a}, \widehat{b}\rangle. && (2.8.23)
\end{aligned}$$

Hence, by density, $JSJT = TJSJ$. □

In the remainder of this section, we show how tracial von Neumann algebras can be defined abstractly, without reference to a particular representation. This is based on the following folklore theorem.

Theorem 2.8.16. *Let τ be a faithful trace on the unital C^*-algebra A. The following are equivalent:*

(i) The unit ball $\{a \in A : \|a\| \le 1\}$ of A is complete with respect to the norm $\|\cdot\|_{2,\tau}$.

(ii) The image of A under the GNS representation π_τ is a von Neumann algebra, i.e. $\pi_\tau(A) = \pi_\tau(A)''$.

(iii) There exists a faithful representation $\pi : A \to B(H)$ under which the image of A is a a von Neumann algebra, i.e. $\pi(A) = \pi(A)''$, and τ defines a normal trace on $\pi(A)$.

Proof. (i) $\Leftrightarrow$ (ii) Let $M = \pi_\tau(A)''$. Since τ is faithful, we can view A as a subset of M by Proposition 2.6.12. Furthermore, by 2.8.13, the vector state $T \mapsto \langle T\xi_\tau, \xi_\tau\rangle$ defines a faithful trace on M which extends τ, so the strong operator topology on bounded subsets of M is induced by the norm $\|\cdot\|_{2,\tau}$ thanks to Proposition 2.7.7.

If the unit ball of A is complete with respect to the $\|\cdot\|_{2,\tau}$-norm, then it is $\|\cdot\|_{2,\tau}$-norm closed, and so closed in the strong operator topology. But then $M = A$ by the Kapansky Density Theorem. Conversely, if $A = M$, then the unit ball is complete with respect to the strong operator topology by Corollary 2.3.2 (see also Remark 2.3.3). Hence, is complete with respect to the $\|\cdot\|_{2,\tau}$-norm.

(ii) $\Rightarrow$ (iii) Immediate.

(iii) $\Rightarrow$ (ii) Identify A with $\pi(A)$. Since τ is normal, so is the GNS representation π_τ by Proposition2.7.5. Hence $\pi_\tau(A) = \pi_\tau(A)''$ by Theorem 2.7.4. □

The completeness condition (i) of Theorem 2.8.16 becomes the definition of an abstract tracial von Neumann algebra, which we record formally for future reference.

Definition 2.8.17. An *abstract tracial von Neumann algebra* is unital C*-algebra A together with a faithful trace τ such that the unit ball $\{a \in A : \|a\| \leq 1\}$ of A is complete with respect to the norm $\|\cdot\|_{2,\tau}$. Morphisms of tracial von Neumann algebras are unital, trace preserving *-homomorphisms between the underlying C*-algebras.

In light of Theorem 2.8.16, we will not distinguish between abstract and concrete tracial von Neumann algebras. By Proposition 2.7.7, the strong topology agrees with the $\|\cdot\|_{2,\tau}$-topology on bounded sets. Hence, normal cpc maps between tracial von Neumann algebras are precisely those with are $\|\cdot\|_{2,\tau}$-continuous restricted to bounded sets. In particular, trace-preserving *-homomorphisms are normal. A simple consequence of the trace being faithful is that morphisms of tracial von Neumann algebras are always injective.

For II$_1$ factors, all unital *-homomorphisms are trace preserving as the trace is unique. So the category of II$_1$ factor with unital *-homomorphisms is a full subcategory of the category of tracial von Neumann algebras.

2.9 The Bidual of a C*-Algebra and its Finite Part

In this section, we recall some of the key results about the bidual of a C*-algebra and its finite part. Good general references for the bidual of a C*-algebra are [82, Section III.2] and [4, Section III.5.2.1].

Fix a unital C*-algebra A. The representation $\pi_U = \oplus_{\phi \in S(A)} \pi_\phi$ obtained by considering the direct sum of all GNS representations is known as the *universal representation* of A.[5] This name is justified by the following theorem.

Theorem 2.9.1. *Any representation of A is sub-representation of an inflation of π_U.*

[5] One can also work with $\oplus_{\phi \in A_+^*} \pi_\phi$, which is an equivalent representation except in the trivial case $A = \mathbb{C}$; see [4, Section III.5.2.3].

Proof (Sketch). By Zorn's lemma any representation π can be decomposed into a direct sum of cyclic representations. Every cyclic representation is equivalent to π_ϕ for some positive linear function ϕ by Theorem 2.6.6. Since we can normalise the cyclic vector, we can take ϕ to be a state. By taking a sufficiently large cardinal λ, the inflation $\pi_u^{(\lambda)}$ contains, up to equivalence, each of the cyclic representations in the decomposition of π including multiplicity. □

The von Neumann algebra $\pi_U(A)''$ is the *universal enveloping von Neumann algebra* of A. The following corollary of Theorem 2.9.1 makes this precise. We say informally that any representation of A has a unique normal extension to the enveloping von Neumann algebra.

Corollary 2.9.2. *Let π be any representation of A. There exists a unique normal *-homomorphism $\Phi : \pi_U(A)'' \to \pi(A)''$ such that $\Phi(\pi_U(a)) = \pi(a)$.*

Proof. Uniqueness is clear as A is ultraweakly dense in $\pi_U(A)''$. By Theorem 2.9.1, there exist an isometry $V : H_\pi \to H_{\pi_U} \otimes \ell^2(\lambda)$, for some cardinal λ, such that $V^*(\pi_U(a) \otimes 1_{B(\ell^2(\lambda))})V = \pi(a)$. The map $\Phi : B(H_{\pi_U}) \to B(H_\pi)$ given by $T \mapsto V^*(T \otimes 1_{B(\ell^2(\lambda))})V$ is a *-homomorphism, as $V^*V = \mathrm{id}_{H_\pi}$, and easily seen to by ultraweakly continuous and thus normal. It follows from normality that $\Phi(\pi_U(A)'') \subseteq \pi(A)''$. □

Every state ϕ on A extends to a normal state on $\pi_U(A)''$, namely the vector state corresponding to the GNS vector $\xi_\phi \in H_{\pi_U} = \oplus_{\psi \in S(A)} H_\psi$. Since A is ultraweakly dense in $\pi_U(A)''$, the normal extension is unique. Developing this idea, one arrives quickly at the following result (see for example [82, Theorem III.2.4]).

Theorem 2.9.3. *There is a isometric isomorphism between $\pi_U(A)''$ and the bidual A^{**} of A (as a Banach space), which maps $\pi_U(A)$ to the canonical copy of A inside A^{**}.*

As a result of this theorem, $\pi_U(A)''$ can be identified with the bidual A^{**}. We shall make this identification in the sequel and identify A with it's image in A^{**}.

Applying Theorem 2.8.9, we have a central decomposition of A^{**} into a finite part A^{**}_{fin} and a properly infinite part A^{**}_{pi}. Let $\iota : A \to A^{**}_{\mathrm{fin}}$ denote the composition of the embedding of A in A^{**} and projection of A^{**} onto the summand A^{**}_{fin}. The map ι is, in general, not injective but, as we shall shortly see, it preserves all the tracial information about A. We begin with a couple of "extension" results.[6] First, an extension result for

[6] By abuse of terminology, we use the word extension here, even though ι may not be injective.

the GNS representations with respect to a trace.

Proposition 2.9.4. *Let τ be a trace on the unital C^*-algebra A. The GNS representation $\pi_\tau : A \to \pi_\tau(A)''$ has a unique normal extension to A^{**}_{fin}, by which we mean that there is a unique normal *-homomorphism $\widetilde{\pi_\tau} : A^{**}_{\mathrm{fin}} \to \pi_\tau(A)''$ such that the diagram*

$$\begin{tikzcd} A^{**}_{\mathrm{fin}} \arrow[dr, "\widetilde{\pi_\tau}"] & \\ A \arrow[u, "\iota"] \arrow[r, "\pi_\tau"'] & \pi_\tau(A)'' \end{tikzcd} \tag{2.9.1}$$

commutes.

Proof. By Theorem 2.9.1, the representation $\pi_\tau : A \to \pi_\tau(A)''$ has a unique normal extension to a map $\widetilde{\pi_\tau} : A^{**} \to \pi_\tau(A)''$. By Proposition 2.8.13, $\pi_\tau(A)''$ is a finite von Neumann algebra, so $\widetilde{\pi_\tau}$ vanishes on A^{**}_{pi}. This proves existence. Uniqueness follows because $\iota(A)$ is ultraweakly dense in A^{**}_{fin}. □

Remark 2.9.5. The extension $\widetilde{\pi_\tau}$ is supported on a direct summand $p_\tau A^{**}_{\mathrm{fin}}$ of A^{**}_{fin}, where p_τ is a central projection in A^{**}_{fin}. Indeed, as $\mathrm{Ker}(\widetilde{\pi_\tau})$ is an ultraweakly closed ideal in A^{**}_{fin}, it is equal to $z_\tau A^{**}_{\mathrm{fin}}$ for some central projection $z_\tau \in A^{**}_{\mathrm{fin}}$. We set $p_\tau = 1 - z_\tau$. One can then identify $\pi_\tau(A)''$ with $p_\tau A^{**}_{\mathrm{fin}}$.

We now state a corresponding extension result for traces.

Corollary 2.9.6. *Let τ be a trace on the unital C^*-algebra A. There is a unique normal extension of τ to A^{**}_{fin}, by which we mean that there is a unique normal trace $\widetilde{\tau}$ on A^{**}_{fin} such that the diagram*

$$\begin{tikzcd} A^{**}_{\mathrm{fin}} \arrow[dr, "\widetilde{\tau}"] & \\ A \arrow[u, "\iota"] \arrow[r, "\tau"'] & \mathbb{C} \end{tikzcd} \tag{2.9.2}$$

commutes.

Proof. For existence, compose $\widetilde{\pi_\tau}$ of Proposition 2.9.4 with the vector state corresponding to ξ_τ. Uniqueness follows because $\iota(A)$ is ultraweakly dense in A^{**}_{fin}. □

The finite part of the bidual A^{**}_{fin} has an alternative characterisation analogous to the characterisation of A^{**} as $\pi_U(A)''$.

Proposition 2.9.7. *Let A be a unital C*-algebra with $T(A) \neq \emptyset$. Then there is an isomorphism $\Phi : A_{\text{fin}}^{**} \to (\oplus_{\tau \in T(A)} \pi_\tau)(A)''$ such that the diagram*

$$\begin{tikzcd} A_{\text{fin}}^{**} \arrow[drr, "\Phi"] & & \\ A \arrow[u, "\iota"] \arrow[rr, "\oplus_{\tau \in T(A)} \pi_\tau"'] & & (\oplus_{\tau \in T(A)} \pi_\tau)(A)'' \end{tikzcd} \tag{2.9.3}$$

commutes.

Proof. Write $N = (\oplus_{\tau \in T(A)} \pi_\tau(A))''$. By Corollary 2.9.2, $\oplus_{\tau \in T(A)} \pi_\tau$ has a unique normal extension to a *-homomorphism $\Phi : A^{**} \to N$. By Theorem 2.7.4, Φ is surjective.

Since $N \subseteq \prod_{\tau \in T(A)} \pi_\tau(A)''$ and each $\pi_\tau(A)''$ is finite by Proposition 2.8.13, N is finite. As N is finite, it has no non-zero properly infinite vectors. Hence, $\Phi(A_{\text{pi}}^{**}) = \{0\}$. Therefore, we can consider Φ as a unital *-homomorphism $A_{\text{fin}}^{**} \to N$ and the diagram (2.9.3) commutes.

It remains only to prove that Φ is injective. Let $a \in A_{\text{fin}}^{**}$ be a non-zero. Then there exists a normal trace τ on A_{fin}^{**} such that $\tau(a^*a) > 0$. Extending τ to be zero on A_{pi}^{**}, we get a normal trace on A^{**}, which we still denote τ. Since $\tau|_A$ has a unique normal extension to A^{**}, we must have that $\tau(a) = \langle \Phi(a)\xi_{\tau|_A}, \xi_{\tau|_A} \rangle$ for all $a \in A^{**}$. Consequently, $\Phi(a^*a) \neq 0$. $\square$

Corollary 2.9.8. *Let A be a unital C*-algebra with $T(A) \neq \emptyset$. The kernel of $\iota : A \to A_{\text{fin}}^{**}$ is $I = \{a \in A : \tau(a^*a) = 0$ for all $\tau \in T(A)\}$.*

Proof. By Proposition 2.9.7, $\text{Ker}(\iota) = \text{Ker}(\oplus_{\tau \in T(A)} \pi_\tau) = \cap_{\tau \in T(A)} \text{Ker}(\pi_\tau)$. Now apply Proposition 2.6.12. $\square$

Remark 2.9.9. Let A be a unital C*-algebra with $T(A) \neq \emptyset$ and $I = \{a \in A : \tau(a^*a) = 0$ for all $\tau \in T(A)\}$. Let $q : A \to A/I$ be the quotient map and $q^* : T(A/I) \to T(A)$ the induced map on traces given by $q^*(\tau) = \tau \circ q$.

The map q^* is continuous, affine and injective. By proposition 2.6.13, we see that q^* is surjective. Since trace spaces are compact and Hausdorff, q^* is an affine homeomorphism. Furthermore, the trace pairing is preserved in the sense that $q^*(\tau)(a) = \tau(q(a))$ for all $a \in A, \tau \in T(A/I)$.

It follows that q induces a unitary map $L^2(A, q^*(\tau)) \to L^2(A/I, \tau)$ via $\widehat{a} \mapsto \widehat{q(a)}$, which we also denote q, and the diagram

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "\pi_{q^*(\tau)}"] & A/I \arrow[d, "\pi_\tau"] \\ B(L^2(A, q^*(\tau))) \arrow[r, "\mathrm{Ad}(q)"'] & B(L^2(A/I, \tau)) \end{tikzcd} \tag{2.9.4}$$

commutes. Hence, q induces a normal isomorphism $\pi_{q^*(\tau)}(A)'' \cong \pi_\tau(A/I)''$. Since q^* is a bijection, we have $A^{**}_{\text{fin}} \cong (A/I)^{**}_{\text{fin}}$ using Proposition 2.9.7.

2.10 Choquet Theory and Trace Simplices

In this section, we first recall some results on the theory of compact convex sets, Choquet simplices and Bauer simplices. We then collect some results about the trace simplex of a unital C*-algebra, which will be used in the thesis.

2.10.1 General Theory

A detailed account of the theory of convex sets can be found in [1] with additional material in [66]. We summarise the basic theory in this subsection.

Let K be a compact, convex subset of locally convex topological space. Let $\mathrm{Aff}_{\mathbb{R}}(K)$ be the space of real-valued, continuous, affine functionals on K. This is a closed subspace of the Banach space $C_{\mathbb{R}}(K)$.

There is a natural partial order on $\mathrm{Aff}_{\mathbb{R}}(K)$ given by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in K$. We write $\mathrm{Aff}_{\mathbb{R}}(K)^+$ for the positive cone, which is easily seen to be closed. The function that takes the value 1 at all points is an order unit for $\mathrm{Aff}_{\mathbb{R}}(K)$; we denote this function by 1. The order unit determines the norm in the sense that

$$\|f\|_\infty = \inf\{t \in \mathbb{R}^+ : -t1 \leq f \leq t1\}. \tag{2.10.1}$$

The additional structure on $\mathrm{Aff}_{\mathbb{R}}(K)$ makes it a *complete order unit space* (See [1, Section II.1] for more details). There is in fact a duality between compact convex subset of locally convex topological spaces and complete order unit spaces due to Kadison [41, Lemma 4.3] (see also [1, Theorem II.1.8]). In particular, we can recover K from $\mathrm{Aff}_{\mathbb{R}}(K)$ as the space of positive linear functionals on $\mathrm{Aff}_{\mathbb{R}}(K)$ of operator norm 1, which are known as states.

We denote the space of complex-valued, continuous, affine functionals on K by $\mathrm{Aff}_{\mathbb{C}}(K)$. This is a closed subspace of the Banach space $C(K)$. Pointwise complex conjugation defines an involution on $\mathrm{Aff}_{\mathbb{C}}(K)$, and we can view $\mathrm{Aff}_{\mathbb{R}}(K)$ as the self-adjoint part of $\mathrm{Aff}_{\mathbb{C}}(K)$. We shall use the terminology *complex complete order unit space* to describe the abstract structure of $\mathrm{Aff}_{\mathbb{C}}(K)$.

A point $x \in K$ is said to be extreme if it cannot be write as $x = \lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$ and $x_1, x_2 \in K \setminus \{x\}$. The set of all extreme points is denoted $\partial_e K$. The Krein–Millman Theorem states that K is the closed convex hull of $\partial_e K$ (see for example [75, Theorem 3.23]). Choquet's theorem strengthens this, asserting that every point of K is the barycentre of some measure concentrated on the boundary. Choquet proved this result in the under the assumption K is metrisable [10, Théorème 1], in which case $\partial_e K$ is a G_δ set. The general case was proven by Bishop and de Leeuw in [3]. We state the result bellow only in the metrisable case. For the full story, we recommend [1, Section I.4].

Theorem 2.10.1. *[1, Corollary I.4.9] Let K be a metrisable, compact, convex subset of locally convex topological space. For each $x_0 \in T(A)$, there is a Borel probability measure μ on $\partial_e T(A)$ such that*

$$f(x_0) = \int_{x\in\partial_e K} f(x)\,d\mu(x) \tag{2.10.2}$$

for all $f \in \mathrm{Aff}_{\mathbb{C}}(K)$.

This brings us to Choquet simplices. There are a number of equivalent definitions of this class of compact convex set (see [1, Section II.3]). We use the one in terms of uniqueness of the measure μ in Theorem 2.10.1.

Definition 2.10.2. Let K be a metrisable, compact, convex subset of locally convex topological space. Then K is *Choquet simplex* if and only if every every $x \in K$ is the barycentre of a unique Borel probability measure concentrated on $\partial_e K$.

Let's look at some examples.

Example 2.10.3 (Finite Dimensional Choquet simplicities)**.** If K is a Choquet simplex, then $\partial_e K$ must be affinely independent. So, if K is a subspace of a finite dimensional space, then $\partial_e K$ is a finite set $\{x_1, \dots x_k\}$ and the map $(\lambda_1, \dots, \lambda_k) \mapsto \sum_{i=1}^k \lambda_i x_i$ is an isomorphism of K with $\triangle^{(k-1)} = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^n : 0 \le \lambda_i \le 1, \sum_{i=1}^k \lambda_i = 1\}$.

Example 2.10.4 (Bauer simplicies)**.** A *Bauer simplex* K is a metrisable Choquet simplex for which $\partial_e K$ is compact [1, Section II.4]. For each metrisable compact space X, there is upto isomorphism a unique Bauer simplex K with $\partial_e K = X$, namely the space $M_1^+(X)$ of Radon probability measures on X [1, Corollary II.4.2].

Example 2.10.5 (A non-Bauer simplex)**.** Let $X = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$, and let $V = C(X)^*$ be the space of Radon measures on X considered with the weak* topology. Let $\mu_1, \mu_2, \mu_3 \ldots$ and μ_∞ be the Dirac measures on X. Let $W = \operatorname{span}\{\mu_\infty - \frac{1}{2}(\mu_1 + \mu_2)\}$. Let $q : V \to V/W$ be the quotient map. Set $K = q(M_1^+(X))$. One can then show that $\partial_e K = \{q(\mu_i) : i \in \mathbb{N}\}$ and K is a Choquet simplex. However, $\lim_{i\to\infty} q(\mu_i) = q(\mu_\infty) = \frac{1}{2}(q(\mu_1) + q(\mu_2))$, so $\partial_e K$ is not closed in K. Hence, $\partial_e K$ is not compact. See [1, Proposition II.7.17] for full details.

By the Krein–Milman Theorem, a continuous, affine functional on K is completely determined by its values at the extreme points. We, therefore, introduce the notation

$$\mathcal{A}\mathrm{ff}_{\mathbb{R}}(K) = \{f|_{\partial_e K} : f \in \mathrm{Aff}_{\mathbb{R}}(K)\}, \tag{2.10.3}$$

$$\mathcal{A}\mathrm{ff}_{\mathbb{C}}(K) = \{f|_{\partial_e K} : f \in \mathrm{Aff}_{\mathbb{C}}(K)\}. \tag{2.10.4}$$

These are subspaces of the Banach spaces $C_{b,\mathbb{R}}(\partial_e K)$ and $C_b(\partial_e K)$ respectively and inherit the additional structure of complete ordered unit spaces from $\mathrm{Aff}_{\mathbb{R}}(K)$ and $\mathrm{Aff}_{\mathbb{C}}(K)$ respectively. In the case of Bauer simplicies, we have the following.

Theorem 2.10.6. *[1, Theorem II.4.3] Let K be a Bauer simplex. Then $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(K) = C(K)$ and $\mathcal{A}\mathrm{ff}_{\mathbb{R}}(K) = C_{\mathbb{R}}(K)$.*

In the further theory of Choquet simplicies, the central affine functionals play an important role [1, Section II.7]. By definition, we have

$$Z(\mathcal{A}\mathrm{ff}_{\mathbb{R}}(K)) = \{f \in \mathcal{A}\mathrm{ff}_{\mathbb{R}}(K) : fg \in \mathcal{A}\mathrm{ff}_{\mathbb{R}}(K) \text{ for all } g \in \mathcal{A}\mathrm{ff}_{\mathbb{R}}(K)\}, \tag{2.10.5}$$

$$Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(K)) = \{f \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(K) : fg \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(K) \text{ for all } g \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(K)\}. \tag{2.10.6}$$

All we shall need in this thesis, is the following corollary of Theorem 2.10.6.

Corollary 2.10.7. *Let K be a Bauer simplex. Then $Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(K)) = C(K)$ and $Z(\mathcal{A}\mathrm{ff}_{\mathbb{R}}(K)) = C_{\mathbb{R}}(K))$.*

2.10.2 The Trace Simplex

Let A be a unital C*-algebra with non-empty trace space $T(A)$. From Section 2.6.2, we know that $T(A)$ is weak*-closed in the state space $S(A)$, so it weak* compact. The trace space is also clearly convex and is metrisable whenever A is separable. By the Krein–Milman Theorem $T(A)$ is the closed convex hull of the set of extreme points $\partial_e T(A)$. In this context, we call elements of $\partial_e T(A)$ extreme traces.

In fact, more is true.

Theorem 2.10.8. *[76, Theorem 3.1.18] Let A be a unital C*-algebra. Then $T(A)$ is a Choquet simplex whenever it is non-empty.*

Since A^{**}_{fin} is also a unital C*-algebra, $T(A^{**}_{\text{fin}})$ is also a Choquet simplex, though typically a non-metrisable one. We now discuss the relationship between these two simplices. By Proposition 2.9.6, the traces on A can be identified with the normal trace on A^{**}_{fin}. Hence, we can view $T(A) \subseteq T(A^{**}_{\text{fin}})$. However, one must be very careful with the topologies. With respect to the weak* topology on $T(A^{**}_{\text{fin}})$ coming from the pairing with A^{**}_{fin}, hereinafter the A^{**}_{fin}-weak* topology, $T(A)$ is a dense subset of $T(A^{**}_{\text{fin}})$. With respect to the A-weak* topology on $T(A)$, $T(A)$ is compact. Therefore, the inclusion $T(A) \subseteq T(A^{**}_{\text{fin}})$ is only a homeomorphism onto its image when $T(A) = T(A^{**}_{\text{fin}})$.

We now consider the extremal traces. If a normal trace $\tau \in T(A^{**}_{\text{fin}})$ is a non-trivial convex combination of traces $\tau = \lambda\tau_1 + (1-\lambda)\tau_2$, then $\tau_1 \leq \lambda^{-1}\tau$ and $\tau_2 \leq (1-\lambda)^{-1}\tau$, so τ_1 and τ_2 are normal too. Hence, we have $\partial_e T(A) \subseteq \partial_e T(A^{**}_{\text{fin}})$. Once again though, we shouldn't expect the topologies to coincide.

We now investigate the centre of A^{**}_{fin} for a separable, unital C*-algebra A with non-empty trace space. Each element $a \in A^{**}_{\text{fin}}$, defines a function $\widehat{a}$ on $\partial_e(T(A))$ via $\widehat{a}(\tau) = \tau(a)$. Since $\widehat{a} = \widehat{\text{ctr}(a)}$ for all $a \in A^{**}_{\text{fin}}$ by Theorem 2.8.11, it suffices to understand $\widehat{z}$ for $z \in Z(A^{**}_{\text{fin}})$. The following theorem of Ozawa gives a partial inverse to the map $z \mapsto \widehat{z}$ for $z \in Z(N)$. In the statement of the theorem, $B(\partial_e T(A))$ denotes the C*-algebra of bounded Borel functions on $\partial_e T(A)$.

Theorem 2.10.9. *[62, Theorem 3] Let A be a unital separable C*-algebra with non-empty trace space $T(A)$. There is a unique unital *-homomorphism $\theta : B(\partial_e T(A)) \to Z(A^{**}_{\text{fin}})$ with ultraweakly dense range such that $\theta(\widehat{a}) = \text{ctr}(a)$ and*

$$\tau(\theta(f)a) = \int_{\lambda \in \partial_e T(A)} f(\lambda)\lambda(a)\,d\mu_\tau(\lambda) \tag{2.10.7}$$

for every $a \in A$, $\tau \in T(A)$ and $f \in B(\partial_e T(A))$.

Proof (Sketch). Ozawa first shows that for every $\tau \in T(A)$, there is a normal *-isomorphism

$\theta_\tau : L^\infty(\partial_e T(A), \mu_\tau) \to Z(\pi_\tau(A)'')$ such that

$$\tau(\theta_\tau(f)a) = \int_{\lambda \in \partial_e T(A)} f(\lambda)\lambda(a) d\mu_\tau(\lambda) \tag{2.10.8}$$

for all $a \in A$ [62, Lemma 10]. The key step is an application of Sakai's non-commutative Radon–Nikodym Theorem, observing that the left hand side of (2.10.8) defines a tracial function of A dominated by $\|f\|_\infty \tau$ in modulus.

One can identify $\pi_\tau(A)''$ with a direct summand $p_\tau A^{**}_{\text{fin}}$ of A^{**}_{fin}; see Remark 2.9.5. The space $T(A)$ becomes a directed set with the direction given by $\tau \precsim \sigma$ if and only if $\tau \leq C\sigma$ for some $C > 1$. An upper bounded for $\tau, \sigma \in T(A)$ is $(\tau + \sigma)/2$. One easily checks that $p_\tau \wedge p_\sigma = p_{(\tau+\sigma)/2}$ and $\sup_\tau p_\tau = 1$.

The idea is to define θ to be the pointwise ultraweak limit of θ_τ as $\tau \to \infty$. This limit exists as a Radon–Nikodym computation shows that $\theta_\tau(f) = p_\tau \theta_\sigma(f)$ for all $f \in B(\partial_e T(A))$ whenever $\tau \precsim \sigma$. The validity of (2.10.7) then follows from (2.10.8). Since $\tau(\theta(\widehat{a})) = \int \widehat{a}(\lambda) d\mu_\tau(\lambda) = \tau(a)$ for $a \in A, \tau \in T(A)$, we get that $\theta(\widehat{a}) = \operatorname{ctr}(a)$. $\square$

2.11 Hilbert-$C(X)$-Modules and Adjointable Operators

Hilbert modules generalise Hilbert spaces by replacing the $\mathbb{C}$-action by scalar multiplication and the $\mathbb{C}$-valued inner product by an A-action and an A-valued inner product for some C*-algebra A. Hilbert modules were first introduce by in [43] in the commutative case and in [63] in general. A good basic reference is [50].

In this thesis, only the commutative case will be required. Therefore, we only consider Hilbert-$C(X)$-modules in this section. For compatibility with the standard conventions for Hilbert spaces, we shall work with left-modules and inner products that are linear in the first place. The results in this section are mostly well-known, but are developed carefully from the first principles for completeness.

2.11.1 Hilbert-$C(X)$-Modules

We begin with the definition of a pre-Hilbert-$C(X)$-module and some of its elementary consequences.

Definition 2.11.1. Let X be a compact Hausdorff space. A *pre-Hilbert-$C(X)$-module* is a left $C(X)$-module H together with a map $\langle \cdot, \cdot \rangle : H \times H \to C(X)$ satisfying the following axioms:

$$\langle fu + gv, w\rangle = f\langle u, w\rangle + g\langle v, w\rangle \qquad (f, g \in C(X), u, v, w \in H), \tag{2.11.1}$$

$$\langle u, v\rangle = \langle v, u\rangle^* \qquad (u, v \in H), \tag{2.11.2}$$

$$\langle u, u\rangle = 0 \implies u = 0 \qquad (u \in H). \tag{2.11.3}$$

Proposition 2.11.2. *Let H be a pre-Hilbert-$C(X)$-module. Set $\|u\|_H = \|\langle u, u\rangle^{1/2}\|_{C(X)}$.*

(a) $\|\cdot\|_H$ defines a norm on H.

(b) Addition on H and the multiplication map $C(X) \times H \to H$ are continuous with respect to the $\|\cdot\|_H$-norm.

(c) The map $\langle\cdot,\cdot\rangle : H \times H \to C(X)$ is continuous with respect to the $\|\cdot\|_H$-norm, and the following identities hold:

$$|\langle u, v\rangle| \le \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2} \qquad (u, v \in H), \tag{2.11.4}$$

$$\|\langle u, v\rangle\|_{C(X)} \le \|u\|_H\|v\|_H \qquad (u, v \in H). \tag{2.11.5}$$

Proof. (a) For all $x \in X$, the map $(u, v) \mapsto \langle u, v\rangle(x)$ is a hermitian form on H, so $u \mapsto \langle u, u\rangle(x)^{1/2}$ is a seminorm. It follows that $\|u\|_H = \sup_{x\in X}\langle u, u\rangle(x)^{1/2}$ is a seminorm and, by axiom (2.11.3), a norm.

(b) Continuity of addition follows from the triangle inequality for $\|\cdot\|_H$. A simple consequence of (2.11.1) and (2.11.2) is that $\|fu\|_H \le \|f\|_{C(X)}\|u\|_H$, from which continuity of the multiplication follows.

(c) The Cauchy-Schwarz inequality for the hermitian form $(u, v) \mapsto \langle u, v\rangle(x)$ gives (2.11.4). Taking suprema in (2.11.4) gives (2.11.5). Continuity of $\langle\cdot,\cdot\rangle$ follows easily. □

We can now give the definition of Hilbert-$C(X)$-modules.

Definition 2.11.3. A *Hilbert-$C(X)$-module* is a pre-Hilbert-$C(X)$-module H for which the norm $\|\cdot\|_H$ is complete.

Remark 2.11.4. Let $X = \{*\}$ be a one point space. Identifying $C(X)$ with $\mathbb{C}$, we see that Hilbert-$C(X)$-modules are precisely Hilbert spaces.

Our first goal is to show that Hilbert-$C(X)$-modules fibre over the base space X with each fibre being a Hilbert spaces. In the following proposition, we construct the fibres.

Proposition 2.11.5. *Let H be a Hilbert-$C(X)$-module and $x \in X$. Set $N_x = \{u \in H : \langle u, u\rangle(x) = 0\}$.*

(a) N_x is a closed subspace of H.

(b) $N_x = C_0(X \setminus \{x\})H$.

(c) H/N_x is a Hilbert space with inner product $\langle u + N_x, v + N_x\rangle = \langle u, v\rangle(x)$.

Proof. (a) The map $(u, v) \mapsto \langle u, v\rangle(x)$ is a hermitian form on H, so $\|u\|_{2,x} = \langle u, u\rangle(x)^{1/2}$ defines a seminorm on H. Clearly $0 \in N_x$. Let $u, v \in N_x$ and $\lambda, \mu \in \mathbb{C}$. Then $\|\lambda u + \mu v\|_{2,x} \le |\lambda|\|u\|_{2,x} + |\mu|\|v\|_{2,x} = 0$ so $\lambda u + \mu v \in N_x$. Since $\langle\cdot, \cdot\rangle$ is continuous with respect to the $\|\cdot\|_H$-norm, N_x is closed.

(b) The inclusion $N_x \supseteq C_0(X \setminus \{x\})H$ follows from (2.11.1). For the reverse inclusion, let $u \in N_x$. Let $f = \langle u, u\rangle^{1/4} \in C_0(X \setminus \{x\})$. We show that the limit

$$\lim_{n\to\infty} \frac{1}{f + \frac{1}{n}} u \tag{2.11.6}$$

exists in H. Let $\epsilon > 0$. Choose $N > \frac{2}{\epsilon}$. Suppose $n, m > N$. Then

$$\left\| \frac{1}{f + \frac{1}{n}} u - \frac{1}{f + \frac{1}{m}} u \right\|_H = \left\| \frac{\frac{1}{m} - \frac{1}{n}}{(f + \frac{1}{n})(f + \frac{1}{m})} f^2 \right\|_{C(X)} \tag{2.11.7}$$

Let $y \in X$. Then we have

$$\left| \frac{\frac{1}{m} - \frac{1}{n}}{(f(y) + \frac{1}{n})(f(y) + \frac{1}{m})} f(y)^2 \right| \le \frac{2}{N}, \tag{2.11.8}$$

noting that when $f(y) = 0$ the left hand side vanishes. Hence,

$$\left\| \frac{1}{f + \frac{1}{n}} u - \frac{1}{f + \frac{1}{m}} u \right\|_H \le \epsilon. \tag{2.11.9}$$

Since H is complete, the limit (2.11.6) exists in H. Denoting this limit by v, we get that $u = fv \in C_0(X \setminus \{x\})H$ by the continuity of the $C(X)$-action.

(c) It's a consequence of the Cauchy-Schwarz inequality for the hermitian form $(u, v) \mapsto \langle u, v\rangle(x)$, that $\langle u, v\rangle(x) = 0$ whenever one of u, v lies in N_x. It follows that $\langle u + N_x, v + N_x\rangle = \langle u, v\rangle(x)$ is a well-defined inner product on H/N_x. We need to show that the quotient norm on the Banach space H/N_x is induced by this inner product, i.e. that, for all $u \in H$,

$$\inf_{v\in N_x} \|u + v\|_H = \langle u, u\rangle(x)^{1/2}. \tag{2.11.10}$$

Let $u \in H$ and $v \in N_x$. Then $\|u+v\|_H^2 \geq \langle u+v, u+v\rangle(x) = \langle u, u\rangle(x)$. Consequently,

$$\inf_{v \in N_x} \|u+v\|_H \geq \langle u, u\rangle(x)^{1/2}. \tag{2.11.11}$$

Conversely, let $u \in H$. Define $f \in C(X)$ by

$$f(y) = \begin{cases} \frac{\langle u,u\rangle(x)^{1/2}}{\langle u,u\rangle(y)^{1/2}} & \text{if } \langle u, u\rangle(y)^{1/2} > \langle u, u\rangle(x)^{1/2}, \\ 1, & \text{if } \langle u, u\rangle(y)^{1/2} \leq \langle u, u\rangle(x)^{1/2}. \end{cases} \tag{2.11.12}$$

Set $v = (1-f)u$. Then $v \in N_x$ and $\|u+v\|_H = \|fu\|_H = \langle u, u\rangle(x)^{1/2}$. Hence, (2.11.10) holds.

□

Given a Hilbert-$C(X)$-module H, we shall write H_x for the Hilbert space H/N_x for each $x \in X$. The canonical quotient map $H \to H_x$ will be denoted by $v \mapsto v(x)$. This notation is justified by the following proposition.

Proposition 2.11.6. *Let H be a Hilbert-$C(X)$-module. Then*

$$\langle u(x), v(x)\rangle_{H_x} = \langle u, v\rangle_H(x) \qquad (u, v \in H, x \in X), \tag{2.11.13}$$

$$\|u\|_H = \sup_{x \in X} \|u\|_{H_x} \qquad (u \in H), \tag{2.11.14}$$

$$(fu)(x) = f(x)u(x) \qquad (u \in H, f \in C(X), x \in X). \tag{2.11.15}$$

Proof. Equation (2.11.13) is just the definition of the inner product on H_x defined in Proposition 2.11.5. Equation (2.11.14) follows from (2.11.13) by taking suprema and square roots. Let $u \in H$, $f \in C(X)$ and $x \in X$. Then $f - f(x) \in C_0(X \setminus \{x\})$, so $(f - f(x))u \in N_x$ by Proposition 2.11.5. Therefore, (2.11.15) holds. □

Next, we introduce a notion of morphism, which is applicable to Hilbert-modules over different base spaces.

Definition 2.11.7. A morphism between the Hilbert-$C(X_1)$-module H_1 and the Hilbert-$C(X_2)$-module H_2 is a bounded linear map $\alpha : H_1 \to H_2$ together with a *-homomorphism $\beta : C(X_1) \to C(X_2)$ such that

$$\langle \alpha(u), \alpha(v)\rangle_{H_2} = \beta(\langle u, v\rangle_{H_1}) \qquad (u, v \in H_1), \tag{2.11.16}$$

$$\alpha(fu) = \beta(f)\alpha(u) \qquad (f \in C(X), u \in H_1). \tag{2.11.17}$$

Remark 2.11.8. By abuse of notation, the same letter will be used to denote the morphism $H_1 \to H_2$ and the underlying bounded linear map.

Example 2.11.9. Let H be a Hilbert-$C(X)$-module and $x \in X$. View the Hilbert space $H_x = H/N_x$ as a Hilbert-$C(\{x\})$-module. Let $\alpha_x : H \to H_x$ be the quotient map and $\beta_x : C(X) \to C(\{x\})$ the transpose of the inclusion $\{x\} \to X$. Then α_x and β_x define a morphism $H \to H_x$ by Proposition 2.11.6.

We now introduce conjugate Hilbert-$C(X)$-modules.

Definition 2.11.10. Let H be a Hilbert-$C(X)$-module. The *conjugate* Hilbert-$C(X)$-module $\overline{H}$ has the same underlying set and addition as H but scalar multiplication and the inner product are defined as follows:

$$f \cdot_{\overline{H}} v = f^* v \qquad (f \in C(X), v \in H), \tag{2.11.18}$$

$$\langle v, w \rangle_{\overline{H}} = \langle w, v \rangle_H^* \qquad (v, w \in H). \tag{2.11.19}$$

We omit the elementary verification that $\overline{H}$ is a Hilbert-$C(X)$-module, remarking only that $\|v\|_{\overline{H}} = \|v\|_H$, so completeness of $\overline{H}$ follows from that of H. Specialising to the case where X is a one point space, we recover the definition of a conjugate Hilbert space.

Definition 2.11.11. Let H be a Hilbert space. The *conjugate* Hilbert space $\overline{H}$ has the same underlying set and addition as H but scalar multiplication and the inner product are defined as follows:

$$\lambda \cdot_{\overline{H}} v = \overline{\lambda} v \qquad (\lambda \in \mathbb{C}, v \in H), \tag{2.11.20}$$

$$\langle v, w \rangle_{\overline{H}} = \overline{\langle w, v \rangle_H} \qquad (v, w \in H). \tag{2.11.21}$$

We now show that passing to conjugate space commutes with passing to fibres.

Proposition 2.11.12. *Let H be a Hilbert-$C(X)$-module and $x \in X$. Then $(\overline{H})_x = \overline{(H_x)}$.*

Proof. Since $\langle u, u \rangle_{\overline{H}}(x) = \langle u, u \rangle_H^*(x) = \overline{\langle u, u \rangle_H(x)}$, we see that $\{u \in H : \langle u, u \rangle_H(x) = 0\} = \{u \in \overline{H} : \langle u, u \rangle_{\overline{H}}(x) = 0\}$. Hence, $(\overline{H})_x = \overline{(H_x)}$ as abelian groups.

Let $u, v \in H$ and $f \in C(X)$. Then

$$\langle u(x), v(x)\rangle_{\overline{(H_x)}} = \overline{\langle v(x), u(x)\rangle_{H_x}} \tag{2.11.22}$$
$$= \overline{\langle v, u\rangle_H(x)} \tag{2.11.23}$$
$$= \langle v, u\rangle_H^*(x) \tag{2.11.24}$$
$$= \langle u, v\rangle_{\overline{H}}(x) \tag{2.11.25}$$
$$= \langle u(x), v(x)\rangle_{(\overline{H})_x} \tag{2.11.26}$$

and $f(x) \cdot_{\overline{(H_x)}} u(x) = \overline{f(x)}u(x) = f^*(x)u(x) = (f^*u)(x) = (f \cdot_{\overline{H}} u)(x) = f(x) \cdot_{(\overline{H})_x} u(x)$. Hence, $(\overline{H})_x = \overline{(H_x)}$ as Hilbert spaces. □

2.11.2 Adjointable Operators

We now turn to the theory of adjointable operators between Hilbert-$C(X)$-modules.

Definition 2.11.13. Let H_1, H_2 be Hilbert-$C(X)$-modules. A bounded linear operator $T : H_1 \to H_2$ is said to be *adjointable* if there is a bounded linear operator $T^* : H_2 \to H_1$ such that

$$\langle Tu, v\rangle_{H_2} = \langle u, T^*v\rangle_{H_1} \qquad (u \in H_1, v \in H_2). \tag{2.11.27}$$

The operator T^* is uniquely determined by T due to (2.11.3) and called the *adjoint* of T. The set of all adjointable functions $H_1 \to H_2$ is denoted $\mathcal{L}(H_1, H_2)$. We write $\mathcal{L}(H_1)$ for $\mathcal{L}(H_1, H_1)$.

We now show that adjointable operators are automatically $C(X)$-linear.

Proposition 2.11.14. *Let $T : H_1 \to H_2$ be an adjointable operator between Hilbert-$C(X)$-modules. Then $T(fv) = fTv$ for all $v \in H_1$ and $f \in C(X)$.*

Proof. Let $T^* : H_2 \to H_1$ be the adjoint of T. Let $v \in H_1$ and $f \in C(X)$. Then

$$\langle T(fu), v\rangle_{H_2} = \langle fu, T^*v\rangle_{H_1} \tag{2.11.28}$$
$$= f\langle u, T^*v\rangle_{H_1} \tag{2.11.29}$$
$$= f\langle Tu, v\rangle_{H_2} \tag{2.11.30}$$
$$= \langle fTu, v\rangle_{H_2} \tag{2.11.31}$$

for all $u \in H_1$, $v \in H_2$. Taking $v = T(fu) - fTu$ and appealing to (2.11.3) gives the result. □

Proposition 2.11.15. *Let H be a Hilbert-$C(X)$-module. Then $\mathcal{L}(H)$ is a C*-algebra.*

Proof. Write $B(H)$ for the Banach algebra of all bounded linear operators $H \to H$. We show that $\mathcal{L}(H)$ is a closed subalgebra of $B(H)$ and the the adjoint structure on $\mathcal{L}(H)$ makes $\mathcal{L}(H)$ a C*-algebra. Let $T, S \in \mathcal{L}(H)$ and $\lambda, \mu \in \mathbb{C}$. Then $\lambda T + \mu S$ is adjointable with adjoint $\overline{\lambda} T^* + \overline{\mu} S^*$, TS is adjointable with adjoint $(S^* T^*)$ and T^* is adjointable with adjoint T. Hence $\mathcal{L}(H)$ is a *-algebra. Moreover,

$$\|T\|^2 = \sup_{\|v\|_H \le 1} \langle Tv, Tv \rangle_H \tag{2.11.32}$$

$$= \sup_{\|v\|_H \le 1} \langle T^*Tv, v \rangle_H \tag{2.11.33}$$

$$\le \sup_{\|v\|_H \le 1} \|T^*Tv\|_H \|v\|_H \tag{2.11.34}$$

$$= \|T^*T\| \tag{2.11.35}$$

$$\le \|T\| \|T^*\|. \tag{2.11.36}$$

We deduce that $\|T\| \le \|T^*\|$, noting that the the case $\|T\| = 0$ is trivial. By replacing T with T^*, we get the reverse inequality; hence, $\|T\| = \|T^*\|$. Substituting this into (2.11.36), we get $\|T\|^2 = \|T^*T\|$. So the C*-identity holds for $\mathcal{L}(H)$.

Suppose $(T_n) \subseteq \mathcal{L}(H)$ converges in operator norm to $T \in B(H)$. Since $\|T_n^* - T_m^*\| = \|T_n - T_m\|$, the sequence $(T_n^*) \subseteq B(H)$ is Cauchy hence convergent. It now follows from the continuity of the $C(X)$-valued inner product that T is adjointable with adjoint $\lim_{n\to\infty} T_n^*$. $\square$

The next Proposition shows how the fibration of a Hilbert-$C(X)$-module H into Hilbert spaces $\{H_x\}_{x \in X}$ induces a fibration of $\mathcal{L}(H)$.

Proposition 2.11.16. *Let H be a Hilbert-$C(X)$-module.*

(a) Let $T \in \mathcal{L}(H)$ and $x \in X$. There exists a uniquely determined bounded linear operator $T_x : H_x \to H_x$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d] & H \arrow[d] \\ H_x \arrow[r, "T_x"] & H_x \end{tikzcd} \tag{2.11.37}$$

commutes, where the vertical arrows are the canonical quotient maps, i.e. $(Tu)(x) = T_x(u(x))$ for all $u \in H$.

(b) The map $T \mapsto T_x$ defined above is a *-homomorphism $\mathcal{L}(H) \to B(H_x)$ for each $x \in X$.

(c) The diagonal map

$$\Phi : \mathcal{L}(H) \to \prod_{x \in X} B(H_x)$$
$$T \mapsto (T_x)_{x \in X}$$

is an isometric *-homomorphism. In particular,

$$\|T\| = \sup_{x \in X} \|T_x\|. \tag{2.11.38}$$

Proof. (a) Fix $x \in X$. By combining Proposition 2.11.14 with Proposition 2.11.5, we see that $T(N_x) \subseteq N_x$. Hence, there is a unique bounded linear operator T_x of norm at most $\|T\|$ such that $T_x(u(x)) = (Tu)(x)$ for all $u \in H$.

(b) Fix $x \in X$. Let $T, S \in \mathcal{L}(H)$, $\lambda, \mu \in \mathbb{C}$ and $u \in H$. Then

$$(\lambda S + \mu T)(u)(x) = (\lambda(Su) + \mu(Tu))(x) \tag{2.11.39}$$
$$= \lambda(Su)(x) + \mu(Tu)(x) \tag{2.11.40}$$
$$= \lambda S_x(u(x)) + \mu T_x((u(x)) \tag{2.11.41}$$
$$= (\lambda S_x + \mu T_x)(u(x)). \tag{2.11.42}$$

Hence, by uniqueness, $(\lambda S + \mu T)_x = (\lambda S_x + \mu T_x)$. The proof that $(ST)_x = S_x T_x$ runs similarly.

Let $u, v \in H$ then

$$\langle T_x(u(x)), v(x)\rangle_{H_x} = \langle (Tu)(x), v(x)\rangle_{H_x} \tag{2.11.43}$$
$$= \langle Tu, v\rangle_H(x) \tag{2.11.44}$$
$$= \langle u, T^*v\rangle_H(x) \tag{2.11.45}$$
$$= \langle u(x), (T^*v)(x)\rangle_{H_x} \tag{2.11.46}$$
$$= \langle u(x), (T^*)_x(v(x))\rangle_{H_x}. \tag{2.11.47}$$

Hence, by uniqueness of the adjoint, $(T^*)_x = T_x^*$.

(c) Since each map $T \mapsto T_x$ is a *-homomorphism, so is Φ. Suppose $\Phi(T) = 0$ for some $T \in \mathcal{L}(H)$. Then, for all $u \in H$ and $x \in X$, $(Tu)(x) = T_x(u(x)) = 0$. Hence, by

(2.11.14) together with (2.11.3), $Tu = 0$. Therefore, $T = 0$. Since Φ is an injective *-homomorphism between C*-algebras, it is norm preserving.

□

We now turn to conjugate-adjointable operators. These are best defined in terms of conjugate Hilbert-$C(X)$-modules.

Definition 2.11.17. Let H be a Hilbert $C(X)$-module. A map $T : H \to H$ is *conjugate-adjointable* if T is adjointable when viewed as a map $T : H \to \overline{H}$.

Specialising to the case where X is a one point space, we get bounded conjugate-linear operators.

It an easy consequence of Definition 2.11.11, conjugate-adjointable maps satisfy

$$\langle Tu, v\rangle_H = \lambda u, T^*v\rangle_H^* \qquad (u, v \in H). \tag{2.11.48}$$

From Proposition 2.11.14 together with Definition 2.11.11, we see that conjugate-adjointable operators are conjugate-$C(X)$-linear. We record this as a proposition for ease of reference.

Proposition 2.11.18. *Let H be a Hilbert $C(X)$-module and $T : H \to H$ a conjugate-adjointable map. Then*

$$T(fv) = f^*Tv. \tag{2.11.49}$$

for all $f \in C(X)$ and $v \in H$.

If $T, S : H \to H$ are both conjugate-adjointable, then the products ST and TS are adjointable maps; if T is conjugate-adjointable and S is adjointable, then the products ST and TS are conjugate-adjointable.

We now state a version of Proposition 2.11.16 conjugate-adjointable operators.

Proposition 2.11.19. *Let H be a Hilbert-$C(X)$-module.*

(a) Let $T : H \to H$ be conjugate-adjointable and $x \in X$. There exists a uniquely determined bounded conjugate-linear operator $T_x : H_x \to H_x$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d] & H \arrow[d] \\ H_x \arrow[r, "T_x"] & H_x \end{tikzcd} \tag{2.11.50}$$

commutes, where the vertical arrows are the canonical quotient maps, i.e. $(Tu)(x) = T_x(u(x))$ for all $u \in H$.

(b) *Let $x \in X$. Suppose $T, S : H \to H$ are either conjugate-agjointable or adjointable. Then $(TS)_x = T_x S_x$.*

Proof. (a) By (2.11.49) together with Propositions 2.11.5 and 2.11.12, we see that $T(N_x) \subseteq N_x$. Hence, there is a unique bounded conjugate-linear operator T_x of norm at most $\|T\|$ such that $T_x(u(x)) = (Tu)(x)$ for all $u \in H$.

(b) Fix $x \in X$ and $u \in H$. Suppose $T, S : H \to H$ are either conjugate-agjointable or adjointable. Then, using (a) and Proposition 2.11.16(a), we get

$$(TS)_x(u(x) = (TSu)(x) \tag{2.11.51}$$
$$= T_x((Su)(x)) \tag{2.11.52}$$
$$= T_x(S_x(u(x)) \tag{2.11.53}$$
$$= (T_x S_x)(u(x)) \tag{2.11.54}$$

Hence, by uniqueness, $(TS)_x = T_x S_x$.

□

2.11.3 The Strict Topology

Definition 2.11.20. Let H be a Hilbert-$C(X)$-module. The strict topology on $\mathcal{L}(H)$ is given by the seminorms

$$\|T\|_v = \|Tv\|_H \qquad (v \in H), \tag{2.11.55}$$
$$\|T\|_{v,*} = \|T^* v\|_H \qquad (v \in H). \tag{2.11.56}$$

In the case $X = \{*\}$ is a one point space, the strict topology is just the strong* topology. A few of useful properties of the strong*-topology carry over to the strict topology with essentially the same proofs.

Proposition 2.11.21. *Let H be a Hilbert-$C(X)$-module.*

(a) *Addition $\mathcal{L}(H) \times \mathcal{L}(H) \to \mathcal{L}(H)$ is strictly continuous.*

(b) *Scalar multiplication $\mathbb{C} \times \mathcal{L}(H) \to \mathcal{L}(H)$ is strictly continuous.*

(c) *The involution $\mathcal{L}(H) \to \mathcal{L}(H)$ is strictly continuous.*

(d) *Multiplication $\mathcal{L}(H) \times \mathcal{L}(H) \to \mathcal{L}(H)$ is strictly continuous when restricted to $\|\cdot\|$-bounded regions.*

(e) *The strict topology is weaker than the norm topology.*

Proof. Addition and scalar multiplication are strictly continuous since the strict topology is defined by a set of seminorms, and the continuity of the involution is built into the definition. For multiplication we have the estimates

$$\|(S_1T_1 - S_2T_2)v\|_H \leq \|S_1\|\|(T_1 - T_2)v\|_H + \|(S_1 - S_2)T_2v\|_H, \tag{2.11.57}$$

$$\|(T_1^*S_1^* - T_2^*S_2^*)v\|_H \leq \|T_1^*\|\|(S_1 - S_2)^*v\|_H + \|(T_1 - T_2)^*S_2^*v\|_H \tag{2.11.58}$$

for all $v \in H$, from which the strict continuity of multiplication on $\|\cdot\|$-bounded regions follows. Finally, since $\|Tv\|$ and $\|T^*v\|$ are both dominated by $\|T\|\|v\|_H$, the strict topology is weaker than the norm topology. □

We now state the analogues of Theorem 2.2.1 and Proposition 2.2.2 for the strict topology.

Proposition 2.11.22. *Let H be a Hilbert-$C(X)$-module. The closed unit ball of $\mathcal{L}(H)$ is complete with respect to the strict topology.*

Proof. Since H is Banach space, we can use Theorem 2.2.1. Suppose $(T_\lambda)_{\lambda\in\Lambda}$ is Cauchy with respect to the strict topology and $\|T_\lambda\| \leq 1$ for all $\lambda \in \Lambda$. Then $(T_\lambda)_{\lambda\in\Lambda}$ and $(T_\lambda^*)_{\lambda\in\Lambda}$ are Cauchy sequences with respect to the strong operator topology on $B(H)$. By Theorem 2.2.1, there exist bounded linear operators T, S in the closed unit ball of $B(H)$ such that $T_\lambda \to T$ and $T_\lambda^* \to S$ with respect to the strong operator topology on $B(H)$.

By Proposition 2.11.2, we have $\langle Tv, w\rangle = \lim_\lambda \langle T_\lambda v, w\rangle = \lim_\lambda \langle v, T_\lambda^* w\rangle = \langle v, Sw\rangle$. Hence $T \in \mathcal{L}(H)$ with $T^* = S$. Since $T_\lambda \to T$ and $T_\lambda^* \to T$ with respect to the strong operator topology on $B(H)$, $T_\lambda \to T$ with respect to the strict topology. □

Proposition 2.11.23. *Let H be a Hilbert-$C(X)$-module and let A be a bounded subset of $\mathcal{L}(H)$. Suppose the $C(X)$-span of S is dense in V. Then the* strict topology *on A is induced by the family of seminorms $\{\|\cdot\|_v, \|\cdot\|_{v,*} : v \in S\}$.*

Proof. It follows from Proposition 2.11.2, that the families of seminorms $\{\|\cdot\|_v, \|\cdot\|_{v,*} : v \in S\}$ and $\{\|\cdot\|_v, \|\cdot\|_{v,*} : v \in \mathrm{span}_{C(X)}(S)\}$ induce the same topology on A. So we may assume without loss of generality that S is dense in H. Now Proposition 2.2.2, can be applied to see that $\{\|\cdot\|_v : v \in S\}$ induces the strong operator topology on $\mathcal{L}(H) \subseteq B(H)$. Therefore, $\{\|\cdot\|_v, \|\cdot\|_{v,*} : v \in S\}$ induces the strict topology on $\mathcal{L}(H)$. □

We now extend Proposition 2.11.16(b) by showing that the map $\mathcal{L}(H) \to \mathcal{L}(H_x)$ given by $T \mapsto T_x$ is continuous from the strict topology on $\mathcal{L}(H)$ to the strong* topology on $B(H_x)$.

Proposition 2.11.24. *Let H be a Hilbert-$C(X)$-module. The map $\mathcal{L}(H) \to B(H_x)$ given by $T \mapsto T_x$ defined in Proposition 2.11.16 is continuous from the strict topology on $\mathcal{L}(H)$ to the strong* topology on $B(H_x)$.*

Proof. Fix $x \in X$. Let $(T^{(\lambda)})_{\lambda\in\Lambda}$ be a net in $\mathcal{L}(H)$ converging strictly to T. Let $v_x \in H_x$. By definition, there is $v \in H$ such that $v(x) = v_x$. By Proposition 2.11.16(b), we have

$$\|(T_x^{(\lambda)} - T_x)v_x\|_{H_x} = \|(T^{(\lambda)} - T)_x v_x\|_{H_x} \tag{2.11.59}$$

$$= \|((T^{(\lambda)} - T)v)(x)\|_{H_x} \tag{2.11.60}$$

$$\leq \|\|((T^{(\lambda)} - T)v\|_H. \tag{2.11.61}$$

Hence $\|(T_x^{(\lambda)} - T_x)v_x\|_{H_x} \to 0$. Similarly $\|(T_x^{(\lambda)} - T_x)^* v_x\|_{H_x} \to 0$. □

In the case that the Hilbert module is countably generated, the strict topology on bounded subsets of $\mathcal{L}(H)$ is described by a $C(X)$-valued metric.

Proposition 2.11.25. *Let H be a Hilbert $C(X)$-module. Suppose $H = \overline{\operatorname{span}}_{C(X)}\{v_i : i \in \mathbb{N}\}$, where $v_i \in H$ and $\|v_i\|_H \leq 1$ for all $i \in \mathbb{N}$. Set*

$$d(T,S) = \sum_{i=0}^{\infty} \frac{1}{2^i}\left(\langle (T-S)v_i, (T-S)v_i\rangle^{1/2} + \langle (T-S)^* v_i, (T-S)^* v_i\rangle^{1/2}\right) \tag{2.11.62}$$

for $T, S \in \mathcal{L}(H)$.

(a) For all $T, S \in \mathcal{L}(H)$, $d(T,S) \in C(X)_+$.

(b) For all $T, S, R \in \mathcal{L}(H)$

  (i) $d(T,S) = 0$ if and only if $S = T$,

  (ii) $d(T,S) = d(T,S)$,

  (iii) $d(T,S) \leq d(T,R) + d(R,T)$.

(c) If (T_λ) is a uniformly bounded net in $\mathcal{L}(H)$ and $T \in \mathcal{L}(H)$, then $T_\lambda \to T$ strictly if and only if $d(T_\lambda, T) \to 0$ uniformly.

Proof. (a) For each $i \in \mathbb{N}$, $\langle (T-S)v_i, (T-S)v_i\rangle^{1/2} + \langle (T-S)^*v_i, (T-S)^*v_i\rangle^{1/2} \in C(X)$ and $\|\langle (T-S)v_i, (T-S)v_i\rangle^{1/2} + \langle (T-S)^*v_i, (T-S)^*v_i\rangle^{1/2}\|_{C(X)} \leq 2\|T-S\|$. Hence, the series defining $d(T,S)$ is absolutely convergent in $C(X)$. All terms of the series are positive by (2.11.3), so $d(T,S) \in C(X)_+$.

(b) (i) Clearly $d(T,T) = 0$. Suppose, $d(T,S) = 0$. Then $\langle (T-S)v_i, (T-S)v_i\rangle = 0$ for all $i \in \mathbb{N}$ by positivity, so $Tv_i = Sv_i$ for all for all $i \in \mathbb{N}$ by (2.11.3). By density, $T = S$.

(ii) We have $\langle (S-T)v_i, (S-T)v_i\rangle = \langle (T-S)v_i, (T-S)v_i\rangle$ and $\langle (S-T)^*v_i, (S-T)^*v_i\rangle = \langle (T-S)^*v_i, (T-S)^*v_i\rangle$ for all $i \in \mathbb{N}$ by (2.11.1).

(iii) Let $x \in X$. The map $(S,T) \mapsto \langle Sv_i, Tv_i\rangle^{1/2}(x)$ is a hermitian form, so $T \mapsto \langle Tv_i, Tv_i\rangle^{1/2}(x)$ is a seminorm. By the triangle inequality, we get $\langle (T-S)v_i, (T-S)v_i\rangle^{1/2}(x) \leq \langle (T-R)v_i, (T-R)v_i\rangle^{1/2}(x) + \langle (R-S)v_i, (R-S)v_i\rangle^{1/2}(x)$ and similarly for the stared terms. Consequently, $d(T,S)(x) \leq d(T,R)(x) + d(R,S)(x)$.

(c) Let (T_λ) be a uniformly bounded net in $\mathcal{L}(H)$ and $T \in \mathcal{L}(H)$. Choose $K > 0$ such that $\|T\|, \|T_\lambda\| \leq K$. Suppose $T_\lambda \to T$ strictly. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{i>N} \frac{1}{2^n} \leq \frac{\epsilon}{8K}$. Since $T_\lambda \to T$ strictly, there exist λ_0 such that

$$\sum_{i=0}^{N} \frac{1}{2^n}\left(\langle (T_\lambda - T)v_i, (T_\lambda - T)v_i\rangle^{1/2} + \langle (T_\lambda - T)^*v_i, (T_\lambda - T)^*v_i\rangle^{1/2}\right) \leq \frac{\epsilon}{2} \tag{2.11.63}$$

whenever $\lambda \geq \lambda_0$. Then

$$d(T_\lambda, T) \leq \frac{\epsilon}{2} + \sum_{i=N+1}^{\infty} \frac{1}{2^n}\left(\langle (T_\lambda - T)v_i, (T_\lambda - T)v_i\rangle^{1/2} + \langle (T_\lambda - T)^*v_i, (T_\lambda - T)^*v_i\rangle^{1/2}\right) \tag{2.11.64}$$

$$\leq \frac{\epsilon}{2} + \sum_{i=N+1}^{\infty} \frac{2\|T_\lambda - T\|}{2^n} \tag{2.11.65}$$

$$\leq \frac{\epsilon}{2} + 4K \sum_{i=N+1}^{\infty} \frac{1}{2^n} \tag{2.11.66}$$

$$< \epsilon \tag{2.11.67}$$

whenever $\lambda \geq \lambda_0$.

Conversely, suppose $d(T_\lambda, T) \to 0$ uniformly. For $i \in \mathbb{N}$, $\langle (T_\lambda - T)v_i, (T_\lambda - T)v_i\rangle^{1/2} \leq 2^i d(T_\lambda, T)$. Hence, $\|(T_\lambda - T)v_i\|_H \to 0$ for all $i \in \mathbb{N}$. It follows that $\|(T_\lambda - T)v_i\|_H \to 0$ for all $v \in \operatorname{span}_{C(X)}\{v_i : i \in \mathbb{N}\}$ by Proposition 2.11.2. Since (T_λ) is uniformly bounded, Proposition 2.2.2 implies that $\|(T_\lambda - T)v_i\|_H \to 0$ for all $v \in H$. Similarly, $\|(T_\lambda - T)^* v_i\|_H \to 0$ for all $v \in H$. Therefore, $T_\lambda \to T$ strictly.

□

Remark 2.11.26. Proposition 2.11.25 allows us to relate in the strict topology on $\mathcal{L}(H)$ to the strong* topology on the fibres. If $H = \overline{\operatorname{span}}_{C(X)}\{v_i : i \in \mathbb{N}\}$, where $v_i \in H$ and $\|v_i\|_H \le 1$. Then $H_x = \overline{\operatorname{span}}_{\mathbb{C}}\{v_i(x) : i \in \mathbb{N}\}$ for all $x \in H$. Hence, the metric

$$d_x(t,s) = \sum_{i=0}^{\infty} \frac{1}{2^i} \left(\|(t-s)v_i(x)\|_{H_x} + \|(t-s)^* v_i(x)\|_{H_x}\right) \tag{2.11.68}$$

for $t, s \in B(H_x)$, induces the strong* topology on bounded subsets. By Propositions 2.11.6 and 2.11.16, we have $d(T,S)(x) = d_x(T_x, S_x)$ for all $T, S \in \mathcal{L}(H)$.

We now turn to the Kaplansky Density Theorem [42, Theorem 1] for the strict topology. This is proved by the same argument as in the Hilbert space case. We adapt the proof from [58, Section 4.3]. Fix a Hilbert $C(X)$-module H.[7] The key definition is the following:

Definition 2.11.27. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is *strictly continuous* if $f(T_\lambda) \to f(T)$ strictly whenever $(T_\lambda) \subseteq \mathcal{L}(H)$ is a net of self-adjoint operators with $T_\lambda \to T$ strictly.

Equivalently, strictly continuous functions are those for which the functional calculus for the function f defines a strictly continuous map $\mathcal{L}(H)_{\mathrm{sa}} \to \mathcal{L}(H)$.

Lemma 2.11.28. *Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. Then f is strictly continuous*

Proof. Let A be the set of strictly continuous functions $\mathbb{R} \to \mathbb{C}$. By Proposition 2.11.21, A is a vector space and closed under complex conjugation. Moreover, if $f, g \in A$ and one of them is bounded, then $fg = gf \in A$ using the estimates (2.11.57) and (2.11.58). Let $A_0 = A \cap C_0(\mathbb{R})$. We shall show, using the Stone–Weierstrass Theorem, that $A_0 = C_0(\mathbb{R})$.

Consider the functions $f, g : \mathbb{R} \to \mathbb{C}$ given by $f(x) = (1+x^2)^{-1}$ and $g(x) = x(1+x^2)^{-1}$. Note that $\|f\|_{C_0(\mathbb{R}}, \|f\|_{C_0(\mathbb{R}} \le 1$. Let $T, S \in \mathcal{L}(H)_{\mathrm{sa}}$. We compute that

$$\begin{aligned} g(T) - g(S) &= T(1+T^2)^{-1} - S(1+S^2)^{-1} && (2.11.69)\\ &= (1+T^2)^{-1}(T(1+S^2) - (1+T^2)S)(1+S^2)^{-1} && (2.11.70)\\ &= (1+T^2)^{-1}(T - S - T(S-T)S)(1+S^2)^{-1}. && (2.11.71) \end{aligned}$$

[7] In fact, the Kaplansky Density Theorem for the strict topology holds for general Hilbert-A-modules by essentially the same proof, but we state the result in the commutative case only.

Therefore, if $v \in H$, then

$$\|g(T) - g(S)\|_v \leq \|(1+T^2)^{-1}(T-S)(1+S^2)^{-1}(v)\|_H \tag{2.11.72}$$
$$+ \|(1+T^2)^{-1}T(S-T)S)(1+S^2)^{-1}(v)\|_H \tag{2.11.73}$$
$$\leq \|(1+T^2)^{-1}\|_{\mathcal{L}(H)}\|T-S\|_{(1+S^2)^{-1}(v)} \tag{2.11.74}$$
$$+ \|(1+T^2)^{-1}T\|_{\mathcal{L}(H)}\|S-T\|_{S(1+S^2)^{-1}(v)} \tag{2.11.75}$$
$$\leq \|T-S\|_{(1+S^2)^{-1}(v)} + \|S-T\|_{S(1+S^2)^{-1}(v)}. \tag{2.11.76}$$

Noting that $\|g(T) - g(S)\|_{v,*} = \|g(T) - g(S)\|_v$, as g is real valued, we see that $g \in A_0$. Since the map $x \mapsto x$ is is strictly continuous, we get that $f = 1 - xg \in A_0$.

The set $\{f, g\}$ separates the points of $\mathbb{R}$ and $f(t) > 0$ for all $t \in \mathbb{R}$. Therefore, f and g generate the C*-algebra $C_0(\mathbb{R})$ by the Stone–Weierstrass Theorem. Thus, $A_0 = C_0(\mathbb{R})$.

Suppose $h \in C_b(\mathbb{R})$. Then $hf, hg \in C_0(\mathbb{R})$, so $hf, hg \in A$. Therefore, $h = hf + xhg \in A$. □

Theorem 2.11.29 (The Kaplansky Density Theorem for the strict topology)**.** *Let H be a Hilbert-$C(X)$-module. Let A be a C^*-subalgebra of $\mathcal{L}(H)$ with strict closure B.*

(i) A_{sa} is strictly dense in B_{sa}.

(ii) The closed unit ball of A_{sa} is strictly dense in the closed unit ball of B_{sa}.

(iii) The closed unit ball of A is strictly dense in the closed unit ball of B.

Proof. (i) Let $b \in B_{sa}$ and (a_λ) be a net in A converging strictly to b. Then $(\frac{1}{2}(a_\lambda + a_\lambda^*))$ is a net in A_{sa} and converges strictly to $\frac{1}{2}(b + b^*) = b$ by Proposition 2.11.21.

(ii) Let $b \in B_{sa}$ with $\|b\| \leq 1$ and (a_λ) be a net in A_{sa} converging strictly to b. Let $f : \mathbb{R} \to \mathbb{C}$ be the bounded, continuous function given by

$$f(x) = \begin{cases} -1, & x \leq 1, \\ x, & -1 \leq x \leq 1, \\ 1, & x \geq 1. \end{cases} \tag{2.11.77}$$

Then $(f(a_\lambda))$ is a net in the closed unit ball of A_{sa} converging strictly to b by Lemma 2.11.28.

(iii) This follows from a matrix inflation trick. The direct sum $H \oplus H$ has a natural Hilbert-$C(X)$-structure where $\langle (u_1, u_2), (v_1, v_2) \rangle_{H \oplus H} = \langle u_1, v_1 \rangle_H + \langle u_2, v_2 \rangle_H$ in analogy with the Hilbert space direct sum. On can easily show that $\mathcal{L}(H \oplus H) \cong M_2(\mathcal{L}(H))$ and that strict convergence on $\mathcal{L}(H \oplus H)$ corresponds to strict convergence in each entry of $M_2(L(H))$. From this, it follows that $M_2(A)$ is a C*-subalgebra of $L(H \oplus H)$ with strict closure $M_2(B)$ and we can apply (ii) to this inflation.

Suppose $b \in B$ has $\|b\| \leq 1$. Then $\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \in M_2(B)_{sa}$ and has norm at most 1. There is a net (a_λ) in the closed unit ball of $M_2(A)_{sa}$ converging strictly to $\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \in M_2(B)_{sa}$. Taking the $(1,2)$-th entries of this net gives a net in the closed unit ball of A converging strictly to b. $\square$

Corollary 2.11.30. *Let H be a Hilbert-$C(X)$-module. Let A be a C*-subalgebra of $\mathcal{L}(H)$. Then A is strictly separable if and only if the closed unit ball of A is strictly separable.*

Proof. If the closed unit ball of A is strictly separable with countable dense subset D, then $\bigcup_{n \in \mathbb{N}} nD$ is a countable dense subset of A for the strict topology.

Suppose A is strictly separable with countable dense subset D. Let B be the C*-algebra generated by D. The unit ball of B is strictly dense in the unit ball of A by Theorem 2.11.29. Since B is countably generated it is $\|\cdot\|$-separable. Hence the unit ball of B is also $\|\cdot\|$-separable. Let D' be a countable $\|\cdot\|$-dense subset of the unit ball of B. Since $\overline{D'}^{\mathrm{st}} \supseteq \overline{D'}^{\|\cdot\|}$, $\overline{D'}^{\mathrm{st}}$ contains the unital ball of B. Therefore, $\overline{D'}^{\mathrm{st}}$ contains the unit ball of A. $\square$

2.12 General Topological Bundles

In this section, we develop a very general notion of a bundle over a topological space. This definition will not require that the bundle is locally trivial or even that the isomorphism class of the fibre is locally constant. The reason for introducing this general notation of bundle is that the bundle-like objects that occur naturally in functional analysis aren't necessarily locally trivial.

This section is heavy inspired by the book [26], in particular its treatment of Banach bundles [26, Chapter 2, Section 13.4]. Unfortunately, tracial von Neumann algebras are not Banach spaces with respect to the $\|\cdot\|_2$-norm because only the $\|\cdot\|$-unit ball is complete in $\|\cdot\|_2$-norm not the whole space. Thus, the results of [26, Chapter 2, Section 13.4] cannot be applied directly to the study of W*-bundles in Section 3.6. We, therefore, work with bundles of normed spaces in Section 2.12.2 and carefully explain what can additionally be proved if the fibres are complete.

2.12.1 Bundles and Sections

Definition 2.12.1. A *bundle* over a Hausdorff topological space X is a pair (B, p) where B is a Hausdorff topological space and $p : B \to X$ is a continuous, open surjection. The fibre at $x \in X$ is the set $p^{-1}(x)$.

We say that the bundle (B_1, p_1) over X_1 is isomorphic to the bundle (B_2, p_2) over X_2 if there are homeomorphisms ψ and φ such that the diagram

$$\begin{tikzcd} B_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] & B_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "\psi"] & X_2 \end{tikzcd} \tag{2.12.1}$$

commutes.

The standard example of a bundle over X is the product space $X \times Y$ for some Hausdorff space Y together with the projection map p on to the first co-ordinate. Note that, in this example, each fibre $p^{-1}(x)$ is canonically homeomorphic to Y via the map $(x, y) \mapsto y$. We call this bundle the *trivial bundle over X with fibre Y*; any bundle isomorphic to this bundle is deemed trivial.

If (B, p) is a bundle over X and $A \subseteq X$ then $(p^{-1}(A), p|_{p^{-1}(A)})$ is also a bundle and is known as the restriction of (B, p) to A. In algebraic topology, one considers almost exclusively bundles that are *locally trivial*, that is bundles (B, p) with the property that for all points x in the base space X have an open neighbourhood U such that the bundle restricted to U is trivial.

In functional analysis, non-locally trivial bundles arise naturally. Indeed, the isomorphism class of the fibre may not be locally constant. The general definition of bundle given above allows for such possibilities.

We now come to the definition of the sections of a bundle.

Definition 2.12.2. Let (B, p) be a bundle over X. A *section* of (B, p) is a map $f : X \to B$ such that $p \circ f = \mathrm{id}_X$.

We are, of course, mostly interested in continuous sections, but will on occasion have need for non-continuous sections. We shall also speak of local sections, that is sections which are defined only on a subset of X or equivalently sections of an appropriate restriction of the bundle.

Definition 2.12.3. A bundle (B, p) over X is said to have *sufficiently many continuous sections* if for every $b \in B$ there is a continuous section $f : X \to B$ with $f(p(b)) = b$.

Note that, if the bundle (B, p) over X is has sufficiently many continuous sections and U is a neigbourhood of $b \in B$, then $p(U) \supseteq f^{-1}(U)$, so $p(U)$ is a neighbourhood of $p(b)$. Consequently, the openness of the map p is necessary for there to be sufficiently map continuous sections.

2.12.2 Bundles of Normed Spaces

The bundles that arise in functional analysis, tend to have fibres which are normed vector spaces (or even Banach spaces). Denoting the bundle (B, p) and the base space X, this gives rise to the following global functions: addition $+ : D \to B$, where $D = \{(b_1, b_2) : B \times B : p(b_1) = p(b_2)\}$; scalar multiplication $\cdot : \mathbb{C} \times B \to B$; and norm $\|\cdot\| : B \to [0, \infty)$. Moreover, each fibre contains a distinguished zero element $0_x \in p^{-1}(x)$, and the map $x \mapsto 0_x$ is a distinguished section of the bundle. We can now state some axioms for such bundles.

Definition 2.12.4. A *bundle of normed spaces* (respectively Banach spaces) is a bundle (B, p) over X where each fibre $p^{-1}(x)$ has the additional structure of a normed vector space (respectively Banach spaces) and the following axioms are satisfied:

(i) The global norm $\|\cdot\| : B \to [0, \infty)$ is continuous.

(ii) The global addition $+ : D \to B$ is continuous.

(iii) For each $\lambda \in \mathbb{C}$, that map $B \to B : b \mapsto \lambda b$ is continuous.

(iv) A net (b_i) in B converges to 0_x provided both $\|b_i\| \to 0$ and $p(b_i) \to x$.

We say that the bundle (B_1, p_1) over X_1 is isomorphic to the bundle (B_2, p_2) over X_2 if there are homeomorphisms ψ and φ such that the diagram

$$\begin{tikzcd} B_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] & B_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "\psi"] & X_2 \end{tikzcd} \tag{2.12.2}$$

commutes, and for all $x_1 \in X_1$, $\varphi|_{p_1^{-1}(x_1)} : p_1^{-1}(x_1) \to p_2^{-1}(\psi(x_1))$ is an isomorphism of normed vector spaces.[8]

[8] One could also consider the stronger notion of isometric isomorphism, where each $\varphi|_{p_1^{-1}(x_1)} : p_1^{-1}(x_1) \to p_2^{-1}(\psi(x_1))$ is required to be norm preserving.

Proposition 2.12.5. *The zero section $B \to B : x \mapsto 0_x$ and the global scalar multiplication map $\cdot : \mathbb{C} \times B \to B$ are continuous.*

Proof. Let (x_i) be a net in X converging to x. Since $p(0_{x_i}) = x_i \to x$ and $\|0_{x_i}\| = 0 \to 0$, Axiom (iv) ensures that $0_{x_i} \to 0_x$.

Let (λ_i) be a net in $\mathbb{C}$ converging to λ and (b_i) a net in B converging to B. (Without loss of generality, we assume they have the same indexing set I.) Then $p(\lambda_i b_i - \lambda b_i) = p(b_i) \to p(b)$ and $\|\lambda_i b_i - \lambda b_i\| = |\lambda_i - \lambda| \|b_i\| \to 0$, since by Axiom (i) $\|b_i\| \to \|b\|$. Hence, by Axiom (iv) $\lambda_i b_i - \lambda b_i \to 0_{p(b)}$. Since $\lambda b_i \to \lambda b$ by Axiom (iii) and since addition is continuous by Axiom (ii), we have $\lambda_i b_i \to \lambda b$. □

Example 2.12.6. If (B, p) is the trivial bundle over of X with fibre Y and Y is a normed vector space, then we get an induced normed vector space structure on all the fibres via the canonical homomorphism. The veracity of the Axioms (i-iv) is an easy consequence of the definition of the product topology. The restriction of a bundle of normed space is also a bundle of normed space, and we can define locally trivial bundles of normed vector spaces.

Remark 2.12.7. Axiom (iv) can be reformulated as follows: Given $x \in X$, the sets $B(U, \epsilon) = \{b \in B : p(b) \in U, \|b\| < \epsilon\}$ as U ranges over open neighbourhood of x and ϵ ranges over positive real numbers are a neighbourhood basis for 0_x. One can view this axiom, as a weakening of local triviality.

We now turn to the sections of a bundle (B, p) of normed spaces over X. A consequence of Axioms (ii) and (iii) together with the continuity of the zero section is that the set $\Gamma(B, p)$ of all continuous section of a bundle of (B, p) is a vector space under fibrewise operations. If one wishes to have a normed space of continuous sections, one must restrict attention to the subspace $\Gamma_b(B, p)$ of bounded, continuous sections, that is sections $f : X \to B$ of a bundle such that $\sup_{x \in X} \|f(x)\| < \infty$. One can then define a uniform norm $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$. In the case X compact, then $\Gamma(B, p) = \Gamma_b(B, p)$.

In the following proposition, we show that continuity of sections is preserved under uniform limits. The proof is essentially that of [26, Corollary II.13.13].

Proposition 2.12.8. *Let (B, p) be a bundle of normed spaces over X. Suppose the sections $f_n : X \to B$ converge uniformly to the section $f : X \to B$. If each f_n is continuous, then so is f.*

Proof. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X converging to $x \in X$. Then $p(f(x_\lambda)) = x_\lambda \to x = p(f(x))$ as $\lambda \to \infty$. Since p is an open map, there exists a subnet $(x'_\mu)_{\mu\in M}$ of $(x_\lambda)_{\lambda\in\Lambda}$ and a net $(b_\mu)_{\mu\in M}$ in B such that

$$p(b_\mu) = x'_\mu \qquad (\mu \in M), \tag{2.12.3}$$

$$\lim_{\mu\to\infty} b_\mu = f(x). \tag{2.12.4}$$

Indeed, let $M = \Lambda \times N_{f(x)}$, where $N_{f(x)}$ is the net of open neighbourhoods of $f(x)$ with the direction given by reverse inclusion. For $\mu = (\lambda, U) \in M$, choose $\lambda' \in \Lambda$ such that $\lambda' \geq \lambda$ and $x_{\lambda'} \in p(U)$, existence of such a λ' being guaranteed as p is open, and set $x'_\mu = x_{\lambda'}$. By construction $(x'_\mu)_{\mu\in M}$ is a subnet of of $(x_\lambda)_{\lambda\in\Lambda}$ and both (2.12.3) and (2.12.4) hold.

Let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\|f_n(y) - f(y)\| < \frac{\epsilon}{2}$ for all $y \in X$. The continuity of f_n together with Axioms (i-iii) ensures that $\|f_n(x'_\mu) - b_\mu\| \to \|f_n(x) - f(x)\|$ as $\mu \to \infty$. Hence, there is $\mu_0 \in M$ such that $\|f_n(x'_\mu) - b_\mu\| < \frac{\epsilon}{2}$ whenever $\mu \geq \mu_0$. Therefore,

$$\|f(x'_\mu) - b_\mu\| \leq \|f(x'_\mu) - f_n(b_\mu)\| + \|f_n(x'_\mu) - b_\mu\| \tag{2.12.5}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \tag{2.12.6}$$

$$= \epsilon \tag{2.12.7}$$

whenever $\mu \geq \mu_0$. Thus, we have shown that $\|f(x'_\mu) - b_\mu\| \to 0$ as $\mu \to \infty$. Invoking Axiom (iv), we find that $f(x'_\mu) - b_\mu \to 0_x$ in B as $\mu \to \infty$. By Axiom (i), $f(x'_\mu) = (f(x'_\mu) - b_\mu) + b_\mu \to f(x)$ as $\mu \to \infty$.

This proves that f is continuous for, if not, there would exist $x \in X$, an open neighbourhood V of $f(x)$ and a net $(x_\lambda)_{\lambda\in\Lambda}$ converging to $x \in X$ such that $f(x_\lambda) \notin V$ for all $\lambda \in \Lambda$; hence, no subnet $(x'_\mu)_{\mu\in M}$ of $(x_\lambda)_{\lambda\in\Lambda}$ could have the property $\lim_{\mu\to\infty} f(x'_\mu) = f(x)$. □

Corollary 2.12.9. *Suppose (B, p) is a bundle of Banach spaces over X. Then $\Gamma_b(B, p)$ is a Banach space.*

Proof. The space of all bounded sections of (B, P), is a Banach space isomorphic to the product $\prod_{x\in X} p^{-1}(x)$. By Proposition 2.12.8, $\Gamma_b(B, p)$ is a closed subspace of the Banach space of all bounded sections. □

Finally, we come to the question of whether a bundle has sufficiently many sections. If a bundle of normed space has sufficiently many continuous sections, then its topology is completely determined by these continuous sections, as the following result shows.

Proposition 2.12.10. *Let (B,p) be a bundle of normed vector spaces over X with sufficiently many continuous sections. Then a basis for the topology of B is given by the the sets $B(U,f,\epsilon) = \{b \in B : p(b) \in U, \|f(p(b)) - b\| < \epsilon\}$ as U ranges over all open sets in X, f ranges over all continuous sections, and ϵ ranges over all positive real numbers.*

Proof. Let $b_0 \in B$ and $x_0 = p(b)$. Since (B,p) has sufficiently many continuous sections, there is a continuous section f_0 such that $f_0(x_0) = b_0$. Since f is continuous, the map $\Phi : B \to B$ given by $b \mapsto b - f(p(b))$ is a homeomorphism with inverse given by $b \mapsto b + f(p(b))$. Since $\Phi(b_0) = 0_{x_0}$, we deduce from Remark 2.12.7 that the sets $B(U, f_0, \epsilon) = \{b \in B : p(b) \in U, \|f_0(p(b)) - b\| < \epsilon\}$ as U ranges over open neighbourhood of x_0, and ϵ ranges over positive real numbers are a neighbourhood basis for b_0. □

Directly from the axioms, one can only deduce the existence of a single continuous section: the zero section. However, if one makes some assumptions on the base space, progress can be made. The follow result is due to Douady and dal Soglio-Herault and appears as an appendix to [25] and in [26, Appendix C].

Theorem 2.12.11. *Let X be either paracompact or locally compact. The any bundle of Banach spaces over X has sufficiently many continuous sections.*

2.12.3 Constructing Continuous Sections

This section is devoted to providing an overview of the proof of Theorem 2.12.11. It is heavily based on the extremely detailed presentation found in [26, Appendix C]. However, since we shall later wish to adapt the construction, special effort will be made to isolate where completeness of the fibres comes into the picture. Hereinafter, let X be a Hausdorff space and (B,p) a bundle of normed spaces over X.

The key concepts introduced by Douady and dal Soglio-Herault are ϵ-thin sets and ϵ-continuous sections.

Definition 2.12.12. Let $\epsilon > 0$. A subset U of B is *ϵ-thin* if $\|b - b'\| < \epsilon$ whenever $b_1, b_2 \in U$ and $p(b_1) = p(b_2)$.

Definition 2.12.13. Let $x \in X$. A section $f : X \to B$ is *ϵ-continuous at x* if there is a neighbourhood V of x and an ϵ-thin neighbourhood U of $f(x)$ such that $f(V) \subseteq U$. A section $f : X \to B$ is *ϵ-continuous* if it is ϵ-continuous at all points $x \in X$.

We begin with the basic existence results.

Proposition 2.12.14. *For all $\epsilon > 0$, every $b \in B$ has an ϵ-thin open neighbourhood.*

Proof. Let $b \in B$. The map $\sigma : D \to B$ given by $(b_1, b_2) \mapsto b_1 - b_2$ is continuous on its domain $D = \{(b_1, b_2) : B \times B : p(b_1) = p(b_2)\}$. Applying the definition of continuity of σ at (b, b), there is an open set U containing b such that $\sigma((U \times U) \cap D) \subseteq B(X, \frac{\epsilon}{2})$. The set U is an open ϵ-thin neighbourhood of b. □

Proposition 2.12.15. *Suppose X is completely regular. For all $\epsilon > 0$, $x_0 \in X$ and $b_0 \in p^{-1}(x_0)$, there is an ϵ-continuous section f with $f(x_0) = b_0$.*

Proof. Let U be an open ϵ-thin neighbourhood of b_0. The set $V = p(U)$ is an open neighbourhood of x_0. By the axiom of choice, there is a local section $f_V : V \to B$ such that $f_V(V) \subseteq U$. Complete regularity of X implies the existence of a continuous bump function $\phi : X \to [0, 1]$ supported on a closed set $F \subseteq V$ and with $\phi(x_0) = 1$. We can define $f : X \to B$ by

$$f(x) = \begin{cases} \phi(x) f_V(x), & x \in V, \\ 0, & x \notin V. \end{cases} \tag{2.12.8}$$

The function f is easily seen to be an ϵ-continuous section. □

The following lemma uses the concept of ϵ-thin sets to describe neighbourhood bases in B.

Lemma 2.12.16. *Let $b \in B$ and let $(U_i)_{i \in I}$ be a decreasing net of open neighbourhoods of b such that*

(i) $\{p(U_i) : i \in I\}$ is a neighbourhood basis for $p(b)$,

(ii) U_i is ϵ_i-thin and $\lim_i \epsilon_i = 0$.

Then $\{U_i : i \in I\}$ is a neighbourhood basis for b.

Proof. Write $x = p(b)$ and $V_i = p(U_i)$ for $i \in I$. Using the notation of Remark 2.12.7, the sets $B(V_i, \epsilon_i)$ form a neighbourhood basis for 0_x.

Let W be an open neighbourhood of b in B. By Axiom (i), there is an open neighbourhood W' of b and an index $i \in I$ such that $b' + b'' \in W$ whenever $b' \in W'$, $b'' \in V(V_i, \epsilon_i)$ and $p(b') = p(b'')$.

Since $p(W' \cap U_i)$ is an open neighbourhood of x, there is $j \geq i$ such that $U_j \subseteq p(W' \cap U_i)$. Since $(U_i)_{i \in I}$ is decreasing, $U_j \subseteq U_i \cap p^{-1} p(W' \cap U_i)$.

Let $c \in U_j$. Then $c \in U_i$ and there exists $c' \in W' \cap U_i$ with $p(c') = p(c)$. As U_i is ϵ_i-thin, $\|c - c'\| < \epsilon_i$, i.e. $c - c' \in B(V_i, \epsilon_i)$. Therefore, $c = c' + (c - c') \in W$, and hence $U_j \subseteq W$. □

We now can give one of the key lemmas of Douady and dal Soglio-Herault.

Lemma 2.12.17. *Suppose the section f is ϵ-continuous for all $\epsilon > 0$. Then f is continuous.*

Proof. Let $x \in X$. For each $n \in \mathbb{N}$, there is an open neighbourhood V_n of x and an n^{-1}-thin open open neighbourhood U_n of $f(x)$ such that $f(V_n) \subseteq U_n$. Replacing U_n with $U_n \cap p^{-1}(V_n)$, we may assume without lose of generality that $f(p(U_n)) \subseteq U_n$. Replacing U_n with $U_1 \cap \cdots \cap U_n$, we may further assume that the U_n are decreasing.

It now follows from Lemma 2.12.16 that the sets $U_n \cap p^{-1}(V)$ form a neighbourhood basis for $f(x)$ as n ranges over the natural numbers and V ranges over a neighbourhood basis for x. Since $f^{-1}(U_n \cap p^{-1}(V)) \supseteq p(U_n) \cap V$, f is continuous at x. □

The following result combined with the previous lemma indicate how one could construct continuous sections.

Lemma 2.12.18. *Let $\epsilon > 0$. Suppose the sections f_n are all ϵ-continuous and converge uniformly to f. Then f is ϵ'-continuous for all $\epsilon' > \epsilon$.*

Proof. Let $\epsilon' > \epsilon$. Choose $n \in \mathbb{N}$ such that $\|f_n(y) - f(y)\| < \frac{\epsilon'-\epsilon}{2}$ for all $y \in X$. Now let $x \in X$. Since f_n is ϵ-continuous there is an ϵ-thin open neighbourhood U of $f_n(x)$ and an open neighbourhood V of x such that $f_n(V) \subseteq U$. Replacing U with $U \cap p^{-1}(V)$, we may assume that $p(U) = V$.

Set $W = \{b + c : b \in U, c \in B(V, \frac{\epsilon'-\epsilon}{2}), p(b) = p(c)\}$. Then $f(x) = f_n(x) + (f(x) - f_n(x)) \in W$ whenever $x \in V$, and the following calculation shows that W is ϵ'-thin:

$$\|(b_1 + c_1) - (b_2 + c_2)\| \leq \|b_1 - b_2\| + \|c_1\| + \|c_2\| \tag{2.12.9}$$

$$< \epsilon + \frac{\epsilon' - \epsilon}{2} + \frac{\epsilon' - \epsilon}{2} \tag{2.12.10}$$

$$= \epsilon', \tag{2.12.11}$$

where $b_1, b_2 \in U$, $c_1, c_2 \in B(V, \frac{\epsilon'-\epsilon}{2})$ and $p(b_1) = p(b_2) = p(c_1) = p(c_2)$.

It remains to show that W is an open set. Write $D = \{(b_1, b_2) : B \times B : p(b_1) = p(b_2)\}$. Then the map $\Phi : D \to D$ given by $(b_1, b_2) \to (b_1 + b_2, b_1 - b_2)$ is a homeomorphism, and the map $\Psi : D \to B$ given by $(b_1, b_2) \mapsto b_1$ is open and continuous. The set W is the image of the open set $D \cap (U \times B(V, \frac{\epsilon'-\epsilon}{2}))$, which is open in D, under the composition $\Psi \circ \Phi$. Hence, W is open in B. $\square$

Key to the proof of Theorem 2.12.11, is a partition of unity argument. This step is isolated in Proposition 2.12.20 and the following technical lemma.

Lemma 2.12.19. *Let $\phi_1, \dots, \phi_n : X \to [0,1]$ be a continuous partition of unity. Then $\sum_{i=1}^n \phi_i f_i$ is ϵ-continuous whenever the f_i are all ϵ-continuous.*

Proof. Let $x_0 \in X$. Then some ϕ_{i_0} is non-zero on a neighbourhood V_0 of x_0. Without loss of generality, assume $i_0 = 1$. For $i = 1, \dots, n$, there is an open neighbourhood V_i of X and an ϵ-thin open neighbourhood U_i of $f_i(x)$ such that $f_i(V_i) \subseteq U_i$. Set $V = V_0 \cap \dots \cap V_n$ and replace each U_i with $U_i \cap p^{-1}(V)$.

Set $W = \{\sum_{i=1}^n \phi_i(x) b_i : x \in V, b_i \in U_i, p(b_i) = x\}$. Then $\sum_{i=1}^n \phi_i(x) f_i(x) \in W$ whenever $x \in V$ and W is ϵ-thin since

$$\|\sum_{i=1}^n \phi_i(x) b_i - \sum_{i=1}^n \phi_i(x) b_i'\| \leq \sum_{i=1}^n \phi_i(x) \|b_i - b_i'\| \tag{2.12.12}$$

$$< \sum_{i=1}^n \phi_i(x) \epsilon \tag{2.12.13}$$

$$= \epsilon, \tag{2.12.14}$$

where $x \in V$, $b_i, b_i' \in U_i$ and $p(b_i) = p(b_i') = x$.

It remains to prove that W is open in B. Write $E = \{(b_1, b_2 \dots, b_n) \in B^n : p(b_1) = p(b_2) = \dots = p(b_n) \in V\}$. Then the map $\Phi : E \to E$ given by $(b_1, b_2, \dots, b_n) \to (\sum_{i=1}^n \phi_i(x) b_i, b_2 \dots, b_n)$, where $x = p(b_1)$, is a homeomorphism[9], and the map $\Psi : E \to p^{-1}(V)$ given by $(b_1, b_2, \dots, b_n) \mapsto b_1$ is open and continuous. The set W is the image of the open set $E \cap (U_1 \times U_2 \times \dots \times U_n)$, which is open in D, under the composition $\Psi \circ \Phi$. Hence, W is open in B. $\square$

Proposition 2.12.20. *Assume X is paracompact. Let $\epsilon > 0$, $x_0 \in X$ and let $f : X \to B$ be an ϵ-continuous section. Then there is an $\frac{\epsilon}{2}$-continuous section $f' : X \to B$ such that $\|f(x) - f'(x)\| < \frac{3}{2}\epsilon$ for all $x \in X$ and $f'(x_0) = f(x_0)$.*

Proof. Let $x \in X$. Let U_1 be an ϵ-thin open neighbourhood of $f(x)$ such that $f(p(U_1)) \subseteq U_1$. By Lemma 2.12.15, there exists an $\frac{\epsilon}{2}$-continuous function $f^{(x)}$ such that $f^{(x)}(x) = f(x)$. Let U_2 be an $\frac{\epsilon}{2}$-thin open neighbourhood of $f^{(x)}(x) = f(x)$ such that $f^{(x)}(p(U_2)) \subseteq U_2$.

[9] The inverse is given by $(b_1, b_2, \dots, b_n) \to (\phi_1(x)^{-1} \left(b_1 - \sum_{i=2}^n \phi_i(x) b_i\right), b_2 \dots, b_n)$

Set $V^{(x)} = p(U_1 \cap U_2)$. Let $y \in V^{(x)}$. Then there exist $b \in U_1 \cap U_2$ with $p(b) = y$ and $\|f(y) - b\| < \epsilon$, since U_1 is ϵ-thin, and $\|f^{(x)}(y) - b\| < \frac{\epsilon}{2}$, since U_2 is $\frac{\epsilon}{2}$-thin. Consequently,

$$\|f(y) - f^{(x)}(y)\| < \frac{3}{2}\epsilon \qquad (y \in V^{(x)}). \tag{2.12.15}$$

Since X is paracompact there is a locally finite refinement $\{V_i : i \in I\}$ of the open cover $\{V^{(x)} : x \in X\}$. Suppose $x_0 \in V_{i_0}$. As X is paracompact, X is regular. Hence, there is an open neighbourhood Z of x_0 such that $\overline{Z} \subseteq V_{i_0}$. Replacing V_i with $V_i \setminus \overline{Z}$ for $i \neq i_0$, we may assume that $x_0 \notin V_i$ for $i \neq i_0$. Let $f_{i_0} = f^{(x_0)}$ and, for $i \neq i_0$, let f_i denote one of the $f^{(x)}$ for which $V_i \subseteq V^{(x)}$.

Let $(\phi_i)_{i\in I}$ be a partition of unity subordinate to $\{V_i : i \in I\}$. Set $f' = \sum_{i\in I} \phi_i f_i$. Since in a neighbourhood of any point only finitely many terms of the sum are non-zero, Lemma 2.12.19 ensures that f' is $\frac{\epsilon}{2}$-continuous. Furthermore, if $y \in X$, then

$$\|f(y) - f'(y)\| = \|\sum_{i\in I} \phi_i(y)(f(y) - f_i(y))\| \tag{2.12.16}$$

$$\leq \sum_{i\in I} \phi_i(y)\|f(y) - f_i(y)\| \tag{2.12.17}$$

$$< \sum_{i\in I} \phi_i(y)\frac{3}{2}\epsilon \tag{2.12.18}$$

$$= \frac{3}{2}\epsilon. \tag{2.12.19}$$

Finally, since $x_0 \notin V_i$ for $i \neq i_0$, $f'(x_0) = f^{(x_0)}(x_0) = f(x_0)$. □

We can now give the proof of Theorem 2.12.11.

Proof of Theorem 2.12.11. We first prove the result for X paracompact. Fix $x_0 \in X$ and $b_0 \in p^{-1}(x)$. By Proposition 2.12.15, there is 1-continuous section f_0 with $f_0(x_0) = b_0$. Using Proposition 2.12.20, we inductively construct a sequence of sections $f_n : X \to B$ with $f_n(x_0) = b_0$ such that f_n is $\frac{1}{2^n}$-continuous and $\|f_n(x) - f_{n-1}(x)\| < \frac{3}{2^n}$ for all $x \in X$ and $n \in \mathbb{N}$.

Since the series $\sum_{i=0}^{\infty} \frac{3}{2^i}$ converges, the sequence of section $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy, i.e. for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_n(x) - f_m(x)\| < \epsilon \tag{2.12.20}$$

whenever $x \in X$ and $m, n \geq N$.

As fibres of the bundle are Banach spaces, f_n converges pointwise to some section f. Fixing x while letting $m \to \infty$ in (2.12.20), we see that f_n converges uniformly to f. This section will by ϵ-continuous for all $\epsilon > 0$ by Lemma 2.12.18 and thus continuous by Lemma 2.12.17.

The result for X locally compact is deduced by working first on a compact neighbourhood of b_0, applying the result for the paracompact case, then multiplying by a suitable bump function. $\square$

Chapter 3

An Introduction to W*-Bundles

W*-bundles were introduced by Ozawa in [62], motivated by work on the Toms–Winter Conjecture in the classification programme for C*-algebras (see Chapter 1). The abstract theory of W*-bundles was then developed further in [5] and [23]. In this chapter, we present the axiomatic definition of W*-bundles, their basic theory and the key examples. We also systematically study the standard form of a W*-bundle and morphisms between W*-bundles, culminating in a bicommutant theorem for W*-bundles in standard form. In the final section of this chapter, we provide an alternative picture of W*-bundles, where one considers bundle spaces rather than algebras of sections.

3.1 Definitions and Examples

The purpose of this section is to present the formal definition of a W*-bundle and illuminate this somewhat abstract definition with the help of some examples. The definition given below is essentially that of [62, Section 5].[1]

Definition 3.1.1. A *tracially continuous W*-bundle* over a compact Hausdorff space X is a unital C*-algebra $\mathcal{M}$ together with a unital embedding of $C(X)$ into the centre $Z(\mathcal{M})$ of $\mathcal{M}$ and a conditional expectation $E : \mathcal{M} \to C(X)$ such that the the following axioms are satisfied:

(T) $E(a_1 a_2) = E(a_2 a_1)$, for all $a_1, a_2 \in \mathcal{M}$.

(F) $E(a^*a) = 0 \Rightarrow a = 0$, for all $a \in \mathcal{M}$.

[1] We follow the conventions of [5] and [23] and do not require X to be a metric space.

(C) The unit ball $\{a \in \mathcal{M} : \|a\| \leq 1\}$ is complete with respect to the norm defined by $\|a\|_{2,u} = \|E(a^*a)^{1/2}\|_{C(X)}$.[2]

We now take the opportunity to fix some terminology and notational conventions that will be used throughout this thesis. We also state formally the definition of a pre-W*-bundle, a concept that will be useful from time to time in this thesis.

Notation and Terminology 3.1.2. We abbreviate *tracially continuous W*-bundle* to *W*-bundle*. In the notation of Definition 3.1.1, we call X the *base space* of the bundle, $\mathcal{M}$ the *section algebra*, and E the *conditional expectation*. The unital embedding of $C(X)$ into $Z(\mathcal{M})$ is typically suppressed in the notation, but will on occasion be denoted by ι. By abuse of notation, the same symbol $\mathcal{M}$ is used to denote both the W*-bundle as a whole and its section algebra. The axioms listed in Definition 3.1.1 are referred to as the tracial axiom, the faithfulness axiom and the completeness axiom respectively. By the unit ball of a W*-bundle $\mathcal{M}$, we always mean $\{a \in \mathcal{M} : \|a\| \leq 1\}$, i.e the closed unit ball with respect to the $\|\cdot\|$-norm.

Definition 3.1.3. A *pre-W*-bundle* over a compact Hausdorff space X is a unital C*-algebra $\mathcal{M}$ together with a unital embedding of $C(X)$ into the centre $Z(\mathcal{M})$ of $\mathcal{M}$ and a conditional expectation $E : \mathcal{M} \to C(X)$ such that the the following axioms are satisfied:

(T) $E(a_1a_2) = E(a_2a_1)$, for all $a_1, a_2 \in \mathcal{M}$.

(F) $E(a^*a) = 0 \Rightarrow a = 0$, for all $a \in \mathcal{M}$.

We now turn to the basic examples of W*-bundles: tracial von Neumann algebras, finite von Neumann algebras, trivial W*-bundles and subtrivial W*-bundles. The motivating examples of W*-bundles, namely those arising from C*-algebras, will be introduced in Section 3.3 after we've developed some of the basic theory of W*-bundles in Section 3.2.

Example 3.1.4 (Tracial von Neumann algebras)**.** A pre-W*-bundle over a one point space $X = \{*\}$ is nothing more than a unital C*-algebra $\mathcal{M} = A$ together with a faithful trace $E = \tau$, provided we make the obvious identifications $\mathbb{C} \cong \mathbb{C}1_A \cong C(\{*\})$. In this case, Axiom (C) holds if and only if (A, τ) is a tracial von Neumann algebra; see Definition 2.8.17 and Theorem 2.8.16.

[2] For each $x \in X$, $a \mapsto E(a)(x)$ is a trace, so $x \mapsto E(a^*a)(x)^{1/2}$ is a seminorm. Taking suprema, we see that $\|\cdot\|_{2,u}$ is a seminorm. Axiom (F) ensures it's a norm.

Example 3.1.5 (Finite von Neumann algebras[3]). Fix a finite von Neumann algebra $\mathcal{M}$. Let X be the hyperstonian space such that $Z(\mathcal{M}) \cong C(X)$ and identify these two algebras. Note, that X is typically non-metrisable. Let E be the centre valued trace (see Theorem 2.8.11). Since all traces on $\mathcal{M}$ factor though the centre valued trace, we have $\|a\|_{2,u} = \|E(a^*a)^{1/2}\|_{C(X)} = \sup_{\tau \in T(M)} \|a\|_{2,\tau}$. A consequence of Corollary 2.8.12 is that the supremum can equally well be taken over the normal traces of M. Axioms (T) and (F), are basic properties of the centre valued trace. Axiom (C) is verified below.

Proposition 3.1.6. *The unit ball of a finite von Neumann algebra $\mathcal{M}$ is complete with respect to the $\|\cdot\|_{2,u}$-norm.*

Proof. Suppose the sequence (a_n) is a $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$ with $\|a_n\| \leq 1$ for all $n \in \mathbb{N}$. Since the $\|\cdot\|_{2,u}$-norm dominates the $\|\cdot\|_{2,\tau}$-norm for any trace $\tau \in T(\mathcal{M})$, (a_n) is Cauchy with respect to the $\|\cdot\|_{2,\tau}$-norm for any trace $\tau \in T(\mathcal{M})$. The family of all normal traces is separating by Corollary 2.8.12, so by Proposition 2.7.7 the family of seminorms $\{\|\cdot\|_{2,\tau} : \tau \in T(\mathcal{M}), \tau \text{ normal}\}$ induces the strong operator topology on bounded subsets of $\mathcal{M}$. Hence, (a_n) is Cauchy with respect to the strong operator topology. However, the unit ball of $\mathcal{M}$ is complete in the strong operator topology by Theorem 2.2.1, so there exists $a \in \mathcal{M}$ with $\|a\| \leq 1$ such that $a_n \to a$ strongly as $n \to \infty$.

Let $\epsilon > 0$. As (a_n) is Cauchy with respect to the $\|\cdot\|_{2,u}$-norm, there is $N \in \mathbb{N}$ such that $\|a_n - a_m\|_{2,u} \leq \epsilon$ whenever $n, m \geq N$. Hence, for all normal traces τ, $\|a_n - a_m\|_{2,\tau} \leq \epsilon$ whenever $n, m \geq N$. However, $a_m \to a$ strongly as $m \to \infty$ and the strong operator topology on bounded sets is induced by family of seminorms $\{\|\cdot\|_{2,\tau} : \tau \in T(\mathcal{M}), \tau \text{ normal}\}$. Thus, for all normal traces τ, $\|a_n - a\|_{2,\tau} \leq \epsilon$ whenever $n \geq N$. Since the normal traces are dense in $T(\mathcal{M})$, we have $\|a_n - a\|_{2,u} \leq \epsilon$. Therefore, (a_n) converges to a with respect to the $\|\cdot\|_{2,u}$-norm. $\square$

Example 3.1.7 (Trivial W*-Bundles c.f. [62, Section 5]). Let X be a compact Hausdorff space and (M, τ) be a tracial von Neumann algebra. The *trivial W*-bundle* over X with fibre M is the unital C*-algebra

$$C_\sigma(X, M) = \{f : X \to (M, \tau) : f \text{ is } \|\cdot\|\text{-bounded}, f \text{ is } \|\cdot\|_{2,\tau}\text{-continuous}\} \tag{3.1.1}$$

together with the embedding $\iota : C(X) \to Z(C_\sigma(X, M))$ and the conditional expectation

[3] This family of examples of W*-bundles was suggested to me by George Elliott during a talk I gave at the Isaac Newton Institute in Cambridge. Later, in Glasgow, I was able to prove that Axiom (C) holds.

$E : C_\sigma(X, M) \to C(X)$ defined by

$$\iota(\phi)(x) = \phi(x)1_M \qquad (x \in X, \phi \in C(X)), \tag{3.1.2}$$

$$E(f)(x) = \tau(f(x)) \qquad (x \in X, f \in C_\sigma(X, M)). \tag{3.1.3}$$

We check the axioms (T), (F) and (C) in the proposition below.

Proposition 3.1.8. *For the trivial W*-bundle over X with fibre M defined above, the axioms (T), (F) and (C) are satisfied.*

Proof. Axioms (T) and (F) follow from the fact that τ is a faithful trace on M. We now show Axiom (C) holds. Let $(f_n) \subseteq C_\sigma(X, M)$ be a sequence such that $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$ and which is a Cauchy sequence with respect to the $\|\cdot\|_{2,u}$-norm. Let $x \in X$. Since $\|f_n(x) - f_m(x)\|_{2,\tau} \leq \|f_n - f_m\|_{2,u}$ for all $n, m \in \mathbb{N}$ and $\|f_n(x)\| \leq \|f_n\|$ for all $n \in \mathbb{N}$, we have that $(f_n(x))$ is a $\|\cdot\|_{2,\tau}$-Cauchy sequence in the unit ball of the tracial von Neumann algebra M and so has a $\|\cdot\|_{2,\tau}$-limit in the unit ball of M. Hence, we can define a pointwise $\|\cdot\|_{2,\tau}$-limit f of the sequence of functions (f_n) and $\sup_{x\in X} \|f(x)\| \leq 1$.

Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that $\|f_n - f_m\|_{2,u} < \epsilon$ whenever $n, m \geq N$. Fix $x \in X$. Then $\|f_n(x) - f_m(x)\|_{2,\tau} < \epsilon$ for $n, m \geq N$. Letting $m \to \infty$ gives $\|f_n(x) - f(x)\|_{2,\tau} \leq \epsilon$. Hence,

$$\sup_{x\in X} \|f_n(x) - f(x)\|_{2,\tau} \leq \epsilon \tag{3.1.4}$$

whenever $n \geq N$. Therefore, (f_n) converges uniformly to f in $\|\cdot\|_{2,\tau}$-norm.

A standard 3ϵ-argument now gives that f is $\|\cdot\|_{2,\tau}$-continuous. Indeed, let $\epsilon > 0$ and $x_0 \in X$. Choose $N \in \mathbb{N}$ as above, so that (3.1.4) holds whenever $n \geq N$. Since f_N is $\|\cdot\|_{2,\tau}$-continuous, there is an open neighbourhood U of x_0 such that $\|f_N(x_0) - f_N(x_1)\| \leq \epsilon$ whenever $x_1 \in U$. Using (3.1.4) twice, we compute that

$$\begin{aligned} \|f(x_0) - f(x_1)\|_{2,\tau} &\leq \|f(x_0) - f_N(x_0)\|_{2,\tau} + \|f_N(x_0) - f_N(x_1)\|_{2,\tau} && (3.1.5)\\ &\quad + \|f_N(x_1) - f(x_1)\|_{2,\tau} && (3.1.6)\\ &\leq 3\epsilon. && (3.1.7)\end{aligned}$$

Hence, f is $\|\cdot\|_{2,\tau}$-continuous. Therefore, f is in the unit ball of $C_\sigma(X, M)$. As (f_n) converges uniformly to f in $\|\cdot\|_{2,\tau}$-norm, (f_n) converges to f in $\|\cdot\|_{2,u}$-norm. This completes the proof of Axiom (C). □

Example 3.1.9 (Subtrivial W*-bundles)**.** Let $\mathcal{M} = C_\sigma(X, M)$ be a trivial W*-bundle. For each $x \in X$, let N_x be a von Neumann subalgebra of M containing the identity. Set

$\mathcal{N} = \{f \in C_\sigma(X, M) : f(x) \in N_x \text{ for all } x \in X\}$, with the embedding and conditional expectation inherited from $C_\sigma(X, M)$. Since each N_x is $\|\cdot\|_{2,\tau}$-closed in M, we get that $\mathcal{M}$ is $\|\cdot\|_{2,u}$-closed in $C_\sigma(X, M)$, and so also $\|\cdot\|$-closed. Hence, $\mathcal{N}$ is a unital C*-algebra and axiom (C) is satisfied. Axioms (T) and (F) are inherited from $C_\sigma(X, M)$. Therefore, $\mathcal{N}$ is a W*-bundle over X.

For each $x \in X$, let $\mathrm{eval}_x : C_\sigma(X, M) \to M$ be the *-homomorphism coming from evaluating a function at the point x. By definition, we have $\mathrm{eval}_x(\mathcal{N}) \subseteq N_x$. In order to have equality, one must make a continuity assumption on the family $\{N_x\}_{x\in X}$. The precise necessary and sufficient condition is given in the following proposition, which is inspired by [19, Proposition 11].

Proposition 3.1.10. *Let $\mathcal{M} = C_\sigma(X, M)$ be a trivial W*-bundle and $\mathcal{N}$ the subtrival bundle defined by the family of von Neumann subalgebras $\{N_x\}_{x\in X}$. Then the following are equivalent:*

(i) For all $x \in X$, $\mathrm{eval}_x(\mathcal{N}) = N_x$.

(ii) For all $b \in M$, the map $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau}}(b, N_x)$ is upper-semicontinuous, where $\mathrm{dist}_{\|\cdot\|_{2,\tau}}(b, N_x) = \inf\{\|b - c\|_{2,\tau} : c \in N_x\}$.

Proof. Hereinafter, we drop subscripts and write $\mathrm{dist}(\cdot, \cdot)$ instead of $\mathrm{dist}_{\|\cdot\|_{2,\tau}}(\cdot, \cdot)$.

(i) $\Rightarrow$ (ii) Let $b \in M, x_0 \in X$, and $\epsilon > 0$. There exists $c \in N_{x_0}$ such that $\|b - c\|_{2,\tau_{x_0}} < \mathrm{dist}(b, N_{x_0}) + \epsilon$. Since $\mathrm{eval}_x(\mathcal{N}) = N_x$ for all $x \in X$, there is $f \in \mathcal{N}$ such that $f(x_0) = c$.

Since $x \mapsto \|b - f(x)\|_{2,\tau}$ is continuous, there is neighbourhood U of x_0 such that $\|b - f(x)\|_{2,\tau} < \mathrm{dist}(b, N_{x_0}) + \epsilon$ for all $x \in U$. Since $f(x) \in N_x$ for all x, it follows that $\mathrm{dist}(b, N_x) < \mathrm{dist}(b, N_{x_0}) + \epsilon$ for all $x \in U$. Hence, $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau}}(b, N_x)$ is upper-semicontinuous.

(ii) $\Rightarrow$ (i) We need to prove that for all $x_0 \in X$ and $b \in N_{x_0}$ there is some $f \in \mathcal{N}$ with $f(x_0) = b$. We construct such an f as the limit of a sequence $(f_n) \subseteq C_\sigma(X, M)$ with the following properties:

$$\|f_n\| \le \|b\|, \tag{3.1.8}$$

$$f_n(x_0) = b, \tag{3.1.9}$$

$$\|f_n - f_{n-1}\|_{2,u} < \frac{1}{2^{n-1}}, \tag{3.1.10}$$

$$\sup_{x\in X} \mathrm{dist}(f_n(x), N_x) < \frac{1}{2^n}. \tag{3.1.11}$$

Assume for now that such a sequence exists. Axiom (C) together with (3.1.8) and (3.1.10) ensures that (f_n) has a $\|\cdot\|_{2,u}$-limit $f \in C_\sigma(X, M)$. Taking limits in (3.1.9) ensures that $f(x_0) = b$. Finally, property (3.1.11) ensures that, for each $x \in X$, $f(x)$ lies in the $\|\cdot\|_{2,\tau}$-closure of N_x, so $f(x) \in N_x$; hence, $f \in \mathcal{N}$.

We now show how to construct the sequence (f_n). This is done by induction. First we construct f_1. Since $x \mapsto \mathrm{dist}(b, N_x)$ is upper-semicontinuous there is an open neighbourhood U of x_0 such that $\sup_{y\in U} \mathrm{dist}(b, N_y) < \frac{1}{2}$. Choose a continuous function $\phi : X \to [0, 1]$ such that $\phi(x_0) = 1$ and $\phi(X \setminus U) \subseteq \{0\}$. Set $f_1 = \phi b$. Properties (3.1.8) and (3.1.9) are clearly satisfied, property (3.1.10) is void, and property (3.1.11) comes from considering the cases $x \in U$ and $x \in X \setminus U$ separately.

Suppose now that $f_1, \ldots, f_{n-1}$ have been constructed with the desired properties. We construct f_n. By (3.1.11), there is, for all $x \in X$, $b^{(x)} \in N_x$ such that $\|f_{n-1}(x) - b^{(x)}\|_{2,\tau} < \frac{1}{2^{n-1}}$. In fact, we can take $b^{(x)} = E_{N_x}(f_{n-1}(x))$, where E_{N_x} is the canonical conditional expectation $M \to N_x$. This ensures that we also have $\|b^{(x)}\| \leq \|f_{n-1}(x)\| \leq \|b\|$ and $b^{(x_0)} = b$.

Fix $x \in X$. By the continuity of f_n and the upper-semicontinuity of $y \mapsto \mathrm{dist}(b^{(x)}, N_y)$, there is an open neighbourhood $U^{(x)} \ni x$ such that

$$\sup_{y\in U^{(x)}} \|f_{n-1}(y) - b^{(x)}\| < \frac{1}{2^{n-1}}, \tag{3.1.12}$$

$$\sup_{y\in U^{(x)}} \mathrm{dist}(b^{(x)}, M_y) < \frac{1}{2^n}. \tag{3.1.13}$$

The open cover $\{U^{(x)} : x \in X\}$ of X has a finite subcover. We write this subcover as $U_1, \ldots, U_m$ and the corresponding elements of $\{b^{(x)} : x \in X\}$ as $b_1, \ldots, b_m$. We may assume that $U_1 = U^{(x_0)}$ and $b_1 = b$. Let $\phi_1, \ldots, \phi_m$ be a partition of unity subordinate to $U_1, \ldots, U_m$ with $\phi_1(x_0) = 1$. Set $f_n = \sum_{i=1}^m \phi_i b_i$. Property (3.1.8) follows since we ensured that $\|b^{(x)}\| \leq \|b\|$ for all $x \in X$ and $\phi_1, \ldots, \phi_m$ is a partition of unity. Property (3.1.9) follows by construction. Properties (3.1.10) (3.1.11) follow from (3.1.12) and (3.1.13) respectively because $\phi_1, \ldots, \phi_m$ is a partition of unity. This completes the proof. □

3.2 The Basic Theory of W*-Bundles

In this section, we develop the basic theory of W*-bundles. First, we consider the fibration of a W*-bundle over its base space, formalising the intuition that the elements of a W*-bundle can be viewed as the sections of a bundle of tracial von Neumann algebras. We shall then show that every W*-bundle has a natural representation on a Hilbert-$C(X)$-module. We call this representation the standard form of a W*-bundle by analogy with the case for tracial von Neumann algebras. Finally, we investigate the strict topology on a W*-bundle coming from the standard form representation and its relation to the $\|\cdot\|_{2,u}$-topology.

The main results on the fibration of a W*-bundle are due to Ozawa [62, Section 5], though the approach in this thesis is different and more easily generalised. The standard form representation and the strict topology are discussed briefly in [62, Section 5] and [5, Section 3.1]; we investigate them fully here.

3.2.1 The Fibres of a W*-Bundle

The idea behind the construction of this section is to generalise the family of *-homomorphisms $\mathrm{eval}_x : C_\sigma(X, M) \to M$ of a trivial W*-bundle given by $f \mapsto f(x)$ to the setting of pre-W*-bundles. This is done by means of a quotient construction.

Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E, and let $x \in X$. By Proposition 2.6.12, $I_x = \{a \in \mathcal{M} : E(a^*a)(x) = 0\}$ is an ideal of $\mathcal{M}$. Set $\mathcal{M}_x = \mathcal{M}/I_x$. By Corollary 2.6.13, the trace $\mathrm{eval}_x \circ E$ induces a faithful trace τ_x on the quotient $\mathcal{M}_x$. The *fibre* of $\mathcal{M}$ at x is the C*-algebra $\mathcal{M}_x$ together with the trace τ_x.[4]

In [62, Section 5], Ozawa defines the fibre of $\mathcal{M}$ at x to be $\pi_x(\mathcal{M})$, where π_x is the GNS representation corresponding to the trace $\mathrm{eval}_x \circ E$. In this thesis, we favour the definition as an abstract quotient so as not to favour a certain representation. In any case, Proposition 2.6.12 together with the first isomorphism theorem for C*-algebras gives us an isomorphism φ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[d, "q_x"'] \arrow[dr, "\pi_x"] & \\ \mathcal{M}_x \arrow[r, "\varphi"'] & \pi_x(\mathcal{M}) \end{tikzcd} \tag{3.2.1}$$

commutes, where q_x is the quotient map. Moreover, under the isomorphism φ, the trace τ_x on $\mathcal{M}_x$ corresponds to the GNS trace on $\pi_x(\mathcal{M})$. Hence, the definition of fibres given here is equivalent to that of Ozawa in [62, Section 5].

Now for some examples.

Example 3.2.1 (Trivial W*-bundles)**.** In the case of a trivial W*-bundle, the evaluation map $\mathrm{eval}_x : C_\sigma(X, M) \to M$ is a *-homomorphisms with kernel $I_x = \{a \in C_\sigma(X, M) :$

[4] However, by abuse of notation, we shall often write $\mathcal{M}_x$ for the fibre instead of $(\mathcal{M}_x, \tau_x)$.

$E(a^*a)(x) = 0\}$, so the first isomorphism theorem gives us an isomorphism φ such that the diagram

$$\begin{tikzcd} C_\sigma(X,M) \arrow[d, "q_x"'] \arrow[dr, "\mathrm{eval}_x"] & \\ C_\sigma(X,M)_x \arrow[r, "\varphi"'] & M \end{tikzcd} \tag{3.2.2}$$

commutes. Moreover, under the isomorphism φ, the trace τ_x corresponds to the trace on M. Hence, the general definition of the fibration of a pre-W*-bundle generalises the evaluation maps of a trivial W*-bundle. In particular, the fibres of $C_\sigma(X, M)$ are all isomorphic to M.

Example 3.2.2 (Subtrivial W*-bundles)**.** If $\mathcal{N} \subseteq C_\sigma(X, M)$ is a subtrivial W*-bundle defined by a family $\{N_x\}_{x\in X}$ of von Neumann subalgebras of M as in Example 3.1.9, then the first isomorphism theorem gives an isomorphism φ such that the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[d, "q_x"'] \arrow[dr, "\mathrm{eval}_x"] & \\ \mathcal{N}_x \arrow[r, "\varphi"'] & \mathrm{eval}_x(\mathcal{N}) \end{tikzcd} \tag{3.2.3}$$

commutes and which sends the trace τ_x to that of M. Hence, Proposition 3.1.10 provides the necessary and sufficient conditions for being able to define a subtrivial W*-bundle with specified fibres.

In particular, one could consider the subtrivial W*-bundle $\mathcal{N} \subseteq C_\sigma([0, 1], \mathcal{R})$ defined by the family of von Neumann subalgebras

$$N_x = \begin{cases} \mathcal{R}, & x \neq 1, \\ \mathbb{C}1_{\mathcal{R}}, & x = 1. \end{cases} \tag{3.2.4}$$

For any $b \in \mathcal{R}$, the map $x \mapsto \mathrm{dist}_{\|\cdot\|_{\tau,2}}(b, N_x)$ is upper-semiconituous. So the fibre of $\mathcal{N}$ at x is N_x. In this way, one can produce many examples of W*-bundle whose fibres are not all isomorphic.

In the following proposition, we give an alternative characterisation of the ideal I_x. Note the appearance of the $\|\cdot\|_{2,u}$-closure. In general, I_x is strictly larger than $C_0(X \setminus \{x\})\mathcal{M}$. Consequently, the fibres of a pre-W*-bundle $\mathcal{M}$ are in general not the same as the fibres of $\mathcal{M}$ when viewed as a $C(X)$-algebra.[5]

[5]Indeed, the map $x \mapsto \|a + I_x\|$ need not be upper-semicontinuous; see (3.2.7).

Proposition 3.2.3. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. Let $x \in X$. Then $I_x = \overline{C_0(X \setminus \{x\})\mathcal{M}}^{\|\cdot\|_{2,u}}$.*

Proof. The inclusion $I_x \supseteq \overline{C_0(X \setminus \{x\})\mathcal{M}}^{\|\cdot\|_{2,u}}$ is clear, since E is a conditional expectation and I_x is $\|\cdot\|_{2,u}$-closed. For the reverse inclusion, let $\epsilon > 0$ and $a \in I_x$. Since $E(a^*a)(x) = 0$, there is an open neighbourhood U of x such that $E(a^*a)(y) \leq \epsilon^2$ for all $y \in U$. Let $f : X \to [0,1]$ be a continuous function with $f(x) = 0$ and $f(X \setminus U) \subseteq \{1\}$. Then

$$\|a - fa\|_{2,u} = \sup_{x \in X} |1 - f(x)| E(a^*a)(x)^{1/2}. \tag{3.2.5}$$

By considering separately the cases $x \in U$ and $x \in X \setminus U$ in this supremum, we obtain $\|a - fa\|_{2,u} \leq \epsilon$. Therefore, $I_x \subseteq \overline{C_0(X \setminus \{x\})\mathcal{M}}^{\|\cdot\|_{2,u}}$. □

Notation and Terminology 3.2.4. An important intuition for pre-W*-bundles is to view an element of a pre-W*-bundle $\mathcal{M}$ as a section of a bundle-like object over X taking the value $a + I_x$ in the fibre $\mathcal{M}_x$. To this end, we introduce the notation $a \mapsto a(x)$ for the canonical quotient map $\mathcal{M} \to \mathcal{M}_x$. It follows immediately from Proposition 3.2.3 that, if $f \in C(X) \subseteq \mathcal{M}$, then $f - f(x)1_{\mathcal{M}} \in I_x$, so the image of f in $\mathcal{M}_x$ is $f(x)1_{\mathcal{M}_x}$. Hence, the notation introduced here is consistent with evaluating elements of $C(X)$ at points of X.

We now verify that an element of a pre-W*-bundle is completely determined its images in all fibres, as is clearly the case for trivial W*-bundles. We also obtain an important formula for the $\|\cdot\|$-norm of elements.

Proposition 3.2.5. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. The map*

$$\begin{aligned} \Phi : \mathcal{M} &\to \prod_{x \in X} \mathcal{M}_x \\ a &\mapsto (a(x))_{x \in X} \end{aligned}$$

*is an isometric *-homomorphism. In particular,*

$$\|a\|_{\mathcal{M}} = \sup_{x \in X} \|a(x)\|_{\mathcal{M}_x}. \tag{3.2.6}$$

Proof. For each $x \in X$, the map $a \mapsto a(x)$ is a *-homomorphism. Hence, Φ is a *-homomorphism. Suppose $a \in \mathcal{M}$ satisfies $a(x) = 0$ for all $x \in X$. Then $a \in I_x$ for all $x \in X$. Hence, $E(a^*a) = 0$. Consequently, $a = 0$ by the faithfulness axiom. Thus, Φ is an injective *-homomorphism of C*-algebras, and so is isometric by [58, Theorem 3.14]. □

Its extremely important to note that the map $x \mapsto \|a(x)\|$ is typically not continuous. For example, consider the element of the trivial W*-bundle $a \in C_\sigma([0,1], L^\infty[0,1])$ given by $a(t) = \chi_{[0,t]}$. Then

$$\|a(x)\| = \begin{cases} 1, & x > 0, \\ 0, & x = 0. \end{cases} \tag{3.2.7}$$

In this respect, the $\|\cdot\|_{2,u}$-norm is better behaved than the $\|\cdot\|$-norm. Not only do we get a formula for the $\|\cdot\|_{2,u}$-norm of an element of a W*-bundle in terms of the $\|\cdot\|_{2,\tau_x}$-norms of its images in the fibres, we also get that the map $x \mapsto \|a(x)\|_{2,\tau_x}$ is continuous.

Proposition 3.2.6. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. For fixed $a \in \mathcal{M}$, the map $x \mapsto \|a(x)\|_{2,\tau_x}$ is continuous. Furthermore, we have*

$$\|a\|_{2,u} = \sup_{x \in X} \|a(x)\|_{2,\tau_x}. \tag{3.2.8}$$

Proof. The proposition follows from the observation that $\|a(x)\|_{2,\tau_x} = E(a^*a)(x)^{1/2}$ and the fact that E takes its values in $C(X)$. □

Next, we collect a number of useful norm estimates. These results can be seen as analogues of the estimates obtained in Proposition 2.6.11. They can be either be deduced from the corresponding results in Proposition 2.6.11 using the formulas (3.2.6) and (3.2.8), or proved by calculations similar to those for Proposition 2.6.11.

Proposition 3.2.7. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X with conditional expectation E. Then for $a, b \in \mathcal{M}$*

(i) $\|a\|_{2,u} \leq \|a\|$,

(ii) $\|E(a)\|_{C(X)} \leq \|a\|_{2,u}$

(iii) $\|a^*\|_{2,u} = \|a\|_{2,u}$,

(iv) $\|ab\|_{2,u} \leq \|a\|\|b\|_{2,u}$,

(v) $\|ab\|_{2,u} \leq \|a\|_{2,u}\|b\|$,

(vi) $\|E(ab)\|_{C(X)} \leq \|a\|_{2,u}\|b\|_{2,u}$.

Proof. (i) We have $\|a\|_{2,u}^2 = \|E(a^*a)\|_{C(X)} \leq \|E\|\|a^*a\| = \|a\|^2$.

(ii) For any $x \in X$, we have $|E(a)(x)| = |\tau_x(a(x))| \leq \|a(x)\|_{2,\tau}$ by Proposition 2.6.11. Now take suprema.

(iii) We have $\|a^*\|_{2,u}^2 = \|E(aa^*)\|_{C(X)} = \|E(a^*a)\|_{C(X)} = \|a\|_{2,u}^2$.

(iv) We have $\|ab\|_{2,u}^2 = \|E(b^*a^*ab)\|_{C(X)} \leq \|E(b^*\|a\|^2 b)\|_{C(X)} = \|a\|^2\|E(b^*b)\|_{C(X)} = \|a\|^2\|b\|_{2,u}^2$.

(v) Using (iii) and (iv), we have $\|ab\|_{2,u} = \|b^*a^*\|_{2,u} \leq \|b^*\|\|a^*\|_{2,u} = \|a\|_{2,u}\|b\|$.

(vi) The Cauchy-Schwarz inequality gives $|\tau_x(a(x)b(x))| \leq \|a(x)\|_{2,\tau_x}\|b(x)\|_{2,\tau_x}$ for all $x \in X$. Now take suprema. □

Corollary 3.2.8. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X with conditional expectation E.*

(i) Addition $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is $\|\cdot\|_{2,u}$-continuous.

(ii) Scalar multiplication $\mathbb{C} \times \mathcal{M} \to \mathcal{L}(H)$ is $\|\cdot\|_{2,u}$-continuous.

(iii) The involution $\mathcal{M} \to \mathcal{M}$ is $\|\cdot\|_{2,u}$-continuous.

(iv) Multiplication $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is $\|\cdot\|_{2,u}$-continuous when restricted to $\|\cdot\|$-bounded sets.

(v) The conditional expectation $\mathcal{M} \to C(X)$ is $\|\cdot\|_{2,u}$-continuous.

Proof. Addition and scalar multiplication are $\|\cdot\|_{2,u}$-continuous since $\|\cdot\|_{2,u}$ is a norm. The $\|\cdot\|_{2,u}$-continuity of the involution follows from Proposition 3.2.7(iii), and the $\|\cdot\|_{2,u}$-continuity of E from Proposition 3.2.7(ii). For multiplication, we have the estimate

$$\begin{aligned} \|a_1b_1 - a_2b_2\|_{2,u} &= \|a_1b_1 - a_1b_2 + a_1b_2 - a_2b_2\|_{2,u} && (3.2.9)\\ &\leq \|a_1b_1 - a_1b_2\|_{2,u} + \|a_1b_2 - a_2b_2\|_{2,u} && (3.2.10)\\ &\leq \|a_1\|\|b_1 - b_2\|_{2,u} + \|a_1 - a_2\|_{2,u}\|b_2\| && (3.2.11) \end{aligned}$$

for $a_1, a_2, b_1, b_2 \in \mathcal{M}$, using 3.2.7(iv) and (v). The $\|\cdot\|_{2,u}$-continuity of multiplication on $\|\cdot\|$-bounded regions follows. □

We end this section with two results due to Ozawa (see [62, Theorem 11]). Both results make crucial use of the completeness axiom. The first is fundamental: the fibres of W*-bundles are tracial von Neumann algebras.

Theorem 3.2.9. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X with conditional expectation E. Let $x \in X$. Then $\mathcal{M}_x$ is a tracial von Neumann algebra.*

Proof. Let $(b_n) \subseteq \mathcal{M}_x$ be a sequence that satisfies $\|b_n\| \leq 1$ for all $n \in \mathbb{N}$ and is a Cauchy sequence with respect to the $\|\cdot\|_{2,\tau_x}$-norm on $\mathcal{M}_x$. We need to find $b \in \mathcal{M}_x$ with $\|b\| \leq 1$ such that (b_n) converges to b in the $\|\cdot\|_{2,\tau_x}$-norm on $\mathcal{M}_x$. Since a Cauchy sequence will converge to the limit of any convergent sub-sequence, we may assume that $\|b_{n+1} - b_n\|_{2,\tau_x} < \frac{1}{2^n}$ without loss of generality.

We shall construct a sequence $(a_n) \subseteq \mathcal{M}$ inductively such that

$$a_n(x) = b_n, \tag{3.2.12}$$

$$\|a_n\| \leq 1, \tag{3.2.13}$$

$$\|a_{n+1} - a_n\|_{2,\tau_x} < \frac{1}{2^n} \tag{3.2.14}$$

for all $n \in \mathbb{N}$. Recall that with C*-algebras we may always lift elements from quotient algebras without increasing the norm [74, Section 2.2.10]. Let a_1 be any such lift of b_1. Suppose now that $a_1, \ldots, a_n$ have been defined and have the desired properties. Let a'_{n+1} be any lift of b_{n+1} with $\|a'_{n+1}\| \leq 1$. Since

$$\|a'_{n+1}(x) - a_n(x)\|_{2,\tau_x} < \frac{1}{2^n}, \tag{3.2.15}$$

we can, by continuity, find an open neighbourhood U of x such that

$$\sup_{y \in U} \|a'_{n+1}(y) - a_n(y)\|_{2,\tau_y} < \frac{1}{2^n}. \tag{3.2.16}$$

We then take a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(X \setminus U) \subseteq \{0\}$, and set $a_{n+1} = f a'_{n+1} + (1-f)a_n$. We have that $a_{n+1}(x) = b_{n+1}$ and, using Proposition 3.2.5, we see that $\|a_{n+1}\| \leq 1$. Finally, we have that

$$\|a_{n+1}(y) - a_n(y)\|_{2,\tau_y} = |f(y)| \|a'_{n+1}(y) - a_n(y)\|_{2,\tau_y} \tag{3.2.17}$$

for $y \in X$. By considering the cases $y \in U$ and $y \in X \setminus U$ separately in (3.2.8) of Proposition 3.2.6, we get that $\|a_{n+1} - a_n\|_{2,u} < \frac{1}{2^n}$. This completes the inductive definition of the sequence (a_n).

The sequence (a_n) convergences to some $a \in \mathcal{M}$ with $\|a\| \leq 1$ because the unit ball of $\mathcal{M}$ is complete in the $\|\cdot\|_{2,u}$-norm. We set $b = a(x)$. The convergence of (b_n) to b follows by Proposition 3.2.6. $\square$

The proof of Theorem 3.2.9 given here, which also appears in [23], is a modification of the original. Ozawa's use of Pedersen's up-down theorem in [62, Theorem 11] is avoided by showing completeness of the unit ball via the argument in [18, Proposition 10.1.12]. This proof is more accessable to generalisations, as we will see later in this thesis.

We now present Ozawa's second result (see also [62, Theorem 11]). This theorem provides us with a useful way of proving the existence of an element of a W*-bundle with certain properties.

Theorem 3.2.10. *Let $\mathcal{M}$ be a W*-bundle over X with conditional expectation E. Let $f : X \to \sqcup_{x\in X}\mathcal{M}_x$ be a function such that $f(x) \in \mathcal{M}_x$ for all $x \in X$. Suppose that $\sup_{x\in X} \|f(x)\| < \infty$ and, for all $x \in X$ and $\epsilon > 0$, there is an open neighbourhood $U^{(x)} \ni x$ and $c^{(x)} \in \mathcal{M}$ such that*

$$\sup_{y\in U^{(x)}} \|f(y) - c^{(x)}(y)\|_{2,\tau_y} < \epsilon. \tag{3.2.18}$$

Then there is $a \in \mathcal{M}$ such that $f(x) = a(x)$ for all $x \in X$.

Proof. For the moment, fix $n \in \mathbb{N}$. Let $x \in X$. Choose $b^{(x)} \in \mathcal{M}$ such that $\|b^{(x)}\| \leq \|f(x)\|$ and $b^{(x)}(x) = f(x)$, and choose $c^{(x)} \in \mathcal{M}$ together with an open neighbourhood $U^{(x)}$ of x such that $\sup_{y\in U^{(x)}} \|f(y) - c^{(x)}(y)\|_{2,\tau_y} < \frac{1}{n}$. Due to the continuity of $y \mapsto \|b^{(x)}(y) - c^{(x)}(y)\|_{2,\tau_y}$, we may, after shrinking $U^{(x)}$, assume that $\sup_{y\in U^{(x)}} \|f(y) - b^{(x)}(y)\| < \frac{1}{n}$. The family of open sets $\{U^{(x)} : x \in X\}$ form an open cover for X. By compactness, a finite subcover exists. Denote the open sets in this finite subcover by $U_1, \dots, U_m$ and the corresponding elements of $\mathcal{M}$ by $b_1, \dots, b_m$. Let $f_1, \dots, f_m$ be a partition of unity subordinate to $U_1, \dots, U_m$. Set $a_n = \sum_{i=1}^m f_i b_i$. Using the fact that $f_1, \dots, f_m$ form a partition of unity, we find that

$$\sup_{y\in X} \|a_n(y) - f(y)\|_{2,\tau_y} < \tfrac{1}{n}. \tag{3.2.19}$$

It follows from Proposition 3.2.6 that (a_n) is a Cauchy sequence with respect to the $\|\cdot\|_{2,u}$-norm. It follows from Proposition 3.2.5 that $\|a_n\| \leq \sup_{x\in X} \|f(x)\|$ for all $n \in \mathbb{N}$. Hence, Axiom (C) implies that (a_n) has a $\|\cdot\|_{2,u}$-norm limit a. That $a(y) = f(y)$ for all $y \in X$ is an easy consequence of (3.2.19). ☐

3.2.2 The Standard Form of a W*-Bundle

In this section, we show that every pre-W*-bundle has a natural representation as an algebra of adjointable operators on a Hilbert-$C(X)$-module. The idea is to perform the GNS construction with respect to the "$C(X)$-valued trace" E. This generalises the standard form of a tracial von Neumann algebra.

Definition 3.2.11. Let $\mathcal{M}$ be a pre-W*-bundle over a compact Hausdorff space X with conditional expectation E. Let $L^2(\mathcal{M})$ be the Banach space obtained by completing $\mathcal{M}$ with respect to the $\|\cdot\|_{2,u}$-norm. Write $\|\cdot\|_{L^2(\mathcal{M})}$ for the norm on $L^2(\mathcal{M})$. Write $\widehat{b}$ for the image of $b \in \mathcal{M}$ in $L^2(\mathcal{M})$ and $\widehat{\mathcal{M}}$ for the image of $\mathcal{M}$ in $L^2(\mathcal{M})$. Define $L_a, R_a : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ (for $a \in \mathcal{M}$), $J : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$, and $\langle \cdot, \cdot \rangle : \widehat{\mathcal{M}} \times \widehat{\mathcal{M}} \to C(X)$ as follows:

$$L_a(\widehat{b}) = \widehat{ab}, \tag{3.2.20}$$

$$R_a(\widehat{b}) = \widehat{ba}, \tag{3.2.21}$$

$$J(\widehat{b}) = \widehat{b^*}, \tag{3.2.22}$$

$$\langle \widehat{b}, \widehat{c} \rangle = E(bc^*), \tag{3.2.23}$$

where $a, b, c \in \mathcal{M}$.

We now show that L_a, R_a (for $a \in \mathcal{M}$), J and $\langle \cdot, \cdot \rangle$ have the desired (conjugate)-linearity properties and extend uniquely to the whole space. To avoid needless repetition, we use the notation of Definition 3.2.11 for the remainder of this section.

Proposition 3.2.12. *The maps L_a, R_a (for $a \in \mathcal{M}$) are linear, J is conjugate-linear, $\langle \cdot, \cdot \rangle$ is linear in the first place and conjugate-linear in the second. Moreover, the following estimates hold:*

$$\|L_a(\widehat{b})\|_{L^2(\mathcal{M})} \leq \|a\| \|\widehat{b}\|_{L^2(\mathcal{M})}, \tag{3.2.24}$$

$$\|R_a(\widehat{b})\|_{L^2(\mathcal{M})} \leq \|a\| \|\widehat{b}\|_{L^2(\mathcal{M})}, \tag{3.2.25}$$

$$\|J(\widehat{b})\|_{L^2(\mathcal{M})} = \|\widehat{b^*}\|_{L^2(\mathcal{M})}, \tag{3.2.26}$$

$$\|\langle \widehat{b}, \widehat{c} \rangle\|_{C(X)} \leq \|\widehat{b}\|_{L^2(\mathcal{M})} \|\widehat{c}\|_{L^2(\mathcal{M})}, \tag{3.2.27}$$

where $a, b, c \in \mathcal{M}$.

Proof. The linearity and conjugate-linearity claims for L_a, R_a (for $a \in \mathcal{M}$), J and $\langle \cdot, \cdot \rangle$ follow from the axioms for a C*-algebra and, in the case of $\langle \cdot, \cdot \rangle$, the linearity of E. The estimates follow from Proposition 3.2.7. □

Corollary 3.2.13. *The maps L_a, R_a extend to bounded linear operators on $L^2(\mathcal{M})$ for all $a \in M$. The map J extends to a isometric, conjugate-linear, self-inverse isomorphism on $L^2(\mathcal{M})$. The map $\langle \cdot, \cdot \rangle$ extends to a bounded sequilinear map (linear in the first variable, conjugate-linear in the second) $L^2(\mathcal{M}) \times L^2(\mathcal{M}) \to C(X)$.*

Finally, we systematically check that $L^2(\mathcal{M})$ becomes a Hilbert-$C(X)$-module and the operators coming from left and right multiplication define representations of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{op}}$ respectively.

Proposition 3.2.14. *For $f \in C(X) \subseteq Z(\mathcal{M})$, we have $L_f = R_f$. This defines a $C(X)$-action on $\mathcal{L}^2(\mathcal{M})$ such that $(L^2(\mathcal{M}), \langle\cdot,\cdot\rangle)$ is a Hilbert-$C(X)$-module. Moreover, we have the following:*

(i) For all $a \in \mathcal{M}$, the map L_a (resp. R_a) is adjointable with adjoint L_{a^*} (resp. R_{a^*}). In particular L_a and R_a are $C(X)$-linear.

(ii) The map J is conjugate-selfadjoint in the sense that $\langle Jv, w\rangle = \langle Jw, v\rangle = \langle v, Jw\rangle^*$ for $v, w \in L^2(\mathcal{M})$. In particular J is $C(X)$-conjugate-linear in the sense that $J(fv) = f^*J(v)$ for $f \in C(X), v \in L^2(\mathcal{M})$.

(iii) The map $L : \mathcal{M} \to \mathcal{L}(L^2(\mathcal{M}))$ defined by $a \mapsto L_a$ and the map $R : \mathcal{M}^{\mathrm{op}} \to \mathcal{L}(L^2(\mathcal{M}))$ defined by $a \mapsto R_a$ are injective *-homomorphisms of C^*-algebras. Moreover, we have

$$L(a)R(b) = R(b)L(a) \qquad (a, b \in \mathcal{M}), \tag{3.2.28}$$

$$JL(a)J = R(a^*) \qquad (a \in \mathcal{M}). \tag{3.2.29}$$

Proof. Let $f \in C(X) \subseteq Z(\mathcal{M})$. For $b \in \mathcal{M}$, $L_f(\widehat{b}) = \widehat{fb} = \widehat{bf} = R_f(\widehat{b})$. Hence, $L_f = R_f$ by density. In the sequel, we write fv instead of $L_f(v)$ or $R_f(v)$. We now check the conditions for a Hilbert-$C(X)$-bundle as in Section 2.11.1.

Let $b, c, d \in \mathcal{M}$ and $f, g \in C(X)$. Since E is a conditional expectation onto $C(X)$,

$$\langle f\widehat{b} + g\widehat{c}, \widehat{d}\rangle = E(fbd^* + gcd^*) \tag{3.2.30}$$

$$= fE(bd^*) + gE(cd^*) \tag{3.2.31}$$

$$= f\langle\widehat{b}, \widehat{d}\rangle + g\langle\widehat{c}, \widehat{d}\rangle. \tag{3.2.32}$$

Since E is *-preserving, $\langle\widehat{b}, \widehat{c}\rangle = E(bc^*) = E(cb^*)^* = \langle\widehat{c}, \widehat{b}\rangle^*$. By density, we get (2.11.1) and (2.11.2).

Since $\|b\|_{2,u} = \|\langle\widehat{b}, \widehat{b}\rangle^{1/2}\|_{C(X)}$ for all $b \in \mathcal{M}$, we get that $\|v\|_{L^2(\mathcal{M})} = \|\langle v, v\rangle^{1/2}\|_{C(X)}$ for all $v \in L^2(\mathcal{M})$ by density. Since $\|\cdot\|_{L^2(\mathcal{M})}$ is a norm on $L^2(\mathcal{M})$ by construction, (2.11.3) follows.

(i) Fix $a \in \mathcal{M}$. We have

$$\langle L_a(\widehat{b}), \widehat{c}\rangle = \langle \widehat{ab}, \widehat{c}\rangle \tag{3.2.33}$$
$$= E(abc^*) \tag{3.2.34}$$
$$= E(bc^*a) \tag{3.2.35}$$
$$= E(b(a^*c)^*) \tag{3.2.36}$$
$$= \langle \widehat{b}, \widehat{a^*c}\rangle \tag{3.2.37}$$
$$= \langle \widehat{b}, L_{a^*}(\widehat{c})\rangle, \tag{3.2.38}$$

where $b, c \in \mathcal{M}$. So, by density, $\langle L_a(v), w\rangle = \langle v, L_{a^*}(w)\rangle$ for $v, w \in L^2(\mathcal{M})$. Hence, L_a is adjointable with $L_a^* = L_{a^*}$. By Proposition 2.11.14, L_a is $C(X)$-linear. The corresponding result for R_a is proved similarly.

(ii) We have

$$\langle J(\widehat{b}), \widehat{c}\rangle = \langle \widehat{b^*}, \widehat{c}\rangle \tag{3.2.39}$$
$$= E(b^*c^*) \tag{3.2.40}$$
$$= E(c^*b^*) \tag{3.2.41}$$
$$= \langle \widehat{c^*}, \widehat{b}\rangle \tag{3.2.42}$$
$$= \langle J(\widehat{c}), \widehat{b}\rangle, \tag{3.2.43}$$

where $b, c \in \mathcal{M}$. So, by density, $\langle J(v), w\rangle = \langle J(w), v\rangle$ for $v, w \in L^2(\mathcal{M})$ and, by (2.11.2), $\langle J(w), v\rangle = \langle v, J(w)\rangle^*$. So J is conjugate-adjointable with $J^* = J$ (see Definition 2.11.17). By Proposition 2.11.18, we see that J is $C(X)$-conjugate-linear.

(iii) We have

$$\langle L(\lambda_1 a_1 + \lambda_2 a_2)(\widehat{b}) = [(\lambda_1 a_1 + \lambda_2 a_2)b]\widehat{\ } \tag{3.2.44}$$
$$= \lambda_1 \widehat{a_1 b} + \lambda_2 \widehat{a_2 b} \tag{3.2.45}$$
$$= (\lambda_1 L(a_1) + \lambda_2 L(a_2))(\widehat{b}), \tag{3.2.46}$$

where $a_1, a_2, b \in \mathcal{M}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. So, by density, L is linear. Furthermore,

$$\langle L(a_1 a_2)(\widehat{b}) = [(a_1 a_2)b]\widehat{\ } \tag{3.2.47}$$
$$= [a_1(a_2 b)]\widehat{\ } \tag{3.2.48}$$
$$= L(a_1)L(a_2)(\widehat{b}), \tag{3.2.49}$$

where $a_1, a_2, b \in \mathcal{M}$. So, by density, L is multiplicative. Finally, by (i), L preserves the involution. Therefore L is a *-homomorphism. If $L(a) = 0$ for some $a \in \mathcal{M}$, then $\widehat{a} = L(a)(\widehat{1_{\mathcal{M}}}) = 0$, so $a = 0$. Hence L is an injective *-homomorphism. The corresponding result for R is proved similarly.

Let $a, b, c \in \mathcal{M}$. Then $L(a)R(b)\widehat{c} = \widehat{acb} = R(b)L(a)\widehat{c}$. Also $JL(a)J(\widehat{b}) = JL(a)(\widehat{b^*}) = J(\widehat{ab^*}) = \widehat{ba^*} = R(a^*)(\widehat{b})$. Hence, by density $L(a)R(b) = R(b)L(a)$ and $JL(a)J = R(a^*)$. □

3.2.3 The Strict Topology on a W*-Bundle

The representation of a pre-W*-bundle $\mathcal{M}$ on the Hilbert module $L^2(\mathcal{M})$ provides us with an additional topology on $\mathcal{M}$: the restriction to $\mathcal{M}$ of the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$ (see Section 2.11.2). In this section, we shall investigate the relation between this topology and the $\|\cdot\|_{2,u}$-topology. The notation of Section 3.2.2 will be used throughout.

Proposition 3.2.15. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. The strict topology and the $\|\cdot\|_{2,u}$-topology agree on any $\|\cdot\|$-bounded subset of $\mathcal{M}$.*

Proof. Fix $K > 0$. Let $(a_\lambda) \subseteq \mathcal{M}$ be a net with $\|a_\lambda\| \leq K$ for all λ. Let $a \in \mathcal{M}$ with $\|a\| \leq K$.

Suppose first that $a_\lambda \to a$ with respect to the $\|\cdot\|_{2,u}$-topology. That is $\|a_\lambda - a\|_{2,u} \to 0$ as $\lambda \to \infty$. We have

$$\|L(a_\lambda)(\widehat{b}) - L(a)(\widehat{b})\|_{L^2(\mathcal{M})} = \|[(a_\lambda - a)b]\widehat{\ }\|_{L^2(\mathcal{M})} \tag{3.2.50}$$
$$= \|(a_\lambda - a)b\|_{2,u} \tag{3.2.51}$$
$$\leq \|a_\lambda - a\|_{2,u}\|b\| \tag{3.2.52}$$

for all $b \in \mathcal{M}$. Hence, $L(a_\lambda)(\widehat{b}) \to L(a)(\widehat{b})$ as $\lambda \to \infty$. Since $(L(a_\lambda))$ is a bounded net, $L(a_\lambda) \to L(a)$ strictly in $\mathcal{L}(L^2(\mathcal{M}))$ by Proposition 2.11.23.

Now let us consider the converse. Suppose $L(a_\lambda) \to L(a)$ in the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$. Then

$$\|a_\lambda - a\|_{2,u} = \|\widehat{a_\lambda} - \widehat{a}\|_{L^2(\mathcal{M})} \tag{3.2.53}$$
$$= \|L(a_\lambda)(\widehat{1}) - L(a)(\widehat{1})\|_{L^2(\mathcal{M})}, \tag{3.2.54}$$

which converges to 0 as $\lambda \to \infty$. □

For unbounded nets, convergence in the strict topology still implies convergence in the $\|\cdot\|_{2,u}$-topology, as can be seen from the proof of Proposition 3.2.15 above. Counterexamples showing that $\|\cdot\|_{2,u}$-convergence need not imply strict convergence can be constructed in the tracial von Neumann algebra $(L^\infty[0,1], \tau_{\text{leb}})$, where τ_{leb} is integration with respect to Lebesgue measure.[6]

Nevertheless, the $\|\cdot\|_{2,u}$-closed *-subalegbras of $\mathcal{M}$ are precisely the strictly closed *-subalgebras. This is because there are versions of the Kaplansky Density Theorem for both the strict topology and the $\|\cdot\|_{2,u}$-topology. The Kaplansky Density Theorem for the strict topology was Theorem 2.11.29. We now prove the corresponding result for $\|\cdot\|_{2,u}$-topology. As for Theorem 2.11.29, we follow the method of proof set out in [58, Section 4.3]. However, the crucial estimates are justified differently.

Lemma 3.2.16 (c.f. Lemma 2.11.28)**.** *Let $\mathcal{M}$ be a pre-W^*-bundle over the compact Hausdorff space X with conditional expectation E and let $f:\mathbb{R}\to\mathbb{C}$ be a bounded continuous function. Suppose (a_n) is a sequence of selfadjoint elements in $\mathcal{M}$ converging to $a\in M$ in $\|\cdot\|_{2,u}$-norm. Then $f(a_n)\to f(a)$ in $\|\cdot\|_{2,u}$-norm.*

Proof. Let A be the set of continuous functions $\mathbb{R}\to\mathbb{C}$ for which the conclusion holds for all sequences (a_n). By Corollary 3.2.8, A is a vector space closed under complex conjugation. Moreover, if $f,g\in A$ and one of them is bounded, then $fg\in A$ using the estimate (3.2.11). Let $A_0 = A\cap C_0(\mathbb{R})$. We shall show, using the Stone–Weierstrass Theorem, that $A_0 = C_0(\mathbb{R})$.

Consider the functions $f,g:\mathbb{R}\to\mathbb{C}$ given by $f(x)=(1+x^2)^{-1}$ and $g(x)=x(1+x^2)^{-1}$. Note that $\|f\|_{C_0(\mathbb{R})}, \|g\|_{C_0(\mathbb{R})}\le 1$. Let $a,b\in\mathcal{M}_{\text{sa}}$. We compute that

$$\begin{aligned} g(a)-g(b) &= a(1+a^2)^{-1} - b(1+b^2)^{-1} && (3.2.55)\\ &= (1+a^2)^{-1}(a(1+b^2)-(1+a^2)b)(1+b^2)^{-1} && (3.2.56)\\ &= (1+a^2)^{-1}(a-b-a(b-a)b)(1+b^2)^{-1}. && (3.2.57)\end{aligned}$$

[6]For example, let $a_n = n\chi_{[0,1/n^3]}\in L^\infty[0,1]$ and $v\in L^2[0,1]$ be given by $v(x)=1/x^{1/4}$. Then $\|a_n\|_2\to 0$ but $\|a_nv\|_2\to\infty$.

Therefore, using Proposition 3.2.7(iv-v),

$$\begin{aligned}\|g(a)-g(b)\|_{2,u} &\le \|(1+a^2)^{-1}(a-b)(1+b^2)^{-1}\|_{2,u} \qquad (3.2.58)\\ &\quad + \|(1+a^2)^{-1}a(b-a)b)(1+b^2)^{-1}\|_{2,u}\\ &\le \|(1+a^2)^{-1}\|\|a-b\|_{2,u}\|(1+b^2)^{-1}\| \qquad (3.2.59)\\ &\quad + \|(1+a^2)^{-1}a\|\|b-a\|_{2,u}\|b(1+b^2)^{-1}\|\\ &\le 2\|a-b\|_{2,u}. \qquad (3.2.60)\end{aligned}$$

Hence, $g \in A_0$. Since the map $x \mapsto x$ is in A, we get that $f = 1 - xg \in A_0$.

The set $\{f, g\}$ separates the points of $\mathbb{R}$ and $f(t) > 0$ for all $t \in \mathbb{R}$. Therefore, f and g generate the C*-algebra $C_0(\mathbb{R})$ by the Stone–Weierstrass Theorem. Thus, $A_0 = C_0(\mathbb{R})$.

Suppose $h \in C_b(\mathbb{R})$. Then $hf, hg \in C_0(\mathbb{R})$, so $hf, hg \in A$. Therefore, $h = hf + xhg \in A$. □

Theorem 3.2.17 (The Kaplansky Density Theorem for $\|\cdot\|_{2,u}$-topology)**.** *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. Let A be a C*-subalgebra of $\mathcal{M}$ with $\|\cdot\|_{2,u}$-closure B. Then we have the following:*

(i) A_{sa} is $\|\cdot\|_{2,u}$-dense in B_{sa}.

(ii) The closed unit ball of A_{sa} is $\|\cdot\|_{2,u}$-dense in the closed unit ball of B_{sa}.

(iii) The closed unit ball of A is $\|\cdot\|_{2,u}$-dense in the closed unit ball of B.

Proof. (i) Let $b \in B_{sa}$ and (a_n) be a sequence in A converging to b in $\|\cdot\|_{2,u}$. Then $(\frac{1}{2}(a_\lambda + a_\lambda^*))$ is a sequence in A_{sa} and converges to $\frac{1}{2}(b+b^*) = b$ in $\|\cdot\|_{2,u}$ by Corollary 3.2.8.

(ii) Let $b \in B_{sa}$ with $\|b\| \le 1$ and (a_n) be a sequence in A_{sa} converging to b in $\|\cdot\|_{2,u}$. Let $f : \mathbb{R} \to \mathbb{C}$ be the bounded, continuous function given by

$$f(x) = \begin{cases} -1, & x \le 1,\\ x, & -1 \le x \le 1,\\ 1, & x \ge 1. \end{cases} \qquad (3.2.61)$$

Then $(f(a_n))$ is a sequence in the closed unit ball of A_{sa} converging to b in $\|\cdot\|_{2,u}$ by Lemma 3.2.16.

(iii) This follows from a matrix inflation trick. For any $n \in \mathbb{N}$, the unital C*-algebra $\mathbb{M}_n(\mathbb{C}) \otimes \mathcal{M}$ with $C(X)$ identified with $1_n \otimes C(X)$ and with conditional expectation $\mathrm{tr}_n \otimes E$ is also a pre-W*-algebra. Indeed, Axiom (T) is clear from computations with elementary tensors, and we compute that

$$(\mathrm{tr}_n \otimes E)((a_{ij})^*(a_{ij})) = \frac{\sum_{1\leq i,j\leq n} E(a_{ij}^* a_{ij})}{n}, \tag{3.2.62}$$

from which Axiom (F) follows and we see that $\|\cdot\|_{2,u}$-convergence in $\mathbb{M}_n(\mathbb{C}) \otimes \mathcal{M}$ is precisely component-wise $\|\cdot\|_{2,u}$-convergence.

It follows that $\mathbb{M}_2(\mathbb{C}) \otimes A$ is a C*-subalgebra of $\mathbb{M}_2(\mathbb{C}) \otimes \mathcal{M}$ with $\|\cdot\|_{2,u}$-closure $\mathbb{M}_2(\mathbb{C}) \otimes B$ and we can apply (ii) to this inflation.

Suppose $b \in B$ has $\|b\| \leq 1$. Then $\left(\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\right) \in (\mathbb{M}_2(\mathbb{C}) \otimes B)_{sa}$ and has norm at most 1. There is a sequence (a_n) in the closed unit ball of $(\mathbb{M}_2(\mathbb{C}) \otimes A)_{sa}$ converging to $\left(\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\right) \in (\mathbb{M}_2(\mathbb{C}) \otimes B)_{sa}$. Taking the $(1,2)$-th entries of this sequence gives a sequence in the closed unit ball of A converging to b in $\|\cdot\|_{2,u}$.

□

Corollary 3.2.18. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. Let $\mathcal{N}$ be a C*-subalgebra of $\mathcal{M}$ then $\overline{\mathcal{N}}^{\|\cdot\|_{2,u}} = \overline{\mathcal{N}}^{\mathrm{st}}$ In particular, $\mathcal{N}$ is $\|\cdot\|_{2,u}$-closed if and only if $\mathcal{N}$ is strictly closed.*

Proof. Suppose $a \in \overline{\mathcal{N}}^{\mathrm{st}}$. Then by the Kaplansky Density Theorem for the strict topology, there is a net (a_λ) converging to a strictly with $\|a_\lambda\| \leq \|a\|$. By Proposition 3.2.15, $a_\lambda \to a$ in $\|\cdot\|_{2,u}$-norm, so $a \in \overline{\mathcal{N}}^{\|\cdot\|_{2,u}}$. For the converse, we use the Kaplansky Density Theorem for the $\|\cdot\|_{2,u}$-topology and Proposition 3.2.15. □

Corollary 3.2.19. *Let $\mathcal{M}$ be a pre-W*-bundle. Then $\mathcal{M}$ is strictly separable if and only if it is $\|\cdot\|_{2,u}$-separable.*

Proof. They are both equivalent to the existence of a countably generated C*-subalgebra whose closure in both the strict topology and the $\|\cdot\|_{2,u}$-topology is $\mathcal{M}$. □

The original definition of a W*-bundle in [62] requires that the base space is a compact metric space. This requirement was relaxed in [5], where the base space was only required to be compact Hausdorff, as in Definition 3.1.1. For strictly separable W*-bundles, metrisability of the base space can be proved.

Proposition 3.2.20. *Let $\mathcal{M}$ be a strictly separable pre-W*-bundle over a compact Hausdorff space X. Then X is metrisable and separable.*

Proof. By Proposition 3.2.15, $\mathcal{M}$ is $\|\cdot\|_{2,u}$-separable. As separability passes to subspaces, so is $C(X) \subseteq \mathcal{M}$. Since $\|f\|_{2,u} = \|f\|$ for $f \in C(X)$, $C(X)$ is separable in $\|\cdot\|$. Hence, X is metrisable (see for example [7, Corollary 12.19]). As compact metric spaces are necessarily separable, the proof is now complete. □

3.3 W*-Bundles from C*-Algebras

In this section, we consider the motivating example of a W*-bundle, namely the strict closure $\overline{A}^{\mathrm{st}}$ of a unital, separable C*-algebra with a non-empty Bauer simplex of traces. This construction is due to Ozawa [62, Theorem 3].

The ambient space for the construction of $\overline{A}^{\mathrm{st}}$, is the finite part of the bidual A^{**}_{fin} (see Section 2.9). For ease of notation, we shall assume hereinafter that the natural map $\iota : A \to A^{**}_{\mathrm{fin}}$ is injective and identify A with its image. If this is not the case, one can always pass to the quotient of A by the kernel of ι; see Remark 2.9.9.

As a finite von Neumann algebra, A^{**}_{fin} can be viewed as a W*-bundle over the spectrum of its centre (Example 3.1.5). In particular, A^{**}_{fin} has a natural strict topology. We set $\overline{A}^{\mathrm{st}}$ to be strict closure of A inside A^{**}_{fin}, which by Corollary 3.2.18 agrees with the $\|\cdot\|_{2,u}$-closure.

Each $a \in A$ defines a continuous affine map $\mathrm{eval}_a \in \mathrm{Aff}_{\mathbb{C}}(T(A))$ via $\mathrm{eval}_a(\tau) = \tau(a)$. In fact, this is also true for $a \in \overline{\mathrm{A}}^{\mathrm{st}}$. We set $\mathrm{eval}_a(\tau) = \widetilde{\tau}(a)$, where $\widetilde{\tau}$ is the unique normal extension of τ to A^{**}_{fin} (see Corollary 2.9.6). Since any $a \in \overline{\mathrm{A}}^{\mathrm{st}}$ is a $\|\cdot\|_{2,u}$-limit of a sequence (a_n) in A, we have that $\mathrm{eval}_{a_n} \to \mathrm{eval}_a$ uniformly on $T(A)$. Hence, $\mathrm{eval}_a \in \mathrm{Aff}_{\mathbb{C}}(T(A))$.

The base space for the W*-bundle $\overline{A}^{\mathrm{st}}$ is the extreme boundary of the trace simplex $X = \partial_e T(A)$ endowed with the weak* topology, which is compact by hypothesis. We define a ucp map $E : \overline{A}^{\mathrm{st}} \to C(X)$ via $a \mapsto \mathrm{eval}_a|_X$. The central embedding of $C(X)$ in $\overline{A}^{\mathrm{st}}$ will be obtained from the map θ of Theorem 2.10.9, but we must check $\theta(C(X)) \subseteq Z(\overline{A}^{\mathrm{st}})$. This follows from the following theorem of Ozawa.

Theorem 3.3.1. *[62, Theorem 3] Let A be a unital, separable C*-algebra with non-empty trace space $T(A)$. Let $\theta : B(\partial_e T(A)) \to Z(A^{**}_{\mathrm{fin}})$ be the unital *-homomorphism constructed in Theorem 2.10.9 with ultraweakly dense range and with $\theta(\widehat{a}) = \mathrm{ctr}(a)$ and*

$$\tau(\theta(f)a) = \int_{\lambda \in \partial_e T(A)} f(\lambda)\lambda(a)\,d\mu_\tau(\lambda) \tag{3.3.1}$$

for $a \in A$, $\tau \in T(A)$ and $f \in B(\partial_e T(A))$. Then, one has

$$\overline{A}^{\mathrm{st}} = \{a \in A^{**}_{\mathrm{fin}} : \mathrm{ctr}(aA) \subseteq \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))), \mathrm{ctr}(a^*a) \in \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))\}, \tag{3.3.2}$$

and, in particular,

$$Z(\overline{A}^{\mathrm{st}}) = \overline{A}^{\mathrm{st}} \cap Z(A^{**}_{\mathrm{fin}}) = \{\theta(f) : f \in Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))\}, \tag{3.3.3}$$

where (c.f. Section 2.10.1)

$$\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)) = \{f|_{\partial_e T(A)} : f \in \mathrm{Aff}_{\mathbb{C}}(T(A))\}, \tag{3.3.4}$$

$$Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))) = \{f \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)) : fg \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)) \text{ for all } g \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))\}. \tag{3.3.5}$$

Proof. The right hand side of (3.3.2) is $\|\cdot\|_{2,u}$-closed and contains A. Hence, it contains $\overline{A}^{\mathrm{st}}$. Proving the reverse inclusion is more involved.

Let a be in the right hand side of (3.3.2). Let $(b_\lambda)_{\lambda\in\Lambda}$ be a net in A converging to a in the ultrastrong* topology of A^{**}_{fin}. The hypotheses on a ensure that, for all $\lambda \in \Lambda$, $\mathrm{ctr}((b_\lambda - a)^*(b_\lambda - a)) \in \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$. Hence, we can identify $\mathrm{ctr}((b_\lambda - a)^*(b_\lambda - a))$ with the function it defines on $\partial_e T(A)$ via the pairing with extreme normal traces.

Since $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)) \cong \mathrm{Aff}_{\mathbb{C}}(T(A))$ as complex complete order unit spaces, the general theory of compact convex sets (see Section 2.10.1) tells us that every positive linear functional on $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$ corresponds to evaluation at some trace $\tau \in T(A)$. Hence, the ultrastrong* convergence of (b_λ) to a implies that $\mathrm{ctr}((b_\lambda - a)^*(b_\lambda - a)) \to 0$ weakly in $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$. We are now in a position to apply the Hahn–Banach Theorem, and deduce that the norm-closed convex hull of $\{\mathrm{ctr}((b_\lambda - a)^*(b_\lambda - a)) : \lambda \in \Lambda\}$ contains 0.[7]

Let $\epsilon > 0$. Then there exist $\lambda_1 \dots \lambda_k \in \Lambda$ and $\alpha_1, \dots, \alpha_k \in [0,1]$ with $\sum_{i=1}^k \alpha_i = 1$ such that

$$\|\sum_{i=1}^k \alpha_i \mathrm{ctr}((b_{\lambda_i} - a)^*(b_{\lambda_i} - a))\| < \epsilon. \tag{3.3.6}$$

Set $b = \sum_{i=1}^k \alpha_i b_{\lambda_i}$. Then $b^*b = c^*r^*rc \leq \|r\|^2 c^*c = \sum_{i=1}^k \alpha_i b_{\lambda_i}^* b_{\lambda_i}$, where $r = \begin{pmatrix}\alpha_1^{1/2} & \cdots & \alpha_k^{1/2}\end{pmatrix}$ and $c = \begin{pmatrix}\alpha_1^{1/2} b_{\lambda_1} & \cdots & \alpha_k^{1/2} b_{\lambda_k}\end{pmatrix}^T$. Therefore,

$$\begin{aligned}\mathrm{ctr}((b-a)^*(b-a)) &= \mathrm{ctr}(b^*b - b^*a - a^*b + a^*a) &(3.3.7)\\ &\leq \mathrm{ctr}\left(\sum_{i=1}^k \alpha_i b_{\lambda_i}^* b_{\lambda_i} - \sum_{i=1}^k \alpha_i b_{\lambda_i}^* a - \sum_{i=1}^k \alpha_i a^* b_{\lambda_i} + a^*a\right) &(3.3.8)\\ &= \mathrm{ctr}\left(\sum_{i=1}^k \alpha_i (b_{\lambda_i} - a)^*(b_{\lambda_i} - a)\right). &(3.3.9)\end{aligned}$$

[7]The norm of $\mathrm{ctr}((b_\lambda - a)^*(b_\lambda - a))$ in $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$ agrees with its $\|\cdot\|$-norm in A^{**}_{fin} by Corollary 2.8.12.

Hence, $\|\mathrm{ctr}((b-a)^*(b-a))\| < \epsilon$. Since ϵ is arbitrary, we have that a is in the $\|\cdot\|_{2,u}$-closure of A, which is $\overline{A}^{\mathrm{st}}$. This proves (3.3.2).

The inclusion $\overline{A}^{\mathrm{st}} \cap Z(A^{**}_{\mathrm{fin}}) \subseteq Z(\overline{A}^{\mathrm{st}})$ is immediate. The reverse inclusion follows as $\overline{A}^{\mathrm{st}}$ is ultrastrong* dense in A^{**}_{fin}. Suppose $a \in \overline{A}^{\mathrm{st}} \cap Z(A^{**}_{\mathrm{fin}})$. Then $a = \mathrm{ctr}(a) \in \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))$ by (3.3.2). So $a = \theta(f)$ for some $f \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$. Since $\mathrm{ctr}(\theta(f)b) = \theta(f)\mathrm{ctr}(b) = \theta(f)\theta(\widehat{b})$ for all $b \in A$ by the bimodule property of conditional expectations, the requirement that $\mathrm{ctr}(aA) \subseteq \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))$ forces $f \in Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))$ because every element of $\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$ is of the form $\widehat{b}$ for some $b \in A$ by [13, Proposition 2.7].

Conversely, suppose $f \in Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))$ and let $b \in B$. Then $f^*f, f\widehat{b} \in \mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))$. Hence, using the bimodule property of conditional expectations, we have

$$
\begin{aligned}
\mathrm{ctr}(\theta(f)b) &= \theta(f)\mathrm{ctr}(b) && (3.3.10)\\
&= \theta(f)\theta(\widehat{b}) && (3.3.11)\\
&= \theta(f\widehat{b}) && (3.3.12)\\
&\in \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))) && (3.3.13)
\end{aligned}
$$

and $\mathrm{ctr}(\theta(f)^*\theta(f)) = \theta(f)^*\theta(f) = \theta(f^*f) \in \theta(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A)))$. Therefore, $\theta(f) \in \overline{A}^{\mathrm{st}} \cap Z(A^{**}_{\mathrm{fin}})$. This proves (3.3.3). □

We now are in a position to check that $\overline{A}^{\mathrm{st}}$ together with the additional structure defined above is a W*-bundle over $X = \partial_e T(A)$.

Theorem 3.3.2. *Let A be a unital, separable C*-algebra with a non-empty Bauer simplex of traces. Let $X = \partial_e T(A)$. Then the map θ of Theorem 2.10.9 restricts to a central embedding of $C(X)$ into $\overline{A}^{\mathrm{st}}$; the map E defines a conditional expectation from $\overline{A}^{\mathrm{st}}$ onto this embedded copy of $C(X)$; and, with this additional structure, $\overline{A}^{\mathrm{st}}$ is a W*-bundle over X.*

Proof. For ease of notation, we shall assume hereinafter that the natural map $\iota : A \to A^{**}_{\mathrm{fin}}$ is injective and identify A with its image. If this is not the case, one can always pass to the quotient of A by the kernel of ι; see Remark 2.9.9.

Since $T(A)$ is a Bauer simplex, $Z(\mathcal{A}\mathrm{ff}_{\mathbb{C}}(T(A))) = C(X)$ by Corollary 2.10.7. Hence, $\theta(C(X)) = Z(\overline{A}^{\mathrm{st}})$ by (3.3.3).

By (3.3.1), we have $\tau(\theta(f)) = f(\tau)$ for all $\tau \in X$, so $E(\theta(f)) = f$. Hence, the ucp map E is a conditional expectation onto the embedded copy of $C(X)$ by Theorem 2.5.8.

Axiom (T) is clear from the definition of E. Suppose $E(a^*a) = 0$ for some $a \in \overline{A}^{\mathrm{st}}$. Then $\mathrm{eval}_{a^*a}|_X = 0$. Since $\mathrm{eval}_{a^*a} \in \mathrm{Aff}_{\mathbb{C}}(T(A))$ and $T(A)$ is the closed convex hull of X, $\mathrm{eval}_{a^*a} = 0$. It follows that $\tau(a^*a) = 0$ for all normal traces $\tau \in T(A^{**}_{\mathrm{fin}})$. Hence, $a = 0$ by Corollary 2.8.12. This proves Axiom (F).

For Axiom (C), we observe that for $a \in \overline{A}^{\mathrm{st}}$

$$\begin{aligned}
\|E(a^*a)^{1/2}\|_{C(X)} &= \sup_{\tau\in X} \widetilde{\tau}(a^*a)^{1/2} && (3.3.14)\\
&= \sup_{\tau\in T(A)} \widetilde{\tau}(a^*a)^{1/2} && (3.3.15)\\
&= \sup_{\tau\in T(A^{**}_{\mathrm{fin}})} \tau(a^*a)^{1/2} && (3.3.16)\\
&= \|\mathrm{ctr}(a^*a)\|_{Z(A^{**}_{\mathrm{fin}})}, && (3.3.17)
\end{aligned}$$

where $\widetilde{\tau}$ denotes the unique normal extension of $\tau \in T(A)$ to a trace on A^{**}_{fin}. Indeed, (3.3.15) holds as $\mathrm{eval}_{a^*a} \in \mathrm{Aff}_{\mathbb{C}}(T(A))$ and $T(A)$ is the closed convex hull of X, (3.3.16) holds by density (Corollary 2.8.12), and (3.3.16) holds as all traces on A^{**}_{fin} factor through the centre valued trace (Theorem 2.8.11).

It follows that the $\|\cdot\|_{2,u}$-norm on the pre-W*-bundle $\overline{A}^{\mathrm{st}}$ coincides with the $\|\cdot\|_{2,u}$-norm on the W*-bundle A^{**}_{fin}. Since $\overline{A}^{\mathrm{st}}$ is $\|\cdot\|_{2,u}$-closed in A^{**}_{fin}, $\overline{A}^{\mathrm{st}}$ satisfies Axiom (C) by virtue of the fact that A^{**}_{fin} does. □

Finally, we determine the fibres of $\overline{A}^{\mathrm{st}}$.

Theorem 3.3.3. *Let A be a unital, separable C*-algebra with non-empty Bauer simplex of traces. The fibre of the W*-bundle $\overline{A}^{\mathrm{st}}$ at $\tau \in \partial_e T(A)$ is isomorphic to $\pi_\tau(A)''$.*

Proof. By Proposition 2.9.4, there exists a unique normal *-homomorphism $\widetilde{\pi_\tau} : A^{**}_{\mathrm{fin}} \to \pi_\tau(A)''$ such that the diagram

$$\begin{tikzcd} A^{**}_{\mathrm{fin}} \arrow[dr, "\widetilde{\pi_\tau}"] & \\ A \arrow[u, "\iota"] \arrow[r, "\pi_\tau"'] & \pi_\tau(A)'' \end{tikzcd} \tag{3.3.18}$$

commutes and, by the proof of Corollary 2.9.6, composing $\widetilde{\pi_\tau}$ with the vector state corresponding to ξ_τ gives the unique normal trace $\widetilde{\tau}$ such that the diagram

$$\begin{tikzcd} A^{**}_{\mathrm{fin}} \arrow[dr, "\widetilde{\tau}"] & \\ A \arrow[u, "\iota"] \arrow[r, "\tau"'] & \mathbb{C} \end{tikzcd} \tag{3.3.19}$$

commutes.

Let $a \in \overline{A}^{\mathrm{st}}$. Then $\widetilde{\pi_\tau}(a) = 0$ if and only if $\langle \widetilde{\pi_\tau}(a)^* \widetilde{\pi_\tau}(a)\xi_\tau, \xi_\tau\rangle = 0$ by Proposition 2.8.13. But $\langle \widetilde{\pi_\tau}(a)^* \widetilde{\pi_\tau}(a)\xi_\tau, \xi_\tau\rangle = \tau(a^*a) = E(a^*a)(\tau)$, so

$$\mathrm{Ker}(\widetilde{\pi_\tau}|_{\overline{A}^{\mathrm{st}}}) = \{a \in \overline{A}^{\mathrm{st}} : E(a^*a)(\tau) = 0\}. \tag{3.3.20}$$

Therefore, by the first isomorphism theorem for C*-algebras, the fibre of the W*-bundle $\overline{A}^{\mathrm{st}}$ at $\tau \in \partial_e T(A)$ is isomorphic to $\widetilde{\pi_\tau}(\overline{A}^{\mathrm{st}})$.

Under this isomorphism, the trace on $(\overline{A}^{\mathrm{st}})_\tau$ corresponds to the GNS trace on $\pi_\tau(A)''$. Hence, the unit ball of $\widetilde{\pi_\tau}(\overline{A}^{\mathrm{st}})$ is $\|\cdot\|_2$-complete in $\pi_\tau(A)''$ by Theorem 3.2.9. Therefore, $\widetilde{\pi_\tau}(\overline{A}^{\mathrm{st}}) = \pi_\tau(A)''$ by Theorem 2.8.16. □

3.4 Further Theory of W*-Bundles

In this section, we return to the abstract theory of W*-bundles, building on and extending the results of [5, 23, 62]. We begin by investigating morphisms between W*-bundles. This leads naturally to an investigation of the ideals of W*-bundles and to quotient W*-bundles. An important application of quotient W*-bundles is to define the restriction of a W*-bundle to an arbitrary closed set, which in turn facilitates the definition of local triviality, which will be investigated in Section 4.7. Finally, we investigate the completion of pre-W*-bundles.

3.4.1 Morphisms

Although the concept of isomorphic W*-bundles appears in [62], the definition of a morphisms between W*-bundles first appears in [5]. The definition was then simplified to its present form in [23].[8] We state this definition in the more general setting of pre-W*-bundles.

Definition 3.4.1. Let $\mathcal{M}_i$ be a pre-W*-bundle over X_i with conditional expectation E_i for $i = 1, 2$. A *morphism* is a unital *-homomorphism $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\alpha(C(X_1)) \subseteq C(X_2)$ and the diagram

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{\alpha} & \mathcal{M}_2 \\ {\scriptstyle E_1}\downarrow & & \downarrow{\scriptstyle E_2} \\ C(X_1) & \xrightarrow{\alpha} & C(X_2) \end{array} \tag{3.4.1}$$

[8] A proof that the definitions of [5, Section 3.1] and [23, Definition 2.3] agree can be found at the end of this section.

commutes.

The key novelty of the definition of a morphism between pre-W*-bundles is that the base spaces of the two pre-W*-bundles are not assumed to be the same. The following example shows that the morphisms between bundles over different spaces occur naturally.

Example 3.4.2. Let $\mathcal{M}$ be a W*-bundle over X with conditional expectation E. For $x \in \mathcal{M}$, view the fibre $\mathcal{M}_x$ as a W*-bundle over the one point space $\{x\}$ by identifying $\mathbb{C}1_{M_x}$ with $C(\{x\})$ and viewing τ_x as a conditional expectation onto $\mathbb{C}1_{M_x}$. Then the quotient map $\mathcal{M} \to \mathcal{M}_x$ is a morphism of W*-bundles.

One reason for introducing morphisms of pre-W*-bundles is to clarify the notion of isomorphism for pre-W*-bundles: two pre-W*-bundles $(\mathcal{M}_i, X_i, E_i)$ for $i = 1, 2$ are isomorphic if there are mutually inverse morphisms $\mathcal{M}_1 \to \mathcal{M}_2$ and $\mathcal{M}_2 \to \mathcal{M}_1$. Equivalently, they are isomorphic if there is an isomorphism of the the C*-algebras $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\alpha(C(X_1)) = C(X_2)$ and the diagram

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "\alpha"] \arrow[d, "E_1"'] & \mathcal{M}_2 \arrow[d, "E_2"'] \\ C(X_1) \arrow[r, "\alpha"] & C(X_2) \end{tikzcd} \tag{3.4.2}$$

commutes.[9] Note that this implies X_1 and X_2 are homeomorphic, via the transpose map $X_2 \to X_1$ of the restriction $\alpha|_{C(X_1)}$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are are isomorphic W*-bundles over the same space X, one might ask whether we can choose α to extend the identity on $C(X)$. In the important case where $\mathcal{M}_2$ is the trivial bundle $C_\sigma(X, M)$, we can always arrange this to be the case. Indeed, if φ is the homeomorphism $X \to X$ transpose to $\alpha|_{C(X)}$, then we define a isomorphism of W*-bundles $\Phi : C_\sigma(X, M) \to C_\sigma(X, M)$ by $f \mapsto f \circ \phi$, then replace α with $\alpha \circ \Phi^{-1}$. However, the following example shows that we cannot always find an isomorphism which extends the identity on $C(X)$.

Example 3.4.3. Consider the two subtrivial W*-bundles $\mathcal{N}, \widetilde{\mathcal{N}} \subseteq C_\sigma([0,1], \mathcal{R})$ with fibres given by

$$\mathcal{N}_x = \begin{cases} \mathcal{R}, & x \neq 1, \\ \mathbb{C}1_{\mathcal{R}}, & x = 1, \end{cases} \qquad \widetilde{\mathcal{N}}_x = \begin{cases} \mathcal{R}, & x \neq 0, \\ \mathbb{C}1_{\mathcal{R}}, & x = 0 \end{cases} \tag{3.4.3}$$

[9] In fact, a bijective morphism α of pre-W*-bundles is an isomorphism. This is well-known at the level of C*-algebras and one easily checks that α^{-1} is a morphism of pre-W*-bundles.

(c.f. Example 3.1.5). Then the map $\alpha : \mathcal{N} \to \widetilde{\mathcal{N}}$ given by $f \mapsto f \circ s$ where $s(x) = 1 - x$ is a (self-inverse) isomorphism of W*-bundles. However, there is no isomorphism which extends the identity on $[0,1]$ as this would imply that that $\mathcal{N}_x \cong \widetilde{\mathcal{N}}_x$ for all $x \in [0,1]$ (see Proposition 3.4.5).

Another reason for introducing morphisms of W*-bundles is to study the functorial nature of some of the constructions that one can do with W*-bundles. In Section 3.5, we'll see that the standard form construction is indeed functorial. Unfortunately, the construction of Section 3.3, which associates a W*-bundle $\overline{A}^{\mathrm{st}}$ to a unital, separable C*-algebra A with a Bauer simplex of traces, is not functorial. This is not a flaw in the definition of morphism but a reflection of the fact that *-homomorphisms of C*-algebras don't necessarily map extremal traces to extremal traces.[10]

We now collect together some important results about morphisms of pre-W*-bundles. To avoid unnecessary repetition, we fix pre-W*-bundles $\mathcal{M}_1$ and $\mathcal{M}_2$, writing X_1 and X_2 for their respective base space and E_1 and E_2 for their respective conditional expectations. We begin with some norm estimates.

Proposition 3.4.4. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Then, for all $a \in \mathcal{M}_1$,*

$$\|\alpha(a)\| \leq \|a\|, \tag{3.4.4}$$

$$\|\alpha(a)\|_{2,u} \leq \|a\|_{2,u}, \tag{3.4.5}$$

with equality when α is injective.

Proof. We recall that a *-homomorphism between C*-algebras is automatically contractive [58, Theorem 2.1.7]. The first inequality is a direct application of this result; the second follows from the computation

$$\begin{aligned}
\|\alpha(a)\|_{2,u}^2 &= \|E_2(\alpha(a)^*\alpha(a))\|_{C(X_2)} && (3.4.6)\\
&= \|E_2(\alpha(a^*a))\|_{C(X_2)} && (3.4.7)\\
&= \|\alpha(E_1(a^*a))\|_{C(X_2)} && (3.4.8)\\
&\leq \|E_1(a^*a)\|_{C(X_1)} && (3.4.9)\\
&= \|a\|_{2,u}^2, && (3.4.10)
\end{aligned}$$

[10] Isomorphism, of course, is preserved: if $A \cong B$ as C*-algebras, then $\overline{A}^{\mathrm{st}} \cong \overline{B}^{\mathrm{st}}$ as W*-bundles. This is clear from the construction of Section 3.3.

where $a \in \mathcal{M}_1$ and the contractivity of the *-homomorphism $\alpha|_{C(X_1)}$ is used in the fourth line. Injective *-homomorphisms between C*-algebras are isometric. Therefore, we get equality in the forth line of the computation above and hence in (3.4.4). $\square$

In particular, we see that morphisms are both $\|\cdot\|$-continuous and $\|\cdot\|_{2,u}$-continuous. We now show that a morphism of W*-bundles induces trace preserving, unital *-homomorphism between the fibres.

Proposition 3.4.5. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Write $\alpha^t : X_2 \to X_1$ for the transpose of the *-homomorphism $\alpha|_{C(X_1)} : C(X_1) \to C(X_2)$. For each $x_1 \in X_1$ and $x_2 \in X_2$ with $\alpha^t(x_2) = x_1$, there is an induced, trace preserving, unital *-homomorphism $\overline{\alpha} : (\mathcal{M}_1)_{x_1} \to (\mathcal{M}_2)_{x_2}$ given by $a(x_1) \mapsto \alpha(a)(x_2)$ for all $a \in \mathcal{M}_1$.*

Proof. Let $x_1 \in X_1, x_2 \in X_2$ with $\alpha^t(x_2) = x_1$. We have $E_2(\alpha(a))(x_2) = \alpha(E_1(a))(x_2) = E_1(a)(\alpha^t(x_2)) = E_1(a)(x_1)$ for all $a \in \mathcal{M}_1$. It follows that $\alpha(I_1) \subseteq I_2$, where $I_i = \{a \in \mathcal{M}_i : E_i(a^*a)(x_i) = 0\}$ for $i = 1, 2$. Hence, we get an induced unital *-homomorphism of the quotient C*-algebras $\overline{\alpha} : (\mathcal{M}_1)_{x_1} \to (\mathcal{M}_2)_{x_2}$ via $a(x_1) \mapsto \alpha(a)(x_2)$ for all $a \in \mathcal{M}_1$. This map is trace preserving since $E_1(a)(x_1) = E_2(\alpha(a))(x_2)$. $\square$

In particular, an isomorphism $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ of pre-W*-bundle induces an homeomorphism $\alpha^t : X_2 \to X_1$ of the base spaces and an isomorphism between the fibres $(\mathcal{M}_1)_{\alpha^t(x)}$ and $(\mathcal{M}_2)_x$ for all $x \in X_2$.

We now turn to results on norm-preserving lifts, which build on the proof techniques of Theorems 3.2.9 and 3.2.10. First, we lift a single element.

Proposition 3.4.6. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Let $b \in \mathcal{M}_2$ be in the image of α. Then there is $a \in \mathcal{M}_1$ such that*

$$\alpha(a) = b \tag{3.4.11}$$

$$\|a\| = \|b\| \tag{3.4.12}$$

$$\|a\|_{2,u} = \|b\|_{2,u}. \tag{3.4.13}$$

Proof. If $b = 0$, then one simply takes $a = 0$, so we assume $\|b\|_{2,u} > 0$ in the sequel. We recall that given a *-homomorphism between C*-algebras is with can lift elements of the image without increasing the norm [74, Section 2.2.10]. Hence we can find $a' \in \mathcal{M}_1$ such that $\|a'\| = \|b\|$ and $\alpha(a') = b$. By the Gluing Lemma, the function $f : X_1 \to \mathbb{C}$ given by

$$f(x_1) = \begin{cases} \|b\|_{2,u}\|a'(x_1)\|_{2,\tau_{x_1}}^{-1}, & \|a'(x_1)\|_{2,\tau_{x_1}} \geq \|b\|_{2,u}, \\ 1, & \|a'(x_1)\|_{2,\tau_{x_1}} \leq \|b\|_{2,u}, \end{cases} \tag{3.4.14}$$

is continuous.[11] Set $a = fa'$. We show by a fibrewise argument that a has the required properties. Let $x_2 \in X_2$. Let α^t and $\overline{\alpha}$ be as in Proposition 3.4.5. We have

$$\begin{aligned} \alpha(a)(x_2) &= \overline{\alpha}(a(\alpha^t(x_2))) &\quad (3.4.15)\\ &= \overline{\alpha}(f(\alpha^t(x_2))a'(\alpha^t(x_2))) &\quad (3.4.16)\\ &= \overline{\alpha}(a'(\alpha^t(x_2))) &\quad (3.4.17)\\ &= \alpha(a')(x_2), &\quad (3.4.18) \end{aligned}$$

where in the third line we've use the fact that

$$\begin{aligned} \|a'(\alpha^t(x_2))\|_{2,\tau_{\alpha^t(x_2)}} &= \|\alpha(a')(x_2)\|_{2,\tau_{x_2}} &\quad (3.4.19)\\ &\leq \|b\|_{2,u}, &\quad (3.4.20) \end{aligned}$$

so $f(\alpha^t(x_2)) = 1$. Hence, by Proposition 3.2.5, we have $\alpha(a) = \alpha(a') = b$.

Furthermore, we have

$$\begin{aligned} \|a(x_1)\| &= |f(x_1)|\|a'(x_1)\| &\quad (3.4.21)\\ \|a(x_1)\|_{2,\tau_{x_1}} &= |f(x_1)|\|a'(x_1)\|_{2,\tau_{x_1}} &\quad (3.4.22) \end{aligned}$$

for $x_1 \in X_1$. By considering the possible values of $f(x_1)$, we see that $\|a(x_1)\| \leq \|b\|$ and $\|a(x_1)\|_{2,\tau_{x_1}} \leq \|b\|_{2,u}$ for all $x_1 \in X_1$, from which $\|a\| \leq \|b\|$ and $\|a\|_{2,u} \leq \|b\|_{2,u}$ follow by (3.2.6) and (3.2.8) respectively. The reverse inequalities are clear from Proposition 3.4.4 as $\alpha(a) = b$. □

This time, we lift a sequence of elements.

Proposition 3.4.7. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Let (b_n) be a sequence in the image of α. Then there exists a sequence (a_n) in $\mathcal{M}_1$ such that*

$$\begin{aligned} \alpha(a_n) &= b_n &\quad (3.4.23)\\ \|a_{n+1} - a_n\|_{2,u} &= \|b_{n+1} - b_n\|_{2,u}. &\quad (3.4.24) \end{aligned}$$

Moreover, if there is $K > 0$ such that $\|b_n\| \leq K$ for all $n \in \mathbb{N}$, then we can take $\|a_n\| \leq K$ for all $n \in \mathbb{N}$.

[11] Let X, Y be topological spaces. Suppose $X = A \cup B$ for closed subsets A, B. A function $f : X \to Y$ is continuous if and only if $f|_A$ and $f|_B$ are continuous. See for example [57, Theorem 18.3].

Proof. We recall that given a *-homomorphism between C*-algebras, we can lift elements of the image without increasing the norm. Let a_1 be any such lift of b_1. Suppose now that $a_1, \ldots, a_n$ have been defined and have the desired properties. If $b_{n+1} = b_n$, then one can simply take $a_{n+1} = a_n$, so we may assume in the sequel that $\|b_{n+1} - b_n\|_{2,u} > 0$. Let a'_{n+1} be any lift of b_{n+1} with $\|a'_{n+1}\| = \|b_{n+1}\|$. By the Gluing Lemma, the function $f : X_1 \to \mathbb{C}$ given by

$$f(x_1) = \begin{cases} \frac{\|b_{n+1}-b_n\|_{2,u}}{\|a'_{n+1}(x_1)-a_n(x_1)\|_{2,\tau_{x_1}}}, & \|a'_{n+1}(x_1) - a_n(x_1)\|_{2,\tau_{x_1}} \geq \|b_{n+1} - b_n\|_{2,u}, \\ 1, & \|a'_{n+1}(x_1) - a_n(x_1)\|_{2,\tau_{x_1}} \leq \|b_{n+1} - b_n\|_{2,u}, \end{cases} \tag{3.4.25}$$

is continuous. Set $a_{n+1} = fa'_{n+1} + (1-f)a_n$.

Let $x_2 \in X_2$. Let α^t and $\overline{\alpha}$ be as in Proposition 3.4.5. We have

$$\begin{aligned} \alpha(a_{n+1})(x_2) &= \overline{\alpha}(a_{n+1}(\alpha^t(x_2))) && (3.4.26)\\ &= \overline{\alpha}(f(\alpha^t(x_2))a'_{n+1}(\alpha^t(x_2)) + (1 - f(\alpha^t(x_2)))a_n(\alpha^t(x_2))) && (3.4.27)\\ &= \overline{\alpha}(a'_{n+1}(\alpha^t(x_2))) && (3.4.28)\\ &= \alpha(a'_{n+1})(x_2), && (3.4.29) \end{aligned}$$

where in the third line we've use the fact that

$$\begin{aligned} \|(a'_{n+1} - a_n)(\alpha^t(x_2))\|_{2,\tau_{\alpha^t(x_2)}} &= \|\alpha(a'_{n+1} - a_n)(x_2)\|_{2,\tau_{x_2}} && (3.4.30)\\ &\leq \|b_{n+1} - b_n\|_{2,u}, && (3.4.31) \end{aligned}$$

so $f(\alpha^t(x_2)) = 1$. Hence, by Proposition 3.2.5, we have $\alpha(a_{n+1}) = \alpha(a'_{n+1}) = b_{n+1}$.

Furthermore, we have that

$$\|a_{n+1}(x_1) - a_n(x_1)\|_{2,\tau_y} = |f(x_1)|\|a'_{n+1}(x_1) - a_n(x_1)\|_{2,\tau_{x_1}} \tag{3.4.32}$$

for $x_1 \in X_1$. By considering the possible values for $f(x_1)$ and applying (3.2.6), we get that $\|a_{n+1} - a_n\|_{2,u} \leq \|b_{n+1} - b_n\|_{2,u}$.

Moreover, if there is $K > 0$ such that $\|b_n\| \leq K$ for all $n \in \mathbb{N}$, then $\|a_{n+1}\| \leq K$ since $a_{n+1}(x_1)$ is a convex combination of two elements of norm at most K for all $x_1 \in X_1$. This completes the inductive construction of the sequence (a_n). □

As an application of Proposition 3.4.7, we show that images of morphism of W*-bundles are $\|\cdot\|_{2,u}$-closed.

Theorem 3.4.8. *A morphism of W*-bundles $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ has $\|\cdot\|_{2,u}$-closed image.*

Proof. Suppose (b_n) is a sequence in $\alpha(\mathcal{M}_1)$ converging to $b \in \mathcal{M}_2$. By the Kaplansky Density Theorem for the $\|\cdot\|_{2,u}$-norm (Theorem 3.2.17), we may assume that $\|b_n\| \leq \|b\|$ for all $n \in \mathbb{N}$. Passing to a subsequence, we may assume that $\|b_{n+1} - b_n\|_{2,u} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. By Lemma 3.4.7, there is a sequence (a_n) in $\mathcal{M}_1$ such that $\alpha(a_n) = b_n$, $\|a_n\| \leq \|b\|$ and $\|a_{n+1} - a_n\|_{2,u} < \frac{1}{2^n}$. It follows that (a_n) is $\|\cdot\|_{2,u}$-Cauchy and so, by Axiom (C), converges to some $a \in \mathcal{M}_1$. By Proposition 3.4.4, we have that $\alpha(a) = b$. $\square$

Equivalent Definitions of Morphisms

The definition of morphisms used in this thesis is that of [23, Definition 2.3]. We show that this definition agrees with the definition found in [5, Section 3.1]. For the benefit of the reader, we recall the definition of [5].

Definition 3.4.9. Given W*-bundles $\mathcal{M}$ and $\mathcal{N}$ over spaces K and L respectively and with conditional expectations $E_\mathcal{M}$ and $E_\mathcal{N}$ respectively. A morphism of W*-bundles $\theta : \mathcal{M} \to \mathcal{N}$ is a *-homomorphism $\mathcal{M} \to \mathcal{N}$ (also denoted θ) together with a continuous map $\sigma : L \to K$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\theta"] \arrow[d, "E_\mathcal{M}"'] & \mathcal{N} \arrow[d, "E_\mathcal{N}"'] \\ C(K) \arrow[r, "\widetilde{\sigma}"] & C(L) \end{tikzcd} \tag{3.4.33}$$

commutes, where $\widetilde{\sigma} : C(K) \to C(L)$ is the transpose of σ.

We formulate the equivalence of Definitions 3.4.9 and 3.4.1 as a proposition.

Proposition 3.4.10. *In Definition 3.4.1, θ is necessarily unital and $\widetilde{\sigma} = \theta|_{C(K)}$. In particular, σ is uniquely determined by θ.*

Proof. The map $\widetilde{\sigma}$ is unital as it is the transpose of a continuous map. Hence, $\theta(1_\mathcal{M})$ is a projection in $\mathcal{N}$ with $E_\mathcal{N}(\theta(1_\mathcal{M})) = \widetilde{\sigma}(E_\mathcal{M}(1_\mathcal{M})) = \widetilde{\sigma}(1_{C(L)}) = 1_{C(K)}$. Therefore, $E_\mathcal{N}((1_\mathcal{N} - \theta(1_\mathcal{M})^*(1_\mathcal{N} - \theta(1_\mathcal{M})) = E_\mathcal{N}(1_\mathcal{N} - \theta(1_\mathcal{M})) = 0$. By Axiom (F), $\theta(1_\mathcal{M}) = 1_\mathcal{N}$.

Now let $a \in \mathcal{M}$ and $x \in L$. By (3.4.1),

$$E_\mathcal{M}(a^*a)(\sigma(x)) = \widetilde{\sigma}(E_\mathcal{M}(a^*a))(x) \tag{3.4.34}$$
$$= E_\mathcal{N}(\theta(a^*a))(x) \tag{3.4.35}$$
$$= E_\mathcal{N}(\theta(a)^*\theta(a))(x). \tag{3.4.36}$$

Hence, θ induces a unital *-homomorphism $\overline{\theta} : \mathcal{M}_{\sigma(x)} \to \mathcal{N}_x$ via $a(\sigma(x)) \mapsto \theta(a)(x)$.

Let $f \in C(K)$. Then

$$\theta(f)(x) = \overline{\theta}(f(\sigma(x))) \tag{3.4.37}$$
$$= \overline{\theta}(f(\sigma(x))1_{\mathcal{M}_{\sigma(x)}}) \tag{3.4.38}$$
$$= f(\sigma(x))\overline{\theta}(1_{\mathcal{M}_{\sigma(x)}}) \tag{3.4.39}$$
$$= f(\sigma(x))1_{\mathcal{N}_x} \tag{3.4.40}$$
$$= f(\sigma(x)). \tag{3.4.41}$$

Hence, $\theta(f) = \widetilde{\sigma}(f)$. □

3.4.2 Ideals and Quotients

In this section, we define the ideals of pre-W*-bundles and construct quotient bundles. It turns out that the ideals of a pre-W*-bundle are in 1-1 correspondence with the closed subspaces of the base space and that passing to the quotient is like restricting sections to the corresponding closed set. Furthermore, we prove that the completeness axiom passes to quotients, so quotients of W*-bundles are W*-bundles. Finally, we discuss the first isomorphism theorem in the setting of W*-bundles.

We begin with the definition of ideals.

Definition 3.4.11. Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. An *ideal* I of the pre-W*-bundle $\mathcal{M}$ is a (two-sided, closed) ideal of the C*-algebra $\mathcal{M}$ which is additionally $\|\cdot\|_{2,u}$-norm closed and satisfies $E(I) \subseteq I$.

The motivation for this definition of the ideals of pre-W*-bundles is that ideals should be precisely the kernels of morphisms. The following result proves one direction of this motivating assertion. The converse will follow once we've defined the quotient of a pre-W*-bundle since passing to the corresponding quotient will be a morphism with the desired kernel.

Proposition 3.4.12. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Then* $\mathrm{Ker}(\alpha)$ *is an ideal of $\mathcal{M}_1$.*

Proof. From the general theory of C*-algebras, $\mathrm{Ker}(\alpha)$ is a norm-closed, two-sided ideal of the C*-algebra $\mathcal{M}_1$. From Proposition 3.4.4, we can deduce that $\mathrm{Ker}(\alpha)$ is $\|\cdot\|_{2,u}$-norm closed. Finally, writing E_i for the conditional expectation of $\mathcal{M}_i$, we have $E_2(\alpha(a)) = \alpha(E_1(a))$ for all $a \in \mathcal{M}_1$. Hence, if $\alpha(a) = 0$, then $\alpha(E_1(a)) = E_2(0) = 0$. □

We now show that ideals are in 1-1 correspondence with closed subspaces of the base space.

Proposition 3.4.13. *Let $\mathcal{M}$ be a pre-W* bundle over the compact Hausdorff space X with conditional expectation E.*

(i) Let Y be a closed subset of X. Then

$$I_Y = \{a \in \mathcal{M} : a(y) = 0 \text{ for all } y \in Y\} \tag{3.4.42}$$

is an ideal of the pre-W-bundle $\mathcal{M}$.*

(ii) Let I be an ideal of the pre-W-bundle $\mathcal{M}$. There exist a closed set $Y \subseteq X$ such that $I = I_Y$.*

Proof. (i) Let $y \in Y$. Since passing to the fibre at y is a morphism of W*-bundles (Example 3.4.2), $I_y = \{a \in \mathcal{M} : a(y) = 0\}$ is an ideal of $\mathcal{M}$ by Proposition 3.4.12. It's straightforward that intersections of ideals are ideals. Hence, $I_Y = \bigcap_{x \in Y} I_x$ is an ideal.

(ii) The intersection $I \cap C(X)$ is an ideal of the C*-algebra $C(X)$. Hence, there is a closed subset $Y \subseteq X$ such that

$$I \cap C(X) = \{f \in C(X) : f(y) = 0 \text{ for all } y \in Y\}. \tag{3.4.43}$$

Let $a \in I$ and $y \in Y$. Then $E(a^*a) \in I \cap C(X)$, so $E(a^*a)(y) = 0$, and so $a(y) = 0$. Hence $I \subseteq I_Y$.

Conversely, suppose $a \in \mathcal{M}$ with $a(y) = 0$ for all $y \in Y$. Let $\epsilon > 0$. Set $Z = \{x \in X : \|a(x)\|_{2,\tau_x} \geq \epsilon\}$. By Proposition 3.2.6, Z is closed. Applying Urysohn's lemma to the closed sets Y and Z, we obtain a continuous function $f : X \to [0, 1]$ such that $f(Y) \subseteq \{0\}$ and $f(Z) \subseteq \{1\}$. In particular, $f \in I$ and so $fa \in I$. For $x \in X$, we have

$$\|(a - fa)(x)\|_{2,\tau_x} = |1 - f(x)| \|a(x)\|_{2,\tau_x}. \tag{3.4.44}$$

By considering the cases $x \in Z$ and $x \in X \setminus Z$ separately, we get that $\|(a - fa)(x)\|_{2,\tau_x} \leq \epsilon$ for all $x \in X$. Hence, by Proposition 3.2.6, $\|a - fa\|_{2,u} \leq \epsilon$. Since ϵ was arbitrary and I is $\|\cdot\|_{2,u}$-norm closed, $a \in I$. □

Next, we construct the quotient of a pre-W*-bundle by an ideal and verify that the quotient is also a pre-W*-bundle. We then prove that the completeness axiom passes to the quotient.

Proposition 3.4.14. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X. Write $\iota : C(X) \to \mathcal{M}$ for the central embedding and $E : \mathcal{M} \to C(X)$ for the conditional expectation of $\mathcal{M}$. Let $I_Y = \{a \in \mathcal{M} : E(a^*a)(y) = 0$ for all $y \in Y\}$ for a closed subset $Y \subseteq X$. Write $\mathcal{M}/I_Y$ for the quotient C*-algebra.*

(i) There is a central embedding $\iota_Y : C(Y) \to \mathcal{M}/I_Y$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "q"] & \mathcal{M}/I_Y \\ C(X) \arrow[u, "\iota"] \arrow[r, "r"] & C(Y) \arrow[u, "\iota_Y"'] \end{tikzcd} \tag{3.4.45}$$

commutes, where q denotes the quotient map and $r : C(X) \to C(Y)$ is restriction of functions.

(ii) Identifying $C(Y)$ with its image under ι_Y, there is a conditional expectation $E_Y : \mathcal{M}/I_Y \to C(Y)$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "q"] \arrow[d, "E"] & \mathcal{M}/I_Y \arrow[d, "E_Y"] \\ C(X) \arrow[r, "r"] & C(Y) \end{tikzcd} \tag{3.4.46}$$

commutes, where q denotes the quotient map and $r : C(X) \to C(Y)$ is restriction of functions.

(iii) The C-algebra $\mathcal{M}/I_Y$ together with the central embedding ι_Y and the conditional expectation E_Y defines a pre-W*-bundle over Y and the quotient map q is a morphism.*

Proof. (i) The composition $q \circ \iota : C(X) \to \mathcal{M}/I_Y$ has kernel $\iota^{-1}(I_Y) = C_0(X \setminus Y)$. By the first isomorphism theorem, $q \circ \iota$ factors through the quotient $C(X)/C_0(X \setminus Y)$. This gives the commuting diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "q"] & \mathcal{M}/I_Y \\ C(X) \arrow[u, "\iota"] \arrow[r, "q'"] & C(X)/C_0(X \setminus Y), \arrow[u, "\iota'"'] \end{tikzcd} \tag{3.4.47}$$

where the horizontal maps are the quotient maps. The *-homomorphism ι' is injective thus an embedding. Since both q and q' are surjective, centrality of ι' follows from that of ι.

The map $r : C(X) \to C(Y)$ given by restriction of functions is surjective by the Tietze Extension Theorem and has kernel $C_0(X \setminus Y)$. Hence, there is an isomorphism of C*-algebras $\varphi : C(Y) \to C(X)/C_0(X \setminus Y)$ which intertwines r and the quotient map $C(X) \to C(X)/C_0(X \setminus Y)$. Set $\iota_Y = \iota' \circ \varphi^{-1}$.

(ii) Let $a \in I$. Then $\iota(E(a)) \in I_Y$. Hence, $E(a) \in C_0(X \setminus Y) = \mathrm{Ker}(r)$. Therefore, there is a ucp map $E_Y : \mathcal{M}/I_Y \to C(Y)$ such that $E \circ r = E_Y \circ q$. This gives us (3.4.46). It remains to show that E_Y is a conditional expectation. Let $f \in C(Y)$. Let $g \in C(X)$ be an extension of f. Then

$$\begin{aligned} f &= r(g) && (3.4.48)\\ &= r(E(\iota(g))) && (3.4.49)\\ &= E_Y(q(\iota(g))) && (3.4.50)\\ &= E_Y(\iota_Y(r(g))) && (3.4.51)\\ &= E_Y(\iota_Y(f)). && (3.4.52) \end{aligned}$$

(iii) Axioms (T) and (F) are verified by simple lifting arguments. Let $b_1, b_2 \in \mathcal{M}/I_Y$. Let $a_1, a_2 \in \mathcal{M}$ be lifts of b_1, b_2. We have $E_Y(b_1 b_2) = E(a_1 a_2) = E(a_2 a_1) = E_Y(b_2 b_1)$. This proves that (T) holds for $\mathcal{M}/I_Y$. Suppose now that $b \in M/I_Y$ is such that $E_Y(b^* b) = 0$. Let $a \in \mathcal{M}$ be a lift of b. Then $r(E(a^* a)) = E_Y(b^* b) = 0$, so $E(a^* a)(y) = 0$ for all $y \in Y$. Hence, $a \in I_Y$ and $b = 0$ in $\mathcal{M}/I_Y$.

It follows that $\mathcal{M}/I_Y$ is a pre-W*-bundle over Y. From the diagram (3.4.45), q induces the restriction map r on the centrally embedded copies of $C(X)$ and $C(Y)$ in $\mathcal{M}$ and $\mathcal{M}/I_Y$ respectively. Consequently, the diagram (3.4.46) shows that q is a morphism of pre-W*-bundles. □

Theorem 3.4.15. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X, and let $I = I_Y$ be the ideal of the W*-bundle $\mathcal{M}$ corresponding to the closed subset Y of X. Then the quotient $\mathcal{M}/I_Y$ is a W*-bundle over Y.*

Proof. It remains to prove Axiom (C). Let (b_n) be a Cauchy sequence in $\mathcal{M}/I_Y$ with respect to the $\|\cdot\|_{2,u}$-norm and with $\|b_n\| \leq 1$ for all $n \in \mathbb{N}$. Pass to a subsequence (b_{n_k}) such that $\|b_{n_{k+1}} - b_{n_k}\|_{2,u} < \frac{1}{2^k}$. By Proposition 3.4.7, we can find (a_{n_k}) in $\mathcal{M}$ with $\|a_{n_k}\| \leq 1$, $q(a_{n_k}) = b_{n_k}$ and such that $\|a_{n_{k+1}} - a_{n_k}\|_{2,u} < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. Since the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, (a_{n_k}) is $\|\cdot\|_{2,u}$-Cauchy in $\mathcal{M}$. Hence, by axiom (C), (a_{n_k}) has a $\|\cdot\|_{2,u}$-limit $a \in \mathcal{M}$ with $\|a\| \leq 1$. By Proposition 3.4.4, $b = q(a)$ satisfies $\|b\| \leq 1$ and (b_{n_k}) has $\|\cdot\|_{2,u}$-limit b. Since (b_n) is a $\|\cdot\|_{2,u}$-norm Cauchy sequence, (b_n) also has $\|\cdot\|_{2,u}$-limit b. □

We now show that one can quotient out the kernel of a morphism and obtain an injective morphism. In the remark following the theorem, we explain how this can be viewed as a first isomorphism theorem in the category of pre-W*-bundles.

Theorem 3.4.16. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. Suppose* $\mathrm{Ker}(\alpha) = I_{Y_1}$ *for the closed subspace $Y_1 \subseteq X_1$. Then there is a unique morphism $\overline{\alpha} : \mathcal{M}_1/I_{Y_1} \to \mathcal{M}_2$ such that $\alpha = \overline{\alpha} \circ q$, where $q : \mathcal{M}_1 \to \mathcal{M}_1/I_{Y_1}$ is the quotient morphism, and this morphism $\overline{\alpha}$ is injective.*

Proof. By the first isomorphism theorem for C*-algebras, there is a unique *-homomorphism $\overline{\alpha} : \mathcal{M}_1/I_{Y_1} \to \mathcal{M}_2$ such that $\alpha = \overline{\alpha} \circ q$ and this *-homomorphism $\overline{\alpha}$ is injective. We need only show that $\overline{\alpha}$ is a morphism of pre-W*-bundles.

Firstly, let $f \in C(Y_1) \subseteq \mathcal{M}_1/I_{Y_1}$ and let $g \in C(X_1) \subseteq \mathcal{M}_1$ be some extension of f. Then $f = q(g)$ by (3.4.45), so $\overline{\alpha}(f) = \overline{\alpha}(q(g)) = \alpha(g) \in C(X_2)$. Hence, $\overline{\alpha}(C(Y_1)) \subseteq C(X_2)$. Secondly, let $a \in \mathcal{M}_1$. Then

$$E_2(\overline{\alpha}(q(a)) = E_2(\alpha(a)) \tag{3.4.53}$$

$$= \alpha(E_1(a)) \tag{3.4.54}$$

$$= \overline{\alpha}(q(E_1(a)) \tag{3.4.55}$$

$$= \overline{\alpha}(E_{Y_1}(q(a)), \tag{3.4.56}$$

where E_{Y_1} denotes the conditional expectation of the quotient bundle. Therefore, $\overline{\alpha}$ is an injective morphism of pre-W*-bundles. □

Remark 3.4.17. If $\mathcal{M}$ is a pre-W*-bundle over X with conditional expectation E, and $\mathcal{N}$ is a unital C*-subalgebra of $\mathcal{M}$ satisfying $E(\mathcal{N}) \subseteq \mathcal{N}$, then $\mathcal{N}$ inherits the structure of a pre-W*-bundle from $\mathcal{M}$ in the following manner. The intersection $\mathcal{N} \cap C(X)$ is a unital, commutative C*-algebra, so can be identified with $C(Y)$ for some compact Hausdorff space Y. Dualising the inclusion $C(Y) \to C(X)$, we see that Y is a Hausdorff quotient of X. Since $E(\mathcal{N}) \subseteq \mathcal{N}$, the conditional expectation E maps $\mathcal{N}$ into $C(Y)$. In this way, the image of a morphism $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ inherits a pre-W*-bundle structure from $\mathcal{M}_2$. It follows from Theorem 3.4.16 that this pre-W*-bundle is isomorphic to the quotient pre-W*-bundle $\mathcal{M}_1/\mathrm{Ker}(\alpha)$.

After all this theory, it's time for an example.

Example 3.4.18 (Restriction for Trivial Bundles)**.** Let M be a tracial von Neumann algebra, X be a compact Hausdorff space and Y a closed subspace of X. Then the map $\mathrm{Rest}_Y : C_\sigma(X, M) \to C_\sigma(Y, M)$ given by $f \mapsto f|_Y$ is a morphism of W*-bundles. Moreover, we have that $\mathrm{Ker}(\mathrm{Rest}_Y) = I_Y$. Hence, by Theorem 3.4.16, there is an injective morphism $\overline{\mathrm{Rest}_Y} : C_\sigma(X, M)/I_Y \to C_\sigma(Y, M)$.

To show that $\overline{\mathrm{Rest}_Y}$ is an isomorphism, it suffices to show that it's surjective. By Theorem 3.4.8, it suffices to show that the image is $\|\cdot\|_{2,u}$-dense. However, this is a simple partition of unity argument. One approximates $f \in C_\sigma(Y, M)$ by a function of the form $y \mapsto \sum_{i=0}^{k} \phi_i(y) b_i$ for some $b_i \in M$ and continuous functions $\phi_1, \ldots, \phi_k \in C(Y)$. Since each ϕ_i has a continuous extension to X, $\sum_{i=0}^{k} \phi_i b_i$ lies in the image of $\overline{\mathrm{Rest}_Y}$.

Motivated by Example 3.4.18, we make the following definition.

Definition 3.4.19. Suppose $\mathcal{M}$ is a pre-W*-bundle over X and Y is a closed subspace of X. The quotient W*-bundle $\mathcal{M}_Y = \mathcal{M}/I_Y$ is called the *restriction of $\mathcal{M}$ to Y*.

This in turn facilitates the definition of a locally trivial W*-bundle, which we will study further in Sections 3.6 and 4.7.

Definition 3.4.20. A W*-bundle $\mathcal{M}$ over the compact Hausdorff space X is *locally trivial* if every point $x \in X$ has a closed neighbourhood Y such that $\mathcal{M}_Y \cong C_\sigma(Y, M)$ for for some tracial von Neumann algebra M.

3.4.3 Completions

In this section, we show that one can always complete a pre-W*-bundle to a W*-bundle over the same base space and that this completion is essentially unique.

The reason that we cannot appeal to standard results on the completion of normed space is that we only want to complete the $\|\cdot\|$-closed unit ball of the pre-W*-bundle with respect to the $\|\cdot\|_{2,u}$-norm not the whole space. Nevertheless, one can modify the standard construction of the completion of a normed space using Cauchy sequences, taking this into account.[12]

The main benefit of this abstract construction of the completion is that the essential uniqueness is easy to prove, as is the fact that morphisms of pre-W*-bundles extend to

[12] This modification is motivated by the definition of the C*-completion $\overline{A}^{\mathrm{u}}$ of a unital C*-algebras A with respect to uniform 2-norm, as appears in [62, Section 1].

morphisms of the completions. A more concrete approach to the completion of a pre-W*-bundle is also possible using the standard form. We include this construction at the end of the section.

Proposition 3.4.21. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. Set*

$$\overline{\mathcal{M}} = \frac{\{(a_i)_{i=1}^\infty \in \ell^\infty(\mathcal{M}) : (a_i)_{i=1}^\infty \text{ is } \|\cdot\|_{2,u}\text{-Cauchy}\}}{\{(a_i)_{i=1}^\infty \in \ell^\infty(\mathcal{M}) : (a_i)_{i=1}^\infty \text{ is } \|\cdot\|_{2,u}\text{-null}\}}. \tag{3.4.57}$$

Embed $\mathcal{M}$, and hence $C(X)$, into $\overline{\mathcal{M}}$ via $a \mapsto [(a)_{i=1}^\infty]$ for $a \in \mathcal{M}$, and define $\overline{E} : \overline{M} \to C(X)$ by $[(a_i)_{i=1}^\infty] \mapsto \lim_{i\to\infty} E(a_i)$, where square brackets denote the equivalence class of the sequence.

(i) The map $\overline{E}$ is a well-defined conditional expectation onto $C(X) \subseteq Z(\overline{\mathcal{M}})$ making $\overline{\mathcal{M}}$ W-bundle. Moreover, the embedding of $\mathcal{M}$ in $\overline{\mathcal{M}}$ has $\|\cdot\|_{2,u}$-dense image.*

(ii) Suppose $\widetilde{\mathcal{M}}$ is a W-bundle over X with a $\|\cdot\|_{2,u}$-dense embedding of $\mathcal{M}$. Then $\widetilde{\mathcal{M}} \cong \overline{\mathcal{M}}$ as W*-bundles and the isomorphism can be taken to be the identity on the embedded copies of $\mathcal{M}$.*

Proof. (i) It follows from the basic properties of the $\|\cdot\|_{2,u}$-norm established in Proposition 3.2.7 that the collection of $\|\cdot\|_{2,u}$-Cauchy sequences forms a unital C*-subalgebra of $\ell^\infty(\mathcal{M})$ which contains the set of $\|\cdot\|_{2,u}$-null sequences as a closed ideal. So $\overline{\mathcal{M}}$ is a well-defined unital C*-algebra.

We have $\|E(a_i) - E(a_j)\|_{C(X)} = \|E(a_i - a_j)\|_{C(X)} \leq \|a_1 - a_j\|_{2,u}$ by Proposition 3.2.7(ii). So, by the completeness of $C(X)$, the limit defining $\overline{E}$ exists whenever $(a_i)_{i=1}^\infty$ is $\|\cdot\|_{2,u}$-Cauchy. If $(a_i)_{i=1}^\infty$ is $\|\cdot\|_{2,u}$-null, then $\lim_{i\to\infty} \|E(a_n)\|_{C(X)} = 0$ by Proposition 3.2.7(ii), so $\overline{E}$ is well-defined on the quotient $\overline{\mathcal{M}}$. The positivity of $\overline{E}$ follows easily from that of E. Indeed, $\overline{E}([(a_i)_{i=1}^\infty]^*[(a_i)_{i=1}^\infty]) = \lim_{i\to\infty} E(a_i^* a_i)$ and the positive cone of a C*-algebra is closed. Complete positivity now follows by Proposition 2.5.2. It is clear that $\overline{E}$ is the identity on $C(X) \subseteq \overline{\mathcal{M}}$, so $\overline{E}$ is a well-defined conditional expectation onto $C(X) \subseteq \overline{\mathcal{M}}$ by Theorem 2.5.8.

The tracial axiom is easily checked and the faithfulness axiom is built into the definition of $\overline{\mathcal{M}}$ because $\overline{E}([(a_i)_{i=1}^\infty]^*[(a_i)_{i=1}^\infty]) = 0$ if and only if $(a_i)_{i=1}^\infty$ is $\|\cdot\|_{2,u}$-null.

Before proving the completeness axiom holds for $\overline{\mathcal{M}}$, we prove the $\|\cdot\|_{2,u}$-density of $\mathcal{M}$ in $\overline{\mathcal{M}}$. This follows from the Cauchy condition. Indeed, if $[(a_i)_{i=1}^\infty] \in \overline{\mathcal{M}}$ and

$\epsilon > 0$ are given, there is $N \in \mathbb{N}$ such that $\|a_i - a_j\|_{2,u} < \epsilon$ whenever $i, j \geq N$. We then have that $\|[(a_i)_{i=1}^\infty] - [(a_N)_{i=1}^\infty]\|_{2,u} = \lim_{i\to\infty} \|a_i - a_N\|_{2,u} \leq \epsilon$. Furthermore, by taking a norm-preserving lift from the quotient [74, Section 2.2.10], we may assume that $\|(a_i)_{i=1}^\infty\| = \|[(a_i)_{i=1}^\infty]\|$. In this case, $\|a_N\| \leq \|[(a_i)_{i=1}^\infty]\|$.

Now, we return to the completeness axiom. Firstly, we show that a $\|\cdot\|_{2,u}$-Cauchy sequence in the unit ball of $\mathcal{M}$ has a $\|\cdot\|_{2,u}$-limit in the unit ball of $\overline{\mathcal{M}}$. Let (a_n) be a $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$ with $\|a_n\| \leq 1$ for all $n \in \mathbb{N}$. Let $a = [(a_i)_{i=1}^\infty] \in \overline{\mathcal{M}}$. Then $\|a\| \leq 1$. Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that $\|a_n - a_m\|_{2,u} \leq \epsilon$ whenever $n, m \geq N$. Hence, in $\overline{\mathcal{M}}$, $\|a_n - a\|_{2,u} = \|[(a_n)_{i=1}^\infty] - [(a_i)_{i=1}^\infty]\|_{2,u} = \lim_{i\to\infty} \|a_n - a_i\|_{2,u} \leq \epsilon$ whenever $n \geq N$. So (a_n) has $\|\cdot\|_{2,u}$-limit a in $\overline{\mathcal{M}}$.

Let (a_n) be a $\|\cdot\|_{2,u}$-Cauchy sequence in $\overline{\mathcal{M}}$ with $\|a_n\| \leq 1$ for all $n \in \mathbb{N}$. Then, by density, we can construct a sequence (a'_n) in $\mathcal{M}$ with $\|a'_n\| \leq 1$ and $\|a_n - a'_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. It follows that (a'_n) is is a $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$, so has a $\|\cdot\|_{2,u}$-limit $a \in \overline{\mathcal{M}}$ with $\|a\| \leq 1$. But $\|a_n - a\|_{2,u} \leq \|a'_n - a\| + \frac{1}{n}$ for all $n \in \mathbb{N}$, so (a_n) also converges to a in $\|\cdot\|_{2,u}$-norm.

(ii) Suppose $\widetilde{\mathcal{M}}$ is a W*-bundle over X and $\mathcal{M}$ is embedded $\|\cdot\|_{2,u}$-densely in $\widetilde{\mathcal{M}}$. Define a *-homomorphism $\{(a_i)_{i=1}^\infty \in \ell^\infty(\mathcal{M}) : (a_i)_{i=1}^\infty \text{ is } \|\cdot\|_{2,u}\text{-Cauchy}\} \to \widetilde{\mathcal{M}}$ by mapping each $\|\cdot\|$-bounded, $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$ to its limit in $\widetilde{\mathcal{M}}$, which exists by Axiom (C). The kernel of this homomorphism is the ideal of $\|\cdot\|_{2,u}$-null sequences and the image is $\widetilde{\mathcal{M}}$ by the Kaplansky Density Theorem for the $\|\cdot\|_{2,u}$-norm (Theorem 3.2.17). Thus, we get a *-homomorphism $\alpha : \overline{\mathcal{M}} \to \widetilde{\mathcal{M}}$, which, after identifying $\mathcal{M}$ with its image in $\overline{\mathcal{M}}$, extends the identity map on $\mathcal{M}$. Since $\mathcal{M}$ is dense in both $\overline{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$, we have $\widetilde{E} \circ \alpha = \alpha \circ \overline{E}$ and $\overline{E} \circ \alpha^{-1} = \alpha^{-1} \circ \widetilde{E}$, where $\widetilde{E}$ denotes the conditional expectation on $\widetilde{\mathcal{M}}$. Therefore, α is an isomorphism of W*-bundles.

□

Proposition 3.4.22. *Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. There is a unique extension of α to a morphism of W*-bundles $\overline{\alpha} : \overline{\mathcal{M}_1} \to \overline{\mathcal{M}_2}$, which is injective whenever α is injective and surjective whenever α is surjective.*

Proof. By Proposition 3.4.4, we have $\|\alpha(a)\| \leq \|a\|$ and $\|\alpha(a)\|_{2,u} \leq \|a\|_{2,u}$ for all $a \in \mathcal{M}_1$. Hence, α maps $\|\cdot\|$-bounded, $\|\cdot\|_{2,u}$-Cauchy sequences in $\mathcal{M}_1$ to $\|\cdot\|$-bounded, $\|\cdot\|_{2,u}$-Cauchy sequences in $\mathcal{M}_2$, and α maps $\|\cdot\|$-bounded, $\|\cdot\|_{2,u}$-null sequences in $\mathcal{M}_1$ to $\|\cdot\|$-bounded, $\|\cdot\|_{2,u}$-null sequences in $\mathcal{M}_2$. This defines a *-homomorphism $\overline{\alpha} : \overline{\mathcal{M}_1} \to \overline{M_2}$. Identifying

$\mathcal{M}_i$ with the image of constant sequences in $\overline{\mathcal{M}_i}$, $\overline{\alpha}$ extends α. Since $\mathcal{M}_1$ is $\|\cdot\|_{2,u}$-dense in $\overline{\mathcal{M}_1}$, a density argument shows that $\overline{\alpha}$ is a morphism of W*-bundles.

If α is injective, then we have $\|\alpha(a)\|_{2,u} = \|a\|_{2,u}$ for all $a \in \mathcal{M}$ by Proposition 3.4.4. Hence,

$$\|\overline{\alpha}([(a_i)_{i=1}^{\infty}])\|_{2,u} = \lim_{i\to\infty} \|\alpha(a_i)\|_{2,u} \tag{3.4.58}$$

$$= \lim_{i\to\infty} \|a_i\|_{2,u} \tag{3.4.59}$$

$$= \|[(a_i)_{i=1}^{\infty}]\|_{2,u}, \tag{3.4.60}$$

which ensures that $\overline{\alpha}$ is injective. If α is surjective, then combining Theorem 3.4.8 and Proposition 3.4.21(a), we find that $\overline{\alpha}$ is surjective. □

We now turn to the construction of the completion of a pre-W*-bundle using the standard form.

Proposition 3.4.23. *Let $\mathcal{M}$ be a pre-W*-bundle over the compact Hausdorff space X with conditional expectation E. Let $L : \mathcal{M} \to \mathcal{L}(L^2(\mathcal{M}))$ be the standard form representation. Set $\widetilde{\mathcal{M}} = \overline{L(\mathcal{M})}^{\mathrm{st}} \subseteq \mathcal{L}(L^2(\mathcal{M}))$. Embed $\mathcal{M}$, and hence $C(X)$, in $\widetilde{\mathcal{M}}$ via L and extend E to $\widetilde{E} : \widetilde{\mathcal{M}} \to C(X)$ via $T \mapsto \langle T\widehat{1}, \widehat{1}\rangle_{L^2(M)}$. Then $\widetilde{\mathcal{M}}$ with the given central embedding and conditional expectation is a W*-bundle completion of $\mathcal{M}$.*

Proof. We must check that $\widetilde{\mathcal{M}}$ with the given central embedding and conditional expectation is a well-defined W*-bundle and show that $\mathcal{M}$ is $\|\cdot\|_{2,u}$-dense in $\mathcal{M}$. Let $T, S \in \widetilde{\mathcal{M}}$ and $\lambda, \mu \in \mathbb{C}$. There are nets (a_i), (b_i) in $\mathcal{M}$ such that $L(a_i) \to T$ and $L(b_i) \to S$ strictly as $i \to \infty$, where we take the indexing set in each case to be a neighbourhood basis of 0. By the Kapansky Density Theorem (see Theorem 2.11.29), we may assume that the nets are bounded. We can now make use of Proposition 2.11.21 to show that $\lambda T + \mu S, TS, T^* \in \widetilde{\mathcal{M}}$. Since the strict topology is weaker than the norm topology, we have that $\widetilde{\mathcal{M}}$ is norm closed. Thus, $\widetilde{\mathcal{M}}$ is a unital C*-algebra.

By Proposition 3.2.14, L is injective, so it defines a embedding of $\mathcal{M}$ and $C(X)$ into $\widetilde{\mathcal{M}}$. Let $f \in C(X)$ and $T \in \widetilde{\mathcal{M}}$. Since $C(X)$ is central in $\mathcal{M}$, we have $L(f)T = \lim_{i\to\infty} L(f)L(a_i) = \lim_{i\to\infty} L(fa_i) = \lim_{i\to\infty} L(a_i f) = \lim_{i\to\infty} L(a_i)L(f) = TL(f)$, where $(L(a_i))$ is a bounded net converging strictly to T. Hence, $C(X)$ embeds centrally in $\widetilde{\mathcal{M}}$.

Let $T \in \widetilde{\mathcal{M}}$. Let (a_i) be a net in $\mathcal{M}$ such that $(L(a_i))$ converges strictly to T. Then $\|L(a_i) - T\|_{2,u}^2 = \langle (L(a_i) - T)^*(L(a_i) - T)\widehat{1}, \widehat{1}\rangle_{L^2(\mathcal{M})} = \|(L(a_i) - T)\widehat{1}\|_{L^2(\mathcal{M})} \to 0$. Hence,

$\mathcal{M}$ embeds densely in $\widetilde{\mathcal{M}}$. In fact, by Theorem 2.11.29, the closed unit ball of $\mathcal{M}$ is dense in the closed unit ball of $\widetilde{\mathcal{M}}$.

For $a \in \mathcal{M}$, we have $\widetilde{E}(a) = \langle L(a)\widehat{1}, \widehat{1}\rangle_{L^2(\mathcal{M})} = \langle \widehat{a}, \widehat{1}\rangle_{L^2(\mathcal{M})} = E(a)$, so $\widetilde{E}$ really does extend E. We have $\widetilde{E}(T^*T) = \langle T^*T\widehat{1}, \widehat{1}\rangle_{L^2(\mathcal{M})} = \langle T\widehat{1}, T\widehat{1}\rangle_{L^2(\mathcal{M})} \geq 0$ for all $T \in \widetilde{\mathcal{M}}$, so E is positive. Complete positivity then follows by Proposition 2.5.2. Thus, $\widetilde{E}$ is a conditional expectation on to $C(X) \subseteq \widetilde{\mathcal{M}}$ by Theorem 2.5.8.

Since the inner product on a Hilbert-$C(X)$-module is continuous, $\widetilde{E}$ is strictly continuous. Let $T, S \in \widetilde{\mathcal{M}}$. Then $\widetilde{E}(TS) = \lim_{i\to\infty} E(a_i b_i) = \lim_{i\to\infty} E(b_i a_i) = \widetilde{E}(ST)$, where $(L(a_i))$ and $(L(b_i))$ are bounded nets converging strictly to T and S respectively. This proves Axiom (T).

Let $T \in \widetilde{M}$ and $a \in \mathcal{M}$. Since $L(\mathcal{M})$ commutes with $R(\mathcal{M})$ by Proposition 3.2.14(iii), we have $TR(a) = \lim_{i\to\infty} L(a_i)R(a) = \lim_{\to\infty} R(a)R(a_i) = R(a)T$, where $(L(a_i))$ is a bounded net converging strictly to T. Hence, $\widetilde{\mathcal{M}}$ commutes with $R(\mathcal{M})$.

Let $T \in \widetilde{M}$ such that $E(T^*T) = 0$. Then $0 = \langle T^*T\widehat{1}, \widehat{1}\rangle_{L^2(\mathcal{M})} = \langle T\widehat{1}, T\widehat{1}\rangle_{L^2(\mathcal{M})}$. By (2.11.3), we have $T\widehat{1} = 0$. Consequently, $T\widehat{a} = TR(a)\widehat{1} = R(a)T\widehat{1} = 0$ for all $a \in \mathcal{M}$. Hence, $T = 0$ since $\widehat{\mathcal{M}}$ is dense in $L^2(\mathcal{M})$. This proves Axiom (F).

Let $(a_n) \subseteq \mathcal{M}$ be a $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$ with $\|a_n\| \leq 1$. Then $\|a_n - a_m\|_{2,u} \to 0$ as $n, m \to \infty$ and $\|a_n - a_m\| \leq 2$ for all $n, m \in \mathbb{N}$. By Proposition 3.2.15, $L(a_n - a_m) \to 0$ strictly as $n, m \to \infty$. Hence, by Proposition 2.11.22, $(L(a_n))$ has a strict limit $T \in \widetilde{\mathcal{M}}$ with $\|T\| \leq 1$. We compute that $\|L(a_n) - T\|_{2,u}^2 = \langle (L(a_n) - T)^*(L(a_n) - T)\widehat{1}, \widehat{1}\rangle_{L^2(\mathcal{M})} = \|(L(a_i) - T)\widehat{1}\|_{L^2(\mathcal{M})}^2 \to 0$. Therefore, $(a_n) \subseteq \mathcal{M}$ has a $\|\cdot\|_{2,u}$-limit in $\widetilde{\mathcal{M}}$.

Now suppose (T_n) is a $\|\cdot\|_{2,u}$-Cauchy sequence in $\widetilde{\mathcal{M}}$ with $\|T_n\| \leq 1$. For each $n \in \mathbb{N}$, there is $a_n \in \mathcal{M}$ with $\|a_n\| \leq 1$ and $\|a_n - T_n\|_{2,u} < \frac{1}{n}$. Then (a_n) is a $\|\cdot\|_{2,u}$-Cauchy sequence in $\mathcal{M}$ with $\|a_n\| \leq 1$, so has a $\|\cdot\|_{2,u}$-limit $T \in \widetilde{\mathcal{M}}$ with $\|T\| \leq 1$. But, $\|T_n - T\|_{2,u} = \|T_n - a_n\|_{2,u} + \|a_n - T\|_{2,u}$, which converges to 0. So (T_n) converges to T in $\|\cdot\|_{2,u}$-norm. □

3.5 Standard Form Revisited

In this section, we return to the topic of the standard form of a pre-W*-bundle. We show that the standard form construction is functorial. In particular, the standard form of a pre-W*-bundle can be understood in terms of the standard form of the fibres. We also prove that the fibration of a pre-W*-bundle $\mathcal{M}$ is consistent with the fibration of

Hilbert-$C(X)$-module $L^2(\mathcal{M})$ and its algebra of adjointable operators $\mathcal{L}(L^2(\mathcal{M}))$. As an application of these techniques, we prove that the left and right regular representation of a W*-bundle are commutants of one another, and so, when in standard form, a W*-bundle equals its bicommutant. This can be viewed as a generalisation of von Neumann's Bicommutant Theorem [92, Satz 8] (see also [58, Theorem 4.1.5]) from the setting of von Neumann algebras to W*-bundles.[13]

3.5.1 Functoriality

We begin by showing that a morphism of pre-W*-bundles induces a morphism between the respective $L^2(\mathcal{M})$ spaces which is compatible with left regular representations, the right regular representations and the involutions of the respective standard forms.

Proposition 3.5.1. *Let $\mathcal{M}_i$ be a pre-W*-bundle over the compact Hausdorff space X_i with conditional expectation E_i for $i = 1, 2$. Let $L^{(\mathcal{M}_i)}, R^{(\mathcal{M}_i)}, J^{(\mathcal{M}_i)}$ denote respectively the left regular representation, right regular representation and the involution for the standard form of $\mathcal{M}_i$.*

(i) A morphism of pre-W-bundles $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ induces a morphism of Hilbert modules $L^2\alpha : L^2(\mathcal{M}_1) \to L^2(\mathcal{M}_2)$ such that*

$$(L^2\alpha)L^{(\mathcal{M}_1)}(a) = L^{(\mathcal{M}_2)}(\alpha(a))(L^2\alpha), \qquad (a \in \mathcal{M}_1), \tag{3.5.1}$$

$$(L^2\alpha)R^{(\mathcal{M}_1)}(a) = R^{(\mathcal{M}_2)}(\alpha(a))(L^2\alpha), \qquad (a \in \mathcal{M}_1), \tag{3.5.2}$$

$$(L^2\alpha)J^{(\mathcal{M}_1)} = J^{(\mathcal{M}_2)}(L^2\alpha) \tag{3.5.3}$$

(ii) If α is injective, then so is $L^2\alpha$.

(iii) If α is surjective, then so is $L^2\alpha$.

Proof. (i) Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of pre-W*-bundles. By Proposition 3.4.4, we have $\|\alpha(a)\|_{2,u} \leq \|a\|_{2,u}$ for all $a \in \mathcal{M}_1$. It follows that α, viewed now as a map $\widehat{\mathcal{M}_1} \to \widehat{\mathcal{M}_2}$, extends to a bounded linear operator $L^2\alpha : L^2(\mathcal{M}_1) \to L^2(\mathcal{M}_2)$.

We now check that $L^2\alpha$, together with $\alpha|_{C(X_1)} : C(X_1) \to C(X_2)$, is a morphism of

[13] However, in the setting of W*-bundles, it is important to use the standard form representation; see Example 5.6.7.

Hilbert modules. We have

$$\langle L^2\alpha(\widehat{a}), L^2\alpha(\widehat{b})\rangle_{L^2(\mathcal{M}_2)} = E_2(\alpha(a)\alpha(b)^*) \tag{3.5.4}$$
$$= E_2(\alpha(ab^*)) \tag{3.5.5}$$
$$= \alpha(E_1(ab^*)) \tag{3.5.6}$$
$$= \alpha(\langle \widehat{a}, \widehat{b}\rangle_{L^2(\mathcal{M}_1)}) \tag{3.5.7}$$

for all $a, b \in \mathcal{M}_1$. So, by density,

$$\langle L^2\alpha(v), L^2\alpha(w)\rangle_{L^2(\mathcal{M}_2)} = \alpha(\langle v, w\rangle_{L^2(\mathcal{M}_1)}) \tag{3.5.8}$$

for all $v, w \in L^2(\mathcal{M}_1)$. Furthermore,

$$L^2\alpha(f\widehat{a}) = L^2\alpha(\widehat{fa}) \tag{3.5.9}$$
$$= \widehat{\alpha(fa)} \tag{3.5.10}$$
$$= \alpha\widehat{(f)\alpha(}a) \tag{3.5.11}$$
$$= \alpha(f)\widehat{\alpha(a)} \tag{3.5.12}$$
$$= \alpha(f)L^2\alpha(\widehat{a}) \tag{3.5.13}$$

for all $a \in \mathcal{M}_1$ and $f \in C(X_1)$. So, by density, $L^2\alpha(fv) = \alpha(f)L^2\alpha(v)$ for all $v \in L^2(\mathcal{M}_1)$ and $f \in C(X_1)$. This completes the proof that $L^2\alpha$, together with $\alpha|_{C(X_1)}$, is a morphism of Hilbert modules.

Let $a, b \in \mathcal{M}_1$. Then

$$(L^2\alpha)L^{(\mathcal{M}_1)}(a)(\widehat{b}) = L^2\alpha(\widehat{ab}) \tag{3.5.14}$$
$$= \widehat{\alpha(ab)} \tag{3.5.15}$$
$$= \alpha\widehat{(a)\alpha(}b) \tag{3.5.16}$$
$$= L^{(\mathcal{M}_2)}(\alpha(a))(\widehat{\alpha(b)}) \tag{3.5.17}$$
$$= L^{(\mathcal{M}_2)}(\alpha(a))(L^2\alpha(\widehat{b})). \tag{3.5.18}$$

This, together with a density argument, gives (3.5.1). The proof of (3.5.2) is similar. The third relation (3.5.3) follows from the following computation and a density argument:

$$(L^2\alpha)J^{(\mathcal{M}_1)}(\widehat{a}) = L^2\alpha(\widehat{a^*}) \tag{3.5.19}$$
$$= \widehat{\alpha(a^*)} \tag{3.5.20}$$
$$= \widehat{\alpha(a)^*} \tag{3.5.21}$$
$$= J^{(\mathcal{M}_2)}(\widehat{\alpha(a)}) \tag{3.5.22}$$
$$= J^{(\mathcal{M}_2)}(L^2\alpha(\widehat{a})), \tag{3.5.23}$$

where $a \in \mathcal{M}_1$.

(ii) Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be an injective morphism of pre-W*-bundles. By Proposition 3.4.4, we have $\|\alpha(a)\|_{2,u} = \|a\|_{2,u}$ for all $a \in \mathcal{M}_1$. It follows that α, viewed now as a map $\widehat{\mathcal{M}_1} \to \widehat{\mathcal{M}_2}$, extends to a isometric linear operator $L^2\alpha : L^2(\mathcal{M}_1) \to L^2(\mathcal{M}_2)$. In particular, $L^2\alpha$ is injective.

(iii) Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a surjective morphism of pre-W*-bundles. We know that $L^2\alpha(\widehat{\mathcal{M}_1}) = \widehat{\mathcal{M}_2}$. Let $v \in L^2(\mathcal{M}_2)$ and (b_n) be a sequence in $\mathcal{M}_1$ such that $\widehat{b_n} \to v$ in $L^2(\mathcal{M}_2)$ as $n \to \infty$. Passing to a subsequence, we may assume that $\|b_{n+1} - b_n\|_{2,u} < \frac{1}{2^n}$. By Proposition 3.4.7, there is a sequence (a_n) in $\mathcal{M}_2$ such that $\alpha(a_n) = b_n$ and $\|a_{n+1} - a_n\|_{2,u} < \frac{1}{2^n}$. The sequence $(\widehat{a_n})$ in $L^2(\mathcal{M}_1)$ is therefore Cauchy and so converges to some $u \in L^2(\mathcal{M}_1)$. By continuity, $L^2\alpha(u) = v$.

□

Remark 3.5.2. Applying Proposition 3.5.1 to the morphism of pre-W*-bundles $\mathrm{eval}_x : \mathcal{M} \to \mathcal{M}_x$ coming from passing from the W*-bundle $\mathcal{M}$ over X to the fibre $\mathcal{M}_x$ at $x \in X$, we see that the standard form of the W*-bundle $\mathcal{M}$ is compatible with the standard form of the fibre $\mathcal{M}_x$.

The remainder of this section is devoted to establishing the connection between the fibration of a pre-W*-bundle $\mathcal{M}$, the fibration of the Hilbert module $L^2(\mathcal{M})$ (see Propositions 2.11.5 and 2.11.6) and the induced fibration of the algebra of adjointable operators $\mathcal{L}(L^2(\mathcal{M}))$ (see Proposition 2.11.16). We fix a W*-bundle $\mathcal{M}$ over a compact Hausdorff space X and an $x \in X$. For simplicity of notation, we identity $C(\{x\})$ with $\mathbb{C}$ and $\mathbb{C}1_{\mathcal{M}_x}$. We also shall make no distinction between Hilbert $C(\{x\})$-modules and Hilbert spaces.

First, we consider the relationship between the morphism of Hilbert modules $L^2(\mathrm{eval}_x) : L^2(\mathcal{M}) \to L^2(\mathcal{M}_x)$ and the morphism $\alpha_x : L^2(\mathcal{M}) \to L^2(\mathcal{M})_x$ defined in Example 2.11.9 by passing from a Hilbert $C(X)$-module to its fibres.

Proposition 3.5.3. *There is a unique isomorphism of Hilbert spaces $\theta_x : L^2(\mathcal{M})_x \to L^2(\mathcal{M}_x)$ such that following diagram*

$$\begin{tikzcd} L^2(\mathcal{M}) \arrow[d, "\alpha_x"'] \arrow[dr, "L^2(\mathrm{eval}_x)"] & \\ L^2(\mathcal{M})_x \arrow[r, "\theta_x"'] & L^2(\mathcal{M}_x) \end{tikzcd} \tag{3.5.24}$$

commutes.

Proof. By (3.5.8), we have

$$\langle L^2(\mathrm{eval}_x)(v), L^2(\mathrm{eval}_x)(w)\rangle_{L^2(\mathcal{M}_x)} = \mathrm{eval}_x(\langle v, w\rangle_{L^2(\mathcal{M})}) \tag{3.5.25}$$

$$= \langle v, w\rangle_{L^2(\mathcal{M})}(x) \tag{3.5.26}$$

for all $u, v \in L^2(\mathcal{M})$, where the last equality makes use of the identification of $C(\{x\})$ with $\mathbb{C}$. From this, we deduce that the kernel of $L^2(\mathrm{eval}_x)$ is precisely $\{v \in L^2(\mathcal{M}) : \langle v, v\rangle(x) = 0\}$. Hence, there is an injective bounded linear map $\theta_x : L^2(\mathcal{M})_x \to L^2(\mathcal{M}_x)$ such that the diagram (3.5.24) commutes. Uniqueness of θ_x is clear from the commuting diagram, since α_x is surjective by construction.

Since eval_x is surjective, so is $L^2(\mathrm{eval}_x)$ by Proposition 3.5.1(iii) and, hence, so is θ_x. Moreover,

$$\langle \theta_x(\alpha_x(v)), \theta_x(\alpha_x(w))\rangle_{L^2(\mathcal{M}_x)} = \langle L^2(\mathrm{eval}_x)(v), L^2(\mathrm{eval}_x)(w)\rangle_{L^2(\mathcal{M}_x)} \tag{3.5.27}$$

$$= \mathrm{eval}_x(\langle v, w\rangle_{L^2(\mathcal{M})}) \tag{3.5.28}$$

$$= \langle v, w\rangle_{L^2(\mathcal{M})}(x) \tag{3.5.29}$$

$$= \langle \alpha_x(v), \alpha_x(w)\rangle_{L^2(\mathcal{M})_x} \tag{3.5.30}$$

for all $v, w \in L^2(\mathcal{M})$. Hence, θ_x is an isomorphism of Hilbert spaces. □

We can now relate the fibration of the pre-W*-bundle $\mathcal{M}$ with the fibration of $\mathcal{L}(L^2(\mathcal{M}))$ coming from Proposition 2.11.16.

Proposition 3.5.4. *Identifying $L^2(\mathcal{M})_x$ with $L^2(\mathcal{M}_x)$ via the cannonical isomorphism θ_x, defined in Proposition 3.5.3, we have the following commuting diagrams for all $a \in \mathcal{M}$ and $x \in X$:*

$$\begin{tikzcd} L^2(\mathcal{M}) \arrow[r, "L^{(\mathcal{M})}(a)"] \arrow[d, "\alpha_x"'] & L^2(\mathcal{M}) \arrow[d, "\alpha_x"'] \\ L^2(\mathcal{M})_x \cong L^2(\mathcal{M}_x) \arrow[r, "L^{(\mathcal{M}_x)}(a(x))"'] & L^2(\mathcal{M}_x) \cong L^2(\mathcal{M})_x, \end{tikzcd} \tag{3.5.31}$$

$$\begin{tikzcd} L^2(\mathcal{M}) \arrow[r, "R^{(\mathcal{M})}(a)"] \arrow[d, "\alpha_x"'] & L^2(\mathcal{M}) \arrow[d, "\alpha_x"'] \\ L^2(\mathcal{M})_x \cong L^2(\mathcal{M}_x) \arrow[r, "R^{(\mathcal{M}_x)}(a(x))"'] & L^2(\mathcal{M}_x) \cong L^2(\mathcal{M})_x, \end{tikzcd} \tag{3.5.32}$$

$$\begin{tikzcd} L^2(\mathcal{M}) \arrow[r, "J^{(\mathcal{M})}"] \arrow[d, "\alpha_x"'] & L^2(\mathcal{M}) \arrow[d, "\alpha_x"'] \\ L^2(\mathcal{M})_x \cong L^2(\mathcal{M}_x) \arrow[r, "J^{(\mathcal{M}_x)}"'] & L^2(\mathcal{M}_x) \cong L^2(\mathcal{M})_x. \end{tikzcd} \tag{3.5.33}$$

Hence, in the notation of Propositions 2.11.16 and 2.11.19, $L^{(\mathcal{M})}(a)_x = L^{(\mathcal{M}_x)}(a(x))$, $R^{(\mathcal{M})}(a)_x = R^{(\mathcal{M}_x)}(a(x))$ *and* $J_x^{(\mathcal{M})} = J^{(\mathcal{M}_x)}$.

Proof. We apply Proposition 3.5.1 to the morphism $\mathrm{eval}_x : \mathcal{M} \to \mathcal{M}_x$ together with the result of Proposition 3.5.3, which says that $L^2(\mathrm{eval}_x) = \theta_x \circ \alpha_x$. We obtain

$$\theta_x \circ \alpha_x \circ L^{(\mathcal{M})}(a) = L^{(\mathcal{M}_x)}(\mathrm{eval}_x(a)) \circ \theta_x \circ \alpha_x, \tag{3.5.34}$$

$$\theta_x \circ \alpha_x \circ R^{(\mathcal{M})}(a) = R^{(\mathcal{M}_x)}(\mathrm{eval}_x(a)) \circ \theta_x \circ \alpha_x, \tag{3.5.35}$$

$$\theta_x \circ \alpha_x \circ J^{(\mathcal{M})} = J^{(\mathcal{M}_x)} \circ \theta_x \circ \alpha_x, \tag{3.5.36}$$

which is what is required because $\mathrm{eval}_x(a) = a(x)$. □

3.5.2 Commutant Theorems

In this section, we use a fibrewise argument to show that the image of the left regular representation of a W*-bundle in standard form is the commutant of the right regular representation and vice versa. We use notation of the previous section but drop the superscripts when they can be inferred from the context. For example, we write $L(a)$ instead of $L^{(\mathcal{M})}(a)$ for $a \in \mathcal{M}$, and we write $L(a(x))$ instead of $L^{(\mathcal{M}_x)}(a(x))$ for $a \in \mathcal{M}$ and $x \in X$.

Theorem 3.5.5. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X. Then* $L(\mathcal{M}) = R(\mathcal{M})'$ *and* $R(\mathcal{M}) = L(\mathcal{M})'$.

Proof. By Proposition 3.2.14(iii), we have $L(\mathcal{M}) \subseteq R(\mathcal{M})'$ and $R(\mathcal{M}) \subseteq L(\mathcal{M})'$. Let $T \in R(\mathcal{M})'$. Then by Proposition 2.11.16 and 3.5.4, $T_x \in R(\mathcal{M}_x)'$ for all $x \in X$. But $R(\mathcal{M}_x)' = L(\mathcal{M}_x)''$ by Theorem 2.8.15 and $L(\mathcal{M}_x)'' = L(\mathcal{M}_x)$ by Theorem 3.2.9 and von Neumann's Bicommutant Theorem [92, Satz 8]. Hence, T defines a function $f : X \to \sqcup_{x\in X}\mathcal{M}_x$ with $f(x) \in \mathcal{M}_x$ for all $x \in X$, as in Theorem 3.2.10, via the condition $T_x = L(f(x))$ for all $x \in X$. Note that $\sup_{x\in X} \|f(x)\| \leq \|T\|$.

Fix $x \in X$. There exists $c^{(x)} \in \mathcal{M}$ such that $f(x) = c^{(x)}(x)$. For $y \in X$ we have,

$$\|f(y) - c^{(x)}(y)\|_{2,\tau_y}^2 = \|\widehat{f(y)} - \widehat{c^{(x)}(y)}\|_{L^2(\mathcal{M}_y)}^2 \tag{3.5.37}$$
$$= \langle [L(f(y)) - L(c^{(x)}(y))]\widehat{1}_y, [L(f(y)) - L(c^{(x)}(y))]\widehat{1}_y \rangle_{L^2(\mathcal{M}_y)} \tag{3.5.38}$$
$$= \langle [T_y - L(c^{(x)})_y]\widehat{1}_y, [T_y - L(c^{(x)}(y))]\widehat{1}_y \rangle_{L^2(\mathcal{M}_y)} \tag{3.5.39}$$
$$= \langle [T - L(c^{(x)})]\widehat{1}, [T - L(c^{(x)})]\widehat{1} \rangle_{L^2(\mathcal{M})}(y). \tag{3.5.40}$$

Therefore, the map $y \mapsto \|f(y) - c^{(x)}(y)\|_{2,\tau_y}^2$ is continuous. Since it vanishes at x, given $\epsilon > 0$, there is a neighbourhood U of x such that $\|f(y) - c^{(x)}(y)\|_{2,\tau_y} < \epsilon$ whenever $y \in U$. Hence, Theorem 3.2.10 applies and there is $a \in \mathcal{M}$ such that $a(x) = f(x)$ for all $x \in X$. Therefore, $L(a) = T$.

This completes the proof that $L(\mathcal{M}) = R(\mathcal{M})'$. The proof that $R(\mathcal{M}) = L(\mathcal{M})'$ is similar. □

Corollary 3.5.6. *Let $\mathcal{M}$ be a W*-bundle represented on $L^2(\mathcal{M})$ in standard form. Then $\mathcal{M}'' = \mathcal{M}$.*

3.6 The Topological Viewpoint

In this section, we shall show how to combine the fibres of a W*-bundle to produce a bundle (B, p) in the sense of Section 2.12. The W*-bundle, more precisely its section algebra, can be recovered as the collection of bounded, continuous sections of (B, p). This builds on known results in the context of continuous fields of Hilbert spaces [19, Section 1.2] and Banach bundles [26, Chapter 2, Section 13.4].

The results of Sections 3.6.1 and 3.6.2 appear in my joint paper with Ulrich Pennig [23, Section 3]. Section 3.6.3 has be added in this thesis to remove a technical hypothesis from Theorem 3.6.11.

3.6.1 Bundles of Tracial von Neumann Algebras

Suppose (B, p) is a bundle in the sense of Definition 2.12.1 over the Hausdorff space X with each fibre $p^{-1}(x)$ having the additional structure of a tracial von Neumann algebra (see Definition 2.8.17). Then addition and multiplication define maps $D \to B$, where $D = \{(b_1, b_2) : B \times B : p(b_1) = p(b_2)\}$; scalar multiplication defines a map $\mathbb{C} \times B \to B$; the involution defines a map $B \to B$; and the trace on each fibre define a map $\tau : B \to \mathbb{C}$. Furthermore, there are two “global norms” $\|\cdot\|, \|\cdot\|_2 : B \to [0, \infty)$, coming from the

C*-norm and the in 2-norm in each fibre respectively. Finally, there are two distinguished sections $x \mapsto 0_x$ and $x \mapsto 1_x$, which pick out the additive and multiplicative identity elements in each fibre.

We can now state some axioms for such bundles. We shall write $B_{\leq r}$ for the subspace $\{b \in B : \|b\| \leq r\}$ of B for $r > 0$.

Definition 3.6.1. A *bundle of tracial von Neumann algebras* over the Hausdorff space X is a bundle (B, p) over X together with operations, norms and traces making each fibre $p^{-1}(x)$ a tracial von Neumann algebra and satisfying the axioms listed below:

(i) Addition, viewed as a map $D \to B$, is continuous.

(ii) Scalar multiplication, viewed as a map $\mathbb{C} \times B \to B$, is continuous.

(iii) The involution, viewed as a map $B \to B$, is continuous.

(iv) The map $X \to B$ which sends x to the to the additive identity 0_x of $p^{-1}(x)$ is continuous and so is the map $X \to B$ which sends x to the to the multiplicative identity 1_x of $p^{-1}(x)$.

(v) The map $\|\cdot\|_2 : B \to \mathbb{C}$ arising from combining the 2-norms from each fibre is continuous, as is the map $\tau : B \to \mathbb{C}$ obtained by combining the traces on each fibre.

(vi) A net $(b_\lambda) \subseteq B$ converges to 0_x whenever $p(b_\lambda) \to x$ and $\|b_\lambda\|_2 \to 0$.

(vii) Multiplication, viewed as a map $D \to B$, is continuous on $\|\cdot\|$-bounded subsets.

(viii) The restriction $p|_{B_{\leq 1}} : B_{\leq 1} \to X$ is open.[14]

We say that two bundles of tracial von Neumann algebras (B_i, p_i) for $i = 1, 2$ are isomorphic if there are homeomorphisms ψ and φ such that the diagram

$$\begin{tikzcd} B_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] & B_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "\psi"] & X_2 \end{tikzcd} \tag{3.6.1}$$

commutes and, for each $x_1 \in X_1$, $\varphi|_{p_1^{-1}(x_1)} : p_1^{-1}(x_1) \to p_2^{-1}(\psi(x_1))$ is an isomorphism of tracial von Neumann algebras.

[14] This is a strengthening of the requirement of Definition 2.12.1, which requires that $p : B \to X$ be open.

Remark 3.6.2. Bundles of tracial von Neumann algebras are bundles of normed spaces in the sense of Definition 2.12.4 with respect to the global $\|\cdot\|_2$-norm. They are not bundles of Banach spaces because one only has $\|\cdot\|_2$-norm completeness of the $\|\cdot\|$-norm closed unit ball in each fibre.

The basic example of a bundle of tracial von Neumann algebras is $(X \times M, \pi_1)$, where X is a Hausdorff space, M is a tracial von Neumann algebra, the topology on $X \times M$ is the product of the topology of X and the 2-norm topology on M, and $\pi_1 : X \times M \to X$ is the projection onto the first coordinate. The veracity of the Axioms (i-viii) is an easy consequence of the definition of the product topology. This is the *trivial bundle* of tracial von Neumann algebras over X with fibre M. We can now define local triviality for bundles of tracial von Neumann algebras.

Definition 3.6.3. Let (B, p) be a bundle of tracial von Neumann algebras over the Hausdorff space X. We say (B, p) is *locally trivial* if every $x \in X$ has an open neighbourhood U such that $(p^{-1}(U), p|_{p^{-1}(U)})$ is isomorphic to a trivial bundle over U.

Remark 3.6.4. Note that when X is compact Hausdorff, we can work with closed neighbourhoods in place of open neighbourhoods. This observation is important when it comes to prove compatibility with local triviality for W*-bundles (Definition 3.4.20).

3.6.2 W*-Bundles vs Bundles of Tracial von Neumann Algebras

Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X. Set $B = \bigsqcup_{x \in X} \mathcal{M}_x$ and define $p : B \to X$ by $p(b) = x$ whenever $b \in \mathcal{M}_x$. Note that, for each $x \in X$, the fibre $p^{-1}(x)$ can be identified with $\mathcal{M}_x$ and, therefore, endowed with operations, a norm and a trace that make it a tracial von Neumann algebra. In the following proposition, we define a topology on B such that (B, p) is a bundle of tracial von Neumann algebras. We then check that isomorphic W*-bundles give rise to isomorphic bundles of tracial von Neumann algebras.

Proposition 3.6.5. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X. Set $B = \bigsqcup_{x \in X} \mathcal{M}_x$ and define $p : B \to X$ by $p(b) = x$ whenever $b \in \mathcal{M}_x$. For $a \in \mathcal{M}$, $\epsilon > 0$ and U open in X, we set $V(a, \epsilon, U) = \{b \in B : p(b) \in U, \|a(p(b)) - b\|_2 < \epsilon\}$.*

(a) The collection $\mathcal{B}$ of all such $V(a, \epsilon, U)$ form a basis for a topology on B. Moreover, if $b \in B$ and $a \in \mathcal{M}$ is chosen with $a(p(b)) = b$, then the collection of $V(a, \epsilon, U)$

as ϵ ranges over positive reals and U ranges over a neighbourhood basis of $p(b)$ is a neighbourhood basis of b.

(b) *When B is endowed with the topology generated by $\mathcal{B}$, (B, p) is a bundle of tracial von Neumann algebras.*

Proof. (a) Given $b \in B$, let $x = p(b)$, so $b \in \mathcal{M}_x$. Let $a \in \mathcal{M}$ be a lift of b. Then, for any open neighbourhood U of x and $\epsilon > 0$, $b \in V(a, \epsilon, U)$. Therefore, $\bigcup_{V \in \mathcal{B}} V = B$.

Suppose $b \in V(a_1, \epsilon_1, U_1) \cap V(a_2, \epsilon_2, U_2)$. Set $x = p(b)$, and let $a \in \mathcal{M}$ be a lift of $b \in \mathcal{M}_x$. We have $x \in U_1 \cap U_2$ and

$$\begin{aligned} \delta_1 &:= \|a(x) - a_1(x)\|_2 < \epsilon_1 \\ \delta_2 &:= \|a(x) - a_2(x)\|_2 < \epsilon_2 \end{aligned} \tag{3.6.2}$$

Choose, by continuity, an open set U such that $x \in U \subseteq U_1 \cap U_2$, and such that

$$\begin{aligned} \|a(x') - a_1(x')\|_2 &< \frac{\epsilon_1 + \delta_1}{2} \\ \|a(x') - a_2(x')\|_2 &< \frac{\epsilon_2 + \delta_2}{2} \end{aligned} \tag{3.6.3}$$

for all $x' \in U$. Set $\epsilon = \min(\frac{\epsilon_1 - \delta_1}{2}, \frac{\epsilon_2 - \delta_2}{2})$. Now, if $b' \in V(a, \epsilon, U)$, then, for $i \in \{1, 2\}$, $x' := p(b') \in U_i$ and

$$\begin{aligned} \|a_i(x') - b'\|_2 &\leq \|a_i(x') - a(x')\|_2 + \|a(x') - b'\|_2 \\ &< \frac{\epsilon_i + \delta_i}{2} + \epsilon \\ &\leq \epsilon_i. \end{aligned} \tag{3.6.4}$$

So, $b' \in V(a_i, \epsilon_i, U_i)$. Hence, $b \in V(a, \epsilon, U) \subseteq V(a_1, \epsilon_1, U_1) \cap V(a_2, \epsilon_2, U_2)$.

This proves that $\mathcal{B}$ does form the basis for a topology on B, and also gives the required neighbourhood basis for $b \in B$.

(b) The topology defined by $\mathcal{B}$ is easily seen to be Hausdorff. Let U be open in X. Let $b \in p^{-1}(U)$ with $x = p(b)$. Choose $a \in \mathcal{M}$ with $a(x) = b$. Then $b \in V(a, 1, U) \subseteq p^{-1}(U)$. So $p^{-1}(U)$ is open in B. Hence p is continuous. It is clearly surjective. We now check the axioms of Definition 3.6.1 in turn, noting that a simple scaling argument shows that axioms (ii) and (viii) imply that the map $p : B \to X$ is open.

(i) Let $b_1, b_2 \in B$ with $x = p(b_1) = p(b_2)$. Let $a_1, a_2 \in \mathcal{M}$ be lifts of $b_1, b_2 \in \mathcal{M}_x$. A basic open neighbourhood of $b_1 + b_2$ has the form $V(a_1 + a_2, \epsilon, U)$ for some $\epsilon > 0$ and open neighbourhood U of x. Let $b_1' \in V(a_1, \frac{\epsilon}{2}, U)$ and $b_2' \in V(a_2, \frac{\epsilon}{2}, U)$, and suppose $x' = p(b_1') = p(b_2') \in U$. We have

$$\begin{aligned} \|(a_1(x') + a_2(x')) - (b_1' + b_2')\|_2 &\leq \|a_1(x') - b_1'\|_2 + \|a_2(x') - b_2'\|_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{3.6.5}$$

So, $b_1' + b_2' \in V(a_1 + a_2, \epsilon, U)$.

(ii) Let $\lambda \in \mathbb{C}$ and $b \in B$ with $x = p(b)$. Choose $a \in \mathcal{M}$ with $a(x) = b$. A basic neighbourhood of λb has the form $V(\lambda a, \epsilon, U)$ for some $\epsilon > 0$ and some open neighbourhood U of x in B. Set $K = \max(\|a\|_{2,u}, |\lambda|) + 1$ and $\delta = \min(\frac{\epsilon}{2K}, 1)$. Let $|\lambda' - \lambda| < \delta$ and $b' \in V(a, \delta, U)$ with $x' = p(b')$. Then

$$\begin{aligned} \|\lambda' b' - \lambda a(x')\|_2 &\leq |\lambda'| \|b' - a(x')\|_2 + |\lambda' - \lambda| \|a(x')\|_2 \\ &\leq (|\lambda| + 1)\|b' - a(x')\|_2 + |\lambda' - \lambda| \|a(x')\|_2 \\ &< K\delta + \delta K \\ &\leq \epsilon. \end{aligned} \tag{3.6.6}$$

(iii) This follows from the observation $V(a, \epsilon, U)^* = V(a^*, \epsilon, U)$.

(iv) For the continuity of the map $x \mapsto 1_x$ it suffices to observe that the open set $V(1, \epsilon, U)$ has preimage U under this map for any $\epsilon > 0$. The continuity of $x \mapsto 0_x$ is similar.

(v) We show the continuity of $\| \cdot \|_2$ on B. Continuity of τ then follows by the polarisation identity together with the continuity of $x \mapsto 1_x$. Let $b \in B$ and $x = p(b)$. Choose $a \in \mathcal{M}$ such that $a(x) = b$. Let $\epsilon > 0$. By Proposition 3.2.6, the map $y \mapsto \|a(y)\|_2$ is continuous. Hence, there is an open set $U \ni x$ such that

$$\Big| \|a(y)\|_2 - \|a(x)\|_2 \Big| < \frac{\epsilon}{2} \tag{3.6.7}$$

for all $y \in U$. Let $b' \in V(a, \frac{\epsilon}{2}, U)$. Writing $x' = p(b')$, we have

$$\begin{aligned} \Big| \|b'\|_2 - \|b\|_2 \Big| &\leq \Big| \|b'\|_2 - \|a(x')\|_2 \Big| + \Big| \|a(x')\|_2 - \|a(x)\|_2 \Big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{3.6.8}$$

(vi) This follows from the fact that a basic open neighbourhood of 0_x has the form $V(0, \epsilon, U)$ for some $\epsilon > 0$ and some open neighbourhood U of x in X.

(vii) Fix $K > 0$. Let $b_1, b_2 \in B$ with $\|\cdot\|$-norm bounded by K. Suppose $x = p(b_1) = p(b_2)$. Let $a_1, a_2 \in \mathcal{M}$ be norm-preserving lifts of $b_1, b_2 \in \mathcal{M}_x$. A basic open neighbourhood of $b_1 b_2$ has the form $V(a_1 a_2, \epsilon, U)$ for some $\epsilon > 0$ and open neighbourhood U of x. Let $b_1' \in V(a_1, \frac{\epsilon}{2K}, U)$ and $b_2' \in V(a_2, \frac{\epsilon}{2K}, U)$. Assume b_1' and b_2' are $\|\cdot\|$-norm bounded by K, and that $x' = p(b_1') = p(b_2') \in U$. We have

$$\begin{aligned}
\|a_1(x')a_2(x') - b_1'b_2'\|_2 &\leq \|a_1(x')\|\|a_2(x') - b_2'\|_2 + \|a_1(x') - b_1'\|_2\|b_2'\| \\
&\leq K\|a_2(x') - b_2'\|_2 + K\|a_1(x') - b_1'\|_2 \\
&< K\left(\frac{\epsilon}{2K} + \frac{\epsilon}{2K}\right) \\
&= \epsilon.
\end{aligned} \tag{3.6.9}$$

So, $b_1'b_2' \in V(a_1 a_2, \epsilon, U)$.

(viii) Let W be open in B with $W \cap B|_{\leq 1} \neq \emptyset$. Let $x \in p(W \cap B|_{\leq 1})$. Choose $b \in W \cap B|_{\leq 1}$ such that $p(b) = x$. Lift $b \in \mathcal{M}_x$ to an element $a \in \mathcal{M}$ of the same norm. The open set W contains a basic open neighbourhood of the form $V(a, \epsilon, U)$, where $\epsilon > 0$ and U is a neighbourhood of x in X. Hence, for all $x' \in U$, it follows that $a(x') \in W$ and $\|a(x')\|_{\mathcal{M}_{x'}} \leq 1$. Therefore $U \subseteq p(W \cap B|_{\leq 1})$ and so $p|_{B_{\leq 1}} : B_{\leq 1} \to X$ is open. □

Proposition 3.6.6. *Let $\mathcal{M}_i$ be a W*-bundle over X_i with conditional expectation E_i for $i = 1, 2$. Let (B_i, p_i) be the corresponding bundle of tracial von Neumann algebras for $i = 1, 2$. If the W*-bundles are isomorphic, then the bundles of tracial von Neumann algebras are isomorphic.*

Proof. Assume $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism of the W*-bundles. Then α restricts to an isomorphism $C(X_1) \to C(X_2)$, so induces a homeomorphism $\alpha^t : X_2 \to X_1$. Since $E_2(\alpha(a))(x_2) = \alpha(E_1(a))(x_2) = E_1(a)(\alpha^t(x_2))$ for all $a \in \mathcal{M}_1$ and $x_2 \in X_2$, α induces an isomorphism between the fibres $(\mathcal{M}_1)_{\alpha^t(x_2)}$ and $(\mathcal{M}_2)_{x_2}$ for each $x_2 \in X_2$. Combining all these isomorphisms, we get a bijection $\varphi : B_1 \to B_2$ such that (3.6.1) holds with $\psi = (\alpha^t)^{-1}$. By considering the basic open neighbourhoods in B_1 and B_2, we see that φ is a homeomorphism. Indeed, $\varphi(V_{\mathcal{M}_1}(a, \epsilon, U)) = V_{\mathcal{M}_2}(\alpha(a), \epsilon, \psi(U))$ for all $a \in \mathcal{M}_1$, $\epsilon > 0$, and U open in X_1. □

In the other direction, given a bundle of tracial von Neumann algebras over a compact Hausdorff space, we can define a W*-bundle by considering sections. Recall from Section 2.12.1 that a *section* of a general topological bundle (B, p) over X is a map $s : X \to B$

such that $p \circ s = \mathrm{id}_X$ (Definition 2.12.2). We now define what it means for a section of a bundle of tracial von Neumann algebras to be bounded.

Definition 3.6.7. A section $s : X \to B$ of a bundle of tracial von Neumann algebras (B, p) over X is said to be *bounded* if $\sup_{x \in X} \|s(x)\| < \infty$.

Remark 3.6.8. Note that boundedness of sections always refers to $\|\cdot\|$-norm. Since it is not required that $\|\cdot\|$ be continuous on B, continuous sections $s : X \to B$ are not automatically bounded in the sense of Definition 3.6.7 even when X is compact.

Let (B, p) be a bundle of tracial von Neumann algebras over the compact Hausdorff space X. The set of bounded sections of (B, p) endowed with fibrewise-defined operations and the uniform norm $\|s\| = \sup_{x \in X} \|s(x)\|$ is a C*-algebra isomorphic to the product $\prod_{x \in X} p^{-1}(x)$. Since the fibres are tracial von Neumann algebras, the uniform 2-norm $\|s\|_{2,u} = \sup_{x \in X} \|s(x)\|_2$ is complete when restricted to the closed unit ball in uniform norm. Let $\mathcal{M}$ be the collection of bounded, continuous sections. Axioms (i)–(viii) ensure that $\mathcal{M}$ is a unital *-subalgebra of the C*-algebra of all bounded sections. It follows from Proposition 2.12.8 that continuity of sections is preserved under uniform-2-norm limits and, a fortiori, under uniform-norm limits. Therefore, $\mathcal{M}$ inherits the completeness properties of the algebra of bounded sections, in particular $\mathcal{M}$ is a C*-algebra.

The additional data for a W*-bundle over X with section algebra $\mathcal{M}$ can now be easily defined and the axioms verified. We identify $f \in C(X)$ with the scalar valued section $x \mapsto f(x)1_x$. Such scalar valued sections are clearly bounded and are continuous since scalar multiplication and the section $x \mapsto 1_x$ are continuous. This gives an inclusion $C(X) \subseteq Z(\mathcal{M})$. We define $E : \mathcal{M} \to C(X)$ by $s \mapsto \tau \circ s$. This is a conditional expectation from $\mathcal{M}$ onto the image of $C(X)$ in $\mathcal{M}$ and induces the uniform 2-norm on $\mathcal{M}$. Axiom (C) follows from Proposition 2.12.8. Axioms (T) and (F) follow fibrewise from the corresponding properties of a faithful trace.

As before, we check that our construction is compatible with our notions of isomorphism.

Proposition 3.6.9. *Let (B_i, p_i) be a bundle of tracial von Neumann algebras over the compact Hausdorff space X_i for $i = 1, 2$. Let $\mathcal{M}_i$ be the W*-bundle over X_i with conditional expectation E_i that comes from (B_i, p_i). If the bundles of tracial von Neumann algebras are isomorphic, then the W*-bundles are isomorphic.*

Proof. If the bundles are isomorphic and φ and ψ are as in (3.6.1) then $s \mapsto \varphi \circ s \circ \psi^{-1}$ defines a bijection between the bounded, continuous sections of $p_1 : B_1 \to X_1$ and those of $p_2 : B_2 \to X_2$, that is a map $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$.

Since for each $x_1 \in X_1$, $\varphi|_{p_1^{-1}(x_1)} : p_1^{-1}(x_1) \to p_2^{-1}(\psi(x_1))$ is an isomorphism of tracial von Neumann algebras, α is a *-homomorphism of C*-algebras. Furthermore, the following computations show that α is a morphism of W*-bundles. Firstly, let $f_1 \in C(X_1) \subseteq Z(\mathcal{M}_1)$ and $x_2 \in X_2$. Then

$$\begin{aligned} \alpha(f_1)(x_2) &= \varphi(f_1(\psi^{-1}(x_2))1_{\psi^{-1}(x_2)}) \\ &= f_1(\psi^{-1}(x_2))1_{x_2}, \end{aligned} \tag{3.6.10}$$

so $\alpha(f_1) = f_1 \circ \psi^{-1} \in C(X_2) \subseteq Z(\mathcal{M}_2)$. Secondly, let $s \in \mathcal{M}_1$ and $x_2 \in X$. Then

$$\begin{aligned} E_2(\alpha(s))(x_2) &= \tau_{p_2^{-1}(x_2)}(\alpha(s)(x_2)) \\ &= \tau_{p_2^{-1}(x_2)}(\varphi(s(\psi^{-1}(x_2)))) \\ &= \tau_{p_1^{-1}(\psi^{-1}(x_2))}(s(\psi^{-1}(x_2))) \\ &= E_1(s)(\psi^{-1}(x_2)) \\ &= \alpha(E_1(s))(x_2), \end{aligned} \tag{3.6.11}$$

so $E_2 \circ \alpha = \alpha \circ E_1$. $\square$

We now investigate the inverse nature of the two constructions considered in this section. The following theorem deals with the case where one starts with a W*-bundle $\mathcal{M}$, constructs the associate bundle of tracial von Neumann algebras (B, p), and then constructs a second W*-bundle from the bounded, continuous sections of (B, p).

Theorem 3.6.10. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X. Let (B, p) be the bundle of tracial von Neumann algebras constructed from $\mathcal{M}$.*

(a) For each $a \in \mathcal{M}$, the map $s_a : X \to B$ given by $x \mapsto a(x) \in \mathcal{M}_x$ defines a bounded, continuous section of (B, p).

(b) Every bounded, continuous section of (B, p) has the form s_a for some $a \in \mathcal{M}$.

(c) The map $a \mapsto s_a$ is an isomorphism between the W*-bundle $\mathcal{M}$ and the W*-bundle constructed from (B, p).

Proof. (a) Let $a \in \mathcal{M}$. By construction s_a is a section of (B, p). We have $\|a(x)\|_{\mathcal{M}_x} \leq \|a\|_{\mathcal{M}}$ for all $x \in X$, so the section s_a is bounded. Let W be open in B and $x \in s_a^{-1}(W)$. Then $s_a(x) = a(x) \in W$. By Proposition 3.6.5(a), there exists $\epsilon > 0$ and an open neighbourhood U of x in X such that $a(x) \in V(a, \epsilon, U) \subseteq W$. It follows that $x \in U \subseteq s_a^{-1}(W)$. Hence, s_a is continuous.

(b) Assume $s : X \to B$ is a continuous and bounded section. Let $x_0 \in X$ and $\epsilon > 0$. Choose $a_0 \in \mathcal{M}$ such that $a_0(x_0) = s(x_0)$. Since the function $x \mapsto \|s(x) - a_0(x)\|_2$ is continuous, there is a neighbourhood U of x_0 such that

$$\sup_{x \in U} \|s(x) - a_0(x)\|_2 < \epsilon. \tag{3.6.12}$$

By Theorem 3.2.10, there exists $a \in \mathcal{M}$ such that $a(x) = s(x)$ for all $x \in X$.

(c) The map $a \mapsto s_a$ is a unital homomorphism of C*-algebras. It is injective by Proposition 3.2.5 and surjective by (b). For $f \in C(X) \subseteq Z(\mathcal{M})$, s_f is the scalar section $x \mapsto f(x)1_x$ and, for arbitrary $a \in \mathcal{M}$ and $x \in X$, $\tau(s_a(x)) = \tau_x(a(x)) = E(a)(x)$. Therefore, $a \mapsto s_a$ is an isomorphism of W*-bundles. $\square$

We now consider the reverse direction. Namely, we start with a bundle of tracial von Neumann algebras (B, p) over a compact Hausdorff space, construct a W*-bundle $\mathcal{M}$ by considering bounded, continuous sections of (B, p), and then construct a second bundle of tracial von Neumann algebras $(\widetilde{B}, \widetilde{p})$ from the fibres of $\mathcal{M}$.

Theorem 3.6.11. *Let (B, p) be a bundle of tracial von Neumann algebras over the compact Hausdorff space X. Let $\mathcal{M}$ be the W*-bundle defined by considering bounded, continuous sections of (B, p). Let $(\widetilde{B}, \widetilde{p})$ be the bundle of tracial von Neumann algebras constructed from the fibres of $\mathcal{M}$. Suppose that*

(∗) *For all $b \in B$, there is $s \in \mathcal{M}$ with $s(p(b)) = b$.*[15]

Then the bundles of tracial von Neumann algebras (B, p) and $(\widetilde{B}, \widetilde{p})$ are isomorphic.

Proof. Write E for the conditional expectation of $\mathcal{M}$. For each $x \in X$, consider the evaluation map $\varphi_x : \mathcal{M} \to p^{-1}(x)$ given by $s \mapsto s(x)$. This is a homomorphism of C*-algebras and, by our assumption, it is surjective. Since $\tau(s(x)) = E(s)(x)$ for all $s \in \mathcal{M}$ and the trace on $p^{-1}(x)$ is faithful, we get an induced isomorphism of tracial von Neumann

[15] We show in the next section that this conditional is in fact automatically satisfied.

algebras $\overline{\varphi}_x : \mathcal{M}_x \to p^{-1}(x)$. Combining all such maps, we get a bijection $\varphi : \widetilde{B} \to B$, such that the diagram

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\varphi"] \arrow[d, "\widetilde{p}"'] & B \arrow[d, "p"] \\ X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd} \tag{3.6.13}$$

commutes. It remains to show that φ is a homeomorphism. Note that, via our convention of writing $s(x)$ for the image of $s \in \mathcal{M}$ in $\mathcal{M}_x$, φ can be viewed as the identity map on B. Thus proving that φ is a homeomorphism amounts to showing that the topology on B, satisfying the axioms for a bundle, has a basis consisting of the sets $V(s, \epsilon, U) = \{b \in B : p(b) \in U, \|s(p(b)) - b\|_2 < \epsilon\}$ for $s \in \mathcal{M}$, $\epsilon > 0$ and U open in X.

Each such set $V(s, \epsilon, U)$ is open in B because the axioms for a bundle ensure that the map $F \colon B \to \mathbb{R} \times X$ given by $b \mapsto (\|s(p(b)) - b\|_2, p(b))$ is continuous. We complete the proof by showing that the set of all such $V(s, \epsilon, U)$ contains a neighbourhood basis for each point of B. Axiom (vi) of Definition 3.6.1 gives that $V(0, \epsilon, U)$ as ϵ ranges over the positive reals and U ranges over a neighbourhood basis for $x \in X$ form a neighbourhood basis for 0_x. Let $b_0 \in B$ and s_0 be a bounded, continuous section with $s_0(p(b_0)) = b_0$. Since the map $G : B \to B$ given by $b \mapsto s_0(p(b)) - b$ is a homeomorphism of B, we see that $V(s_0, \epsilon, U)$ as ϵ ranges over the positive reals and U ranges over a neighbourhood basis for $p(b_0)$ form a neighbourhood basis for b_0. □

We observe that the bundle of tracial von Neumann algebras corresponding to a trivial W*-bundle $C_\sigma(X, M)$, where M is a fixed tracial von Neumann algebra and X is a compact Hausdorff space, is $(X \times M, \pi_1)$, where the topology on $X \times M$ is the product of the topology of X and the 2-norm topology on M and $\pi_1 : X \times M \to X$ is the projection onto the first coordinate. Thus, the notion of triviality for a bundle of tracial von Neumann algebras matches up with that for a W*-bundle. We show below that the notions of restriction to a closed subset also match up and, therefore, so do the natural notions of local triviality (Definitions 3.4.20 and 3.6.3).

Proposition 3.6.12. *Let X be a compact Hausdorff space and Y a closed subset.*

(a) Let $\mathcal{M}$ be a W-bundle over X and (B, p) the corresponding bundle of tracial von Neumann algebras. Let (B_Y, p_Y) be the bundle of tracial von Neumann algebras corresponding to the quotient W*-bundle $\mathcal{M}_Y = \mathcal{M}/I_Y$ (Definition 3.4.19). There*

exists a homeomorphism φ such that the diagram

$$\begin{tikzcd} B_Y \arrow[r, "\varphi"] \arrow[d, "p_Y"'] & p^{-1}(Y) \arrow[d, "p|_{p^{-1}(Y)}"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd}$$

commutes, which induces an isomorphism of tracial von Neumann algebras in each fibre.

(b) *Let (B, p) be a bundle of tracial von Neumann algebras over X. Let $\mathcal{M}$ be the W*-bundle arising from bounded, continuous sections of (B, p). Then $\mathcal{M}_Y$ is isomorphic to the W*-bundle $\widetilde{\mathcal{M}}$ of bounded, continuous sections of $(p^{-1}(Y), p|_{p^{-1}(Y)})$.*

Proof. (a) Write $a \mapsto a|_Y$ for the quotient morphism $\mathcal{M} \to \mathcal{M}_Y$. For $y \in Y$, the fibre $(\mathcal{M}_Y)_y$ of $\mathcal{M}_Y$ can be identified with the fibre $\mathcal{M}_y$ of $\mathcal{M}$, via the map $a|_Y(y) \mapsto a(y)$. Combining all these maps, we obtain a bijection φ such that the diagram commutes. Considering basic open neighbourhoods we see that φ is a homeomorphism. Indeed, $\varphi(V_{\mathcal{M}_Y}(a|_Y, \epsilon, U \cap Y)) = V_{\mathcal{M}}(a, \epsilon, U) \cap p^{-1}(Y)$ for all $a \in \mathcal{M}$, $\epsilon > 0$, and U open in X.

(b) Write E for the conditional expectation of $\mathcal{M}$ and $\widetilde{E}$ for the conditional expectation on $\widetilde{\mathcal{M}}$. Restricting a bounded, continuous section $s : X \to B$ of p to Y gives a continuous bounded section of $(p^{-1}(Y), p|_{p^{-1}(Y)})$. This defines a homomorphism of C*-algebras $\mathcal{M} \to \widetilde{\mathcal{M}}$. The kernel of this homomorphism is the ideal $I_Y = \{s \in \mathcal{M} : E(s^*s)(y) = 0 \text{ for all } y \in Y\}$. So we get an induced isometric homomorphism of C*-algebras $\alpha : \mathcal{M}_Y \to \widetilde{\mathcal{M}}$. This homomorphism restricts to the identity map on the central copies of $C(Y)$ in $\mathcal{M}_Y$ and $\widetilde{\mathcal{M}}$, and the diagram

$$\begin{tikzcd} \mathcal{M}_Y \arrow[r, "\alpha"] \arrow[d, "E_Y"'] & \widetilde{\mathcal{M}} \arrow[d, "\widetilde{E}"'] \\ C(Y) \arrow[r, "\mathrm{id}"] & C(Y) \end{tikzcd} \tag{3.6.14}$$

commutes. In particular, α preserves the uniform 2-norm. The argument to show that α is surjective has two parts. First, using a partition of unity argument as in [18, Lemma 10.1.11], one shows that, for any continuous section $s : Y \to B$ with $\|s(y)\| \leq 1$ for all $y \in Y$ and any $\epsilon > 0$, there is a bounded, continuous section $\overline{s} : X \to B$ with $\|\overline{s}(x)\| \leq 1$ for all $x \in X$ and $\|s(y) - \overline{s}(y)\|_2 < \epsilon$. This implies that the $\|\cdot\|$-norm closed unit ball of $\mathcal{M}_Y$ has $\|\cdot\|_{2,u}$-dense image in the $\|\cdot\|$-norm closed unit ball of $\widetilde{M}$. The completeness of the $\|\cdot\|$-norm closed units balls in $\|\cdot\|_{2,u}$-norm then implies that α is surjective. $\square$

3.6.3 Constructing Continuous Sections

This section is devoted to proving that Axioms (i-viii) for a bundle of tracial von Neumann algebras ensure the existence of sufficiently many bounded, continuous sections under suitable assumptions on the base space. The main result of this section, Theorem 3.6.18, implies that the additional hypothesis $(*)$ of Theorem 3.6.11 is, in fact, automatically satisfied. Hence, there is a perfect 1-1 correspondence between isomorphism classes of W*-bundles and isomorphism classes of bundles of tracial von Neumann algebras over compact Hausdorff spaces.

The methodology is to adapt the proof of the analogous result for bundles of Banach spaces, due to Douady and dal Soglio-Herault, presented in Section 2.12.3, to account for the fact that the $\|\cdot\|_2$-norm is only complete on the $\|\cdot\|$-norm unit ball of each fibre.

We begin by establishing some notation and terminology for the remainder of this section. Let (B,p) denote a bundle of tracial von Neumann algebras over the Hausdorff space X. Since (B,p) is a bundle of normed vector spaces with respect to the global $\|\cdot\|_2$-norm, the definitions of ϵ-thin sets and ϵ-continuous sections (Definitions 2.12.12 and 2.12.13) carry over to bundles of tracial von Neumann algebra as $\|\cdot\|_2$-norm properties. For the benefit of the reader, we repeat the definitions here.

Definition 3.6.13. Let $\epsilon > 0$. A subset U of (B,p) is *ϵ-thin* if $\|b-b'\|_2 < \epsilon$ whenever $b_1, b_2 \in U$ and $p(b_1) = p(b_2)$.

Definition 3.6.14. Let $x \in X$. A section $f : X \to B$ is *ϵ-continuous at x* if there is a neighbourhood V of x and an ϵ-thin neighbourhood U of $f(x)$ such that $f(V) \subseteq U$. A section $f : X \to B$ is *ϵ-continuous* if it is ϵ-continuous at all points $x \in X$.

We supplement the definition of ϵ-continuity, a $\|\cdot\|_2$-norm property, with that of M-boundedness, a $\|\cdot\|$-norm property.

Definition 3.6.15. Let $M \geq 0$. A section $f : X \to B$ is *M-bounded* if $\|f(x)\| \leq M$ for all $x \in X$.

We now state analogues of Propositions 2.12.15 and 2.12.20 for bundles of tracial von Neumann algebras.

Proposition 3.6.16. *Suppose X is completely regular. For all $\epsilon > 0$, $x_0 \in X$ and $b_0 \in p^{-1}(x_0)$, there is an ϵ-continuous, $\|b_0\|$-bounded section f with $f(x_0) = b_0$.*

Proof. If $\|b_0\| = 0$, then take f to be the zero section. Hereinafter, assume $\|b_0\| > 0$. By Proposition 2.12.14, there is an ϵ-thin, open neighbourhood U of b_0. Set $M = \|b_0\|$. By Axiom (viii), the set $V = p(U \cap B_{\leq M})$ is an open neighbourhood of x_0. By the axiom of choice, there is a local section $f_V : V \to B$ such that $f_V(V) \subseteq U \cap B_{\leq M}$. Complete regularity of X implies the existence of a continuous bump function $\phi : X \to [0,1]$ supported on a closed set $F \subseteq V$ and with $\phi(x_0) = 1$. We can define $f : X \to B$ by

$$f(x) = \begin{cases} \phi(x) f_V(x), & x \in V, \\ 0, & x \notin V. \end{cases} \tag{3.6.15}$$

The function f is an ϵ-continuous section because U is ϵ-thin and is M-bounded by construction. ☐

Proposition 3.6.17. *Assume X is paracompact. Let $\epsilon > 0, M > 0$ and $x_0 \in X$. Suppose $f : X \to B$ is an M-bounded, ϵ-continuous section. Then there is an M-bounded, $\frac{\epsilon}{2}$-continuous section $f' : X \to B$ such that $\|f(x) - f'(x)\|_2 < \frac{3}{2}\epsilon$ for all $x \in X$ and $f'(x_0) = f(x_0)$.*

Proof. The proof is essentially the same as that of Proposition 2.12.20 except that we use Proposition 3.6.16 to ensure that each $f^{(x)}$ is M-bounded. With this amendment the function f', as constructed in the proof of Proposition 2.12.20, will also be M-bounded. Indeed, $f' = \sum_{i \in I} \phi_i f_i$, where the ϕ_i are a partition of unity and each f_i is some $f^{(x)}$. Therefore, as $\|f_i(x)\| \leq M$ for all $x \in X$ and $i = 1, \dots, n$, we have

$$\begin{aligned} \|f(y)\| &= \|\sum_{i \in I} \phi_i(y) f_i(y)\| && (3.6.16) \\ &\leq \sum_{i \in I} \phi_i(y) \|f_i(y)\| && (3.6.17) \\ &\leq \sum_{i \in I} \phi_i(y) M && (3.6.18) \\ &= M. \end{aligned}$$

☐

Finally, we state and prove the analogue of Theorem 2.12.11 for bundles of tracial von Neumann algebras.

Theorem 3.6.18. *Let (B, p) be a bundle of tracial von Neumann algebras over a space X which is either paracompact or locally compact. Then for every $b_0 \in B$, there exists a bounded, continuous section $f : X \to B$ with $f(p(b_0)) = b_0$.*

Proof. We first prove the result for X paracompact. Fix $b_0 \in B$. Set $x_0 = p(b)$ and $M = \|b_0\|$. By Proposition 3.6.16, there is an M-bounded, 1-continuous section f_0 with $f_0(x_0) = b_0$. Using Proposition 3.6.17, we inductively construct a sequence of M-bounded sections $f_n : X \to B$ with $f_n(x_0) = b_0$ such that f_n is $\frac{1}{2^n}$-continuous and $\|f_n(x) - f_{n-1}(x)\|_2 < \frac{3}{2^n}$ for all $x \in X$ and $n \in \mathbb{N}$.

Since the series $\sum_{i=0}^{\infty} \frac{3}{2^i}$ converges, the sequence of sections $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy, i.e. for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_n(x) - f_m(x)\|_2 < \epsilon \tag{3.6.19}$$

whenever $x \in X$ and $m, n \geq N$. Moreover, $\|f_n(x)\| \leq M$ for all $x \in X$ and $n \in \mathbb{N}$.

As fibres of the bundle are tracial von Neumann algebras, f_n converges pointwise to some section f in $\|\cdot\|_2$-norm. Fixing x while letting $m \to \infty$ in (3.6.19), we see that f_n converges uniformly to f in $\|\cdot\|_2$-norm. This section will be ϵ-continuous for all $\epsilon > 0$ by Lemma 2.12.18 and thus continuous by Lemma 2.12.17.

The result for X locally compact is deduced by working first on a compact neighbourhood of b_0, applying the result for the paracompact case, then multiplying by a suitable bump function. $\square$

Chapter 4

The Triviality Problem for W^*-Bundles

The key problem in the field of W*-bundles is the following: Is every W*-bundle with all fibres isomorphic to the hyperfinite II_1 factor $\mathcal{R}$ a trivial W*-bundle?

The reason for the special interest in the case where the fibres are $\mathcal{R}$ comes from the classification programme for C*-algebras. This is because if A is a simple, separable, unital, nuclear, infinite-dimensional C*-algebra with a non-empty Bauer simplex of traces, then $\overline{A}^{\mathrm{st}}$ is a W*-bundle with all fibres isomorphic to the hyperfinite II_1 factor $\mathcal{R}$. Moreover, triviality of the W*-bundle $\overline{A}^{\mathrm{st}}$ enables one to prove that strict comparison implies tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$, which is one part of the Toms–Winter Conjecture (see Chapter 1).

In this chapter, we present Ozawa's Triviality Theorem [62, Theorem 15] and its corollaries. This theorem provides equivalent conditions for a strictly separable W*-bundle with fibres $\mathcal{R}$ to be trivial. As observed by Ozawa [62, Corollary 12], one can show that these conditions are satisfied when the base space has finite covering dimension using the techniques of [47, Section 7] (or [77, 88]). It also follows from Ozawa's Triviality Theorem that $\overline{A}^{\mathrm{st}}$ is trivial whenever A is $\mathcal{Z}$-stable in addition to the aforementioned hypotheses, an observation first made in [5, Theorem 3.15].

Using Ozawa's Triviality Theorem, we can also prove that $\overline{A}^{\mathrm{st}}$ is trivial for some C*-algebras not covered by the results mentioned above, for example when A is a Villadsen algebra.[1]. This is discussed in Section 4.6.

[1] These results stem from an idea suggested by Aaron Tikuisis

In the final section of this chapter, we consider the case of locally trivial W*-bundles, proving that local triviality implies triviality in the case where the fibres are all isomorphic to $\mathcal{R}$. This result comes from my joint work with Ulrich Pennig and appears in our paper [23].

4.1 Ozawa's Triviality Theorem

We begin our presentation of Ozawa's Triviality Theorem by stating the theorem in its original form.

Theorem 4.1.1. *[62, Theorem 15] Let $\mathcal{M}$ be a strictly separable W*-bundle over X with $\mathcal{M}_x \cong \mathcal{R}$ for all $x \in X$. Then the following are equivalent:*

(i) $\mathcal{M}$ is isomorphic to the trivial W-bundle $C_\sigma(X, \mathcal{R})$.*

(ii) There is a sequence of positive contractions (p_n) in $\mathcal{M}$ such that, as $n \to \infty$,

$$\|[p_n, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}), \tag{4.1.1}$$

$$\|p_n - p_n^2\|_{2,u} \to 0, \tag{4.1.2}$$

$$\|E(p_n) - \tfrac{1}{2}\|_{C(X)} \to 0. \tag{4.1.3}$$

(iii) For each $k \in \mathbb{N}$, there is a sequence of cpc maps $\varphi_n : \mathbb{M}_k(\mathbb{C}) \to \mathcal{M}$ such that, as $n \to \infty$,

$$\|[\varphi_n(b), a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}, b \in \mathbb{M}_k(\mathbb{C})), \tag{4.1.4}$$

$$\|\varphi_n(b_1 b_2) - \varphi_n(b_1)\varphi_n(b_2)\|_{2,u} \to 0 \qquad (b_1, b_2 \in \mathbb{M}_k(\mathbb{C}), \tag{4.1.5}$$

$$\|\varphi_n(1) - 1\|_{2,u} \to 0. \tag{4.1.6}$$

The proof of Ozawa's Triviality Theorem is deferred until Section 4.5. Beforehand, we shall discuss the conditions (ii) and (iii) of the theorem.

Condition (ii) can be viewed as the W*-bundle analogue of Murray and von Neumann's property Γ for II_1 factors. It asserts the existence of an approximately central sequence of approximate projections of trace $\frac{1}{2}$ in a uniform sense. In Section 4.3, after we've defined the ultrapower of a W*-bundle, we will be able to make this reformulation precise.

Condition (iii) can be viewed as the W*-bundle analogue of the McDuff property for II_1 factors. It asserts the existence of approximately central embeddings of matrix algebras. This condition is studied extensively in [5, Section 3.2] and we shall present the key results in Section 4.4. In particular, one has that condition (iii) is equivalent to tensorial absorption of $\mathcal{R}$ in a W*-bundle sense.

4.2 Tensor Products and Ultrapowers of W*-bundles

In this section, we recall the definitions and key properties of two very useful W*-bundle constructions: tensor products and ultrapowers. Both these constructions were first introduced in [5, Section 3.2]. In addition, we define inductive limits of W*-bundles and infinite tensor products.

4.2.1 Tensor Products

The key to the construction of the tensor product of W*-bundles is the observation that the minimal tensor product of C*-algebras is compatible with the additional structure of a pre-W*-bundle.[2] We formulate this as a proposition.

Proposition 4.2.1. *Let $\mathcal{M}_i$ be a pre-W*-bundle over X_i with conditional expectation E_i for $i = 1, 2$. Write ι_i for the embedding of $C(X_i)$ in $Z(\mathcal{M}_i)$. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ together with the embedding $\iota_1 \otimes \iota_2$ and the conditional expectation $E_1 \otimes E_2$ is a pre-W*-bundle over $X_1 \times X_2$.*

Proof. Identify $C(X_1) \otimes C(X_2)$ with $C(X_1 \times X_2)$. First, note that $\iota_1 \otimes \iota_2 : C(X_1 \times X_2) \to \mathcal{M}_1 \otimes \mathcal{M}_2$ remains an embedding since we are working with the minimal tensor product [6, Proposition 3.6.1]. The embedding is clearly central. Next, note that $E_1 \otimes E_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to C(X_1 \times X_2)$ is a unital completely positive map [6, Theorem 3.5.3]. Computing with elementary tensors shows that $E_1 \otimes E_2$ is a conditional expectation and satisfies the tracial property (T).

To prove the faithfulness property (F), we use Kirchberg's Slice Lemma [71, Lemma 4.1.9]. The set $I = \{a \in \mathcal{M}_1 \otimes \mathcal{M}_2 : (E_1 \otimes E_2)(a^*a) = 0\}$ is an ideal because $E_1 \otimes E_2$ satisfies the tracial axiom (T).[3] In particular, I is a hereditary subalgebra of $\mathcal{M}_1 \otimes \mathcal{M}_2$. Assume I is non-zero. Then, by Kirchberg's Slice Lemma, there exists a non-zero $z \in \mathcal{M}_1 \otimes \mathcal{M}_2$ such that $zz^* \in I$ and $z^*z = a_1 \otimes a_2$ for some $a_1 \in \mathcal{M}_1$ and $a_2 \in \mathcal{M}_2$. By taking absolute values, we may assume a_1 and a_2 are positive. Since I is an ideal $(zz^*)^{1/2} \in I$. Hence,

[2] We write $\otimes$ for the minimal tensor product of C*-algebras.

[3] For example, compose $E_1 \otimes E_2$ with evaluation maps to get traces, and apply Proposition 2.6.12.

we have $(E_1 \otimes E_2)(zz^*) = 0$. By Axiom (T), $0 = (E_1 \otimes E_2)(z^*z) = E(a_1) \otimes E(a_2)$. Since $E(a_1)$ and $E(a_2)$ are positive, this forces either $E_1(a) = 0$ or $E_2(a_2) = 0$. Since a_1 and a_2 are positive, Axiom (F) for $\mathcal{M}_1$ and $\mathcal{M}_2$ implies either $a_1 = 0$ or $a_2 = 0$. Therefore, $z = 0$. This contradiction implies that I must be zero. Thus, axiom (F) holds for $\mathcal{M}_1 \otimes \mathcal{M}_2$. □

We can now make the following definition.

Definition 4.2.2. Let $\mathcal{M}_i$ be W*-bundle over X_i with conditional expectation E_i and central embedding ι_i for $i = 1, 2$. The *tensor product* of the W*-bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ is the completion of the pre-W*-bundle with section algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$, conditional expectation $E_1 \otimes E_2$ and central embedding $\iota_1 \otimes \iota_2$. We denote this tensor product by $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ and write $E_1 \overline{\otimes} E_2$ for the conditional expectation.

This definition agrees with [5, Definition 3.4.5], where the construction of the completion using standard form is preferred (see Proposition 3.4.23). The Hilbert-$C(X_1 \times X_2)$-module $L^2(\mathcal{M}_1 \otimes \mathcal{M}_2)$, which is the same as $L^2(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)$, is isomorphic to the external tensor product (see [50, Chapter 4]) of the Hilbert modules $L^2(\mathcal{M}_1)$ and $L^2(\mathcal{M}_2)$ as can be verified from the identity

$$\langle \widehat{a_1 \otimes a_2}, \widehat{b_1 \otimes b_2} \rangle_{L^2(\mathcal{M}_1 \otimes \mathcal{M}_2)} = \langle \widehat{a_1}, \widehat{b_1} \rangle_{L^2(\mathcal{M}_1)} \langle \widehat{a_2}, \widehat{b_2} \rangle_{L^2(\mathcal{M}_2)}. \tag{4.2.1}$$

Remark 4.2.3. In fact, one could construct $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ by starting with the external tensor product $L^2(\mathcal{M}_1) \otimes L^2(\mathcal{M}_2)$, considering the tensor product action of $\mathcal{M}_1 \otimes \mathcal{M}_2$ and taking the strict closure. This construction avoids the use of Kirchberg's Slice Lemma at the expense of showing that the inner product for the external tensor product is positive definite, which is equally non-trivial.

Now for some examples.

Example 4.2.4 (Tracial von Neumann Algebras)**.** If we view tracial von Neumann algebras as W*-bundles over a one point space, then the tensor product of the W*-bundles agrees with that of the tracial von Neumann algebras. Indeed, if (M_1, τ_1) and (M_2, τ_2) are tracial von Neumann algebras, then the minimal tensor product $M_1 \otimes M_2$ embeds in the von Neumann tensor product $M_1 \overline{\otimes} M_2$ because the minimal tensor product is the spatial tensor product. Moreover, $M_1 \otimes M_2$ is $\|\cdot\|_{2,\tau_1 \overline{\otimes} \tau_2}$-dense in $M_1 \overline{\otimes} M_2$, so its completion is isomorphic to $M_1 \overline{\otimes} M_2$.

Example 4.2.5 (Trivial Bundles)**.** The trivial W*-bundle $C_\sigma(X, M)$ can be viewed as the W*-bundle tensor product of $C(X)$, viewed a W*-bundle over X, with M viewed as a

W*-bundle over a one point space. Indeed, since $C(X)$ is nuclear, the map $f \otimes a \mapsto f(\cdot)a$ defines an embedding $C(X) \otimes M \to C_\sigma(X, M)$. A simple partition of unity argument shows that this embedding is $\|\cdot\|_{2,u}$-dense. Verifying that this gives an isomorphism of W*-bundles is straightforward.

Next, we record the following result of [5], which shows that the tensor product of W*-bundles is compatible with the construction of Section 3.3. A particularly nice corollary of this proposition is that, if additionally A is $\mathcal{Z}$-stable, then the W*-bundle $\overline{A}^{\mathrm{st}}$ will be McDuff.

Proposition 4.2.6. *[5, Proposition 3.6] Let A and B be simple, separable, unital, stably finite C*-algebras with compact extreme tracial boundaries $\partial_e T(A)$ and $\partial_e T(B)$ respectively. Then $\overline{A \otimes B}^{\mathrm{st}} \cong \overline{A}^{\mathrm{st}} \overline{\otimes} \overline{B}^{\mathrm{st}}$ as W*-bundles over $\partial_e T(A \otimes B) \cong \partial_e T(A) \times \partial_e T(B)$.*

We now turn to the tensor product of morphisms. The following result shows that the tensor product of W*-bundles behaves like the the minimal tensor product of C*-algebra.

Proposition 4.2.7. *Let $\alpha_i : \mathcal{M}_i \to \mathcal{N}_i$ be a morphism of W*-bundles for $i = 1, 2$. The tensor product map $\alpha_1 \otimes \alpha_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{N}_2$ extends to a morphism of W*-bundles $\alpha_1 \overline{\otimes} \alpha_2 : \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \to \mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$. This morphism is injective if the α_i are injective and surjective if the α_i are surjective.*

Proof. Let $\mathcal{M}_i$ have base space X_i and conditional expectation E_i. Let $\mathcal{N}_i$ have base space Y_i and conditional expectation F_i. Computing with elementary tensors shows that $\alpha_1 \otimes \alpha_2$ maps $C(X_1) \otimes C(X_2)$ into $C(Y_1) \otimes C(Y_2)$ and that the diagram

$$\begin{tikzcd} \mathcal{M}_1 \otimes \mathcal{M}_2 \arrow[r, "\alpha_1 \otimes \alpha_2"] \arrow[d, "E_1 \otimes E_2"'] & \mathcal{N}_1 \otimes \mathcal{N}_2 \arrow[d, "F_1 \otimes F_2"'] \\ C(X_1) \otimes C(X_2) \arrow[r, "\alpha_1 \otimes \alpha_2"] & C(Y_1) \otimes C(Y_2) \end{tikzcd} \tag{4.2.2}$$

commutes. Hence, $\alpha_1 \otimes \alpha_2$ defines a morphism of pre-W*-bundles. This extends to a morphism of the completions $\alpha_1 \overline{\otimes} \alpha_2 : \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \to \mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$ by Proposition 3.4.22.

If α_1 and α_2 are injective then $\alpha_1 \otimes \alpha_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{N}_2$ is also injective since we are using the minimal tensor product [6, Proposition 3.6.1]. Hence, $\alpha_1 \overline{\otimes} \alpha_2$ is injective by Proposition 3.4.22.

If α_1 and α_2 are surjective then $(\alpha_1 \otimes \alpha_2)(\mathcal{M}_1 \otimes_{\mathrm{alg}} \mathcal{M}_2) = \mathcal{N}_1 \otimes_{\mathrm{alg}} \mathcal{N}_2$ by considering elementary tensors. Since the image of a C*-algebra under a *-homomorphism is closed, $\alpha_1 \otimes \alpha_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{N}_2$ is surjective. Hence, $\alpha_1 \overline{\otimes} \alpha_2$ is surjective by Proposition 3.4.22. □

We can now verify that the fibres of a tensor product of W*-bundles are the tensor products of the fibres. We remark that in the setting of continuous $C(X)$-algebras the analogous result can fail in the absence of exactness (see [48]).

Proposition 4.2.8. *Let $\mathcal{M}_i$ be a W*-bundle over X_i with conditional expectation E_i for $i = 1, 2$. Let $(x_1, x_2) \in X_1 \times X_2$. Then $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_{(x_1,x_2)} \cong (\mathcal{M}_1)_{x_1} \overline{\otimes} (\mathcal{M}_2)_{x_2}$ as tracial von Neumann algebras.*

Proof. Let $\alpha_i : \mathcal{M}_i \to (\mathcal{M}_i)_{x_i}$ denote the fibre evaluation $a \mapsto a(x_i)$. Viewing $(\mathcal{M}_i)_{x_i}$ as a W*-bundle over a one point space, α_i is a surjective morphism of W*-bundles. We apply Proposition 4.2.7 to obtain a surjective morphism $\alpha_1 \overline{\otimes} \alpha_2$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \arrow[r, "\alpha_1 \overline{\otimes} \alpha_2"] \arrow[d, "E_1 \overline{\otimes} E_2"'] & (\mathcal{M}_1)_{x_1} \overline{\otimes} (\mathcal{M}_2)_{x_2} \arrow[d, "\tau_{x_1} \overline{\otimes} \tau_{x_2}"'] \\ C(X_1 \times X_2) \arrow[r, "\mathrm{eval}_{(x_1,x_2)}"] & \mathbb{C}. \end{tikzcd} \tag{4.2.3}$$

commutes.

From the commuting diagram, we have, for $a \in \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$,

$$(\alpha_1 \overline{\otimes} \alpha_2)(a) = 0 \Leftrightarrow (\tau_{x_1} \overline{\otimes} \tau_{x_2})(\alpha_1 \overline{\otimes} \alpha_2)(a^* a) = 0 \tag{4.2.4}$$

$$\Leftrightarrow (E_1 \overline{\otimes} E_2)(a^* a)(x_1, x_2) = 0. \tag{4.2.5}$$

So $\alpha_1 \overline{\otimes} \alpha_2$ induces an injective *-homomorphism $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_{(x_1,x_2)} \to (\mathcal{M}_1)_{x_1} \overline{\otimes} (\mathcal{M}_2)_{x_2}$. This map is trace preserving by (4.2.3) and is surjective as $\alpha_1 \overline{\otimes} \alpha_2$ is surjective. □

4.2.2 Inductive Limits and Infinite Tensor Products

In this section, we define the inductive limit of a system of W*-bundles and morphisms. The main application of this is to facilitate the definition of the infinite tensor product, though the construction may be of independent interest.

As with tensor products, the key to the construction is the observation that inductive limits in the category of unital C*-algebras with *-homomorphisms as morphisms are compatible with the additional structure of a pre-W*-bundle. We assume familiarity, with inductive limits of C*-algebras as outlined in [74, Section 6.2].

Proposition 4.2.9. *Let $\mathcal{M}_i$ be a W*-bundle over X_i with conditional expectation E_i and central embedding ι_i for $i \in \mathbb{N}$. Let $\alpha_i : \mathcal{M}_i \to \mathcal{M}_{i+1}$ be a morphism for $i \in \mathbb{N}$. Set $\mathcal{M} = \lim_{\rightarrow}(\mathcal{M}_i, \alpha_i)$ be the C*-inductive limit and $X = \lim_{\leftarrow}(X_i, \alpha_i^t)$ the projective* *limit of compact Hausdorff spaces. Then there exist a unique unital* **-homomorphism* $\iota : C(X) \to \mathcal{M}$ *and a unique ucp map* $E : \mathcal{M} \to X$ *such that the following diagrams commute:*

$$\begin{array}{ccccccccc} \mathcal{M}_1 & \xrightarrow{\alpha_1} & \mathcal{M}_2 & \xrightarrow{\alpha_2} & \mathcal{M}_3 & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & \mathcal{M} \\ \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle \iota_3} & & & & \uparrow{\scriptstyle \iota} \\ C(X_1) & \xrightarrow{\alpha_1} & C(X_2) & \xrightarrow{\alpha_2} & C(X_3) & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & C(X), \end{array} \tag{4.2.6}$$

$$\begin{array}{ccccccccc} \mathcal{M}_1 & \xrightarrow{\alpha_1} & \mathcal{M}_2 & \xrightarrow{\alpha_2} & \mathcal{M}_3 & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & \mathcal{M} \\ {\scriptstyle E_1}\downarrow & & {\scriptstyle E_2}\downarrow & & {\scriptstyle E_3}\downarrow & & & & {\scriptstyle E}\downarrow \\ C(X_1) & \xrightarrow{\alpha_1} & C(X_2) & \xrightarrow{\alpha_2} & C(X_3) & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & C(X). \end{array} \tag{4.2.7}$$

Moreover, $\mathcal{M}$ *together with* ι *and* E *is a pre-W*-bundle over* X *and the canonical* **-homomorphism* $\mu_i : \mathcal{M}_i \to \mathcal{M}$ *is a morphism of pre-W*-bundles.*

Proof. The existence and uniqueness of ι such that (4.2.6) commutes follows from the universal property for the inductive limits in the category of unital C*-algebras with *-homomorphisms. Since each ι_i is isometric, the map ι is isometric, so is an embedding. A simple density argument shows that $\iota(C(X))$ lies in the centre of $\mathcal{M}$.

One could similarly deduce the existence and uniqueness of E from the universal property for the inductive limits in the category of operator systems with ucp maps as the morphism (see for example [54]), but we favour a direct proof.

Let $\mu_i : \mathcal{M}_i \to \mathcal{M}$ and $\nu_i : C(X_i) \to C(X)$ denote the canonical maps into the inductive limit. We define E on the dense subspace $\bigcup_{i\in\mathbb{N}} \mu_i(\mathcal{M}_i)$ by $\mu_i(a_i) \mapsto \nu_i(E_i(a_i))$ for $a_i \in \mathcal{M}_i$. This is well defined as $E_{i+1} \circ \alpha_i = \alpha_{i+1} \circ E_i$. On this dense subspace, E is clearly linear. To prove that E is bounded, we use that fact that, if $a \in \mu_i(\mathcal{M}_i)$, then there is $a_i \in \mathcal{M}_i$ with $\|a_i\| = \|a\|$ and $\mu_i(a_i) = a$ [74, Section 2.2.10]. We then have $\|E(a)\| = \|\nu_i(E_i(a_i))\| \leq \|a_i\| \leq \|a\|$. Hence, E has a unique extension to a bounded linear operator $\mathcal{M} \to C(X)$.

We now show that E is a ucp map. By construction, $E(1) = 1$. By Proposition 2.5.2, it suffices to show that E is positive. However, this follows since $\bigcup_{i\in\mathbb{N}} \mu_i((\mathcal{M}_i)_+)$ is dense in $\mathcal{M}_+$ and each E_i is positive. This completes the proof of the existence of a ucp map such that (4.2.7) commutes. Uniqueness follows by a simple density argument.

Identify $C(X)$ with its image under ι. A simple density argument shows that E is a conditional expectation onto $C(X)$ and that the tracial axiom (T) holds. It only remains to show that the faithfulness axiom (F) holds. Here, the key observation is that $I = \{a \in \mathcal{M} : E(a^*a) = 0\}$ is an ideal of the C*-algebra $\mathcal{M}$ because E satisfies the tracial axiom (T).[4] Hence, we have $I = \overline{\bigcup_{i\in\mathbb{N}} \mu_i(\mathcal{M}_i) \cap I}$ (see for example [4, II.8.2.3]). However, $\mu_i(\mathcal{M}_i) \cap I = \{0\}$ for all $i \in \mathbb{N}$ by the faithfulness axiom (F) for the W*-bundle $\mathcal{M}_i$. This completes the proof that $\mathcal{M}$ endowed with the central embedding ι and the conditional expectation E is a pre-W*-bundle. It follows from the commuting diagrams (4.2.6) and (4.2.7) that $\mu_i : \mathcal{M}_i \to \mathcal{M}$ is a morphism of pre-W*-bundles for all $i \in \mathbb{N}$. □

We can now define the inductive limit of an inductive system of W*-bundles and prove that it satisfies the appropriate universal property.

Definition 4.2.10. The *inductive limit* of a inductive system of W*-bundles and morphisms

$$\mathcal{M}_1 \xrightarrow{\alpha_1} \mathcal{M}_2 \xrightarrow{\alpha_2} \mathcal{M}_3 \xrightarrow{\alpha_3} \cdots \tag{4.2.8}$$

is the completion $\overline{\mathcal{M}}$ of the pre-W*-bundle $\mathcal{M}$ constructed in Proposition 4.2.9 together with the canonical morphisms $\mu_i : \mathcal{M}_i \to \mathcal{M} \subseteq \overline{\mathcal{M}}$.

Proposition 4.2.11. *Let*

$$\mathcal{M}_1 \xrightarrow{\alpha_1} \mathcal{M}_2 \xrightarrow{\alpha_2} \mathcal{M}_3 \xrightarrow{\alpha_3} \cdots \tag{4.2.9}$$

be an inductive system of W-bundles and morphisms. Let $\overline{\mathcal{M}}$ be the completion of the pre-W*-bundle $\mathcal{M}$ constructed in Proposition 4.2.9 and let $\mu_i : \mathcal{M}_i \to \mathcal{M} \subseteq \overline{\mathcal{M}}$ be the canonical morphisms.*

Suppose $\mathcal{N}$ is a W-bundle and there are morphisms $\beta_i : \mathcal{M}_i \to \mathcal{N}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "\alpha_1"] \arrow[drrr, "\beta_1"'] & \mathcal{M}_2 \arrow[r, "\alpha_2"] \arrow[drr, "\beta_2"] & \mathcal{M}_3 \arrow[r, "\alpha_3"] \arrow[dr, "\beta_3"] & \cdots \\ & & & \mathcal{N}. \end{tikzcd} \tag{4.2.10}$$

Then there exists a unique morphism $\beta : \overline{\mathcal{M}} \to \mathcal{N}$ such that $\beta \circ \mu_i = \beta_i$ for all $i \in \mathbb{N}$.

Proof. Uniqueness follows by diagram chasing and a density argument. We shall focus on proving the existence of β. We shall use the notation of Proposition 4.2.9 for the conditional expectations and base spaces of the $\mathcal{M}_i$ and $\mathcal{M}$ but treat the embeddings

[4] For example, compose E with evaluation maps to get traces, and apply Proposition 2.6.12.

as identifications. Furthermore, we shall write Y for the base space of $\mathcal{N}$ and F for the conditional expectation $F : \mathcal{N} \to C(Y)$.

By the universal property for C*-inductive limits, there is a *-homomorphism $\beta : \mathcal{M} \to \mathcal{N}$ such that $\beta \circ \mu_i = \beta_i$. Since each β_i is a morphism of W*-bundles, we have $\beta(\mu_i(C(X_i))) = \beta_i((C(X_i)) \subseteq C(Y)$ and

$$F(\beta(\mu_i(a_i))) = F(\beta_i(a_i)) \tag{4.2.11}$$

$$= \beta_i(E_i(a_i)) \tag{4.2.12}$$

$$= \beta(\mu_i(E_i(a_i))) \tag{4.2.13}$$

$$= \beta(E(\mu_i(a_i)) \tag{4.2.14}$$

for $a_i \in \mathcal{M}_i$. Hence, by density, $\beta(C(X)) \subseteq C(Y)$ and $F \circ \beta = \beta \circ E$. Therefore, $\beta : \mathcal{M} \to \mathcal{N}$ is a morphism of pre-W*-bundles. By Proposition 3.4.22, β extends to a morphism $\overline{\mathcal{M}} \to \overline{\mathcal{N}} = \mathcal{N}$. We still have that $\beta \circ \mu_i = \beta_i$ for all $i \in \mathbb{N}$. □

Finally, we can give the definition of the infinite tensor product of W*-bundles.

Definition 4.2.12. The *infinite tensor product* of W*-bundles $\overline{\bigotimes}_{i=1}^{\infty}\mathcal{M}_i$ is defined to be the inductive limit of the system

$$\mathcal{M}_1 \xrightarrow{\mathrm{id}\otimes 1} \overline{\bigotimes}_{i=1}^{2}\mathcal{M}_i \xrightarrow{\mathrm{id}\otimes 1} \overline{\bigotimes}_{i=1}^{3}\mathcal{M}_i \xrightarrow{\mathrm{id}\otimes 1} \cdots . \tag{4.2.15}$$

4.2.3 Ultrapowers

Ultrapowers are an extremely useful technical tool in operator algebras. They allow us to turn approximate properties, that is conditions satisfied in the limit, into exactly satisfied properties at the expense of working in a larger algebra. In order to define ultrapowers, we require a free ultrafilter on the natural numbers $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. The existence of such an ultrafilter follows from the Axiom of Choice.

We first recall the definition of the ultrapower of a C*-algebra and the (tracial) ultrapower of a tracial von Neumann algebra before moving onto the ultrapowers of W*-bundles.

Definition 4.2.13. Let A be a C*-algebra. The *ultrapower* of A with respect to $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ is defined to be

$$A_\omega = \frac{\ell^\infty(A)}{\{(a_n) \in \ell^\infty(A) : \lim_{n\to\omega} \|a_n\| = 0\}}. \tag{4.2.16}$$

Definition 4.2.14. Let (M,τ) be a tracial von Neumann algebra. The *(tracial) ultrapower* of M with respect to $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ is defined to be

$$M^\omega = \frac{\ell^\infty(M)}{\{(a_n)\in\ell^\infty(M) : \lim_{n\to\omega}\|a_n\|_{2,\tau}=0\}} \tag{4.2.17}$$

together with the trace τ_ω induced on the quotient by the map $(a_n)\mapsto \lim_{n\to\omega}\tau(a_n)$.

For a proof that the tracial ultrapower (M^ω,τ_ω) is a tracial von Neumann algebra, i.e. that the unit ball is $\|\cdot\|_{2,\tau_\omega}$-complete, see [81, Theorem A.3.5].

Notation and Terminology 4.2.15. We follow the now standard conventions of using a superscript for the tracial ultrapower and a subscript for the the C*-ultrapower. The adjective tracial is dropped when there is no chance of confusion. We shall often speak of *the* ultrapower, ignoring the potential dependence on the choice of ultrafilter, since we shall not be interested in properties of ultrapowers that depend on the choice of ultrafilter. We write $[(a_n)]$ for the element in an ultrapower coming from the sequence (a_n). We identify A (respectively M) with the images of constant sequence in A_ω (respectively M^ω).

We now turn to the ultrapowers of W*-bundles. We first consider the base space. If X is a compact Hausdorff space then $C(X)_\omega$ is a commutative unital C*-algebra, so there exists a compact Hausdorff space $\sum_\omega X$, unique up to homeomorphism, such that $C(X)_\omega \cong C(\sum_\omega X)$. The space $\sum_\omega X$ is the ultra-copoduct of X (see [2, Section 1]). We shall identify $C(\sum_\omega X)$ with $C(X)_\omega$.

A dense subspace of $\sum_\omega X$ can be identified with the set-theoretic ultrapower $\prod_\omega X$ defined to be the set of sequences in X modulo the relation $(x_n)\sim(y_n)$ if and only if $\{n\in\mathbb{N} : x_n=y_n\}\in\omega$. Indeed, given a point $[(x_n)]_\sim$ in $\prod_\omega X$, we can define a character on $C(X)_\omega$ via $[(f_n)]\mapsto \lim_{n\to\omega} f_n(x_n)$, and the collection of all such characters determines the norm of $[(f_n)]\in C(X)_\omega$, so forms a dense subspace of $\sum_\omega X$.

We can now give the definition of the ultrapower of a W*-bundle, which is a special case of [5, Definition 3.7].

Definition 4.2.16. Let $\mathcal{M}$ be a W*-bundle over X with conditional expectation E and central embedding ι. The ultrapower of $\mathcal{M}$ with respect to $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$ is the W*-bundle over the ultra-coproduct $\sum_\omega X$ with section algebra

$$\mathcal{M}^\omega = \frac{\ell^\infty(\mathcal{M})}{\{(a_n)\in\ell^\infty(\mathcal{M}) : \lim_{n\to\omega}\|a_n\|_{2,u}=0\}}, \tag{4.2.18}$$

the central embedding $\iota_\omega : C(X)_\omega \to \mathcal{M}^\omega$ induced by the product map $\prod_{n=1}^\infty \iota : \ell^\infty(C(X)) \to \ell^\infty(\mathcal{M})$, and the conditional expectation $E_\omega : \mathcal{M}^\omega \to C(X)_\omega$ induced by the product map $\prod_{n=1}^\infty E : \ell^\infty(\mathcal{M}) \to \ell^\infty(C(X))$.

For a proof that $\mathcal{M}^\omega$ is a W*-bundle, see [5, Proposition 3.9].

4.3 Property Γ for W*-Bundles

In this section, we consider condition (ii) of Ozawa's Triviality Theorem (Theorem 4.1.1). This condition and its consequences are sufficiently important to merit the following definition.

Definition 4.3.1. Let $\mathcal{M}$ be a strictly separable W*-bundle over the compact Hausdorff space X with conditional expectation E. We say that $\mathcal{M}$ has *property* Γ if there is a sequence of positive contractions (p_n) in $\mathcal{M}$ such that, as $n \to \infty$,

$$\|[p_n, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}), \tag{4.3.1}$$

$$\|p_n - p_n^2\|_{2,u} \to 0, \tag{4.3.2}$$

$$\|E(p_n) - \tfrac{1}{2}\|_{C(X)} \to 0. \tag{4.3.3}$$

Remark 4.3.2. Murray and von Neumann introduced property Γ for II_1 factors in [61, Chapter VI] in order to prove that there exist separably acting II_1 factors other than the hyperfinite II_1 factor (see Example 4.3.3 below). They used a slightly different formulation of property Γ, requiring a sequence of trace zero unitaries (u_n) in the separably acting II_1 factor (M, τ) such that $\|[u_n, a]\|_{2,\tau} \to 0$ for all $a \in M$. Thanks to the work of Dixmier [17, Proposition 1.10], this is equivalent to the existence of a sequence of positive contractions (p_n) such that $\|[p_n, a]\|_{2,\tau} \to 0$ for all $a \in M$, $\|p_n^2 - p\|_{2,\tau} \to 0$ and $\tau(p_n) \to 1/2$, so is compatible with Definition 4.3.1.

Example 4.3.3 (Tracial von Neumann algebras)**.** The hyperfinite II_1 factor $\mathcal{R}$ has property Γ because, writing $\mathcal{R} = \overline{\bigotimes}_{i=1}^\infty (\mathbb{M}_2(\mathbb{C}), \mathrm{tr}_2)$, we can take p_n to be a projection of trace 1/2 in the n-th tensor factor. In contrast, the group algebras of non-abelian free groups $L(\mathbb{F}_n)$ do not have property Γ by the 14ϵ-Argument (see for example [81, Theorem A.7.2]).

Example 4.3.4 (Trivial Bundles)**.** If the tracial von Neumann algebra (M, τ) has property Γ, then so does the trivial bundle $C_\sigma(X, M)$. Indeed, if (p_n) is a sequence of positive contractions in M with $\|[p_n, a]\|_{2,\tau} \to 0$ for all $a \in M$, $\|p_n^2 - p\|_{2,\tau} \to 0$ and $\tau(p_n) \to 1/2$, then the same sequence, viewed as constant functions in $C_\sigma(X, M)$, satisfies the conditions of Definition 4.3.1 as the following argument shows.

Let $f \in C_\sigma(X, M)$ and $\epsilon > 0$. By continuity of f and the compactness of X, there exists a continuous partition of unity $\phi_1, \ldots, \phi_k : X \to [0,1]$ and $x_1, \ldots, x_k \in X$ such that $\|f - g\|_{2,u} < \epsilon$, where $g(x) = \sum_{i=1}^k \phi_i(x) f(x_i)$ for $x \in X$. Furthermore, there exists $N \in \mathbb{N}$ such that $\|[p_n, f(x_i)]\|_{2,\tau} < \epsilon$ whenever $n \geq N$. Let $n \geq N$ and $x \in X$. Then

$$\begin{aligned}
\|[p_n, f(x)]\|_{2,\tau} &\leq \|[p_n, f(x) - g(x)]\|_{2,\tau} + \|[p_n, g(x)]\|_{2,\tau} && (4.3.4)\\
&< 2\|p_n\|\|f(x) - g(x)\|_{2,\tau} + \sum_{i=1}^k \phi_i(x)\|[p_n, f(x_i)]\|_{2,\tau} && (4.3.5)\\
&< 2\epsilon + \sum_{i=1}^k \phi_i(x)\epsilon && (4.3.6)\\
&= 3\epsilon. && (4.3.7)
\end{aligned}$$

Therefore, $\|[p_n, f]\|_{2,u} \to 0$.

The goal of this section is to show that for strictly separable W*-bundles with factorial fibres property Γ implies the existence of a large number of central sequences. The hypothesis of factorial fibres is used in the next lemma to approximate the conditional expectation by a $C(X)$-convex combination of unitaries.

Lemma 4.3.5. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X with conditional expectation E. Suppose $\mathcal{M}_x$ is a factor for all $x \in X$. Then, for all $a \in \mathcal{M}$ and $\epsilon > 0$, there exist $k \in \mathbb{N}$, unitaries $u_1, \ldots, u_k \in \mathcal{M}$, and a continuous partition of unity $g_1, \ldots, g_k : X \to [0,1]$ such that*

$$\|E(a) - \sum_{i=1}^k g_i u_i a u_i^*\|_{2,u} < \epsilon. \tag{4.3.8}$$

Proof. Let $a \in \mathcal{M}$ and $\epsilon > 0$. Fix $x \in X$. By the Dixmier Approximation Theorem [16, Theorem III.5.1], there exist $k \in \mathbb{N}$ and unitaries $w_1, \ldots, w_k \in \mathcal{M}_x$ such that

$$\|\tau_x(a(x))1_x - \frac{1}{k}\sum_{j=1}^k w_j a(x) w_j^*\| < \epsilon. \tag{4.3.9}$$

Furthermore, as $\mathcal{M}_x$ is a von Neumann algebra, we can lift the unitaries $w_1, \ldots, w_k \in \mathcal{M}_x$ to unitaries $u_1, \ldots, u_k \in \mathcal{M}$.[5] We have

$$\|E(a)(x) - \frac{1}{k}\sum_{j=1}^k u_j(x) a(x) u_j(x)^*\|_{2,\tau_x} < \epsilon \tag{4.3.10}$$

[5]Using Borel functional calculus, all unitaries in a von Neumann algebra are of the form e^{ih} for self-adjoint h, and one can lift self-adjoint elements to self-adjoint elements.

since the $\|\cdot\|$ norm on $\mathcal{M}_x$ dominates the $\|\cdot\|_{2,\tau_x}$-norm. By Proposition 3.2.6, there is an open neighbourhood $V^{(x)}$ of x such that

$$\|E(a)(y) - \frac{1}{k}\sum_{j=1}^{k} u_j(y)a(y)u_j(y)^*\|_{2,\tau_x} < \epsilon \tag{4.3.11}$$

whenever $y \in V^{(x)}$.

As x varies, the $V^{(x)}$ form an open cover of X. By compactness, there is a finite subcover. Denote this subcover $V_1, \ldots, V_n$. Each V_i comes with a $k_i \in \mathbb{N}$ and unitaries $u_{i1}, \ldots, u_{ik_i} \in \mathcal{M}$. Let $\psi_1, \ldots, \psi_n : X \to [0,1]$ be a continuous partition of unity subordinate to $V_1, \ldots, V_n$. We form a second partition of unity

$$1 = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \frac{1}{k_i}\psi_i, \tag{4.3.12}$$

i.e. the functions occurring in the partition of unitary are $\frac{1}{k_i}\psi_i$ for $i = 1, \ldots, n$ but $\frac{1}{k_i}\psi_i$ occurs with multiplicity k_i. Set

$$b = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \frac{1}{k_i}\psi_i u_{ij} a u_{ij}^*. \tag{4.3.13}$$

Let $x \in X$. Then

$$\begin{aligned}
\|E(a)(x) - b(x)\|_{2,\tau_x} &= \left\|\sum_{i=1}^{n}\psi_i(x)\left(E(a)(x) - \frac{1}{k_i}\sum_{j=1}^{k_i} u_{ij}(x)a(x)u_{ij}(x)^*\right)\right\|_{2,\tau_x} && (4.3.14)\\
&\leq \sum_{i=1}^{n}\psi_i(x)\left\|E(a)(x) - \frac{1}{k_i}\sum_{j=1}^{k_i} u_{ij}(x)a(x)u_{ij}(x)^*\right\|_{2,\tau_x} && (4.3.15)\\
&< \sum_{i=1}^{n}\psi_i(x)\epsilon && (4.3.16)\\
&= \epsilon. && (4.3.17)
\end{aligned}$$

Therefore,

$$\left\|E(a) - \sum_{i=1}^{n}\sum_{j=1}^{k_i}\frac{1}{k_i}\psi_i u_{ij} a u_{ij}^*\right\|_{2,u} < \epsilon. \tag{4.3.18}$$

After reindexing the sum, this gives (4.3.8). □

Using Lemma 4.3.5, we are now able to prove an important result about tracial factorisation.

Proposition 4.3.6. *Let $\mathcal{M}$ be a W*-bundle over the compact Hausdorff space X with conditional expectation E. Suppose $\mathcal{M}_x$ is a factor for all $x \in X$. Suppose (b_n) is a $\|\cdot\|$-bounded sequence in $\mathcal{M}$ such that $\|[b_n, a]\|_{2,u} \to 0$ for all $a \in \mathcal{M}$. Then*

$$\|E(b_n a) - E(b_n)E(a)\|_{C(X)} \to 0 \tag{4.3.19}$$

for all $a \in \mathcal{M}$.

Proof. Suppose $\|b_n\| \leq M$ for all $n \in \mathbb{N}$. In the sequel, we shall use the notation $a_1 \approx_\eta a_2$ to denote $\|a_1 - a_2\|_{2,u} \leq \eta$.

Let $a \in \mathcal{M}$ and $\epsilon > 0$. By Lemma 4.3.5, there are $k \in \mathbb{N}$, unitaries $u_1, \ldots, u_k \in \mathcal{M}$, and a continuous partition of unity $g_1, \ldots, g_k : X \to [0,1]$ such that

$$E(a) \approx_\epsilon \sum_{i=1}^k g_i u_i a u_i^*. \tag{4.3.20}$$

We choose $N \in \mathbb{N}$ such that

$$\sum_{i=1}^k g_i u_i b_n a u_i^* \approx_\epsilon b_n \sum_{i=1}^k g_i u_i a u_i^* \tag{4.3.21}$$

whenever $n \geq N$.

Let $n \geq N$. Since E is tracial, it is unitary invariant. Using this and Proposition 3.2.7(ii), we compute that

$$\begin{aligned} E(b_n a) &= \sum_{i=1}^k g_i E(u_i b_n a u_i^*) && (4.3.22)\\ &= E\left(\sum_{i=1}^k g_i u_i b_n a u_i^*\right) && (4.3.23)\\ &\approx_\epsilon E\left(b_n \left(\sum_{i=1}^k g_i u_i a u_i^*\right)\right) && (4.3.24)\\ &\approx_{M\epsilon} E(b_n E(a)) && (4.3.25)\\ &= E(b_n)E(a). && (4.3.26) \end{aligned}$$

Hence $E(b_n a) \approx_{(M+1)\epsilon} E(b_n)E(a)$. Therefore, $\|E(b_n a) - E(b_n)E(a)\|_{C(X)} \to 0$. $\square$

Next, we reformulate property Γ and the tracial factorisation result in the language of ultrapowers. This will require some additional notation. Firstly, if S is $\|\cdot\|_{2,u}$-separable subalgebra of the W*-bundle $\mathcal{M}^\omega$ we write $\mathcal{M}^\omega \cap S'$ for the commutant of S in $\mathcal{M}^\omega$. The algebra $\mathcal{M}^\omega \cap S'$ inherits the structure of a W*-bundle from $\mathcal{M}^\omega$. Secondly, we denote by $\tau_{1/2}$ the trace on $\mathbb{C}^2$ with $\tau_{1/2}((1,0)) = \tau_{1/2}((0,1)) = \frac{1}{2}$.

Proposition 4.3.7. *Let $\mathcal{M}$ be a strictly separable W*-bundle over the compact Hausdorff space X with conditional expectation E. Then the following are equivalent:*

(i) $\mathcal{M}$ has property Γ.

(ii) For any $\|\cdot\|_{2,u}$-separable subalgebra $S \subseteq \mathcal{M}^\omega$, there is a morphism of W-bundles $\varphi : (\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$.*

Moreover, if $\mathcal{M}_x$ is a factor for all $x \in \mathcal{M}$, then φ can be taken to have the tracial factorisation property

$$E_\omega(\varphi(b)a) = \tau_{1/2}(b)E_\omega(a) \tag{4.3.27}$$

for all $b \in \mathbb{C}^2$ and $a \in S$.

Proof. Firstly, assume that $\mathcal{M}$ satisfies (ii). Take $S = \mathcal{M}$. Let $p = \varphi((1,0))$. Since φ is a morphism of W*-bundles, we have that $p \in \mathcal{M}^\omega \cap \mathcal{M}$, $p^2 = p^* = p$ is a projection, and $E_\omega(p) = \tau_{1/2}((1,0)) = \frac{1}{2}$.

We lift p to a positive contraction $(p_n)_{n=1}^\infty$ in $\ell^\infty(\mathcal{M})$. The properties of p established above imply that, as $n \to \omega$,

$$\|[p_n, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}), \tag{4.3.28}$$

$$\|p_n - p_n^2\|_{2,u} \to 0, \tag{4.3.29}$$

$$\|E(p_n) - \tfrac{1}{2}\|_{C(X)} \to 0. \tag{4.3.30}$$

At the expense of passing to a subsequence, we can replace the limits as $n \to \omega$ above with limits as $n \to \infty$. Although this argument is fairly standard, we present it in full on this occasion for the benefit of the reader.

Since $\mathcal{M}$ is strictly separable, there is a $\|\cdot\|_{2,u}$-dense sequence (a_i) in $\mathcal{M}$ by Corollary 3.2.19. We inductively choose a subsequence $(p_{n_k})_{k=1}^\infty$ of $(p_n)_{n=1}^\infty$ such that

$$\|[p_{n_k}, a_i]\|_{2,u} < \tfrac{1}{k} \qquad (i = 1, \dots, k), \tag{4.3.31}$$

$$\|p_{n_k} - p_{n_k}^2\|_{2,u} < \tfrac{1}{k}, \tag{4.3.32}$$

$$\|E(p_{n_k}) - \tfrac{1}{2}\|_{C(X)} < \tfrac{1}{k}. \tag{4.3.33}$$

then, as $k \to \infty$,

$$\|[p_{n_k}, a_i]\|_{2,u} \to 0 \qquad (i \in \mathbb{N}), \tag{4.3.34}$$

$$\|p_{n_k} - p_{n_k}^2\|_{2,u} \to 0, \tag{4.3.35}$$

$$\|E(p_{n_k}) - \tfrac{1}{2}\|_{C(X)} \to 0. \tag{4.3.36}$$

Let $a \in \mathcal{M}$ and $\epsilon > 0$. Choose $i \in \mathbb{N}$ such that $\|a - a_i\|_{2,u} < \epsilon$. Then, using Proposition 3.2.7,

$$\|[p_{n_k}, a - a_i]\|_{2,u} \leq \|p_{n_k}\|\|a - a_i\|_{2,u} + \|a - a_i\|_{2,u}\|p_{n_k}\| \tag{4.3.37}$$

$$< 2\epsilon. \tag{4.3.38}$$

Let $K \in \mathbb{N}$ be chosen such that $\|[p_{n_k}, a_i]\|_{2,u} < \epsilon$ whenever $k \geq K$. Then

$$\|[p_{n_k}, a]\|_{2,u} \leq \|[p_{n_k}, a_i]\|_{2,u} + \|[p_{n_k}, a - a_i]\|_{2,u} \tag{4.3.39}$$

$$< 3\epsilon \tag{4.3.40}$$

whenever $k \geq K$.

Therefore, as $k \to \infty$,

$$\|[p_{n_k}, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}), \tag{4.3.41}$$

$$\|p_{n_k} - p_{n_k}^2\|_{2,u} \to 0, \tag{4.3.42}$$

$$\|E(p_{n_k}) - \tfrac{1}{2}\|_{C(X)} \to 0. \tag{4.3.43}$$

Hence, $\mathcal{M}$ has property Γ.

Conversely, suppose $\mathcal{M}$ has property Γ. Since S is $\|\cdot\|_{2,u}$-separable, there is a sequence (s_i) such that $\{s_i : i \in \mathbb{N}\}$ is $\|\cdot\|_{2,u}$-dense in S. For each $i \in \mathbb{N}$, $s_i \in \mathcal{M}_\omega$, so can be represented by a sequence $(s_{in})_{n=1}^\infty \in \ell^\infty(\mathcal{M})$. For each $n \in \mathbb{N}$, let $p_n \in \mathcal{M}$ be chosen with $0 \leq p_n \leq 1$ such that

$$\|[p_n, s_{in}]\|_{2,u} < \tfrac{1}{n} \qquad (i = 1, \ldots, n), \tag{4.3.44}$$

$$\|p_n^2 - p_n\|_{2,u} < \tfrac{1}{n}, \tag{4.3.45}$$

$$\|E(p_n) - \tfrac{1}{2}\|_{C(X)} < \tfrac{1}{n}. \tag{4.3.46}$$

Let p be the element of $\mathcal{M}^\omega$ corresponding to the sequence (p_n). Firstly, p commutes with s_i for all $i \in \mathbb{N}$. Hence, $p \in \mathcal{M}^\omega \cap S'$. Secondly, $p^2 = p$, so p is a projection. Thirdly, $E_\omega(p) = \frac{1}{2}$. Indeed, for any point $x = [(x_n)]_\sim \in \prod_\omega X$, we have $E_\omega(p)(x) = \lim_{n\to\omega} E(p_n)(x_n) = \frac{1}{2}$ because $E(p_n)$ converges uniformly to $\frac{1}{2}$. Since $\prod_\omega X$ is dense in $\sum_\omega X$, we have $E_\omega(p) = \frac{1}{2}$.

We define $\varphi : (\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$ by $(\lambda, \mu) \mapsto \lambda p + \mu(1 - p)$. The properties of p established above ensure that this is a morphism of W*-bundles. Hence, $\mathcal{M}$ satisfies (ii).

Suppose additionally that $\mathcal{M}_x$ is a factor for all $x \in \mathcal{M}$. By Proposition 4.3.6, we can additionally take p_n to satisfy

$$\|E(p_n s_{in}) - E(p_n)E(s_{in})\|_{C(X)} < \tfrac{1}{n} \qquad (i = 1, \ldots, n). \tag{4.3.47}$$

Now $p = [p_n] \in \mathcal{M}^\omega$, satisfies $E_\omega(ps_i) = E_\omega(p)E_\omega(s_i)$ for all $i \in \mathbb{N}$. Indeed, if $x = [(x_n)]_\sim \in \prod_\omega X$, we have $E_\omega(ps_i)(x) = \lim_{n\to\omega} E(p_n s_{in})(x_n) = \lim_{n\to\omega} E(p_n)(x_n)E(s_{in})(x_n)$ because $E(p_n s_{in}) - E(p_n)E(s_{in})$ converges uniformly to zero. Since $\prod_\omega X$ is dense in $\sum_\omega X$, we have $E_\omega(ps_i) = E_\omega(p)E_\omega(s_i)$ for all $i \in \mathbb{N}$.

Since p and $1-p$ span $\varphi(\mathbb{C}^2)$ and $\{s_i : i \in \mathbb{N}\}$ is $\|\cdot\|_{2,u}$-dense in S, we have

$$E_\omega(\varphi(b)a) = \tau_{1/2}(b)E_\omega(a) \tag{4.3.48}$$

for all $b \in \mathbb{C}^2$ and $a \in S$. □

We can now deduce that a W*-bundle with property Γ and factorial fibres has a large number of central sequences.

Theorem 4.3.8. *Let $\mathcal{M}$ be a strictly separable W*-bundle over the compact Hausdorff space X. Suppose $\mathcal{M}_x$ is a factor for all $x \in X$ and that $\mathcal{M}$ has property Γ. Then for any $\|\cdot\|_{2,u}$-separable subalgebra $S \subseteq \mathcal{M}^\omega$, there exists a embedding $\varphi : C_\sigma(X, L^\infty[0,1]) \to \mathcal{M}^\omega \cap S'$. Moreover, we have the tracial factorisation property*

$$E_\omega(\varphi(b)a) = E_\sigma(b)E_\omega(a) \tag{4.3.49}$$

for all $b \in C_\sigma(X, L^\infty[0,1])$ and $a \in S$, where E_σ denote the conditional expectation of the trivial bundle $C_\sigma(X, L^\infty[0,1])$.[6]

Proof. Fix a $\|\cdot\|_{2,u}$-separable subalgebra $S \subseteq \mathcal{M}^\omega$. By Proposition 4.3.7, there exists an embedding $\varphi_1 : (\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$ such that $E_\omega(\varphi(b)a) = \tau_{1/2}(b)E_\omega(a)$ for all $a \in S$ and $b \in \mathbb{C}^2$.

We shall recursively define embeddings $\varphi_n : \overline{\bigotimes}_{i=1}^n(\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$ such that $\varphi_{n+1} \circ \iota_n = \varphi_n$, where $\iota_n : \overline{\bigotimes}_{i=1}^n(\mathbb{C}^2, \tau_{1/2}) \to \overline{\bigotimes}_{i=1}^{n+1}(\mathbb{C}^2, \tau_{1/2})$ is the canonical inclusion $a \mapsto a \otimes 1$, and

$$E_\omega(\varphi_n(b)a) = \tau_n(b)E_\omega(a) \tag{4.3.50}$$

for all $a \in S$ and $b \in \overline{\bigotimes}_{i=1}^n \mathbb{C}^2$, where τ_n is the n-fold tensor product of $\tau_{1/2}$.

Indeed, if φ_n has already been constructed and has the desired properties, then let S_n be the $\|\cdot\|_{2,u}$-separable subalgebra of $\mathcal{M}^\omega$ generated by S and $\varphi_n(\overline{\bigotimes}_{i=1}^n \mathbb{C}^2)$. By Proposition 4.3.7, there exists an embedding $\varphi^{(n+1)} : (\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S_n'$ such that

$$E_\omega(\varphi^{(n+1)}(c)a) = \tau_{1/2}(c)E_\omega(a) \tag{4.3.51}$$

[6] In (4.3.49), note that $C(X)$ is identified with the subalgebra of $C(\sum_\omega X) \cong C(X)_\omega$ coming from constant sequences.

for all $a \in S_n$ and $c \in \mathbb{C}^2$.

We can then define φ_{n+1} by $b \otimes c \mapsto \varphi^{(n+1)}(c)\varphi_n(b)$ for $b \in \overline{\bigotimes}_{i=1}^n \mathbb{C}^2$ and $c \in \mathbb{C}^2$. Let $a \in S$, $b \in \overline{\bigotimes}_{i=1}^n \mathbb{C}^2$ and $c \in \mathbb{C}^2$. Since $\varphi_n(b)a \in S_n$, it follows readily from (4.3.51) and (4.3.50) that

$$E_\omega(\varphi_{n+1}(b \otimes c)a) = E_\omega(\varphi^{(n+1)}(c)\varphi_n(b)a) \tag{4.3.52}$$

$$= \tau_{1/2}(c)E_\omega(\varphi_n(b)a) \tag{4.3.53}$$

$$= \tau_{1/2}(c)\tau_n(b)E_\omega(a) \tag{4.3.54}$$

$$= \tau_{n+1}(b \otimes c)E_\omega(a). \tag{4.3.55}$$

This shows that φ_{n+1} is a morphism of W*-bundles and satisfies the appropriate tracial factorisation property.

By the universal property of inductive limits (Proposition 4.2.11), we get an induced map $\varphi_\infty : \overline{\bigotimes}_{i=1}^\infty (\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$. By density, we have

$$E_\omega(\varphi_\infty(b)a) = \tau_\infty(b)E_\omega(a) \tag{4.3.56}$$

for all $a \in S$ and $b \in \overline{\bigotimes}_{i=1}^\infty (\mathbb{C}^2, \tau_{1/2})$, where τ_∞ denote the trace of the infinite tensor product.

The commutative tracial von Neumann algebra $\overline{\bigotimes}_{i=1}^\infty (\mathbb{C}^2, \tau_{1/2})$ is well known to be isomorphic to $L^\infty[0,1]$ with the Lebesgue trace and we identify these tracial von Neumann algebras.

Since the image of φ_∞ commutes with $C(X) \subseteq C(X)_\omega \subseteq Z(\mathcal{M}^\omega)$, and $C(X)$ is nuclear, we get a *-homomorphism $\varphi : C(X) \otimes L^\infty[0,1] \to \mathcal{M}^\omega \cap S'$. Moreover, we have

$$E_\omega(\varphi(f \otimes b)a) = E_\omega(f\varphi_\infty(b)a) \tag{4.3.57}$$

$$= fE_\omega(\varphi_\infty(b)a) \tag{4.3.58}$$

$$= f\tau_\infty(b)E_\omega(a) \tag{4.3.59}$$

for all $a \in S$, $b \in L^\infty[0,1]$ and $f \in C(X)$. Taking $a = 1$, we see that φ is a morphism of the pre-W*-bundles, where $C(X) \otimes L^\infty[0,1]$ is viewed as a pre-W*-bundle via the construction of Proposition 4.2.1. Hence, φ extends to a morphism of the W*-bundle tensor product $C(X)\overline{\otimes}L^\infty[0,1]$ into $\mathcal{M}^\omega \cap S'$, which we also denote by φ. By Example 4.2.5, the W*-bundle tensor product $C(X)\overline{\otimes}L^\infty[0,1]$ is isomorphic to $C_\sigma(X, L^\infty[0,1])$. Finally, by (4.3.59) and density, we obtain (4.3.49). □

The main use of Theorem 4.3.8 will be to facilitate orthogonal partition of unity arguments. We formulate this as a corollary for ease of reference.

Corollary 4.3.9. *Let $\mathcal{M}$ be a strictly separable W^*-bundle over the compact Hausdorff space X with conditional expectation E. Suppose $\mathcal{M}_x$ is a factor for all $x \in X$ and that $\mathcal{M}$ has property Γ. Let $g_1, \ldots, g_k : X \to [0,1]$ be a continuous partition of unity. Then there exist sequences of mutually orthogonal positive contractions $(e_1^{(n)}), \ldots, (e_k^{(n)})$ in $\mathcal{M}$ such that*

$$\|[e_i^{(n)}, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}, i = 1, \ldots, k), \tag{4.3.60}$$

$$\|(e_i^{(n)})^2 - e_i\|_{2,u} \to 0 \qquad (i = 1, \ldots, k), \tag{4.3.61}$$

$$\left\| \sum_{i=1}^{k} (e_i^{(n)})^2 - 1 \right\|_{2,u} \to 0 \tag{4.3.62}$$

as $n \to \infty$, and, for any $a_1, \ldots, a_k \in \mathcal{M}$,

$$\left\| \sum_{i=1}^{k} (e_i^{(n)} a_i e_i^{(n)})(x) \right\|_{2,\tau_{\mathcal{M}_x}}^2 \to \sum_{i=1}^{k} g_i(x) \|a_i(x)\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.3.63}$$

uniformly over $x \in X$ as $n \to \infty$.

Proof. Let $h_i = \sum_{j=1}^{i} g_j$ for $i = 1, \ldots, k$ and set $h_0 = 0$. Let $p_i \in C_\sigma(X, L^\infty[0,1])$ be given by $x \mapsto \chi_{[h_{i-1}(x), h_i(x)]}$ for $i = 1, \ldots, k$, where we write χ_S for the indicator function of the set S. By construction $p_1, \ldots, p_k$ are mutually orthogonal projections in $C_\sigma(X, L^\infty[0,1])$ which sum to 1 and satisfy $E_\sigma(p_i) = g_i$ for $i = 1, \ldots, k$.

By Theorem 4.3.8, there exists a morphism $\varphi : C_\sigma(X, L^\infty[0,1]) \to \mathcal{M}^\omega \cap \mathcal{M}'$ such that $E_\omega(\varphi(b)a) = E_\sigma(b)E_\omega(a)$ for all $b \in C_\sigma(X, L^\infty[0,1])$ and $a \in \mathcal{M}$. In particular, we have $E_\omega(\varphi(p_i)a) = g_i E_\omega(a)$ for all $a \in \mathcal{M}$. We compute that

$$\begin{aligned} E_\omega\left(\left| \sum_{i=1}^{k} \varphi(p_i) a_i \varphi(p_i) \right|^2 \right) &= \sum_{i=1}^{k} E_\omega(\varphi(p_i) a_i^* a_i) & (4.3.64) \\ &= \sum_{i=1}^{k} g_i E_\omega(a_i^* a_i) & (4.3.65) \end{aligned}$$

for any $a_1, \ldots, a_k \in \mathcal{M}$.

Restricting φ to the span of $p_1, \ldots, p_n$, we get a *-homomorphism $\mathbb{C}^n \to \mathcal{M}^\omega$. By Proposition 2.5.11, we may take a cpc order zero lifting $\varphi' : \mathbb{C}^n \to \ell^\infty(\mathcal{M})$ and get sequences of mutually orthogonal positive contractions $(e_1^{(n)}), \ldots, (e_k^{(n)}) \in \ell^\infty(\mathcal{M})$ representing $\varphi(p_1), \dots, \varphi(p_n) \in \mathcal{M}^\omega$. By the definition of the ultrapower, we have

$$\|[e_i^{(n)}, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}, i = 1, \dots, k), \tag{4.3.66}$$

$$\|(e_i^{(n)})^2 - e_i(n)\|_{2,u} \to 0 \qquad (i = 1, \dots, k), \tag{4.3.67}$$

$$\left\| \sum_{i=1}^k e_i^{(n)} - 1 \right\|_{2,u} \to 0. \tag{4.3.68}$$

as $n \to \omega$, and, for any $a_1, \dots, a_k \in \mathcal{M}$,

$$\left\| \sum_{i=1}^k (e_i^{(n)} a_i e_i^{(n)})(x) \right\|_{2,\tau_{\mathcal{M}_x}}^2 \to \sum_{i=1}^k g_i(x) \|a_i(x)\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.3.69}$$

uniformly for $x \in X$ as $n \to \omega$. At the expense of passing to a subsequence, we can replace the limits along the ultrafilter ω with limits as $n \to \infty$. □

4.4 The McDuff Property for W*-Bundles

McDuff was able to prove the existence of uncountably many isomorphism classes of II_1 factors with separable predual by a detailed study of their central sequences [55]. In the course of her work, she showed that II_1 factors with separable predual that absorb the hyperfinite II_1 factor tensorially can be identified by their central sequence algebras [56]. Indeed, a II_1 factor M with separable predual satisfies $M \overline{\otimes} \mathcal{R} \cong M$ if and only if the matrix algebra $\mathbb{M}_k(\mathbb{C})$ embeds unitally in $M^\omega \cap M'$ for all $k \in \mathbb{N}$.

Condition (iii) of Ozawa's Triviality Theorem, can therefore be viewed as a W*-bundle analogue of the McDuff property. This observation was made in [5, Section 3], where the McDuff property for W*-bundles was first defined. The many equivalent definitions of the McDuff property in the setting of II_1 factors carry over to the setting of W*-bundles. A proof of the following proposition can be found in [5].

Proposition 4.4.1. *[5, Proposition 3.11] Let $\mathcal{M}$ be a strictly separable W*-bundle. Then the following are equivalent:*

(i) $\mathcal{M} \overline{\otimes} \mathcal{R} \cong \mathcal{M}$ as W-bundles.*

(ii) $\mathcal{R}$ embeds unitally into $\mathcal{M}^\omega \cap \mathcal{M}'$.

(iii) $\mathbb{M}_k(\mathbb{C})$ embeds unitally into $\mathcal{M}^\omega \cap \mathcal{M}'$ for some $k \geq 2$

(iii') $\mathbb{M}_k(\mathbb{C})$ embeds unitally into $\mathcal{M}^\omega \cap \mathcal{M}'$ for all $k \geq 2$.

(iv) There exists $k \geq 2$ such that, for any $\|\cdot\|_{2,u}$-separable subalgebra $S \subseteq \mathcal{M}^\omega$, there exists a unital embedding $\mathbb{M}_k(\mathbb{C}) \to \mathcal{M}^\omega \cap S'$.

(iv') For all $k \geq 2$ and for any $\|\cdot\|_{2,u}$-separable subalgebra $S \subseteq \mathcal{M}^\omega$, there exists a unital embedding $\mathbb{M}_k(\mathbb{C}) \to \mathcal{M}^\omega \cap S'$.

Definition 4.4.2. [5, Definition 3.12] We call a strictly separable W*-bundle *McDuff* if it satisfies any of the equivalent conditions of Proposition 4.4.1 .

Remark 4.4.3. Condition (iii') of Proposition 4.4.1 is an ultrapower formulation of condition (iii) of Ozawa's Triviality Theorem (Theorem 4.1.1). A sequence of cpc maps $\varphi_n : \mathbb{M}_k(\mathbb{C}) \to \mathcal{M}$ satisfying condition (iii) of Ozawa's Triviality Theorem induce a unital *-homomorphism $\varphi : \mathbb{M}_k(\mathbb{C}) \to \mathcal{M}^\omega \cap \mathcal{M}'$. Conversely, applying the Choi–Effros Lifting Theorem to a unital *-homomorphism $\varphi : \mathbb{M}_k(\mathbb{C}) \to \mathcal{M}^\omega \cap \mathcal{M}'$, one gets a sequence of cpc maps $\varphi_n : \mathbb{M}_k(\mathbb{C}) \to \mathcal{M}$ satisfying condition (iii) of Ozawa's Triviality Theorem as $n \to \omega$. At the expense of passing to a subsequence, we can replace limits as $n \to \omega$ with limits as $n \to \infty$.[7]

We now prove that the McDuff property implies property Γ. In particular, this proves that (iii) $\Rightarrow$ (ii) in Ozawa's Triviality Theorem.

Proposition 4.4.4. *Let $\mathcal{M}$ be a strictly separable, McDuff W*-bundle. Then $\mathcal{M}$ has property Γ.*

Proof. By Proposition 4.4.1, there is a untial *-homomorphism $\mathbb{M}_2(\mathbb{C}) \to \mathcal{M}^\omega \cap S'$ for any $\|\cdot\|_{2,u}$ separable subalgebra $S \subseteq \mathcal{M}^\omega$. Since $\mathbb{M}_2(\mathbb{C})$ has a unique trace, this will be a morphism of W*-bundles. Restricting to the diagonal, gives a morphism of W*-bundles $(\mathbb{C}^2, \tau_{1/2}) \to \mathcal{M}^\omega \cap S'$. Therefore, $\mathcal{M}$ has property Γ by Proposition 4.3.7. □

In [5], the McDuff property is used to facilitate orthogonal partition of unity arguments and thereby deduce properties of the W*-bundle from properties of the fibres (see [5, Lemma 3.16]). In this thesis, we have taken an alternative approach, using Corollary 4.3.9, which assumes property Γ and factorial fibres, as the basis for orthogonal partition of unity arguments.

Using such an orthogonal partition of unity argument, we prove that property Γ together with all fibres being McDuff II_1 factors implies that the W*-bundle is McDuff. In particular, we prove that (ii) $\Rightarrow$ (iii) in Ozawa's Triviality Theorem.

[7] See the proof of Proposition 4.3.7 for an explicit example of replacing limits as $n \to \omega$ with limits as $n \to \infty$.

Proposition 4.4.5. *Let $\mathcal{M}$ be a strictly separable W^*-bundle over the compact Hausdorff space X with conditional expectation E. Suppose that $\mathcal{M}$ has property Γ and that $\mathcal{M}_x$ is a McDuff factor for each $x \in X$. Then $\mathcal{M}$ is McDuff.*

Proof. Let $\mathcal{F}_0$ be a finite set of contractions in $\mathcal{M}$ and $\mathcal{G}_0$ be a finite set of contractions in $\mathbb{M}_2(\mathbb{C})$. Let $\epsilon > 0$.

For the moment fix $x \in X$. Since $\mathcal{M}_x$ is McDuff, there exists a cpc map $\varphi_x : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}_x$ such that

$$\|[\varphi_x(b), a(x)]\|_{2,\tau_x} < \epsilon \qquad (a \in \mathcal{F}_0, b \in \mathcal{G}_0), \tag{4.4.1}$$

$$\|\varphi_x(b_1 b_2) - \varphi_x(b_1)\varphi_x(b_2)\|_{2,\tau_x} < \epsilon \qquad (b_1, b_2 \in \mathcal{G}_0), \tag{4.4.2}$$

$$\|\varphi_x(1) - 1\|_{2,\tau_x} < \epsilon. \tag{4.4.3}$$

By the Choi-Effros Lifting Theorem, we can lift φ_x to a cpc map $\Phi_x : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}$. By Proposition 3.2.6, there exists an open neighbourhood $V^{(x)}$ of x such that

$$\|[\Phi_x(b)(y), a(y)]\|_{2,\tau_y} < \epsilon \qquad (y \in V^{(x)}, a \in \mathcal{F}_0, b \in \mathcal{G}_0), \tag{4.4.4}$$

$$\|\Phi_x(b_1 b_2)(y) - \Phi_x(b_1)(y)\Phi_x(b_2)(y)\|_{2,\tau_y} < \epsilon \qquad (y \in V^{(x)}, b_1, b_2 \in \mathcal{G}_0), \tag{4.4.5}$$

$$\|\Phi_x(1)(y) - 1_y\|_{2,\tau_y} < \epsilon \qquad (y \in V^{(x)}). \tag{4.4.6}$$

As x varies, the collection of all $V^{(x)}$ forms an open cover of X. By compactness, there is a finite subcover. Denote the finite subcover $V_1, \ldots, V_k$ and let $\Phi_1, \ldots, \Phi_k$ be the corresponding cpc maps $\mathbb{M}_2(\mathbb{C}) \to \mathcal{M}$. Let $g_1, \ldots, g_k : X \to [0,1]$ be a continuous partition of unity subordinate to $V_1, \ldots, V_k$. By Corollary 4.3.9, there are sequences of mutually orthogonal contractions $(e_1^{(n)}), \ldots, (e_k^{(n)})$ in $\mathcal{M}$ such that

$$\|[e_i^{(n)}, a]\|_{2,u} \to 0 \qquad (a \in \mathcal{M}, i = 1, \ldots, k), \tag{4.4.7}$$

$$\|(e_i^{(n)})^2 - e_i\|_{2,u} \to 0 \qquad (i = 1, \ldots, k), \tag{4.4.8}$$

$$\left\|\sum_{i=1}^{k} (e_i^{(n)})^2 - 1\right\|_{2,u} \to 0 \tag{4.4.9}$$

as $n \to \infty$, and, for any $a_1, \ldots, a_k \in \mathcal{M}$,

$$\left\|\sum_{i=1}^{k} (e_i^{(n)} a_i e_i^{(n)})(x)\right\|_{2,\tau_{\mathcal{M}_x}}^2 \to \sum_{i=1}^{k} g_i(x)\|a_i(x)\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.4.10}$$

uniformly over $x \in X$ as $n \to \infty$.

Define $\Phi^{(n)} : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}$ by $b \mapsto \sum_{i=1}^k e_i^{(n)}\Phi_i(b)e_i^{(n)}$. As each Φ_i is cpc and $e_1^{(n)}, \ldots, e_k^{(n)}$ are orthogonal, $\Phi^{(n)}$ is cpc for all $n \in \mathbb{N}$. We claim that

$$\limsup_{n\to\infty} \|[\Phi^{(n)}(b), a]\|_{2,u} < \epsilon \qquad (a \in \mathcal{F}_0, b \in \mathcal{G}_0), \tag{4.4.11}$$

$$\limsup_{n\to\infty} \|\Phi^{(n)}(b_1b_2) - \Phi^{(n)}(b_1)\Phi^{(n)}(b_2)\|_{2,u} < \epsilon \qquad (b_1, b_2 \in \mathcal{G}_0), \tag{4.4.12}$$

$$\limsup_{n\to\infty} \|\Phi^{(n)}(1) - 1\|_{2,u} < \epsilon. \tag{4.4.13}$$

Each of these estimates will now be justified in turn. We begin with (4.4.11). Let $a \in \mathcal{F}_0$ and $b \in \mathcal{G}_0$. We have

$$\begin{aligned}
\limsup_{n\to\infty} \|[\Phi^{(n)}(b), a]\|_{2,u}^2 &= \limsup_{n\to\infty} \left\| \sum_{i=1}^k [e_i^{(n)}\Phi_i(b)e_i^{(n)}, a] \right\|_{2,u}^2 && (4.4.14)\\
&\overset{(4.4.7)}{=} \limsup_{n\to\infty} \left\| \sum_{i=1}^k e_i^{(n)}[\Phi_i(b), a]e_i^{(n)} \right\|_{2,u}^2 && (4.4.15)\\
&\overset{(3.2.8)}{=} \limsup_{n\to\infty} \sup_{x\in X} \left\| \sum_{i=1}^k (e_i^{(n)}[\Phi_i(b), a]e_i^{(n)})(x) \right\|_{2,\tau_{\mathcal{M}_x}}^2 && (4.4.16)\\
&\overset{(4.4.10)}{=} \sup_{x\in X} \sum_{i=1}^k g_i(x) \|([\Phi_i(b), a])(x)\|_{2,\tau_{\mathcal{M}_x}}^2 && (4.4.17)\\
&\overset{(4.4.4)}{<} \sup_{x\in X} \sum_{i=1}^k g_i(x)\epsilon^2 && (4.4.18)\\
&= \epsilon^2. && (4.4.19)
\end{aligned}$$

Next, we turn to (4.4.12). Let $b_1, b_2 \in \mathcal{G}_0$. Since $e_1^{(n)}, \ldots, e_k^{(n)}$ are orthogonal, we have

$$\begin{aligned}
\Phi^{(n)}(b_1)\Phi^{(n)}(b_2) &= \left(\sum_{i=1}^k e_i^{(n)}\Phi_i(b_1)e_i^{(n)} \right) \left(\sum_{j=1}^k e_j^{(n)}\Phi_j(b_2)e_j^{(n)} \right) && (4.4.20)\\
&= \sum_{i,j} e_i^{(n)}\Phi_i(b_1)e_i^{(n)}e_j^{(n)}\Phi_j(b_2)e_j^{(n)} && (4.4.21)\\
&= \sum_{i=1}^k e_i^{(n)}\Phi_i(b_1)(e_i^{(n)})^2\Phi_i(b_2)e_i^{(n)}. && (4.4.22)
\end{aligned}$$

Therefore, writing $\gamma^{(n)} := \|\Phi^{(n)}(b_1 b_2) - \Phi^{(n)}(b_1)\Phi^{(n)}(b_2)\|_{2,u}$, we have

$$\limsup_{n\to\infty}(\gamma^{(n)})^2 = \limsup_{n\to\infty}\left\|\sum_{i=1}^k e_i^{(n)}\Phi_i(b_1b_2)e_i^{(n)} - e_i^{(n)}\Phi_i(b_1)(e_i^{(n)})^2\Phi_i(b_2)e_i^{(n)}\right\|_{2,u}^2 \tag{4.4.23}$$

$$\overset{(4.4.7),(4.4.8)}{=} \limsup_{n\to\infty}\left\|\sum_{i=1}^k e_i^{(n)}(\Phi_i(b_1b_2) - \Phi_i(b_1)\Phi_i(b_2))e_i^{(n)}\right\|_{2,u}^2 \tag{4.4.24}$$

$$\overset{(3.2.8)}{=} \limsup_{n\to\infty}\sup_{x\in X}\left\|\sum_{i=1}^k (e_i^{(n)}(\Phi_i(b_1b_2) - \Phi_i(b_1)\Phi_i(b_2))e_i^{(n)})(x)\right\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.4.25}$$

$$\overset{(4.4.10)}{=} \sup_{x\in X}\sum_{i=1}^k g_i(x)\|(\Phi_i(b_1b_2) - \Phi_i(b_1)\Phi_i(b_2))(x)\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.4.26}$$

$$\overset{(4.4.5)}{<} \sup_{x\in X}\sum_{i=1}^k g_i(x)\epsilon^2 \tag{4.4.27}$$

$$= \epsilon^2. \tag{4.4.28}$$

Finally, we justify (4.4.13). We have

$$\limsup_{n\to\infty}\|\Phi^{(n)}(1) - 1\|_{2,u}^2 = \limsup_{n\to\infty}\left\|\sum_{i=1}^k e_i^{(n)}\Phi_i(1)e_i^{(n)} - 1\right\|_{2,u}^2 \tag{4.4.29}$$

$$\overset{(4.4.9)}{=} \limsup_{n\to\infty}\left\|\sum_{i=1}^k e_i^{(n)}(\Phi_i(1) - 1)e_i^{(n)}\right\|_{2,u}^2 \tag{4.4.30}$$

$$\overset{(3.2.8)}{=} \limsup_{n\to\infty}\sup_{x\in X}\left\|\sum_{i=1}^k (e_i^{(n)}(\Phi_i(1) - 1)e_i^{(n)})(x)\right\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.4.31}$$

$$\overset{(4.4.10)}{=} \sup_{x\in X}\sum_{i=1}^k g_i(x)\|(\Phi_i(1) - 1)(x)\|_{2,\tau_{\mathcal{M}_x}}^2 \tag{4.4.32}$$

$$\overset{(4.4.6)}{<} \sup_{x\in X}\sum_{i=1}^k g_i(x)\epsilon^2 \tag{4.4.33}$$

$$= \epsilon^2. \tag{4.4.34}$$

Therefore, taking $\Phi = \Phi^{(n)}$ for sufficiently large n, we have

$$\|[\Phi(b), a]\|_{2,u} < \epsilon \qquad (a \in \mathcal{F}_0, b \in \mathcal{G}_0), \tag{4.4.35}$$

$$\|\Phi(b_1b_2) - \Phi(b_1)\Phi(b_2)\|_{2,u} < \epsilon \qquad (b_1, b_2 \in \mathcal{G}_0), \tag{4.4.36}$$

$$\|\Phi(1) - 1\|_{2,u} < \epsilon, \tag{4.4.37}$$

from which we deduce that $\mathcal{M}$ is McDuff. □

4.5 A Proof of Ozawa's Triviality Theorem

Now, at last, we come to the proof of Ozawa's Triviality Theorem (Theorem 4.1.1). The easier implications of this theorem, have already been dealt with. In Example 4.3.4, we observed that a trivial W*-bundle $C_\sigma(X, M)$ has property Γ whenever the tracial von Neumann algebra M has property Γ. Since $\mathcal{R}$, has property Γ, as verified in Example 4.3.3, it follows that $C_\sigma(X, \mathcal{R})$ has property Γ. This completes the proof that (i) $\Rightarrow$ (ii) in Ozawa's Triviality Theorem. The implications (ii) $\Leftrightarrow$ (iii) follow from Propositions 4.4.4 and 4.4.5, noting that $\mathcal{R}$ is McDuff.

This leaves the implication (iii) $\Rightarrow$ (i). In this section, we shall discuss Ozawa's proof of this. The proof is relatively long and technical, so we will sketch the main argument first before filling in some of the more technical details in two technical appendices at the end of the section.

A key tool in the proof is the following pseudometric.

Definition 4.5.1. Let P and Q be two copies of the hyperfinite II_1 factor $\mathcal{R}$. Suppose $a_1, \ldots, a_n \in P$ and $b_1, \ldots, b_n \in Q$. We define

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \inf_{\varphi,\psi} \max_i \|\varphi(a_i) - \psi(b_i)\|_{2,\mathrm{tr}_\mathcal{R}}, \tag{4.5.1}$$

where the infimum is taken over all isomorphisms $\varphi : P \to \mathcal{R}$ and $\psi : Q \to \mathcal{R}$.

Note, we are intentionally refraining from identifying P with Q because typically there will be no canonical choice of isomorphism $P \cong Q$. Thus, our use of the word pseduometric is a slight abuse of terminology. Nevertheless, d satisfies the properties one expects of a pseudometric. Positivity and symmetry of d follow easily from the fact that $\|\cdot\|_{2,\mathrm{tr}_\mathcal{R}}$ is a norm. The triangle inequality also holds for the d-pseudometric but we defer the proof to Technical Appendix A (Proposition 4.5.3).

Crucially, the d-pseudometric is compatible with the tracial continuity of W*-bundles (Theorem 4.5.9, Technical Appendix A) and, if $d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = 0$, then there is a trace-preserving isomorphism between the tracial von Neumann algebras $W^*\{a_1, \ldots, a_n\}$ and $W^*\{b_1, \ldots, b_n\}$ sending a_i to b_i for $i = 1, \ldots, n$. (Proposition 4.5.5, Technical Appendix A).

An orthogonal partition of unity argument, using the assumption that $\mathcal{N}$ is McDuff, gives the following lemma.

Lemma 4.5.2 (c.f. Lemma 14 in [62])**.** *Let $\mathcal{N}$ and $\mathcal{M}$ be W^*-bundles over a compact Hausdorff space X with fibres all isomorphic to $\mathcal{R}$. Suppose that $\mathcal{N}$ is McDuff. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1$ be finite subsets of the unit ball of $\mathcal{M}$, and let $\epsilon > 0$. Assume there exists a map θ_0 from $\mathcal{F}_0$ into the unit ball of $\mathcal{N}$ such that*

$$\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_0}, \{\theta_0(a)(x)\}_{a\in\mathcal{F}_0}) < \epsilon.^{8} \tag{4.5.2}$$

Then, for any $\delta > 0$, there is a map θ_1 from $\mathcal{F}_1$ into the unit ball of $\mathcal{N}$ such that

$$\max_{a\in\mathcal{F}_0} \|\theta_1(a) - \theta_0(a)\|_{2,u} < \epsilon, \tag{4.5.3}$$

$$\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_1}, \{\theta_1(a)(x)\}_{a\in\mathcal{F}_1}) < \delta. \tag{4.5.4}$$

We sketch a proof of Lemma 4.5.2 in Technical Appendix B. Beforehand, we show how Lemma 4.5.2 is used in the proof of Ozawa's Triviality Theorem.

Proof of Ozawa's Triviality Theorem 4.1.1 (iii) $\Rightarrow$ (i). Let (a_n) be a strictly dense sequence in the unit ball of $\mathcal{M}$, and (b_n) be a strictly dense sequence in the unit ball of $\mathcal{N} = C_\sigma(X,\mathcal{R})$. We shall construct, by recursion, finite sets $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subset \cdots$ of the unit ball of $\mathcal{M}$ and maps θ_n from $\mathcal{F}_n$ into the unit ball of $\mathcal{N}$ such that

$$\{a_1, \ldots, a_n\} \subseteq \mathcal{F}_n, \tag{4.5.5}$$

$$\{b_1, \ldots, b_n\} \subseteq \theta_n(\mathcal{F}_n), \tag{4.5.6}$$

$$\max_{a\in\mathcal{F}_{n-1}} \|\theta_n(a) - \theta_{n-1}(a)\|_{2,u} < 2^{-(n-1)}, \tag{4.5.7}$$

$$\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_n}, \{\theta_n(a)(x)\}_{a\in\mathcal{F}_n}) < 2^{-n} \tag{4.5.8}$$

for all $n \in \mathbb{N}$. The recursion begins with $\mathcal{F}_0 = \emptyset$. Suppose now that $\mathcal{F}_i$ and θ_i for $i \leq n-1$ have been constructed. We shall apply Lemma 4.5.2 twice in total interchanging the roles of $\mathcal{N}$ and $\mathcal{M}$, noting that both are McDuff. Firstly, we set $\mathcal{F}_n' = \mathcal{F}_{n-1} \cup \{a_n\}$ and apply Lemma 4.5.2 to get $\theta_n' : \mathcal{F}_n' \to \mathcal{N}$ such that

$$\max_{a\in\mathcal{F}_{n-1}} \|\theta_n'(a) - \theta_{n-1}(a)\|_{2,u} < 2^{-(n-1)}, \tag{4.5.9}$$

$$\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_n'}, \{\theta_n'(a)(x)\}_{a\in\mathcal{F}_n'}) < 2^{-(n+1)}. \tag{4.5.10}$$

By perturbing θ_n' if necessary, we may assume that θ_n' is injective and $\theta_n'(\mathcal{F}_n')$ doesn't contain any of $b_1 \ldots, b_n$. We now apply Lemma 4.5.2 to $\widetilde{\mathcal{F}_n} = \theta_n'(\mathcal{F}_n') \cup \{b_1, \ldots, b_n\}$ and

[8] For notational convience, we apply the d-pseudometric to finite sequences indexed by $\mathcal{F}_0$ instead of fixing an enumeration of $\mathcal{F}_0$.

$(\theta'_n)^{-1}$. The result is a map $\psi : \widetilde{\mathcal{F}_n} \to \mathcal{M}$ such that

$$\max_{a\in\mathcal{F}'_n} \|a - \psi(\theta'_n(a))\|_{2,u} < 2^{-(n+1)}, \tag{4.5.11}$$

$$\sup_{x\in X} d(\{b(x)\}_{b\in\widetilde{\mathcal{F}_n}}, \{\psi(b)(x)\}_{b\in\widetilde{\mathcal{F}_n}}) < 2^{-(n+1)}. \tag{4.5.12}$$

Finally, we set $\mathcal{F}_n = F'_n \cup \{\{\psi(b_1), \ldots, \psi(b_n)\}$, perturbing ψ slightly if necessary to ensure that this union is disjoint, and we define $\theta_n : \mathcal{F}_n \to \mathcal{N}$ to be equal to θ'_n on $\mathcal{F}'_n$ and by $\theta_n(\psi(b_i)) = b_i$ for $i = 1, \ldots, n$. The required properties (4.5.5) and (4.5.6) hold by construction. Property (4.5.7) is just (4.5.9). The last property (4.5.8) follows since

$$\begin{aligned}
\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_n}, \{\theta_n(a)(x)\}_{a\in\mathcal{F}_1}) \leq{} & \sup_{x\in X} d(\{b(x)\}_{b\in\widetilde{\mathcal{F}_n}}, \{\psi(b)(x)\}_{b\in\widetilde{\mathcal{F}_n}}) \\
& + \max_{a\in\mathcal{F}'_n} \|a - \psi(\theta'_n(a))\|_{2,u} && (4.5.13) \\
<{} & 2^{-(n+1)} + 2^{-(n+1)} && (4.5.14) \\
={} & 2^{-n}. && (4.5.15)
\end{aligned}$$

With the inductive definition of the θ_n now complete, we observe that, for any $a \in \mathcal{F}_i$ the sequence $(\theta_n(a))_{n=i}^{\infty}$ is a $\|\cdot\|_{2,u}$-Cauchy sequence in the unit ball of $\mathcal{N}$ by (4.5.7), so converges by the completeness axiom. Hence, we have a map θ from $\mathcal{D} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ into the unit ball of $\mathcal{N}$, coming from taking the pointwise limit of (θ_n).

Moreover, by taking limits in (4.5.8), for any $n \in \mathbb{N}$ and $x \in X$,

$$d(\{a(x)\}_{a\in\mathcal{F}_n}, \{\theta(a)(x)\}_{a\in\mathcal{F}_n}) = 0. \tag{4.5.16}$$

Thus, θ induces a trace-preserving isomorphism in each fibre between the von Neumann algebra generated by $\{a(x) : a \in \mathcal{F}_n\}$ and that generated by $\{\theta(a)(x) : a \in \mathcal{F}_n\}$ for each $n \in \mathbb{N}$ (see Proposition 4.5.5, Technical Appendix A). Hence, it induces a trace-preserving isomorphism in each fibre between the von Neumann algebra generated by $\{a(x) : a \in \mathcal{D}\}$ and that generated by $\{\theta(a)(x) : a \in \mathcal{D}\}$. Therefore, using Propositions 3.2.5 and 3.2.6, we see that θ extends to a $\|\cdot\|_{2,u}$-preserving, $\|\cdot\|$-preserving *-homomorphism from $^*\mathrm{alg}(\mathcal{D})$ into $\mathcal{N}$. Since $\mathcal{D} \supseteq \{a_i : i \in \mathbb{N}\}$, $\mathcal{D}$ is dense in $\mathcal{M}$, so θ extends further to *-homomorphism $\mathcal{M} \to \mathcal{N}$, which we shall also denote θ.

That θ is an injective morphism of W*-bundles follows from the fact that it induces a traces-preserving injective *-homomorphism $\mathcal{M}_x \to \mathcal{N}_x$ for each x. Surjectivity of θ follows from (4.5.6) and the density of $\theta(D) \supseteq \{b_i : i \in \mathbb{N}\}$ in $\mathcal{N}$. $\square$

Technical Appendix A

In this technical appendix, we establish some of the important properties of the d-pseudometric, and show that it is compatible with the tracial continuity of W*-bundles.

We begin by verifying the triangle inequality for the d-pseudometric.

Proposition 4.5.3. *Let P, Q, R be copies of the hyperfinite II_1 factor $\mathcal{R}$. Let $\mathbf{a} = (a_1, \dots, a_n) \in P^n$, $\mathbf{b} = (b_1, \dots, b_n) \in Q^n$ and $\mathbf{c} = (c_1, \dots, c_n) \in R^n$. Then*

$$d(\mathbf{a}, \mathbf{c}) \leq d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) \tag{4.5.17}$$

Proof. Let $\epsilon > 0$. There exist isomorphisms $\varphi : P \to \mathcal{R}, \psi_1 : Q \to \mathcal{R}$ such that

$$\max_i \|\varphi(a_i) - \psi_1(b_i)\|_{2,\mathrm{tr}_{\mathcal{R}}} \leq d(\mathbf{a}, \mathbf{b}) + \epsilon. \tag{4.5.18}$$

and there exist isomorphisms $\psi_2 : Q \to \mathcal{R}, \chi : R \to \mathcal{R}$ such that

$$\max_i \|\psi_2(b_i) - \chi(c_i)\|_{2,\mathrm{tr}_{\mathcal{R}}} \leq d(\mathbf{b}, \mathbf{c}) + \epsilon. \tag{4.5.19}$$

The map $\psi_1 \circ \psi_2^{-1} : \mathcal{R} \to \mathcal{R}$ is an automorphism, and hence is trace preserving. Therefore

$$\max_i \|\psi_1(b_i) - \psi_1(\psi_2^{-1}(\chi(c_i)))\|_{2,\mathrm{tr}_{\mathcal{R}}} \leq d(\mathbf{b}, \mathbf{c}) + \epsilon. \tag{4.5.20}$$

Hence, by the triangle inequality for the $\|\cdot\|_{2,\mathrm{tr}_{\mathcal{R}}}$-norm, we have

$$\begin{aligned}
d(\mathbf{a}, \mathbf{c}) &\leq \max_i \|\varphi(a_i) - \psi_1(\psi_2^{-1}(\chi(c_i)))\|_{2,\mathrm{tr}_{\mathcal{R}}} && (4.5.21)\\
&\leq \max_i \|\varphi(a_i) - \psi_1(b_i)\|_{2,\mathrm{tr}_{\mathcal{R}}} + \max_i \|\psi_1(b_i) - \psi_1(\psi_2^{-1}(\chi(c_i)))\|_{2,\mathrm{tr}_{\mathcal{R}}} && (4.5.22)\\
&\leq d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) + 2\epsilon. && (4.5.23)
\end{aligned}$$

As ϵ was arbitrary, this completes the proof. □

We now investigate what the vanishing of the d-pseudometric means. For this, we need the following folklore lemma.

Lemma 4.5.4. *Let (A, tr_A) be a tracial von Neumann algebra generated by $a_1, \dots, a_n$. Let (B, tr_B) be another tracial von Neumann algebra. Suppose $b_1, \dots, b_n \in B$, satisfy*

$$\mathrm{tr}_B(p(b_1, \dots, b_n)) = \mathrm{tr}_A(p(a_1, \dots, a_n)) \tag{4.5.24}$$

for all non-commutative *-polynomials p in n variables. Then there exists a trace-preserving *-homomorphism $\varphi : A \to B$ satisfying $\varphi(a_i) = b_i$ for $i = 1, \dots, n$.

Proof. Let $\mathbb{P}_n = \mathbb{C}\langle X_1, \ldots, X_n, X_1^*, \ldots, X_n^*\rangle$ be the *-algebra of non-commutative *-polynomials in n variables. Let $p_1, p_2 \in \mathbb{P}_n$. Since tr_A is a faithful trace, we have that $p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n)$ if and only if $\mathrm{tr}_A(p(a_1, \ldots, a_n)) = 0$, where $p = (p_1 - p_2)^*(p_2 - p_1)$. Applying the same reasoning to B, we see that there is a well-defined *-homomorphism $\varphi_0 : D_A \to B$, where D_A is the dense *-subalgebra generated by $a_1, \ldots, a_n$, given by $p(a_1, \ldots, a_n) \mapsto p(b_1, \ldots, b_n)$ for all $p \in \mathbb{P}_n$. Moreover, φ_0 is an isomorphism onto its image.

The map φ_0 is trace preserving by (4.5.24). We show that it also preserves the $\|\cdot\|$-norm. The key observation here is that

$$\|a\| = \sup\{\|av\|_{2,\mathrm{tr}_A} : v \in D_A, \|v\|_{2,\mathrm{tr}_A} \leq 1\} \tag{4.5.25}$$

for all $a \in D_A$. This follows by considering the standard form representation of (A, tr_A) and the density of $\widehat{D_A}$ in $L^2(A, \mathrm{tr}_A)$. Working in the tracial von Neumann subalgebra generated by $b_1, \ldots, b_n$, by same reasoning,

$$\|b\| = \sup\{\|bw\|_{2,\mathrm{tr}_B} : w \in D_B, \|w\|_{2,\mathrm{tr}_B} \leq 1\}, \tag{4.5.26}$$

for all $b \in D_B$, where D_B is the *-subalgebra generated by $b_1, \ldots, b_n$.

Since φ_0 trace preserving, it is $\|\cdot\|_2$-norm preserving. It now follows from (4.5.25) and (4.5.26) that $\|\varphi_0(a)\| = \|\varphi_0(b)\|$. Therefore, φ_0 extends uniquely to a trace preserving *-homomorphism $\varphi : A \to B$, where $\varphi(a) = \lim_\lambda \varphi_0(a_\lambda)$ for any $\|\cdot\|$-bounded net converging to a in $\|\cdot\|_2$. □

We now show that the conditions of Lemma 4.5.4 are satisfied, when the d-pseudometric vanishes.

Proposition 4.5.5. *Let P and Q be two copies of the hyperfinite II_1 factor $\mathcal{R}$. Let $a_1, \ldots, a_n \in P$ and $b_1, \ldots, b_n \in Q$. Suppose*

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = 0. \tag{4.5.27}$$

Then there exists a trace-preserving isomorphism between the tracial von Neumann algebras $W^*\{a_1, \ldots, a_n\}$ and $W^*\{b_1, \ldots, b_n\}$ sending a_i to b_i.

Proof. There exist sequences of isomorphisms $\varphi_k : P \to \mathcal{R}$ and $\psi_k : Q \to \mathcal{R}$ such that

$$\|\varphi_k(a_i) - \psi_k(b_i)\|_{2,\mathrm{tr}_{\mathcal{R}}} \to 0 \tag{4.5.28}$$

as $k \to \infty$ for $i = 1, \ldots, n$. Let p be a non-commutative *-polynomial in n variables. Since isomorphisms of II_1 factors are trace-preserving.

$$\mathrm{tr}_P(p(a_1, \ldots, a_n)) = \lim_{k\to\infty} \mathrm{tr}_{\mathcal{R}}(p(\varphi_k(a_1), \ldots, \varphi_k(a_n))) \tag{4.5.29}$$

$$= \lim_{k\to\infty} \mathrm{tr}_{\mathcal{R}}(p(\psi_k(b_1), \ldots, \psi_k(b_n))) \tag{4.5.30}$$

$$= \lim_{k\to\infty} \mathrm{tr}_Q(p(b_1, \ldots, b_n)). \tag{4.5.31}$$

The result now follows from Lemma 4.5.4. □

To prove that the d-pseudometric is compatible with the tracial continuity of a W*-bundle, the additional flexibility of working in the ultrapower $\mathcal{R}^\omega$ instead of $\mathcal{R}$ is important. Here, the following uniqueness result is crucial. This result is well known to experts and is discussed, together with a partial converse, in [40]. For completeness, we include a proof.

Theorem 4.5.6. *Let (N, tr_N) be a hyperfinite tracial von Neumann algebra with separable predual. Suppose $\varphi, \psi : N \to \mathcal{R}^\omega$ are trace preserving *-homomorphisms. Then φ and ψ are unitary equivalent in $\mathcal{R}^\omega$.*

Proof. We first consider the case where N is a finite-dimensional von Neumann algebra. Say $N = \bigoplus_{k=1}^K \mathbb{M}_{n_k}(\mathbb{C})$ and write $e_{ij}^{(k)}$ for the matrix units of the k-th summand for $k = 1, \ldots, K$ and $i, j = 1, \ldots, n_k$. Since φ and ψ are trace-preserving, $\mathrm{tr}_{\mathcal{R}^\omega}(\varphi(e_{ii}^{(k)})) = \mathrm{tr}_{\mathcal{R}^\omega}(\psi(e_{ii}^{(k)}))$ for all $k = 1, \ldots, K$ and $i = 1, \ldots, n_k$. Since $\mathcal{R}^\omega$ is a II_1 factor [81, Lemma A.4.2], $\varphi(e_{11}^{(k)})$ and $\psi(e_{11}^{(k)})$ are Murray–von Neumann equivalent in $\mathcal{R}^\omega$. Let $v_1^{(k)}$ be a partial isometry implementing this equivalence. Then $u = \sum_{k=1}^K \sum_{i=1}^{n_k} \varphi(e_{i1}^{(k)}) v_1^{(k)} \psi(e_{1i}^{(k)})$ is the required unitary.

Now suppose that N is hyperfinite. Then $N = \overline{\bigcup_{i=1}^\infty F_i}^{\|\cdot\|_2}$ for an increasing sequence of finite dimensional von Neumann algebras $1_N \in F_1 \subseteq F_2 \subseteq \cdots$. For each i, there exists a unitary u_i which implements a unitary equivalence between $\varphi|_{F_i}$ and $\psi|_{F_i}$. It follows that φ and ψ are $\|\cdot\|_2$-approximately unitary equivalent via the sequence of unitaries (u_i). A standard reindexing argument, now gives that φ and ψ are unitary equivalent in $\mathcal{R}^\omega$. □

We now show that in the definition of the d-pseudometric, we can work with trace-preserving embeddings into $\mathcal{R}^\omega$ instead of isomorphisms with $\mathcal{R}$.

Proposition 4.5.7. *Let P and Q be two copies of the hyperfinite II_1 factor $\mathcal{R}$. Suppose $a_1, \ldots, a_n \in P$ and $b_1, \ldots, b_n \in Q$. Then*

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \inf_{\varphi,\psi} \max_i \|\varphi(a_i) - \psi(b_i)\|_{2,\mathrm{tr}_{\mathcal{R}^\omega}}, \tag{4.5.32}$$

where the infimum is taken over all trace preserving embeddings $\varphi : W^*\{a_1, \ldots, a_n\} \to \mathcal{R}^\omega$ *and* $\psi : W^*\{b_1, \ldots, b_n\} \to \mathcal{R}^\omega$

Proof. For the moment, denote the right hand side of (4.5.32), by $\rho((a_1, \ldots, a_n), (b_1, \ldots, b_n))$. By composing any isomorphisms $P \cong \mathcal{R}$ and $Q \cong \mathcal{R}$ with the canonical, trace-preserving embedding $j : \mathcal{R} \to \mathcal{R}^\omega$ and restricting, we see that

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \geq \rho((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \tag{4.5.33}$$

Fix isomorphisms $\iota_P : P \to \mathcal{R}$ and $\iota_Q : Q \to \mathcal{R}$. Let $A = W^*\{a_1, \ldots, a_n\} \subseteq P$ and $B = W^*\{b_1, \ldots, b_n\} \subseteq Q$. By Connes' Theorem, A and B are hyperfinite.

By Theorem 4.5.6, any trace-preserving embeddings $\varphi : A \to \mathcal{R}^\omega$ has the form $\mathrm{Ad}(u) \circ j \circ \iota_P$, where $j : \mathcal{R} \to R^\omega$ is the canonical embedding and u is a unitary in $\mathcal{R}^\omega$, and similarly for $\psi : B \to \mathcal{R}^\omega$. Hence, we have

$$\rho((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \inf_{u,v \in U(\mathcal{R}^\omega)} \max_i \|uj(\iota_P(a_i))u^* - vj(\iota_Q(b_i))v^*\|_{2,\mathrm{tr}_{\mathcal{R}^\omega}} \tag{4.5.34}$$

$$= \inf_{u,v \in U(\mathcal{R}^\omega)} \max_i \|v^*uj(\iota_P(a_i))u^*v - j(\iota_Q(b_i))\|_{2,\mathrm{tr}_{\mathcal{R}^\omega}} \tag{4.5.35}$$

$$= \inf_{u \in U(\mathcal{R}^\omega)} \max_i \|uj(\iota_P(a_i))u^* - j(\iota_Q(b_i))\|_{2,\mathrm{tr}_{\mathcal{R}^\omega}}. \tag{4.5.36}$$

Since every unitary $u \in \mathcal{R}^\omega$ can be represented by a sequence of unitaries (u_k) in $\mathcal{R}$ [81, Theorem A.5.2], we have

$$\rho((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \inf_{u \in U(\mathcal{R})} \max_i \|u\iota_P(a_i)u^* - \iota_Q(b_i)\|_{2,\mathrm{tr}_{\mathcal{R}}} \tag{4.5.37}$$

$$\leq d((a_1, \ldots, a_n), (b_1, \ldots, b_n)). \tag{4.5.38}$$

Therefore, $d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \rho((a_1, \ldots, a_n), (b_1, \ldots, b_n))$. □

We isolate the main technical step in the proof that the d-pseudometric is compatible with the tracial continuity of W*-bundles in the following proposition.

Proposition 4.5.8. *Fix* $K > 0$ *and* $a_1, \ldots, a_n \in \mathcal{R}$ *with each* $\|a_i\| \leq K$*. Let* $\epsilon > 0$*. Then there exist* $\delta > 0$ *and a finite set* $\mathcal{G}$ *of non-commutative* **-polynomials in* n *variables such that, for any* $b_1, \ldots, b_n \in \mathcal{R}$ *with each* $\|b_i\| \leq K$ *that satisfy*

$$|\mathrm{tr}_{\mathcal{R}}(p(b_1, \ldots, b_n)) - \mathrm{tr}_{\mathcal{R}}(p(a_1, \ldots, a_n))| < \delta \tag{4.5.39}$$

for all $p \in \mathcal{G}$*, there exists a unitary* $u \in \mathcal{R}$ *such that*

$$\max_i \|ua_iu^* - b_i\|_2 < \epsilon. \tag{4.5.40}$$

Proof. Let N be the von Neumann subalgebra of $\mathcal{R}$ generated by $a_1, \ldots, a_n$. By the work of Connes, hyperfiniteness passes from $\mathcal{R}$ to N. Since N is finitely generated, it has a separable predual.

For each $k \in \mathbb{N}$, let $\delta_k = \frac{1}{k}$ and $\mathcal{G}_k$ be the set of all non-commutative *-mononomials in n variables of length at most k.

Suppose the proposition doesn't hold. Then there is $\epsilon > 0$ such that for each $k \in \mathbb{N}$ one can find $b_1^{(k)}, \ldots, b_n^{(k)} \in \mathcal{R}$ with each $\|b_i^{(k)}\| \leq K$ that satisfy

$$|\mathrm{tr}_{\mathcal{R}}(p(b_1^{(k)}, \ldots, b_n^{(k)})) - \mathrm{tr}_{\mathcal{R}}(p(a_1, \ldots, a_n))| < \delta_k \tag{4.5.41}$$

for all $p \in \mathcal{G}_k$ yet, for any unitary $u \in \mathcal{R}$,

$$\|ua_iu^* - b_i^{(k)}\|_2 \geq \epsilon \tag{4.5.42}$$

for some $i \in \{1, \ldots, n\}$.

Let $b_i = [(b_i^{(k)})_{k=1}^\infty] \in \mathcal{R}^\omega$ for $i = 1, \ldots, n$. Then

$$\mathrm{tr}_{\mathcal{R}^\omega}(p(b_1, \ldots, b_n)) = \mathrm{tr}_{\mathcal{R}}(p(a_1, \ldots, a_n)) \tag{4.5.43}$$

for all non-commutative *-polynomials in n variables.

By Lemma 4.5.4, there is a trace-preserving embedding $\psi : N \to \mathcal{R}^\omega$ which satisfies $\varphi(a_i) = b_i$ for $i = 1, \ldots, n$. Let $\psi : N \to \mathcal{R}^\omega$ be the embedding coming from the inclusion $N \subseteq \mathcal{R}$ and the canonical inclusion $\mathcal{R} \to \mathcal{R}^\omega$. By Theorem 4.5.6, there exists a unitary $w \in \mathcal{R}^\omega$ such that $\varphi = \mathrm{Ad}(u) \circ \psi$. Let (u_k) be a sequence of unitaries in $\mathcal{R}$ representing w. Then, for sufficiently large k,

$$\|u_ka_iu_k^* - b_i^{(k)}\|_2 < \epsilon \tag{4.5.44}$$

for all $i \in \{1, \ldots, n\}$. This contradiction completes the proof. $\square$

Finally, we arrive that the main result of this Appendix.

Theorem 4.5.9. *Let $\mathcal{M}$ be a W^*-bundle over a compact Hausdorff space X with conditional expectation E. Suppose all the fibres of $\mathcal{M}$ are isomorphic to $\mathcal{R}$. Let $a_1, \ldots, a_n \in \mathcal{M}$. The map $x \mapsto (a_1(x), \ldots, a_n(x))$ is continuous with respect to the d-pseduometric in the following sense: for any $x_0 \in X$ and $\epsilon > 0$, there exists an open neighbourhood U of x_0 such that*

$$d((a_1(x), \ldots, a_n(x)), (a_1(x_0), \ldots, a_n(x_0))) < \epsilon \tag{4.5.45}$$

whenever $x \in U$.

Proof. Set $K = \max_i \|a_i\|$. Let $x_0 \in X$ and $\epsilon > 0$. In order to apply Proposition 4.5.8, fix an isomorphism $\mathcal{M}_{x_0} \cong \mathcal{R}$ and view $a_1(x_0), \ldots, a_n(x_0) \in \mathcal{R}$. Let $\delta > 0$ and $\mathcal{G}$ be a finite set of *-polynomials such that the conclusion of Proposition 4.5.8 holds. For each $p \in \mathcal{G}$, $E(p(a_1, \ldots, a_n))$ is continuous. Since $\mathcal{G}$ is finite, there is an open neighbourhood U of x_0 such that, for all $p \in \mathcal{G}$,

$$|\tau_x(p(a_1(x), \ldots, a_n(x)) - \tau_{x_0}(p(a_1(x_0), \ldots, a_n(x_0))| < \delta \tag{4.5.46}$$

whenever $x \in U$.

Let $x \in U$. Let $\iota : \mathcal{M}_x \to \mathcal{R}$ be an isomorphism. Then there exists a unitary $u \in \mathcal{R}$ such that

$$\max_i \|ua_i(x_0)u^* - \iota(a_i(x))\|_2 < \epsilon. \tag{4.5.47}$$

It follows that $d((a_1(x), \ldots, a_n(x)), (a_1(x_0), \ldots, a_n(x_0))) < \epsilon$. □

Corollary 4.5.10. *Let $\mathcal{M}$ be a W*-bundle over a compact Hausdorff space X with all fibres $\mathcal{M}_x$ isomorphic to $\mathcal{R}$. Let $a_1, \ldots, a_n \in \mathcal{M}$. Let $\mathcal{N}$ be a W*-bundle over a compact Hausdorff space Y with all fibres $\mathcal{N}_y$ isomorphic to $\mathcal{R}$. Let $b_1, \ldots, b_n \in \mathcal{N}$. Then the map $(x, y) \mapsto d((a_1(x), \ldots, a_n(x)), (b_1(y), \ldots, b_n(y))$ is continuous.*

Proof. For brevity, we shall use vector notation for the n-tuples, for example we write $\mathbf{a}$ for $(a_1, \ldots, a_n)$ and $\mathbf{a}(x)$ for $(a_1(x), \ldots, a_n(x))$.

Fix $(x_0, y_0) \in X \times Y$. Let $\epsilon > 0$. By Theorem 4.5.9, there are open neighbourhoods U of x and V of y such that

$$d(\mathbf{a}(x), \mathbf{a}(x_0)) < \tfrac{\epsilon}{2} \tag{4.5.48}$$

$$d(\mathbf{b}(y), \mathbf{b}(y_0)) < \tfrac{\epsilon}{2} \tag{4.5.49}$$

whenever $x \in U$ and $y \in V$. Let $(x, y) \in U \times V$. Using Proposition 4.5.3, we have

$$\begin{aligned} |d(\mathbf{a}(x), \mathbf{b}(y)) - d(\mathbf{a}(x_0), \mathbf{b}(y_0))| &< d(\mathbf{a}(x), \mathbf{a}(x_0)) + d(\mathbf{b}(y_0), \mathbf{b}(y)) && (4.5.50)\\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} && (4.5.51)\\ &= \epsilon. && \square \end{aligned}$$

Technical Appendix B

For completeness, we sketch a proof of Lemma 4.5.2 in this technical appendix. Lemma 4.5.2 is essentially the same as [62, Lemma 14] and the proof sketched here is essentially the same as occurs in that paper. Notation has be changed to reflect that of this thesis and a few additional comments have been added to explain aspects of the proof.

Sketch proof of Lemma 4.5.2. Since $\mathcal{N}$ is McDuff, there exists, for each $k \in \mathbb{N}$, a sequence of cpc maps $\varphi_{n,k} : \mathbb{M}_k(\mathbb{C}) \to \mathcal{N}$ such that

$$\lim_{n\to\infty} \|[\varphi_{n,k}(b), a]\|_{2,u} = 0 \qquad (a \in \mathcal{N}, b \in \mathbb{M}_k(\mathbb{C})), \tag{4.5.52}$$

$$\lim_{n\to\infty} \|\varphi_{n,k}(b_1b_2) - \varphi_{n,k}(b_1)\varphi_{n,k}(b_2)\|_{2,u} = 0 \qquad (b_1, b_2 \in \mathbb{M}_k(\mathbb{C}), \tag{4.5.53}$$

$$\lim_{n\to\infty} \|\varphi_{n,k}(1) - 1\|_{2,u} = 0. \tag{4.5.54}$$

In fact, since the unit ball of $\mathbb{M}_k(\mathbb{C})$ is compact, the convergence in (4.5.52) is uniform over the unit ball of $\mathbb{M}_k(\mathbb{C})$, i.e

$$\lim_{n\to\infty} \sup_{\|b\|\leq 1} \|[\varphi_{n,k}(b), a]\|_{2,u} = 0 \qquad (a \in \mathcal{N}). \tag{4.5.55}$$

Fix $x \in X$. Since $d(\{a(x)\}_{a\in\mathcal{F}_0}, \{\theta_0(a)(x)\}_{a\in\mathcal{F}_0}) < \epsilon$, there is an isomorphism $\rho_x : \mathcal{M}_x \to \mathcal{N}_x$ such that

$$\max_{a\in\mathcal{F}_0} \|\rho_x(a(x)) - \theta_0(a)(x)\|_{2,\tau_{\mathcal{N}_x}} < \epsilon. \tag{4.5.56}$$

For each $a \in \mathcal{F}_1$, let $\theta^{(x)}(a) \in \mathcal{N}$ be a norm-preserving lift of $\rho_x(a(x)) \in \mathcal{N}_x$. By Proposition 3.2.6 and Corollary 4.5.10, there is an open neighbourhood $U^{(x)}$ of x such that

$$\max_{a\in\mathcal{F}_0} \|\theta^{(x)}(a)(y) - \theta_0(a)(y)\|_{2,\tau_{\mathcal{N}_x}} < \epsilon, \tag{4.5.57}$$

$$d(\{a(y)\}_{a\in\mathcal{F}_1}, \{\theta^{(x)}(a)(y)\}_{a\in\mathcal{F}_1}) < \delta \tag{4.5.58}$$

whenever $y \in U^{(x)}$.

By compactness of X, finitely many of the $U^{(x)}$ cover X. Denote these sets $U^{(1)}, \ldots, U^{(\ell)}$ and the corresponding maps by $\theta^{(1)}, \ldots, \theta^{(\ell)}$. Let $g_1, \ldots, g_\ell : X \to [0,1]$ be a continuous partition of unity subordinate to $U^{(1)}, \ldots, U^{(\ell)}$. Set $h_0 = 0$ and $h_j = \sum_{i=1}^{j} g_i$ for $j = 1 \ldots, \ell$.

Define $p : [0,1] \to \mathbb{M}_k(\mathbb{C})$ by setting $p(i/k) = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, the diagonal matrix with i ones and $k - i$ zeros on the diagonal, for $i = 0, \ldots, k$ and interpolating linearly. Then $t \mapsto p(t)$ is continuous, $\mathrm{tr}_k(p(t)) = t$, and $\mathrm{tr}_k(p(t) - p(t)^2) \leq (4k)^{-1}$, where tr_k denotes the normalised trace on $\mathbb{M}_k(\mathbb{C})$.

For $k, n \in \mathbb{N}$ and $j = 1, \ldots, \ell$. Let $f_{k,n,j} \in \mathcal{N}$ be the element such that $f_{k,n,j}(x) = \varphi_{n,k}(p(h_j) - p(h_{j-1}))(x)$ for all $x \in X$. The existence of $f_{k,n,j}$ follows from Theorem

3.2.10. The point of the construction is that for large k and n, $f_{k,n,1}, \ldots, f_{k,n,\ell}$ are approximately orthogonal projections summing to 1 commuting with a given finite set and we have the (approximate) tracial factorisation $\tau_{\mathcal{N}_x}(f_{k,n,j}(x)a(x)) \approx \mathrm{tr}_k(f_{k,n,j}(x))\tau_{\mathcal{N}_x}(a(x)) = f_j(x)\tau_{\mathcal{N}_x}(a(x))$ for $a \in \mathcal{N}$ due to the fact that $\mathbb{M}_k(\mathbb{C})$ has a unique trace.

Set $\theta_1^{k,n}(a) = \sum_{j=1}^{\ell} f_{k,n,j}^{1/2}\theta^{(j)}(a)f_{k,n,j}^{1/2}$ for $a \in \mathcal{F}_1$. Since the set $\{\theta^{(j)}(a) : a \in \mathcal{F}_0, j = 1, \ldots, \ell\}$ is finite, we get

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\max_{a\in\mathcal{F}_0}\|\theta_1^{k,n}(a) - \theta_0(a)\|_{2,u} < \epsilon \tag{4.5.59}$$

using the asymptotic properties of the $f_{n,k,j}$.[9] The more subtle estimate is

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\sup_{x\in X} d(\{a(x)\}_{a\in\mathcal{F}_1}, \{\theta_1^{n,k}(a)(x)\}_{a\in\mathcal{F}_1}) < \delta, \tag{4.5.60}$$

which is derived step by step in [62, Lemma 14]. Hence, $\theta_1 = \theta_1^{k,n}$ will have the required properties for sufficiently large k, n. □

4.6 Applications of the Triviality Theorem

Ozawa's Triviality Theorem is an extremely powerful tool for proving that certain strictly separable W*-bundles with fibres $\mathcal{R}$ are trivial. Indeed, the reason why there are no known examples of non-trivial, strictly separable W*-bundles with fibres $\mathcal{R}$ is because Ozawa's Triviality Theorem rules out most candidates. In particular, we have the following corollaries:

Corollary 4.6.1. *[62, Corollary 16] Let $\mathcal{M}$ be a strictly separable W*-bundle over a compact Hausdorff space X. Suppose that $\mathcal{M}_x \cong \mathcal{R}$ for each $x \in X$ and X has finite covering dimension. Then $\mathcal{M}$ is trivial.*

Corollary 4.6.2. *[5, Theorem 3.15] Let A be a separable, nuclear, unital C*-algebra with a Bauer simplex of traces and no finite dimensional quotients. If additionally A tensorialy absorbs the Jiang–Su algebra $\mathcal{Z}$, then $\overline{A}^{\mathrm{st}}$ is trivial.*

We shall briefly discus these two corollaries but the main purpose of this section is to prove the triviality of $\overline{A}^{\mathrm{st}}$ for some C*-algebras not covered by these results by directly verifying that $\overline{A}^{\mathrm{st}}$ has property Γ.

[9] The estimates used here are, in fact, very similar to those used in Proposition 4.4.5.

4.6.1 Finite-Dimensional Base Spaces and Jiang–Su Stability

There are a number of approaches to defining the dimension of a topological space X. The most common are the small inductive dimension, the large inductive dimension, and the covering dimension. Whilst there are topological spaces for which these different dimension theories give different values for $\dim(X)$, they agree for all separable metrisable spaces, and give the expected value when X is, say, a CW-complex. A good reference for dimension theory is [65].

The most natural dimension theory to work with in the context of operator algebras is the covering dimension, which we define below.

Definition 4.6.3. Let X be a set. The *order* of a collection $\mathcal{U}$ of subsets of X is defined to be the largest integer n for which there exist distinct elements $U_0, U_1, \ldots, U_n \in \mathcal{U}$ such that $\bigcap_{k=0}^{n} U_k \neq \emptyset$. If there is no largest such integer, then the order of $\mathcal{U}$ is infinite.

Definition 4.6.4. The *covering dimension* of a topological space X, is the smallest integer n such that every open cover of X has an open refinement of order at most n. If no such integer exists, then the covering dimension of X is said to be infinite. Write $\dim(X)$ for the covering dimension of X.

Corollary 4.6.1 follows by combining Ozawa's Triviality Theorem with the following proposition.

Proposition 4.6.5. *Let $\mathcal{M}$ be a strictly separable W*-bundle over a compact Hausdorff space X. Suppose that $\mathcal{M}_x$ is a McDuff II$_1$ factor for each $x \in X$ and $\dim(X) < \infty$. Then $\mathcal{M}$ is McDuff.*

Proposition 4.6.5, appears in [62] as Corollary 12 with the remark that it is essentially the same as [47, Proposition 7.7]. The full details of the proof of Proposition 4.6.5 require techniques beyond the scope of this thesis. We shall give a proof just in the zero-dimensional case and discuss briefly how it generalises to the finite dimensional case.

Proof of Proposition 4.6.5 (zero-dimensional case). Let $\mathcal{F}$ be a finite subset of $\mathcal{M}$, $\mathcal{G}$ be a finite subset of $\mathbb{M}_2(\mathbb{C})$, and $\epsilon > 0$. Fix $x \in X$. Since $\mathcal{M}_x$ is McDuff, there is a ucp map $\varphi_x : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}_x$ such that

$$\|[\varphi_x(b), a(x)]\|_{2,\tau_x} < \epsilon \qquad (a \in \mathcal{F}, b \in \mathcal{G}), \tag{4.6.1}$$

$$\|\varphi_x(b_1 b_2) - \varphi_x(b_1)\varphi_x(b_2)\|_{2,\tau_x} < \epsilon \qquad (b_1, b_2 \in \mathcal{G}). \tag{4.6.2}$$

By the Choi–Effros lifting theorem, φ_x has a ucp lift $\Phi_x : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}$. By Proposition 3.2.6, there is an open neighbourhood $U^{(x)}$ of x such that

$$\|[\Phi_x(b)(y), a(y)]\|_{2,\tau_y} < \epsilon \qquad (y \in U^{(x)}, a \in \mathcal{F}, b \in \mathcal{G}), \tag{4.6.3}$$

$$\|\Phi_x(b_1 b_2)(y) - \Phi_x(b_1)(y)\Phi_x(b_2)(y)\|_{2,\tau_y} < \epsilon \qquad (y \in U^{(x)}, b_1, b_2 \in \mathcal{G}). \tag{4.6.4}$$

As x varies, the $U^{(x)}$ form an open cover of X. By compactness, there is a finite subcover $U_1, \ldots, U_k$. Because X is assumed to be zero dimensional, these sets can be assumed disjoint. Let $\Phi_1, \ldots, \Phi_k$ denote the ucp maps corresponding to the finite subcover $U_1, \ldots, U_k$. As $U_1, \ldots, U_k$ are disjoint open sets that cover X, the indicator functions χ_{U_i} form a continuous partition of unity.

Define a ucp map $\Phi : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}$ by $\Phi = \sum_{i=1}^k \chi_{U_i}\Phi_i$. Each $x \in X$ lies in exactly one U_i and $\Phi(x) = \Phi_i(x)$ for this i. Therefore, we have

$$\|[\Phi(b), a]\|_{2,u} < \epsilon \qquad (a \in \mathcal{F}, b \in \mathcal{G}), \tag{4.6.5}$$

$$\|\Phi(b_1 b_2) - \Phi(b_1)\Phi(b_2)\|_{2,u} < \epsilon \qquad (b_1, b_2 \in \mathcal{G}). \tag{4.6.6}$$

Since $\mathcal{F}, \mathcal{G}$ and ϵ where arbitrary, there is a unital *-homomorphism $\mathbb{M}_2(\mathbb{C}) \to \mathcal{M}^\omega \cap \mathcal{M}'$. Hence, $\mathcal{M}$ is McDuff. □

Remark 4.6.6. If X has finite but non-zero covering dimension, one cannot assume the open sets $U_1, \ldots, U_k$ in the proof above are disjoint. However, after refinement, they can be coloured by $n+1$ colours such that any two open sets with the same colour are disjoint. This allows one to construct $n+1$ completely positive *order zero* maps $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n)} : \mathbb{M}_2(\mathbb{C}) \to \mathcal{M}^\omega \cap \mathcal{M}'$ with commuting images and $\sum_{i=0}^n \Phi^{(i)}(1) = 1$. One can then construct a *-homomorphism $\mathbb{M}_2(\mathbb{C}) \to \mathcal{M}^\omega \cap \mathcal{M}'$ using the theory of order zero maps (see [47, Lemma 7.6]).

We now turn to Corollary 4.6.2. As mentioned in Chapter 1, the Jiang–Su algebra $\mathcal{Z}$ is a simple, separable, unital, nuclear, infinite-dimensional C*-algebra with the same K-theory and traces as the complex numbers, and we shall only use these properties of the Jiang–Su algebra in the sequel. The details of the construction of the Jiang–Su algebra are beyond the scope of this thesis (see [37]). A C*-algebra A is said to absorb the Jiang–Su algebra tensorially if $A \cong A \otimes \mathcal{Z}$. Absorbing the Jiang–Su algebra tensorially is one of the regularity properties of the Toms–Winter Conjecture (see Chapter 1).

We now proceed with the proof of Corollary 4.6.2.

Proof of Proposition 4.6.2. By Theorem 3.3.3, the fibres of $\overline{A}^{\mathrm{st}}$ are $\pi_\tau(A)''$ for $\tau \in \partial_e T(A)$. The hypotheses that A is separable, nuclear and has no finite dimensional quotients ensure that $\pi_\tau(A)''$ is an injective II_1 factor with separable predual and, hence, isomorphic to $\mathcal{R}$ by Connes' Theorem. The separability of A ensures the strict separably of $\overline{A}^{\mathrm{st}}$. Hence, Ozawa's Triviality Theorem applies.

Since the Jiang–Su algebra has a unique trace $\tau_{\mathcal{Z}}$, $\overline{\mathcal{Z}}^{\mathrm{st}}$ is a W*-bundle over a one point space and can be identified with $\pi_{\tau_{\mathcal{Z}}}(\mathcal{Z})''$. Since the Jiang–Su algebra is separable and nuclear with no finite-dimensional quotients[10], $\pi_{\tau_{\mathcal{Z}}}(\mathcal{Z})'' \cong \mathcal{R}$ by Connes' Theorem.

Using Proposition 4.2.6, we compute that

$$\overline{A}^{\mathrm{st}} \cong \overline{A \otimes \mathcal{Z}}^{\mathrm{st}} \tag{4.6.7}$$

$$\cong \overline{A}^{\mathrm{st}} \bar{\otimes} \overline{\mathcal{Z}}^{\mathrm{st}} \tag{4.6.8}$$

$$\cong \overline{A}^{\mathrm{st}} \bar{\otimes} \mathcal{R}. \tag{4.6.9}$$

Hence, $\overline{A}^{\mathrm{st}}$ is McDuff. By Ozawa's Triviality Theorem, $\overline{A}^{\mathrm{st}}$ is trivial. □

4.6.2 AH Algebras with Diagonal Connecting Maps

In this section, we show that if A is an AH algebra with particularly simple connecting maps, then there is an approximately central sequence of projections (p_n) in A such that $\tau(p_n) \to \frac{1}{2}$ uniformly for $\tau \in T(A)$. It follows that, whenever $T(A)$ is a Bauer simplex, the W*-bundle $\overline{A}^{\mathrm{st}}$ has property Γ. Ozawa's Triviality Theorem will then imply that $\overline{A}^{\mathrm{st}}$ is trivial whenever A has no finite-dimensional quotients. The class of AH algebras studied contains the class of Villadsen algebras of the first type defined in [89], which includes some non-$\mathcal{Z}$-stable C*-algebras with $\partial_e T(A)$ compact and infinite dimensional.

The allowed connecting maps for the class of AH algebras that we study are the diagonal maps, which we now define.

Definition 4.6.7. A *-homomorphism

$$\phi : C(X) \to \mathbb{M}_k(\mathbb{C}) \otimes C(Y) \tag{4.6.10}$$

is called *diagonal of multiplicity k* if it has the form

$$f \mapsto \mathrm{diag}(f \circ \lambda_1, \ldots, f \circ \lambda_k), \tag{4.6.11}$$

where $\lambda_1, \ldots, \lambda_k : Y \to X$ are continuous maps. We call the λ_i the *eigenvalue maps* of ϕ. Matrix amplifications of diagonal maps are also said to be diagonal.

[10] by virtue of being simple and infinite dimensional

We now proceed to the main proposition of this section.

Proposition 4.6.8. *Let A be the inductive limit of the an inductive system*

$$\mathbb{M}_{m_1}(\mathbb{C}) \otimes C(X_1) \xrightarrow{\phi_1} \mathbb{M}_{m_2}(\mathbb{C}) \otimes C(X_2) \xrightarrow{\phi_2} \mathbb{M}_{m_3}(\mathbb{C}) \otimes C(X_3) \xrightarrow{\phi_3} \cdots \tag{4.6.12}$$

with each ϕ_i a diagonal map and $m_i \to \infty$. Then there is a sequence of projections (p_n) in A such that, as $n \to \infty$,

$$\|[p_n, a]\| \to 0 \qquad\qquad (a \in A), \tag{4.6.13}$$

$$\sup_{\tau \in T(A)} |\tau(p_n) - \tfrac{1}{2}| \to 0. \tag{4.6.14}$$

Proof. Since the composition of diagonal maps is a diagonal map, we may refine the inductive system without loss of generality. The connecting maps are unital, so one obtains a unital homomorphism $\mathbb{M}_{m_i}(\mathbb{C}) \to \mathbb{M}_{m_{i+1}}(\mathbb{C})$ by restricting ϕ_i to $\mathbb{M}_{m_i}(\mathbb{C}) \otimes 1$ and composing with a evaluation map. Hence, $m_i | m_{i+1}$ for each $i \in \mathbb{N}$. Since additionally $m_i \to \infty$, we can pass to subsequence of (m_i), refine the inductive system to this subsequence, and assume that $k_i = \frac{m_{i+1}}{m_i} \to \infty$ as $i \to \infty$.

We now introduce some notation. Let $A_i = \mathbb{M}_{m_i}(\mathbb{C}) \otimes C(X_i)$ and write $\phi_{i,\infty} : A_i \to A$ for the canonical map into the inductive limit for each $i \in \mathbb{N}$.

Fix $i \in \mathbb{N}$. Since ϕ_i is diagonal, it has the form

$$\phi_i : C(X_i, \mathbb{M}_{m_i}(\mathbb{C})) \to \mathbb{M}_{k_i}(\mathbb{C}) \otimes C(X_{i+1}, \mathbb{M}_{m_i}(\mathbb{C})) \tag{4.6.15}$$

$$F \mapsto \mathrm{diag}(F \circ \lambda_1^{(i)}, \ldots, F \circ \lambda_{k_i}^{(i)}), \tag{4.6.16}$$

for some continuous functions $\lambda_1^{(i)}, \ldots, \lambda_{k_i}^{(i)} : X_{i+1} \to X_i$, where we have made the identification $A_{i+1} \cong \mathbb{M}_{k_i}(\mathbb{C}) \otimes C(X_{i+1}, \mathbb{M}_{m_i}(\mathbb{C}))$. Let $q_i = \mathrm{diag}(I_{m_i}, \ldots, I_{m_i}, 0_{m_i}, \ldots, 0_{m_i}) \in A_{i+1}$ where I_{m_i} has multiplicity $\lfloor \frac{k_i}{2} \rfloor$ and 0_{m_i} has multiplicity $\lceil \frac{k_i}{2} \rceil$. Set $p_i = \phi_{i+1,\infty}(q_i)$.

By construction, q_i is a projection in A_{i+1} that commutes with $\phi_i(A_i)$. Hence, p_i is a projection in A that commutes with $\phi_{i,\infty}(A_i)$. It follows that $\lim_{i\to\infty} \|[p_i, a]\| = 0$ for all $a \in \bigcup_{i=1}^{\infty} \phi_{i,\infty}(A_i)$. Since $\|p_i\| \leq 1$ for all $i \in \mathbb{N}$, a simple density argument gives that $\|[p_i, a]\| \to 0$ for all $a \in A$.

We now consider traces. Let $\tau \in T(A_i)$. Then there exists a Radon probability measure μ_τ on X_{i+1} such that

$$\tau(F) = \frac{1}{m_{i+1}} \int_{x \in X_{i+1}} \mathrm{Tr}(F(x)) \mathrm{d}\mu_\tau(x) \tag{4.6.17}$$

for all $F \in A_{i+1} \cong C(X_{i+1}, \mathbb{M}_{m_{i+1}}(\mathbb{C}))$. Therefore, $\tau(q_i) \in [\frac{1}{2} - \frac{1}{k_i}, \frac{1}{2}]$ for all $\tau \in T(A_{i+1})$. Since every trace on $T(A)$ pulls back to a trace on $C(X_{i+1}) \otimes \mathbb{M}_{m_{i+1}}(\mathbb{C})$, we have $\tau(p_i) \in [\frac{1}{2} - \frac{1}{k_i}, \frac{1}{2}]$ for all $\tau \in T(A)$. Therefore, $\sup_{\tau \in T(A)} |\tau(p_i) - \frac{1}{2}| \to 0$ as $i \to \infty$. □

Corollary 4.6.9. *Let A be as in Proposition 4.6.8. Suppose $T(A)$ is a non-empty Bauer simplex. Then $\overline{A}^{\mathrm{st}}$ is a W*-Bundle over $\partial_e T(A)$ with property Γ. Furthermore, if A has no finite dimensional quotients, then $\overline{A}^{\mathrm{st}}$ is a trivial W*-Bundle over $\partial_e T(A)$.*

Proof. It follows from the inductive limit structure, that A is separable, unital. So, under the additional assumption that $T(A)$ is Bauer, $\overline{A}^{\mathrm{st}}$ has the structure of a W*-bundle by Theorem 3.3.2.

Let $\iota : A \to \overline{A}^{\mathrm{st}}$ be the canonical map and (p_n) be the sequence of projections constructed in Proposition 4.6.8. Then $(\iota(p_n))$ is a sequence of projections in $\overline{A}^{\mathrm{st}}$. Since $\iota(A)$ is $\|\cdot\|_{2,u}$-dense in $\overline{A}^{\mathrm{st}}$, we get that $\|[\iota(p_n), a]\|_{2,u} \to 0$ for all $a \in A$, using the fact that $\|\iota(p_n)\| \leq 1$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \sup_{\tau \in T(A)} |\tau(p_n) - \frac{1}{2}| = 0$, we have that $\|E(\iota(p_n)) - \frac{1}{2}\|_{C(\partial_e T(A))} \to 0$ as $n \to \infty$.

Since A is separable, $\overline{A}^{\mathrm{st}}$ is strictly separable. So in order to apply Ozawa's Triviality Theorem to $\overline{A}^{\mathrm{st}}$, we just need to know that the fibres of $\overline{A}^{\mathrm{st}}$ are all isomorphic to $\mathcal{R}$. By Theorem 3.3.3, the fibre of $\overline{A}^{\mathrm{st}}$ at $\tau \in \partial_e T(A)$ is $\pi_\tau(A)''$. Since τ is an extreme trace, this is a finite factor. Since A is separable and nuclear, $\pi_\tau(A)''$ is injective and has separable predual. If A has no finite quotients, then $\pi_\tau(A)''$ cannot be a matrix algebra, so must be a II$_1$ factor. Therefore, $\pi_\tau(A)'' \cong \mathcal{R}$ by Connes' Theorem. □

We now apply the results obtained in this section to the Villadsen algebras of the first type.

Definition 4.6.10. [89, Definition 3.1] A *-homomorphism

$$\phi : \mathbb{M}_m(\mathbb{C}) \otimes C(X) \to \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) \otimes C(X^n) \tag{4.6.18}$$

is called a *$\mathcal{V}I$ map* if it is a diagonal map of multiplicity k whose eigenvalue maps $\lambda_1, \ldots, \lambda_k : X^n \to X$ are either coordinate projections or have range equal to a single point.

Definition 4.6.11. [89, Definition 3.2] A C*-algebra A is a *Villadsen algebra of the first type* or a *$\mathcal{V}I$ algebra* if it can be written as an inductive limit

$$A \cong \lim_{i\to\infty} (\mathbb{M}_{m_i}(\mathbb{C}) \otimes C(X^{n_i}), \phi_i), \tag{4.6.19}$$

where (n_i) and (m_i) are sequence of natural numbers, X is a compact Hausdorff space, and each ϕ_i is a $\mathcal{V}$I map. We call the inductive system $(\mathbb{M}_{m_i}(\mathbb{C})\otimes C(X^{n_i}), \phi_i)$ the *standard decomposition* of A with *seed space* X.

By virtue of their inductive limit structure, all $\mathcal{V}$I algebras are separable, nuclear, unital, stably finite with at least one trace. If the point evaluations are chosen appropriately, the resulting Villadsen algebra will be simple (see [91]). If the seed space is a CW-complex of dimension at least one, simplicity of the inductive limit algebra forces $m_i \to \infty$.

A simple $\mathcal{V}$I algebra will fail to have strict comparison, and therefore not be $\mathcal{Z}$-stable, whenever the seed space is a CW-complex of dimension at least one and, informally speaking, the connecting maps contain vastly more coordinate projections than point evaluations. More precisely (c.f. [89, Lemma 4.1]), suppose the $\mathcal{V}$I algebra has standard decomposition $(\mathbb{M}_{m_i}(\mathbb{C})\otimes C(X^{n_i}), \phi_i)$ with seed space X. Let $M_{i,j}$ be the multiplicity of the $\mathcal{V}$I map $\phi_{j-1}\circ\cdots\circ\phi_{i+1}\circ\phi_i$ and let $N_{i,j}$ be the number of distinct coordinate projections occurring as eigenvalue functions. Then A will fail to be $\mathcal{Z}$-stable whenever the seed space is a CW complex of dimension at least one and

$$\lim_{i\to\infty}\lim_{j\to\infty}\frac{N_{i,j}}{M_{i,j}} = 1. \tag{4.6.20}$$

Under the condition (4.6.20), the trace simplex of A will be the Bauer simplex with extreme boundary $\prod_{i\in\mathbb{N}} X$ (See [89, Section 8] together with the computations in [86, Theorem 4.1]).

Corollary 4.6.9, applies to the simple, non-$\mathcal{Z}$-stable $\mathcal{V}$I algebras described above, and we deduce that $\overline{A}^{\mathrm{st}}$ is a trivial bundle for these C*-algebras A, even though neither Corollary 4.6.1 nor Corollary 4.6.2 applies.

4.6.3 Non-Trivial $C(X)$-Algebras

In [34, Example 4.7], a non-trivial $C(X)$-algebra A with all fibres isomorphic to the CAR algebra $M_{2^\infty} = \bigotimes_{i\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ is constructed. The base space is $X = \prod_{i\in\mathbb{N}} S^2$ and the algebra A is constructed as an infinite tensor product of 2-homogeneous C*-algebras. In [34, Example 4.8], the construction is modified, producing a $C(X)$-algebra A that is not $\mathcal{Z}$-stable, even though all its fibres are.

It's reasonable to expect that such example in the $C(X)$-algebra setting could give rise to non-trivial W*-bundles over $X = \prod_{i\in\mathbb{N}} S^2$ with all fibres isomorphic to $\mathcal{R}$. Alas, this is not the case. As in the previous section, one can, in each case, directly construct an asymptotically central sequence of projections (p_n) in A such that $\tau(p_n) = \frac{1}{2}$ for all $n \in \mathbb{N}$ and $\tau \in T(A)$. It then follows that the W*-bundle $\overline{A}^{\mathrm{st}}$ have property Γ. Triviality of the W*-bundles then follows by Ozawa's Triviality Theorem.

We sketch the argument for [34, Example 4.7] below. The argument for the case of [34, Example 4.8] is similar.

Example 4.6.12 (c.f. Example 4.7 in [34])**.** Let $B = (e+f)M_3(C(S^2))(e+f)$ be the 2-homogeneous C*-algebra where e and f are orthogonal projections in $M_3(C(S^2))$ with e a trivial projection and f equivalent to the Bott projection. Set $A = \bigotimes_{i=1}^{\infty} B$. By [34, Lemma 1.8], A is a $C(X)$-algebra over $X = \prod_{i\in\mathbb{N}} S^2$ with all fibres isomorphic to the CAR algebra M_{2^∞}. In [34, Example 4.7], A is shown to be not isomorphic to the trivial $C(X)$-algebra $C(X, M_{2^\infty})$ by a K-theoretic computation.

Since B is a locally trivial 2-homogeneous algebra, every normalised trace τ on B has the form

$$\tau(b) = \frac{1}{2}\int_{x\in S^2} \mathrm{Tr}(b(x))\,\mathrm{d}\mu_\tau(x) \tag{4.6.21}$$

for $b \in B \subseteq M_3(C(S^2))$, where μ_τ is a Radon probability measure on S^2. So $T(B)$ is the Bauer simplex with extreme boundary S^2. By [5, Proposition 3.5], $\partial_e T(\bigotimes_{i=1}^n B) \cong \prod_{i=1}^n \partial_e T(B)$ with $(\tau_1,\ldots,\tau_n) \in \prod_{i=1}^n \partial_e T(B)$ corresponding to trace $b_1 \otimes \cdots \otimes b_n \mapsto \tau_1(b_1)\cdots\tau_n(b_b)$ on $\bigotimes_{i=1}^n B$. Thus, the canonical inclusion of $\bigotimes_{i=1}^n B$ in $\bigotimes_{i=1}^{n+1} B$ induces the canonical projection $\prod_{i=1}^{n+1} S^2 \to \prod_{i=1}^n S^2$ at the level of extreme traces. Therefore, $T(A)$ is the Bauer simplex with extreme boundary $\prod_{i=1}^{\infty} S^2$.

By (4.6.21), $\tau(e) = \tau(f) = \frac{1}{2}$ for all $\tau \in B$. Let $p_n = 1_B \otimes \cdots \otimes 1_B \otimes e \in \bigotimes_{i=1}^{n+1} B \subseteq A$, where 1_B occurs in the first n tensor factors. Then p_n is a projection in A commuting with $\bigotimes_{i=1}^n B \subseteq A$. Since $\|p_n\| = 1$ for all $n \in \mathbb{N}$, a simple density argument gives $\|[p_n, a]\| \to 0$ for all $a \in A$. If $\tau \in T(A)$ then the map $b \mapsto \tau(1_B \otimes \cdots \otimes 1_B \otimes b)$, where 1_B occurs n times, defines a trace on B, so $\tau(p_n) = \frac{1}{2}$. The argument of Corollary 4.6.9 gives that $\overline{A}^{\mathrm{st}}$ has property Γ. Therefore, A is trivial by Ozawa's Triviality Theorem.

4.7 Locally Trivial W*-Bundles

A discussion of the triviality problem for W*-bundles would not be complete without consideration of the locally trivial case, which was solved by myself and Pennig. Here, the powerful methods of algebraic topology can be brought to bear. The main result is the following.

Theorem 4.7.1. *A locally trivial W*-bundle with all fibres isomorphic to the hyperfinite II_1 factor $\mathcal{R}$ is trivial.*

In fact, the only property of the II_1 factor $\mathcal{R}$ that is needed is that its automorphism group is contractible, which was proved in [70, Theorem 4]. After a brief discussion of topologies on automorphism groups in Section 4.7.1, we turn to proving Theorem 4.7.1 in Section 4.7.2. In Section 4.7.3, we show that there are II_1 factors M for which there exist non-trivial, locally trivial W*-bundles with fibres all isomorphic to M.

The results of this section are from joint work with Ulrich Pennig and appear in our paper [23].

4.7.1 The Automorphism Group of a II_1 Factor

In this short subsection, we discuss some of the possible topologies on the automorphism group $\mathrm{Aut}(M)$ of a tracial von Neumann algebra M. We then show that for II_1 factors they coincide.

Definition 4.7.2. [31, Definition 3.4] Let M be a von Neumann algebra with a faithful, normal trace $\tau\colon M \to \mathbb{C}$. Let $\mathcal{B}_*(M)$ be the set of bounded σ-weakly continuous operators on M.

- The *u-topology* on $\mathcal{B}_*(M)$ is the topology generated by the seminorms $\|T\|^u_\varphi = \|\varphi\circ T\|$ for all $\varphi \in M_*$.
- The *p-topology* on $\mathcal{B}_*(M)$ is defined via the seminorms $\|T\|^p_{\varphi,a} = |(\varphi \circ T)(a)|$ for all $a \in M$ and $\varphi \in M_*$.
- The *pointwise 2-norm topology* on $\mathcal{B}_*(M)$ is induced by the seminorms $\|T\|^{2,\tau}_a = \tau(T(a)^*T(a))^{1/2}$ for all $a \in M$.

Lemma 4.7.3. *Let M be a II_1-factor and denote the faithful, normal trace by τ. The three topologies from Definition 4.7.2 agree on* $\mathrm{Aut}(M)$.

Proof. It was proven in [31, Corollary 3.8] that the p- and the u-topology coincide on $\mathrm{Aut}(M)$. By Proposition 2.7.7, the $\|\cdot\|_{2,\tau}$-topology agrees with the strong operator topology on bounded sets. In fact, as the involution $\|\cdot\|_{2,\tau}$-continuous, the $\|\cdot\|_{2,\tau}$-topology agrees with the strong* on bounded sets, which in turn agrees with the ultrastrong* topology on bounded sets.

Since an automorphism maps bounded subsets of M to bounded subsets, the pointwise 2-norm topology agrees with the pointwise ultrastrong* topology, which in turn agrees with the p-topology on $\mathrm{Aut}(M)$ as stated in [93, Section 1.4]. □

4.7.2 Locally Trivial W*-Bundles and Principal Bundles

Now fix a locally trivial W*-bundle $\mathcal{M}$ over X with all fibres isomorphic to a II_1 factor M. Let (B, p) be the bundle of tracial von Neumann algebras associated to $\mathcal{M}$. By the definition of local triviality (Definition 3.4.20) together with Proposition 3.6.12 and the discussion preceding it, we have the following: for any $x \in X$ there is a closed neighbourhood $Y \ni x$ such that $\mathcal{M}_Y$ is trivial. Hence, there are homeomorphisms φ and ψ such that the diagram

$$\begin{tikzcd} p^{-1}(Y) \arrow[r, "\varphi"] \arrow[d, "p"'] & Y \times M \arrow[d, "\pi"] \\ Y \arrow[r, "\psi"] & Y \end{tikzcd} \tag{4.7.1}$$

commutes, where $\pi : Y \times M \to Y$ is the projection onto the first coordinate. By replacing φ with $\varphi \circ (\psi^{-1} \times \mathrm{id}_M)$ and Y with $U = Y^\circ$, we get a commuting diagram of the form

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[d, "p"'] & U \times M \arrow[d, "\pi"] \\ U \arrow[r, "\mathrm{id}_U"] & U. \end{tikzcd} \tag{4.7.2}$$

We call such a U a trivialising neighbourhood for $B \to X$. We shall use these trivialising neighbourhoods to associate a principal $\mathrm{Aut}(M)$-bundle $P_B \to X$ to our locally trivial bundle. The following lemma will be crucial.

Lemma 4.7.4. *Let U be a topological space and let M be a II_1-factor. Consider M to be equipped with the 2-norm topology. Then there is a bijection between the continuous maps $\varphi\colon U \times M \to M$, such that $a \mapsto \varphi(x, a)$ is an automorphism of M for all $x \in U$ and the continuous maps $\hat{\varphi}\colon U \to \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is equipped with the u-topology. It is defined by $\hat{\varphi}(x) = \varphi(x, \cdot\,)$*

Proof. It is clear that the construction yields a bijection. The only issue to check is continuity. By Lemma 4.7.3, the u-topology agrees with the pointwise 2-norm topology. Suppose first that $\hat{\varphi}$ is continuous, i.e. $\hat{\varphi}(x_n)$ converges to $\hat{\varphi}(x)$ pointwise in 2-norm for every net (x_n) in U that converges to $x \in U$. Let (a_m) be a net in M converging to $a \in M$

in 2-norm. We have

$$\begin{aligned}\|\varphi(x_n, a_m) - \varphi(x, a)\|_2 &\leq \|\varphi(x_n, a_m - a)\|_2 + \|\hat{\varphi}(x_n)(a) - \hat{\varphi}(x)(a)\|_2 \\ &\leq \|a_m - a\|_2 + \|\hat{\varphi}(x_n)(a) - \hat{\varphi}(x)(a)\|_2\end{aligned}$$

where we used that an automorphism preserves the trace and is therefore isometric for the 2-norm. This proves that φ is continuous. Now suppose that φ is continuous. Let (x_n) be net in U converging to x and let $a \in M$. Then we have that $\|\hat{\varphi}(x_n)(a) - \hat{\varphi}(x)(a)\|_2 = \|\varphi(x_n, a) - \varphi(x, a)\|_2 \to 0$. Therefore, $\hat{\varphi}$ is continuous. □

We will now construct the principal $\mathrm{Aut}(M)$-bundle $P_B \to X$ associated to the locally trivial bundle of tracial von Neumann algebras (B, p). Since we do not assume that the reader is familiar with the notion of a principal G-bundles for a topological group G, we highlight the main points below. A good reference for this material is [35, Section 4].

Definition 4.7.5. Let X be a topological space and let G be a topological group. A (right) G-space P together with a continuous G-map $q\colon P \to X$ (where G acts trivially on X) is called a *principal G-bundle*, if every point $x \in X$ has a neighbourhood $U \ni x$, such that there exists a G-equivariant homeomorphism $\phi_U\colon q^{-1}(U) \to U \times G$ with $\mathrm{pr}_U \circ \phi_U = q|_{q^{-1}(U)}$.

Let (B, p) be a locally trivial bundle of tracial von Neumann algebras over X with fibre M. Consider $\mathrm{Aut}(M)$ as a topological group equipped with the u-topology. The principal $\mathrm{Aut}(M)$-bundle P_B is obtained by replacing the fibre M of B by the group $\mathrm{Aut}(M)$ while preserving the transition maps. Write $B_x = p^{-1}(x)$ for the fibre at x and $\mathrm{Iso}(M_1, M_2)$ for the set of isomorphisms between two von Neumann algebras. As a set we define

$$P_B = \coprod_{x \in X} \mathrm{Iso}(M, B_x).$$

Denote the canonical quotient map $P_B \to X$ by q. A local trivialisation $\varphi_U\colon U \times M \to p^{-1}(U)$ induces a bijection

$$\psi_U\colon U \times \mathrm{Aut}(M) \to q^{-1}(U) = P_B|_U = \coprod_{x \in U} \mathrm{Iso}(M, B_x).$$

Let $V \subseteq X$ be another subset with $U \cap V \neq \emptyset$ and such that there is a local trivialisation $\varphi_V\colon V \times M \to p^{-1}(V)$. Note that

$$\varphi_V^{-1} \circ \varphi_U\big|_{(U \cap V) \times M}\colon (U \cap V) \times M \to (U \cap V) \times M$$

is of the form $(x, a) \mapsto (x, \varphi_{UV}(a))$ for a continuous map $\varphi_{UV} \colon (U \cap V) \times M \to M$ and $\varphi_{VU}^{-1}(x, a) = \varphi_{UV}(x, a)$. We have

$$\psi_V^{-1} \circ \psi_U\big|_{(U\cap V)\times \mathrm{Aut}(M)}(x, \alpha) = (x, \hat{\varphi}_{UV}(x) \circ \alpha).$$

By Corollary 4.7.4 and the continuity of composition these maps are homeomorphisms.

Now equip P_B with the following topology: Cover X by trivialising neighbourhoods $(U_i)_{i\in I}$ for B. A set $V \subseteq P_B$ is open if and only if for every point $y \in V$ there exists an $i \in I$ and a subset $y \in V' \subseteq V \cap q^{-1}(U_i)$, such that $\psi_{U_i}^{-1}(V') \subseteq U_i \times \mathrm{Aut}(M)$ is an open neighbourhood of $\psi_{U_i}^{-1}(y)$. Since the transition maps $\psi_{U_j}^{-1} \circ \psi_{U_i} \colon (U_i \cap U_j) \times \mathrm{Aut}(M) \to (U_i \cap U_j) \times \mathrm{Aut}(M)$ are homeomorphisms, this definition is consistent. With this topology all maps $\psi_{U_i} \colon U_i \times \mathrm{Aut}(M) \to q^{-1}(U_i)$ become homeomorphisms. It is straightforward to check that this topology does not depend on the choice of trivialising cover and that $q \colon P_B \to X$ is a principal $\mathrm{Aut}(M)$-bundle.

Conversely, given a principal $\mathrm{Aut}(M)$-bundle $q \colon P \to X$ the quotient $(P \times M)/\sim$ with respect to the equivalence relation $(p \cdot \alpha, a) \sim (p, \alpha(a))$ for $\alpha \in \mathrm{Aut}(M)$ is called the associated bundle of tracial von Neumann algebras.

We shall show that these two constructions are inverse to one another. We need the following well-known fact about principal bundles.

Lemma 4.7.6. *Let X be a topological space and let G be a topological group. Let $q \colon P \to X$ be a principal G-bundle. Suppose there exists a continuous section $\sigma \colon X \to P$. Then P is isomorphic to the trivial principal G-bundle $X \times G$.*

Proof. The trivialisation of P is given by $\psi \colon X \times G \to P$ with $\psi(x, g) = \sigma(x)g$, which is clearly G-equivariant. To construct an inverse, let $P \times_q P = \{(p_1, p_2) \in P \times P \mid q(p_1) = q(p_2)\} \subseteq P \times P$ and note that the map $\kappa : P \times_q P \to G$ given by $\kappa(p_1, p_2) = g_{12}$ with $p_1 g_{12} = p_2$ is well-defined and continuous, which can be checked using the local triviality of P. The inverse of ψ is then $\phi : P \to X \times G$ where $\phi(p) = (q(p), \kappa(\sigma(q(p)), p))$. □

Remark 4.7.7. In a similar fashion, one can show that any G-equivariant map $\varphi \colon P \to P'$ between principal bundles $q \colon P \to X$ and $q' \colon P' \to X$ such that $q' \circ \varphi = q$ is in fact an isomorphism. Such a map is said to cover the identity on X.

Proposition 4.7.8. *Let M be a II_1-factor and let X be a topological space. The associated bundle construction yields a bijection between isomorphism classes of locally trivial bundles* *of tracial von Neumann algebras with fibre M over X and isomorphism classes of principal* $\mathrm{Aut}(M)$*-bundles over X.*

Proof. Let (B,p) be a locally trivial bundle of tracial von Neumann algebras and denote by P_B the corresponding principal $\mathrm{Aut}(M)$-bundle. We need to check that the bundle of tracial von Neumann algebras associated to P_B agrees with B. Consider the map $(P_B \times M)/{\sim} \to B$ given by $[r,a] \mapsto r(a)$, where $r \in \mathrm{Iso}(M, B_{q(r)})$ and $a \in M$. To see that this is a homeomorphism, it suffices to check that it is a bijective local homeomorphism. It is straightforward to see that it is bijective. Any choice of local trivialisation of B, over $U \subseteq X$ say, induces a corresponding trivialisation of P_B and we have

$$\begin{tikzcd}
(P_B \times M)/{\sim}|_U \arrow[r] \arrow[d, "\cong"'] & B|_U \arrow[d, "\cong"] \\
U \times (\mathrm{Aut}(M) \times M)/{\sim} \arrow[r, "\cong"'] & U \times M
\end{tikzcd}$$

where the inverse of the lower horizontal map is given by $(x,a) \mapsto (x, [\mathrm{id}_M, a])$.

Let P be a principal $\mathrm{Aut}(M)$-bundle. We have to check that the principal $\mathrm{Aut}(M)$-bundle P_B obtained from $B = (P \times M)/{\sim}$ agrees with P. By Remark 4.7.7 it suffices to construct a continuous $\mathrm{Aut}(M)$-equivariant map $P \to P_B$ covering the identity on X. This is defined by sending $r \in P$ to the isomorphism in $\mathrm{Iso}(M, B_{q(r)})$ that maps a to $[r,a] \in B$. Continuity is again easy to check in local trivialisations. □

Now that we have rephrased the classification of locally trivial W*-bundles in terms of principal G-bundles, we can now make use of tools from algebraic topology and sheaf theory for classifying principal G-bundles. For a general overview of such methods, see for example [35, Section 12]. For our purpose, we need only the following theorem.

Theorem 4.7.9. *Let X be a paracompact Hausdorff space and let G be a contractible topological group. Let $q\colon P \to X$ be a principal G-bundle. Then P is trivialisable.*

Proof. The assumptions about P, X and G imply that $P \to X$ has a global section by [19, Lemma 4]. Now apply Lemma 4.7.6. □

Corollary 4.7.10. *Let $\mathcal{M}$ be a locally trivial W*-bundle with all fibres isomorphic to the II_1 factor M. Assume* $\mathrm{Aut}(M)$ *is contractible with respect to the u-topology. Then $\mathcal{M}$ is trivial.*

Proof. By the results of Section 3.6.2 and Proposition 4.7.8, it suffices to show the corresponding principal $\mathrm{Aut}(M)$-bundle is trivial. This follows from Theorem 4.7.9, since by assumption $\mathrm{Aut}(M)$ is contractible. □

Corollary 4.7.10 together with Popa and Takesaki's result that $\mathrm{Aut}(\mathcal{R})$ is contractible in the u-topology [70, Theorem 4] gives Theorem 4.7.1.

4.7.3 Non-Trivial, Locally Trivial W*-Bundles

In this section, we construct examples of non-trivial, but still locally trivial, W*-bundles over the circle S^1. This construction is due to myself and Ulrich Pennig and appears in [23, Section 5].

The construction is based on the idea that the isomorphism classes of locally trivial bundles of tracial von Neumann algebras with fibre the II_1 factor M are in bijection with the homotopy classes of continuous maps $S^1 \to B\mathrm{Aut}(M)$, where $B\mathrm{Aut}(M)$ denotes the classifying space of the automorphism group. This space of homotopy classes is in turn isomorphic to the set of conjugacy classes in $\pi_0(\mathrm{Aut}(M))$. Since $\pi_0(\mathrm{Aut}(M))$ surjects onto $\pi_0(\mathrm{Out}(M))$, it suffices to find II_1 factors for which $\mathrm{Out}(M)$ is not path-connected to obtain non-trivial examples. Examples of II_1 factors with $\mathrm{Out}(M)$ isomorphic to a prescribed compact group have been constructed by Ioana, Peterson and Popa in [36] in the abelian case and by Vaes and Falguières in [24] for general compact groups.

We shall use the construction from [24]. Let G be a non-trivial finite group. As sketched at the end of [24, Section 2], there exists a minimal action of G on $\mathcal{R}$. By [24, Corollary 2.2] the group $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ acts on the fixed point algebra $\mathcal{R}^G$. Let $M = (\mathcal{R}^G \rtimes \Gamma) *_{\mathcal{R}^G} \mathcal{R}$. The natural map $G \to \mathrm{Aut}(M)$ induces an isomorphism $G \cong \mathrm{Out}(M)$ by [24, Corollary 2.2]. Since M is full, $\mathrm{Out}(M)$ is Hausdorff. Therefore, the bijection $G \to \mathrm{Out}(M)$ induced by the action is a homeomorphism. Let $\theta\colon \mathrm{Aut}(M) \to \mathrm{Out}(M) \cong G$ be induced by the quotient map and the above identification.

Fix $g \in G$ and let $\alpha \in \mathrm{Aut}(M)$ be an automorphism with $\theta(\alpha) = g$. This choice induces a group homomorphism $\mathbb{Z} \to \mathrm{Aut}(M)$, which will also be denoted by α. Let

$$B = \mathbb{R} \times_\alpha M \tag{4.7.3}$$

that is, take the product $\mathbb{R} \times M$ modulo the equivalence relation $(t+n, m) \sim (t, \alpha(n)(m))$ for all $n \in \mathbb{Z}$. Together with the canonical quotient map $B \to S^1$, this is a bundle of tracial von Neumann algebras over S^1 in the sense of Definition 3.6.1 with trivialising neighbourhoods as in (4.7.2). We can, therefore, via Theorem 3.6.11, define a locally trivial W*-bundle $\mathcal{M}$ which induces B.

Lemma 4.7.11. *Let G be a non-trivial finite group, let M be the II_1-factor with $\mathrm{Out}(M) \cong G$ constructed above, let $\alpha \in \mathrm{Aut}(M)$ and $g = \theta(\alpha)$, such that $g \neq e$. Then the W*-bundle $\mathcal{M}$ associated to the bundle of tracial von Neumann algebras B given by (4.7.3) is non-trivial.*

Proof. Let $q\colon P \to S^1$ be the principal $\mathrm{Aut}(M)$-bundle of B. Suppose for a contradiction that $\mathcal{M}$ is trivial. By the results of Section 3.6.2 and Proposition 4.7.8, P is trivialisable. Consider $Q = P \times_\theta G$ defined as the quotient of the product $P \times G$ with respect to the equivalence relation $(p \cdot \beta, g) \sim (p, \theta(\beta) \cdot g)$ for $\beta \in \mathrm{Aut}(M)$. If P is trivialisable, so is Q, but $Q \to S^1$ is a principal G-bundle over S^1 for the finite group G. By elementary covering space theory, the isomorphism classes of these are in correspondence with the conjugacy classes of G.

More precisely, the conjugacy class associated to Q can be obtained as follows. Choose a basepoint $q_0 \in Q$ and lift the quotient map $[0,1] \to S^1$ to a continuous path $\gamma\colon [0,1] \to Q$ with $\gamma(0) = q_0$. By the path lifting property such a lift exists and is unique. Since $\gamma(0)$ and $\gamma(1)$ lie in the same fibre, there is a unique $h \in G$, such that $\gamma(0) = \gamma(1) \cdot h$. The conjugacy class of $h \in G$ is independent of q_0.

The bundle Q constructed above corresponds to the class of $g \in G$, whereas the trivial bundle corresponds to the conjugacy class of the neutral element $e \in G$, which only contains e, in contradiction with $g \neq e$. Therefore $\mathcal{M}$ can not be trivial. $\square$

The above construction can easily be extended to construct non-trivial, but locally trivial, W*-bundles over more general spaces than S^1; see [23, Remark 5.2].

Chapter 5

The Theory of Sub-W*-Bundles

Vaughan Jones' discovery of the rich combinatorial structure arising from a pair of II_1 factors $N \subset M$ earned him a Fields Medal in 1990 and founded a new branch of operator algebras: subfactor theory.

Subsequently, a variety of methods for constructing subfactors were developed and investigated. Often, such constructions produce not just one subfactor but a whole parametrised family of subfactors. In this chapter, we use W*-bundles to encode a family of subfactors as a single inclusion of W*-bundles $\mathcal{N} \subset \mathcal{M}$. As we develop the theory of these sub-W*-bundles, following the path set by Jones in [38] for subfactors, we shall see that tracial continuity properties of the family of subfactors will play a crucial role.

5.1 A Primer on Subfactor Theory

Before beginning our study of sub-W*-bundles, we recall the landscape of subfactor theory as developed by Jones in [38]. We begin by defining the object of study.

Definition 5.1.1. A subfactor is a unital inclusion of II_1 factors $N \subset M$.[1]

Simple examples of subfactors arise from matrix inflations $1 \otimes N \subset \mathbb{M}_k(\mathbb{C}) \otimes N$, where N is a II_1 factor; from cross products $N \subset N \rtimes_\alpha G$, for suitable group actions $\alpha : G \curvearrowright N$; and by considering the inclusion of the group von Neumann algebras $L(H) \subset L(G)$ coming from an inclusion of ICC groups $H \subset G$. See [38, Section 2.3] for more details.

The first and most fundamental invariant of a subfactor is its index. The index is defined using the theory of Murray–von Neumann coupling constants, which goes back

[1]Type III subfactors can also be studied (see [49]) but they will not be considered here. We follow common practice and use the symbol $\subset$ for subfactors instead of $\subseteq$. The inclusion need not be strict.

to [60]. In [38, Section 2.1], Jones defines the index of a subfactor $N \subset M$ by considering the ratio of the Murray–von Neumann coupling constants $\dim_M(H)$ and $\dim_N(H)$ for a normal representation H of M and its restriction to N. He then shows that it suffices to consider the the Murray–von Neumann coupling constant for the representation of N on $L^2(M)$. We take this as our definition.

Definition 5.1.2. [38, Section 2.1] The index of a subfactor $N \subset M$ is given by

$$[M : N] = \dim_N L^2(M) \tag{5.1.1}$$

Using the basic properties of Murray–von Neumann coupling constants, it not hard to verify that $[\mathbb{M}_k(\mathbb{C}) \otimes N : 1 \otimes N] = k^2$. Moreover, we have $[N \ltimes_\alpha G : N] = |G|$, under reasonable assumptions on that action α, and $[L(G) : L(H)] = |G : H|$, the index of H in the group G. See [38, Section 2.3] for more details. The index of a subfactor can be infinite, for example $1 \otimes \mathcal{R} \subset \mathcal{R} \overline{\otimes} \mathcal{R}$, and can also be non-integral, as will see shortly.

Given a subfactor, $N \subset M$ there is a unique trace-preserving conditional expectation $E_N : M \to N$ [90, Theorem 1]. This conditional expectation extends to a self-adjoint projection $e_N \in B(L^2(M))$ [38, Section 3.1]. The von Neumann algebra generated by M together with e_N is called the *basic construction* for the subfactor $N \subset M$ and denoted $\langle M, e_N \rangle$. The key result of Jones, which makes the general study of subfactors possible, is the following.

Proposition 5.1.3. *[38, Proposition 3.1.7] Let $N \subset M$ is a subfactor of finite index, then the basic construction $\langle M, e_N \rangle$ is a II_1 factor containing M and $[\langle M, e_N \rangle : M] = [M : N]$.*

It follows that, starting with a subfactor $N \subset M$ of finite index, the basic construction can be iterated, producing a nested sequence of II_1 factors

$$N \subset M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots , \tag{5.1.2}$$

where $M_0 = M$, $M_1 = \langle M, e_N \rangle$ and $M_{i+1} = \langle M_i, e_{M_{i-1}} \rangle$ for $i \geq 1$. We call (5.1.2) the *Jones tower* of the subfactor $N \subset M$. The sequence of projections with $e_0 = e_N$ and $e_i = e_{M_{i-1}}$ for $i \geq 1$ are called the *Jones projections*. We can view the Jones projections as living in the II_1 factor $\mathcal{M}_\infty = (\bigcup_{i=1} M_i)''$.[2]

[2] Since II_1 factors have a unique trace, there is a unique trace on the union $\bigcup_{i=1}^\infty M_i$. We take the bicommutant in the GNS representation corresponding to this trace.

The Jones projections satisfy the following relations

$$e_i = e_i{}^* = e_i{}^2 \qquad (i \in \mathbb{N}_0) \tag{5.1.3}$$

$$e_i e_j = e_j e_i \qquad (i, j \in \mathbb{N}_0, |i-j| \geq 2), \tag{5.1.4}$$

$$e_i e_{i\pm 1} e_i = \beta^{-1} e_i \qquad (i \in \mathbb{N}_0), \tag{5.1.5}$$

$$\mathrm{tr}(we_n) = \beta^{-1}\mathrm{tr}(w) \qquad (n \in \mathbb{N}_0, w \in \mathrm{Alg}\{1, e_0, \ldots, e_{n-1}\}), \tag{5.1.6}$$

where $\beta = [M : N]$. These relations are sometimes refereed to as the Temperly–Lieb relations [83], although conventions differ. To avoid ambiguity, we shall use the terminology *Jones relations.*

The Jones relations together with the requirement that the trace on a II_1 factor is a positive functional lead to a restriction on the possible values for the index of a subfactor. This is the main result of [38].

Theorem 5.1.4. *[38, Theorem 4.3.1] If $N \subset M$ is a subfactor, then either $[M : N] > 4$ or $[M : N] = 4\cos^2(\frac{\pi}{n})$ for some $n \geq 3$.*

We call $\{4\cos^2(\frac{\pi}{n}) : n = 3, 4, \ldots\} \cup [4, \infty]$ the *Jones set* of allowed indices. The Jones set can be viewed as the union of a discrete part $\{4\cos^2(\frac{\pi}{n}) : n = 3, 4, \ldots\}$ and a continuous part $[4, \infty]$.

If $\beta < \infty$ is in the Jones set, then one can construct a subfactor $N \subset M$ of index β starting with the Jones relations for that β. The idea is that M is the von Neumann algebra generated by all the Jones projections $\{e_i : i \geq 0\}$ and N is the von Neumann algebra generated by all the Jones projections bar the zero-th $\{e_i : i \geq 1\}$ (see [38, Theorem Theorem 4.1.1]).

The difficult part is to show that the Jones relations can be realised in a tracial von Neumann algebra whenever β is in the Jones set. This is shown in [38, Theorem 4.3.2] by considering inclusions of finite dimensional algebras in the case $\beta < 4$. A more direct approach is set out in [30, Sections 2.8-9].

5.2 Basic Definitions

We now begin the study of sub-W*-bundles. Informally, a sub-W*-bundle over the compact Hausdorff space X will be a pair of W*-bundles $\mathcal{N} \subset \mathcal{M}$ with the same base space X sharing a common, centrally-embedded copy of $C(X)$ and with the same conditional expectation E, by which we mean that the conditional expectation $E^{(\mathcal{N})}$ of the W*-bundle $\mathcal{N}$ is the restriction to $\mathcal{N}$ of the conditional expectation $E^{(\mathcal{M})}$ of the W*-bundle $\mathcal{M}$.

In what follows, we give an a priori more general definition, that of a $C(X)$-preserving inclusion of W*-bundles, and show that it agrees with the informal definition of sub-W*-bundles given above.

Definition 5.2.1. Let $\mathcal{M}, \mathcal{N}$ be W*-bundles over a common compact Hausdorff space X. A $C(X)$*-preserving inclusion* is a morphism of W*-bundles $\iota : \mathcal{N} \to \mathcal{M}$ which extends the identity on $C(X)$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\iota"] \arrow[d, "E^{(\mathcal{N})}"'] & \mathcal{M} \arrow[d, "E^{(\mathcal{M})}"'] \\ C(X) \arrow[r, "\mathrm{id}"] & C(X), \end{tikzcd} \tag{5.2.1}$$

where $E^{(\mathcal{N})}$ and $E^{(\mathcal{M})}$ are the conditional expectations of the respective W*-bundles.

Proposition 5.2.2. *A $C(X)$-preserving inclusion is injective.*

Proof. This follows from (5.2.1) and the faithfulness of $E^{(\mathcal{N})}$. Indeed, for $a \in \mathcal{N}$,

$$\begin{aligned} \iota(a) = 0 &\Leftrightarrow \iota(a)^*\iota(a) = 0 && (5.2.2)\\ &\Leftrightarrow E^{(\mathcal{M})}(\iota(a)^*\iota(a)) = 0 && (5.2.3)\\ &\Leftrightarrow E^{(\mathcal{M})}(\iota(a^*a)) = 0 && (5.2.4)\\ &\Leftrightarrow E^{(\mathcal{N})}(a^*a) = 0 && (5.2.5)\\ &\Leftrightarrow a = 0. && (5.2.6) \end{aligned}$$

□

As a consequence of Proposition 5.2.2, given a $C(X)$-preserving inclusion $\iota : \mathcal{N} \to \mathcal{M}$, we can identify $\mathcal{N}$ with a $\|\cdot\|_{2,u}$-closed subalgebra of $\mathcal{M}$ containing $C(X)$ and view $E^{(\mathcal{N})}$ the restriction of $E^{(\mathcal{M})}$ to $\mathcal{N}$. This shows that $C(X)$-preserving inclusions and sub-W*-bundles are essentially the same thing.

The next proposition shows how a sub-W*-bundle encodes a family of inclusions of tracial von Neumann algebras parametrised by the base space.

Proposition 5.2.3. *A $C(X)$-preserving inclusion $\iota : \mathcal{N} \to \mathcal{M}$ induces a unital, trace-preserving inclusion $\iota_x : \mathcal{N}_x \to \mathcal{M}_x$ for all $x \in X$.*

Proof. Fix $x \in X$. From (5.2.1) it follows that $E^{(\mathcal{M})}(\iota(a)^*\iota(a))(x) = E^{(\mathcal{N})}(a^*a)(x)$ for all $a \in \mathcal{N}$. Hence, the inclusion $\iota : \mathcal{N} \to \mathcal{M}$ descends to a trace preserving map between the corresponding quotients $\iota_x : \mathcal{N}_x \to \mathcal{M}_x$ (see Section 3.2.1). $\square$

We end this section with some notational conventions for the rest of the chapter.

Notation and Terminology 5.2.4. Fix a compact Hausdorff space X to be the base space for all W*-bundles. We write $\mathcal{N} \subset \mathcal{M}$ to denote a sub-W*-bundle and typically write E for the conditional expectations of $\mathcal{N}$ and $\mathcal{M}$ onto $C(X)$, saving the notation $E^{(\mathcal{N})}$ and $E^{(\mathcal{M})}$ for cases where additional clarity is helpful. Since in later sections there will be a plethora of conditional expectations, we will refer to the conditional expectation E as the $C(X)$*-valued trace.* We write $\mathcal{N}_x \subset \mathcal{M}_x$ for the induced trace-preserving inclusion of the fibres.

5.3 An Existence Theorem

We have already constructed many examples of sub-W*-bundles, namely the subtrivial W*-bundles of Section 3.1. The main result of this section is a generalisation of Proposition 3.1.10 to the case where $\mathcal{M}$ is not necessarily a trivial bundle.

Proposition 5.3.1. *Let $\mathcal{M}$ be a W*-bundle over X. For each $x \in X$, let N_x be a von Neumann subalgebra of $\mathcal{M}_x$ containing the identity. Set $\mathcal{N} = \{a \in \mathcal{M} : a(x) \in N_x \text{ for all } x \in X\}$. Then $\mathcal{N} \subset \mathcal{M}$ is a sub-W*-bundle and the following are equivalent:*

(i) For all $x \in X$, $\mathcal{N}_x = N_x$.

(ii) For all $a \in \mathcal{M}$, the map $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(a(x), N_x)$ is upper-semicontinuous.

Proof. Hereinafter, we drop subscripts and write $\mathrm{dist}(\cdot,\cdot)$ instead of $\mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(\cdot,\cdot)$. The fibre $\mathcal{M}_x$ will be clear from the context.

Using the fact that passing to fibres of a W*-bundle is a *-homomorphism, it's straightforward to show that $\mathcal{N}$ is a $\|\cdot\|_{2,u}$-closed *-subalgebra of $\mathcal{M}$ containing $C(X)$, so is itself a W*-bundle when endowed with the restriction of the conditional expectation $E^{(\mathcal{M})}$.

(i) $\Rightarrow$ (ii) Let $a \in \mathcal{M}, x_0 \in X$, and $\epsilon > 0$. There exists $c \in N_{x_0}$ such that $\|a(x_0) - c\|_{2,\tau_{x_0}} < \mathrm{dist}(a(x_0), N_{x_0}) + \epsilon$. Since $\mathcal{N}_x = N_x$ for all $x \in x$, there is $b \in \mathcal{N}$ such that $b(x_0) = c$.

Since $x \mapsto \|a(x) - b(x)\|_{2,\tau_x}$ is continuous by Proposition 3.2.6, there is neighbourhood U of x_0 such that $\|a(x) - b(x)\|_{2,\tau_x} < \mathrm{dist}(a(x_0), N_{x_0}) + \epsilon$ for all $x \in U$. Since $b(x) \in N_x$

for all x, it follows that $\mathrm{dist}(a(x), N_x) < \mathrm{dist}(a(x_0), N_{x_0}) + \epsilon$ for all $x \in U$. Therefore, the map $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(a(x), N_x)$ is upper-semicontinuous.

(ii) $\Rightarrow$ (i) By definition $\mathcal{N}_x \subseteq N_x$. We need to show equality. In other words, we need to prove that for all $x_0 \in X$ and $c \in N_{x_0}$ there is some $a \in \mathcal{N}$ with $a(x_0) = c$.

We construct such an a as the limit of a sequence $(a_n) \subseteq \mathcal{M}$ with the following properties:

$$\|a_n\| \leq \|c\|, \tag{5.3.1}$$

$$a_n(x_0) = c, \tag{5.3.2}$$

$$\|a_n - a_{n-1}\|_{2,u} < \frac{1}{2^{n-1}}, \tag{5.3.3}$$

$$\sup_{x \in X} \mathrm{dist}(a_n(x), N_x) < \frac{1}{2^n}. \tag{5.3.4}$$

Assuming for now that such a sequence exists. Axiom (C) together with (5.3.1) and (5.3.3) ensure that (a_n) has a $\|\cdot\|_{2,u}$-limit $a \in \mathcal{M}$. Taking limits in (5.3.2) ensures that $a(x_0) = c$. Finally, property (5.3.4) ensures that, for each $x \in X$, $a(x)$ lies in the $\|\cdot\|_{2,\tau_x}$-closure of N_x, so $a(x) \in N_x$; hence, $a \in \mathcal{N}$.

We now construct the sequence (a_n). This is done by induction. First, we construct a_1. Let $b \in \mathcal{M}$ be any norm preserving lift of $c \in \mathcal{M}_{x_0}$. Since $x \mapsto \mathrm{dist}(b(x), N_x)$ is upper-semicontinuous, there is an open neighbourhood $U \ni x_0$ such that $\sup_{y \in U} \mathrm{dist}(b(y), N_y) < \frac{1}{2}$. Choose a continuous function $\phi : X \to [0,1]$ such that $\phi(x_0) = 1$ and $\phi(X \setminus U) \subseteq \{0\}$. Set $a_1 = \phi b$. Properties (5.3.1) and (5.3.2) are clearly satisfied, property (5.3.3) is void, and property (5.3.4) comes from considering the cases $x \in U$ and $x \in X \setminus U$ separately.

Suppose now that $a_1, \ldots, a_{n-1}$ have been constructed with the desired properties. We construct a_n. By (5.3.4), there is, for all $x \in X$, $c^{(x)} \in N_x$ such that $\|a_{n-1}(x) - c^{(x)}\|_{2,\tau_x} < \frac{1}{2^{n-1}}$. In fact we can take $c^{(x)} = E_{N_x}(a_{n-1}(x))$, where E_{N_x} is the canonical conditional expectation $\mathcal{M}_x \to N_x$. This ensures that we also have $\|c^{(x)}\| \leq \|a_{n-1}(x)\| \leq \|c\|$ and $c^{(x_0)} = c$. Let $b^{(x)} \in M$ be any norm preserving lift of $c^{(x)} \in \mathcal{M}_x$.

By Proposition 3.2.6 and the upper-semicontinuity of $y \mapsto \mathrm{dist}(b^{(x)}(y), N_y)$, there is an open neighbourhood $U^{(x)} \ni x$ such that

$$\sup_{y \in U^{(x)}} \|a_{n-1}(y) - b^{(x)}(y)\|_{2,\tau_y} < \frac{1}{2^{n-1}}, \tag{5.3.5}$$

$$\sup_{y \in U^{(x)}} \mathrm{dist}(b^{(x)}(y), N_y) < \frac{1}{2^n}. \tag{5.3.6}$$

The open cover $\{U^{(x)} : x \in X\}$ of X has a finite subcover by compactness of X. We write this subcover as $U_1, \ldots, U_m$ and the corresponding elements of $\{b^{(x)} : x \in X\}$ as

$b_1, \ldots, b_m$. We may assume that $U_1 = U^{(x_0)}$ and $b_1 = b^{(x_0)}$. Let $\phi_1, \ldots, \phi_m$ be a partition of unity subordinate to $U_1, \ldots, U_m$ with $\phi_1(x_0) = 1$. Set $a_n = \sum_{i=1}^m \phi_i b_i$. Property (5.3.1) follows since we ensured that $\|b^{(x)}\| \leq \|c\|$ for all $x \in X$ and $\phi_1, \ldots, \phi_m$ is a partition of unity. Property (5.3.2) follows by construction. Properties (5.3.3) and (5.3.4) follow from (5.3.5) and (5.3.6) respectively because $\phi_1, \ldots, \phi_m$ is a partition of unity. This completes the proof. □

Remark 5.3.2. Since $\operatorname{dist}(a_1(x), N_x) \leq \operatorname{dist}(a_2(x), N_x) + \|a_1(x) - a_2(x)\|_{2,\tau_x}$ for $a_1, a_2 \in \mathcal{M}$, it suffices to check condition (ii) in Proposition 5.3.1 for all $a \in \Gamma$, where Γ is a subset of $\mathcal{M}$ with the property that for all $x \in X$ and $b \in \mathcal{M}_x$ there is $a \in \Gamma$ such that $a(x) = b$. In particular, when $\mathcal{M} = C_\sigma(X, M)$ is a trivial bundle, we may take Γ to be the constant functions and, thereby, deduce Proposition 3.1.10.

5.4 Expected Sub-W*-Bundles

For a unital inclusion of tracial von Neumann algebras $N \subset M$, there is always a unique trace-preserving conditional expectation $E_N : M \to N$. In this section, we provide a necessary and sufficient condition on an sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ for the existence of a conditional expectation $\mathcal{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ which preserves the $C(X)$-valued trace E.

The following proposition implies that there is only one candidate for a $C(X)$-trace–preserving conditional expectation $\mathcal{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$: we must have $\mathcal{E}_\mathcal{N}(a)(x) = E_{\mathcal{N}_x}(a(x))$, where $E_{\mathcal{N}_x}$ is the canonical trace-preserving conditional expectation $\mathcal{M}_x \to \mathcal{N}_x$.

Proposition 5.4.1. *Let $\mathcal{N} \subset \mathcal{M}$ be a sub-W*-bundle over X. Suppose there is a conditional expectation $\mathcal{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ such that $E^{(\mathcal{M})} = E^{(\mathcal{N})} \circ \mathcal{E}_\mathcal{N}$. Then for each $x \in X$, there is an induced trace-preserving conditional expectation $(\mathcal{E}_\mathcal{N})_x : \mathcal{M}_x \to \mathcal{N}_x$ given by $a(x) \mapsto \mathcal{E}_\mathcal{N}(a)(x)$.*

Proof. Let $a \in \mathcal{M}$. Since $\mathcal{E}$ is a cpc map, we have the Schwarz inequality $0 \leq \mathcal{E}(a)^*\mathcal{E}(a) \leq \mathcal{E}(a^*a)$ (see Corollary 2.5.4). Suppose now that $E^{(\mathcal{M})}(a^*a)(x) = 0$ for some $x \in X$. Then

$$0 \leq E^{(\mathcal{N})}(\mathcal{E}_\mathcal{N}(a)^*\mathcal{E}_\mathcal{N}(a))(x) \leq E^{(\mathcal{N})}(\mathcal{E}_\mathcal{N}(a^*a))(x) = E^{(M)}(a^*a) = 0. \tag{5.4.1}$$

So $E^{(\mathcal{N})}(\mathcal{E}_\mathcal{N}(a)^*\mathcal{E}_\mathcal{N}(a))(x) = 0$. Hence, there is a bounded linear map between the respective quotients $(\mathcal{E}_{\mathcal{N}})_x : \mathcal{M}_x \to \mathcal{N}_x$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathcal{E}"] \arrow[d, "\mathrm{eval}_x"'] & \mathcal{N} \arrow[d, "\mathrm{eval}_x"'] \\ \mathcal{M}_x \arrow[r, "(\mathcal{E}_{\mathcal{N}})_x"] & \mathcal{N}_x, \end{tikzcd} \tag{5.4.2}$$

commutes. Basic lifting arguments show that $(\mathcal{E}_{\mathcal{N}})_x$ is a conditional expectation. Since $E^{(\mathcal{M})}(a)(x) = E^{(\mathcal{N})}(\mathcal{E}_{\mathcal{N}}(a))(x)$, we see that $(\mathcal{E}_{\mathcal{N}})_x$ is trace preserving. □

Now that we know that there is only one candidate for an E-preserving conditional expectation $\mathcal{E}_{\mathcal{N}}$, we simply have to determine when the fibrewise definition $\mathcal{E}_{\mathcal{N}}(a)(x) = E_{\mathcal{N}_x}(a(x))$ is well defined. The point here is that, for a given $a \in \mathcal{M}$, there need not be an element $\mathcal{E}_N(a) \in \mathcal{N}$ with $\mathcal{E}_N(a)(x) = E_{\mathcal{N}_x}(a(x))$.

Theorem 5.4.2. *Let $\mathcal{N} \subset \mathcal{M}$ be a sub-W*-bundle over X. The following are equivalent:*

(i) There exists a conditional expectation $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ such that $E^{(\mathcal{M})} = E^{(\mathcal{N})} \circ \mathcal{E}_{\mathcal{N}}$.

(ii) For all $a \in \mathcal{M}$, the map $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(a(x), \mathcal{N}_x)$ is continuous.

Proof. Hereinafter, we drop subscripts and write $\mathrm{dist}(\cdot,\cdot)$ instead of $\mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(\cdot,\cdot)$. The fibre $\mathcal{M}_x$ will be clear from the context.

(i) ⇒ (ii) Suppose such a conditional expectation exists. Then by Proposition 5.4.1, $\mathcal{E}_N(a)(x) = E_{\mathcal{N}_x}(a(x))$ for all $a \in \mathcal{M}$ and $x \in X$. Hence, for all $a \in \mathcal{M}$ and $x \in X$, we have

$$\mathrm{dist}(a(x), \mathcal{N}_x) = \|a(x) - E_{\mathcal{N}_x}(a(x))\|_{2,\tau_x} = \|(a - \mathcal{E}_{\mathcal{N}}(a))(x)\|_{2,\tau_x}, \tag{5.4.3}$$

which is continuous in x by Proposition 3.2.6.

(ii) ⇒ (i) This makes use of Theorem 3.2.10. Let $a \in \mathcal{M}$, $x_0 \in X$ and $\epsilon > 0$. Let $b \in \mathcal{N}$ be a norm-preserving lift of $E_{\mathcal{N}_{x_0}}(a(x_0)) \in \mathcal{N}_{x_0}$. Let $x \in X$. Since $a(x) - E_{\mathcal{N}_x}(a(x))$ and $E_{\mathcal{N}_x}(a(x)) - b(x) \in \mathcal{N}_x$ are orthogonal in $L^2(\mathcal{M}_x)$, we have

$$\begin{aligned} \|a(x) - b(x)\|_{2,\tau_x}^2 &= \|a(x) - E_{\mathcal{N}_x}(a(x))\|_{2,\tau_x}^2 + \|E_{\mathcal{N}_x}(a(x)) - b(x)\|_{2,\tau_x}^2 && (5.4.4) \\ &= \mathrm{dist}(a(x), \mathcal{N}_x)^2 + \|E_{\mathcal{N}_x}(a(x)) - b(x)\|_{2,\tau_x}^2. && (5.4.5) \end{aligned}$$

Suppose the map $x \mapsto \mathrm{dist}(a(x), \mathcal{N}_x)$ is continuous. Since $x \mapsto \|a(x) - b(x)\|_{2,\tau_x}$ is also continuous by Proposition 3.2.6, it follows from (5.4.5) that the map $x \mapsto \|E_{\mathcal{N}_x}(a(x)) - b(x)\|_{2,\tau_x}$ is continuous. Hence, there exists an open neighbourhood U of x_0 such that

$$\|E_{\mathcal{N}_x}(a(x)) - b(x)\|_{2,\tau_x} < \epsilon \tag{5.4.6}$$

for all $x \in U$. By Theorem 3.2.10, there exists $\mathcal{E}_{\mathcal{N}}(a) \in \mathcal{N}$ such that $\mathcal{E}_{\mathcal{N}}(a)(x) = E_{\mathcal{N}_x}(a(x))$ for all $x \in X$. That the map $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, thus defined, is a conditional expectation satisfying $E^{(\mathcal{M})} = E^{(\mathcal{N})} \circ \mathcal{E}_{\mathcal{N}}$ follows from the fact that for each $x \in X$, $E_{\mathcal{N}_x}$ is a trace-preserving conditional expectation $\mathcal{M}_x \to \mathcal{N}_x$ together with Propositions 3.2.5 and 3.2.6.

□

It is noteworthy that condition (ii) of Theorem 5.4.2 is missing half of continuity from condition (ii) of Proposition 5.3.1. Indeed, in order to define a sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ by specifying a family of $\{N_x\}_{x \in X}$ of von Neumann subalgebras, one only needs upper-semicontinuity of $x \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau_x}}(a(x), N_x)$ for all $a \in \mathcal{M}$. For the bundle to have a conditional expectation $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ such that $E^{(\mathcal{M})} = E^{(\mathcal{N})} \circ \mathcal{E}_{\mathcal{N}}$, however, lower-semicontinuity is also needed.

In light of this, it is easy to give an example of a sub-W*-bundle with no E-preserving conditional expectation.

Example 5.4.3. Let M be a tracial von Neumann algebra and N a von Neumann subalgebra containing 1_M. We can define a sub-W*-bundle $\mathcal{N}$ of the trivial bundle $\mathcal{M} = C_\sigma([0,1], M)$ with $\mathcal{N}_x = M$ for $x \in [0,1)$ and $\mathcal{N}_1 = N$. Indeed, we have

$$\mathrm{dist}(a(x), \mathcal{N}_x) = \begin{cases} 0 & x \neq 1 \\ \mathrm{dist}(a(1), N) & x = 1 \end{cases} \tag{5.4.7}$$

for $a \in \mathcal{M}$, which is upper semicontinuous. When $N \neq M$, there will not be a conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ since, choosing $a \in \mathcal{M}$ with $a(1) \notin N$, $x \mapsto \mathrm{dist}(a(x), \mathcal{N}_x)$ is not continuous.

The existence of an E-preserving conditional expectation is vital for the further development of a Jones theory for sub-W*-bundles. We therefore make the following definition.

Definition 5.4.4. An *expected sub-W*-bundle* is a sub-W*-bundle for which an E-preserving conditional expectation exists. We write $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ for an expected sub-W*-bundle.

5.5 Standard Form and the Basic Construction

In this section, we mimic Jones' basic construction from [38, Section 3] for an expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ (over a base space X which is compact Hausdorff and fixed for this section).

Jones' basic construction relies on the standard form for a tracial von Neumann algebra; hence, we shall be using the results of Sections 3.2.2 and 3.5. In order to cut down on excessive notation, we make the following conventions. If $\mathcal{M}$ is a W*-bundle over X, then we represent $\mathcal{M}$ on the Hilbert-$C(X)$-module $L^2(\mathcal{M})$ via left multiplication (see Definition 3.2.11) and identify $\mathcal{M}$ with its image under this representation. We obtain induced representations of the fibres $\mathcal{M}_x$ on $L^2(\mathcal{M}_x)$ (see Remark 3.5.2), which we also view as identifications. By Proposition 3.5.4, the fibration of $\mathcal{M}$ is consistent with that of $L^2(\mathcal{M})$ and $\mathcal{L}(L^2(\mathcal{M}))$. We use the notation of Section 3.2.2, writing $\widehat{b}$ for the image of $b \in \mathcal{M}$ in $L^2(\mathcal{M})$ and $\widehat{\mathcal{M}}$ for the image of $\mathcal{M}$ in $L^2(\mathcal{M})$.

Now suppose $\mathcal{N} \subset \mathcal{M}$ is sub-W*-bundle. Since $E^{(\mathcal{M})}$ extends $E^{(\mathcal{N})}$, we simply write E for the $C(X)$-valued trace. Since the inclusion $\mathcal{N} \subset \mathcal{M}$ is $\|\cdot\|_{2,u}$ preserving, it induces an inclusion of Hilbert-$C(X)$-modules $L^2(\mathcal{N}) \subset L^2(\mathcal{M})$. We write $\langle \cdot, \cdot \rangle$ for the $C(X)$–inner product on $L^2(\mathcal{N})$ and its extension to $L^2(\mathcal{M})$.

The first thing to do is to show that the conditional expectation $\mathcal{E}_{\mathcal{N}}$ of an expected sub-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ defines a self-adjoint projection on $L^2(\mathcal{M})$.

Proposition 5.5.1. *Let $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ be an expected sub-bundle. The conditional expectation $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ extends to a self-adjoint projection $e_{\mathcal{N}} : L^2(\mathcal{M}) \to L^2(\mathcal{N}) \subset L^2(\mathcal{M})$.*

Proof. Since $\mathcal{E}_{\mathcal{N}}$ is a cpc map, we have the Schwarz inequality $0 \leq \mathcal{E}_{\mathcal{N}}(a)^*\mathcal{E}_{\mathcal{N}}(a) \leq \mathcal{E}_{\mathcal{N}}(a^*a)$ for all $a \in \mathcal{M}$ (see Corollary 2.5.4). It follows that $\|\mathcal{E}_{\mathcal{N}}(a)\|_{2,u} \leq \|a\|_{2,u}$ for all $a \in \mathcal{M}$. Hence, the densely defined operator $e_{\mathcal{N}} : \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ given by $\widehat{a} \mapsto \widehat{\mathcal{E}_{\mathcal{N}}(a)}$ is bounded and, therefore, extends to a bounded operator $L^2(\mathcal{M}) \to L^2(\mathcal{N}) \subset L^2(\mathcal{M})$. Since $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation, we have

$$
\begin{aligned}
\langle e_{\mathcal{N}}\widehat{a}, \widehat{b} \rangle &= E(\mathcal{E}_{\mathcal{N}}(a)b^*) && (5.5.1)\\
&= E(\mathcal{E}_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}(a)b^*)) && (5.5.2)\\
&= E(\mathcal{E}_{\mathcal{N}}(a)\mathcal{E}_{\mathcal{N}}(b)^*) && (5.5.3)\\
&= E(\mathcal{E}_{\mathcal{N}}(a\mathcal{E}_{\mathcal{N}}(b)^*)) && (5.5.4)\\
&= E(a\mathcal{E}_{\mathcal{N}}(b)^*) && (5.5.5)\\
&= \langle \widehat{a}, e_{\mathcal{N}}\widehat{b} \rangle, && (5.5.6)
\end{aligned}
$$

where $a, b \in \mathcal{M}$. So, by density, $e_{\mathcal{N}}$ is an adjointable operator on $L^2(\mathcal{M})$ with $e_{\mathcal{N}}^* = e_{\mathcal{N}}$. As $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation, $\mathcal{E}_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}(a)) = a$. By density, it follows that $e_{\mathcal{N}}^2 = e_{\mathcal{N}}$. Hence, $e_{\mathcal{N}}$ is a self-adjoint projection. ∎

Inspired by Jones, we make the following definition.

Definition 5.5.2. Let $\mathcal{N} \subset_{\mathcal{E}_\mathcal{N}} \mathcal{M}$ be an expected sub-bundle. The *basic construction* is the strictly closed *-subalgebra $\mathcal{M}_1$ of $\mathcal{L}(L^2(\mathcal{M}))$ generated by $\mathcal{M}$ and $e_\mathcal{N}$ of Proposition 5.5.1. We write $\mathcal{M}_1 = \langle \mathcal{M}, e_\mathcal{N} \rangle$.

Some of the algebraic features of Jones' basic construction from [38, Section 3.1] carry over to the new setting with the same proof. We record them in the following proposition.

Proposition 5.5.3. *Let $\mathcal{N} \subset_{\mathcal{E}_\mathcal{N}} \mathcal{M}$ be an expected sub-W*-bundle and $\mathcal{M}_1 = \langle \mathcal{M}, e_\mathcal{N} \rangle$ be the basic construction.*

(i) Let $a \in N$. Then $e_\mathcal{N} a e_\mathcal{N} = \mathcal{E}_\mathcal{N}(a) e_\mathcal{N}$.

(ii) Let $a \in \mathcal{M}$. Then $a \in \mathcal{N}$ if and only if $e_\mathcal{N} a = a e_\mathcal{N}$.

(iii) Let J be the involution on $L^2(\mathcal{M})$. Then $J e_\mathcal{N} = e_\mathcal{N} J$.

(iv) The set $\{a + \sum_{\text{finite}} a_i e_\mathcal{N} b_i : a, a_i, b_i \in \mathcal{M}\}$ is a strictly dense subalgebra of $\mathcal{M}_1$.

Proof. (i) Let $a, b \in \mathcal{M}$. Then

$$e_\mathcal{N} a e_\mathcal{N} \widehat{b} = e_\mathcal{N} a (\widehat{\mathcal{E}_\mathcal{N}(b)}) \tag{5.5.7}$$

$$= e_\mathcal{N} (a \mathcal{E}_\mathcal{N}(b))\widehat{\ } \tag{5.5.8}$$

$$= (\mathcal{E}_\mathcal{N}(a \mathcal{E}_\mathcal{N}(b)))\widehat{\ } \tag{5.5.9}$$

$$= (\mathcal{E}_\mathcal{N}(a) \mathcal{E}_\mathcal{N}(b))\widehat{\ } \tag{5.5.10}$$

$$= \mathcal{E}_\mathcal{N}(a) e_\mathcal{N} \widehat{b}. \tag{5.5.11}$$

The result now follows by density.

(ii) Let $x \in \mathcal{M}$. Suppose $e_\mathcal{N} a = a e_\mathcal{N}$. Then $\widehat{a} = a e_\mathcal{N} \widehat{1} = e_\mathcal{N} a \widehat{1} = \widehat{\mathcal{E}_\mathcal{N}(a)}$. So, $a = \mathcal{E}_\mathcal{N}(a) \in \mathcal{N}$. Conversely, suppose $a \in \mathcal{N}$. Then $e_\mathcal{N} a \widehat{b} = \widehat{\mathcal{E}_\mathcal{N}(ab)} = a\widehat{\mathcal{E}_\mathcal{N}(b)} = a e_\mathcal{N} \widehat{b}$ for all $b \in \mathcal{M}$. Hence, by density, $e_\mathcal{N} a = a e_\mathcal{N}$.

(iii) Let $a \in \mathcal{M}$. Then $J e_\mathcal{N} \widehat{a} = \widehat{\mathcal{E}_\mathcal{N}(a)^*} = \widehat{\mathcal{E}_\mathcal{N}(a^*)} = e_\mathcal{N} J \widehat{a}$. Hence, by density, $e_\mathcal{N} J = J e_\mathcal{N}$.

(iv) It follows from (i) that $\{a + \sum_{\text{finite}} a_i e_\mathcal{N} b_i : a, a_i, b_i \in \mathcal{M}\}$ is a *-subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$ containing $\mathcal{M}$ and $e_\mathcal{N}$. It is clearly contained in $\mathcal{M}_1$. Hence, $\mathcal{M}_1$ is the strict closure of this subalgebra. $\square$

The basic construction $\mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle$ corresponding to an expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ is a subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$, so inherits a fibration over the base space X. We shall typically write $a \mapsto a(x)$ for evaluating at the fibre $(\mathcal{M}_1)_x$ as for W*-bundles.

One would expect that $(\mathcal{M}_1)_x$ is the basic construction corresponding to the inclusion $\mathcal{N}_x \subset \mathcal{M}_x$. Half of this intuition is dealt with by the following proposition.

Proposition 5.5.4. *Let $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ be an expected sub-bundle over the base space X and $\mathcal{M}_1$ the basic construction. Then $(\mathcal{M}_1)_x \subseteq \langle \mathcal{M}_x, e_{\mathcal{N}_x} \rangle$ for any $x \in X$.*

Proof. Let $x \in X$. By Proposition 5.4.1, we have $e_{\mathcal{N}}(x)\widehat{a(x)} = (e_{\mathcal{N}}\widehat{a})(x) = \widehat{\mathcal{E}(a)(x)} = E_{N_x}(a(x))\widehat{\ }$ for $a \in \mathcal{M}$. So, by density, $e_{\mathcal{N}}(x) = e_{\mathcal{N}_x}$.

By Propositions 2.11.16 and 2.11.24, passing to fibres is a *-homomorphism which is continuous for the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$ and the strong* topology on $\mathcal{L}(L^2(\mathcal{M}_x))$. Hence, $(\mathcal{M}_1)_x$ is contained in the von Neumann subalgebra of $\mathcal{L}(L^2(\mathcal{M}_x))$ generated by $\mathcal{M}_x$ and $e_{\mathcal{N}_x}$, which is $\langle \mathcal{M}_x, e_{\mathcal{N}_x} \rangle$ by definition. □

The reverse inclusion would follow if one could show that $(\mathcal{M}_1)_x$ is a von Neumann algebra. In the next section, we prove this under the additional hypothesis that $\mathcal{M}$ is strictly separable. Under the same additional hypothesis, we are able to show that $\mathcal{M}_1$ and $J\mathcal{N}J$ are commutants of each other.

5.6 Generalised W*-Bundles

Investigating the basic construction for an expected sub-W*-bundle forces us to consider a generalisation of W*-bundles.

Definition 5.6.1. Let X be a compact Hausdorff space. A *generalised W*-bundle* over X is a strictly closed subalgebra of $\mathcal{L}(H)$, for some Hilbert-$C(X)$-module H, containing the canonical copy of $C(X) \subseteq \mathcal{L}(H)$.[3]

This definition of generalised W*-bundles encompasses the original by identifying a W*-bundle with its standard form.

Proposition 5.6.2. *Let $\mathcal{M}$ be a W*-bundle over X. Write $L : \mathcal{M} \to \mathcal{L}(L^2(\mathcal{M})$ for the standard form representation. Then $L(\mathcal{M})$ is a strictly closed subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$ containing the canonical copy of $C(X)$.*

[3]The canonical copy of $C(X) \subseteq \mathcal{L}(H)$ means the operators $f\mathrm{Id}_{\mathrm{H}}$ for $f \in C(X)$.

Proof. It follows from Proposition 3.2.14 that $L(\mathcal{M})$ is a C*-subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$ containing the canonical copy of $C(X)$. We show that it is strictly closed.

By the Kaplansky Density Theorem, it suffices to show that the unit ball of $L(\mathcal{M})$ is strictly closed. Hence, it suffices to show that the unit ball $L(\mathcal{M})$ is complete with respect to the strict topology. However, this follows from Proposition 3.2.15 and the fact that the unit ball of $\mathcal{M}$ is complete with respect to the $\|\cdot\|_{2,u}$-norm because, if $(a_\lambda)_{\lambda\in\Lambda}$ is a bounded net in $\mathcal{M}$, so is $(a_\lambda - a_\mu)_{(\lambda,\mu)\in\Lambda\times\Lambda}$. □

The fibration of $\mathcal{L}(H)$, where H is a Hilbert-$C(X)$-module, defined in Proposition 2.11.16 induces a fibration of a generalised W*-bundle. By Proposition 3.5.4, this definition of the fibres of a generalised W*-bundle extends that of W*-bundles. We denote the fibre of a generalised W*-bundle $\mathcal{M}$ at the point $x \in X$ by $\mathcal{M}_x$ as with standard W*-bundles.

For Definition 5.6.1 to be an appropriate definition, it should be required that the fibres of a generalised W*-bundle are von Neumann algebras. This can be shown in the case that the Hilbert-$C(X)$-module is countably generated using the $C(X)$-valued metrics introduced in Proposition 2.11.25. The proof is closely modelled on that of Theorem 3.2.9.

Theorem 5.6.3. *Let $\mathcal{M} \subseteq \mathcal{L}(H)$ be a generalised W*-bundle over X with H countably generated. Then $\mathcal{M}_x$ is a von Neumann algebra for all $x \in X$.*

Proof. Let $x \in X$. Suppose $H = \overline{\operatorname{span}}_{C(X)}\{v_i : i \in \mathbb{N}\}$, where $v_i \in H$ and $\|v_i\|_H \le 1$ for all $i \in \mathbb{N}$. Define a $C(X)$-valued metric on $\mathcal{L}(H)$ by

$$d(T,S) = \sum_{i=0}^{\infty} \frac{1}{2^i}\left(\langle (T-S)v_i, (T-S)v_i\rangle^{1/2} + \langle (T-S)^*v_i, (T-S)^*v_i\rangle^{1/2}\right) \tag{5.6.1}$$

for $T, S \in \mathcal{L}(H)$ as in Proposition 2.11.25, and consider also the metric on $\mathcal{L}(H_x)$ given by

$$d_x(t,s) = \sum_{i=0}^{\infty} \frac{1}{2^i}\left(\|(t-s)v_i(x)\|_{H_x} + \|(t-s)^*v_i(x)\|_{H_x}\right), \tag{5.6.2}$$

for $t, s \in L(H_x)$ as in Remark 2.11.26.

We need to show that $\mathcal{M}_x$ is strong* closed in $\mathcal{L}(L^2(\mathcal{M})_x)$. By the Kaplansky Density Theorem, it suffices to show that the unit ball of $\mathcal{M}_x$ is closed in the unit ball of $\mathcal{L}(L^2(\mathcal{M})_x)$. Since d_x induces the strong* topology on bounded sets, it suffices to show that d_x restricted to the unit ball of $\mathcal{M}_x$ is a complete metric.

Let $(t_n) \subseteq \mathcal{M}_x$ be a sequence that satisfies $\|t_n\| \le 1$ for all $n \in \mathbb{N}$ and is a Cauchy sequence with respect to the d_x metric on $\mathcal{M}_x$. We need to find $t \in \mathcal{M}_x$ with $\|t\| \le 1$ such that (t_n) converges to t with respect to the d_x metric on $\mathcal{M}_x$. Since a Cauchy sequence will converge to the limit of any convergent sub-sequence, we may assume that $d_x(t_{n+1}, t_n) < \frac{1}{2^n}$ without loss of generality.

We shall construct a sequence $(T^{(n)}) \subseteq \mathcal{M}$ inductively such that

$$T^{(n)}_x = t_n, \tag{5.6.3}$$

$$\|T^{(n)}\| \leq 1, \tag{5.6.4}$$

$$d(T^{(n)}, T^{(n+1)}) < \frac{1}{2^n} \tag{5.6.5}$$

for all $n \in \mathbb{N}$. Recall that with C*-algebras we may always lift elements from quotient algebras without increasing the norm [74, Section 2.2.10]. Let $T^{(1)}$ be any such lift of t_1. Suppose now that $T^{(1)}, \dots, T^{(n)}$ have been defined and have the desired properties. Let $T'^{(n+1)}$ be any lift of t_{n+1} with $\|T'^{(n+1)}\| \leq 1$. The map $y \mapsto d_y(T'^{(n+1)}_y, T^{(n)}_y) = d(T'^{(n+1)}, T^{(n)})(y)$ is continuous by Proposition 2.11.25(a). Since

$$d_x(T'^{(n+1)}_x, T^{(n)}_x) < \frac{1}{2^n}, \tag{5.6.6}$$

we can find an open neighbourhood U of x such that

$$\sup_{y \in U} d_y(T'^{(n+1)}_y, T^{(n)}_y) < \frac{1}{2^n}. \tag{5.6.7}$$

We then take a continuous function $\phi : X \to [0,1]$ such that $\phi(x) = \{1\}$ and $\phi(X \setminus U) \subseteq \{0\}$, and set $T^{(n+1)} = \phi T'^{(n+1)} + (1-\phi)T^{(n)}$. We have that $T^{(n+1)}_x = t_{n+1}$ and, using Proposition 2.11.16, we see that $\|T^{(n+1)}\| \leq 1$. Finally, we have that

$$d_y(T^{(n+1)}_y, T^{(n)}_y) = \sum_{i=0}^{\infty} \frac{\|(T^{(n+1)}_y - T^{(n)}_y)v_i(y)\|_{H_y} + \|(T^{(n+1)}_y - T^{(n)}_y)^* v_i(y)\|_{H_y}}{2^i} \tag{5.6.8}$$

$$= \sum_{i=0}^{\infty} \frac{\|\phi(y)(T'^{(n+1)}_y - T^{(n)}_y)v_i(y)\|_{H_y} + \|\phi(y)(T'^{(n+1)}_y - T^{(n)}_y)^* v_i(y)\|_{H_y}}{2^i} \tag{5.6.9}$$

$$= |\phi(y)| d_y(T'^{(n+1)}_y, T^{(n)}_y) \tag{5.6.10}$$

for $y \in X$. By considering the cases $y \in U$ and $y \in X \setminus U$ separately, we get that $d(T^{(n)}, T^{(n+1)}) < \frac{1}{2^n}$. This completes the inductive definition of the sequence $(T^{(n)})$.

Since the $C(X)$-valued metric d induces the strict topology on bounded sets, $(T^{(n)})$ is strictly Cauchy, so convergences to some $T \in \mathcal{L}(H)$ with $\|T\| \leq 1$ because the unit ball of $\mathcal{L}(H)$ is complete with respect to the strict topology (Proposition 2.11.22). Since $\mathcal{M}$ is strictly closed $T \in \mathcal{M}$. We set $t = T_x$. The convergence of (t_n) to t follows since $d_x(t_n, t) = d(T^{(n)}, T)(x)$. $\square$

Corollary 5.6.4. *Let $\mathcal{N} \subset_{\mathcal{E}} \mathcal{M}$ be an expected sub-W*-bundle over X with $\mathcal{M}$ strictly separable. Then the basic construction $\mathcal{M}_1$ is a generalised W*-bundle and $(\mathcal{M}_1)_x$ is the basic construction $\langle \mathcal{M}_x, e_{\mathcal{N}_x}\rangle$ for $\mathcal{N}_x \subset \mathcal{M}_x$ for a $x \in X$.*

Proof. By Proposition 5.5.4, $(\mathcal{M}_1)_x \subseteq \langle \mathcal{M}_x, e_{\mathcal{N}_x}\rangle$. Strict separability of $\mathcal{M}$ implies $\|\cdot\|_{2,u}$-separability of $\mathcal{M}$ by Corollary 3.2.19 and in turn that $L^2(\mathcal{M})$ is countably generated. Hence, we may apply Theorem 5.6.3 to get that $(\mathcal{M}_1)_x$ is as von Neumann algebra. Since $(\mathcal{M}_1)_x$ contains $\mathcal{M}$ and $e_{\mathcal{N}_x} = e_{\mathcal{N}}(x)$, it follows that $(\mathcal{M}_1)_x \supseteq \langle \mathcal{M}_x, e_{\mathcal{N}_x}\rangle$. □

We can also prove a analogue of Theorem 3.2.10 for generalised W*-bundles represented on countably generated Hilbert modules by mimicking the proof of Theorem 3.2.10 but using a $C(X)$-valued metric instead of the $C(X)$-valued trace E.

Theorem 5.6.5. *Let $\mathcal{M} \subseteq \mathcal{L}(H)$ be a generalised W*-bundle over X with $H = \overline{\operatorname{span}}_{C(X)}\{v_i : i \in \mathbb{N}\}$, where $v_i \in H$ and $\|v_i\|_H \leq 1$ for all $i \in \mathbb{N}$.*

Let $f : X \to \sqcup_{x\in X}\mathcal{M}_x$ be a function such that $f(x) \in \mathcal{M}_x$ for all $x \in X$. Suppose that $\sup_{x\in X}\|f(x)\| < \infty$ and, for all $x \in X$ and $\epsilon > 0$, there is an open neighbourhood $U^{(x)} \ni x$ and $T^{(x)} \in \mathcal{M}$ such that

$$\sup_{y\in U^{(x)}} d_y(f(y), T^{(x)}_y) < \epsilon, \tag{5.6.11}$$

where d_y is the metric on $L(H_y)$ defined in Remark 2.11.26. Then there is $T \in \mathcal{M}$ such $f(x) = T_x$ for all $x \in X$.

Proof. Fix $n \in \mathbb{N}$. Let $x \in X$. Choose $S^{(x)} \in \mathcal{M}$ such that $\|S^{(x)}\| \leq \|f(x)\|$ and $S^{(x)}_x = f(x)$, and choose $T^{(x)} \in \mathcal{M}$ together with an open neighbourhood $U^{(x)}$ of x such that $\sup_{y\in U^{(x)}} d_y(f(y), T^{(x)}_y) < \frac{1}{n}$. Due to the continuity of $y \mapsto d_y(T^{(x)}_y, S^{(x)}_y)$, we may, after shrinking $U^{(x)}$, assume that $\sup_{y\in U^{(x)}} d_y(f(y), S^{(x)}_y) < \frac{1}{n}$. The family of open sets $\{U^{(x)} : x \in X\}$ form an open cover for X. By compactness, a finite subcover exists. Denote the open sets in this finite subcover by $U_1, \ldots, U_m$ and the corresponding elements of $\mathcal{M}$ by $S_1, \ldots, S_m$. Let $\phi_1, \ldots, \phi_m$ be a partition of unity subordinate to $U_1, \ldots, U_m$. Set

$R^{(n)} = \sum_{j=1}^m \phi_j S^{(j)}$. Using the fact that $\phi_1, \dots, \phi_m$ form a partition of unity, we compute

$$d_y(f(y), R_y^{(n)}) = \sum_{i=0}^{\infty} \frac{1}{2^i}(\| \sum_{j=1}^m \phi_j(y)(f(y) - S_y^{(j)})v_i(y)\|_H + \| \sum_{j=1}^m \phi_j(y)(f(y) - S_y^{(j)})^* v_i(y)\|_H) \tag{5.6.12}$$

$$= \sum_{j=1}^m \phi_j(y)(\sum_{i=0}^{\infty} \frac{1}{2^i}(\|(f(y) - S_y^{(j)})v_i(y)\|_H + \|(f(y) - S_y^{(j)})^* v_i(y)\|_H) \tag{5.6.13}$$

$$= \sum_{j=1}^m \phi_j(y) d_y(f(y), S_y^{(j)}) \tag{5.6.14}$$

$$< \frac{1}{n}. \tag{5.6.15}$$

It follows that $(R^{(n)})$ is a Cauchy sequence with respect to the $C(X)$-metric d. It follows from Proposition 2.11.16 that $\|R^{(n)}\| \leq \sup_{x\in X} \|f(x)\|$ for all $n \in \mathbb{N}$. Now Propositions 2.11.22 and 2.11.25 together ensure that that $(R^{(n)})$ has a strict limit $T \in \mathcal{L}(H)$. As $\mathcal{M}$ is strictly closed, $T \in \mathcal{M}$. That $T_y = f(y)$ for all $y \in Y$ follows from uniqueness of limits as $R_y^{(n)} \to T_y$ and $R_y^{(n)} \to f(y)$. □

Corollary 5.6.6. *Let $\mathcal{N} \subset_{\mathcal{E}} \mathcal{M}$ be an expected sub-W*-bundle over X with $\mathcal{M}$ strictly separable. Let $\mathcal{M}_1$ be the basic construction. Then $\mathcal{M}_1$ and $J\mathcal{N}J$ are commutants of one another.*

Proof. The algebra $J\mathcal{N}J \subseteq J\mathcal{M}J = R(\mathcal{M})$ commutes with $\mathcal{M} = L(\mathcal{M})$ by Proposition 3.2.14 together with Theorem 3.5.5. Furthermore, $J\mathcal{N}J$ commutes with $e_{\mathcal{N}}$ by Proposition 5.5.3(ii-iii). Hence, $\mathcal{M}_1 \subseteq (J\mathcal{N}J)'$ and $J\mathcal{N}J \subseteq \mathcal{M}_1'$, making use of the fact that commutants are strictly closed. The reverse inclusions use Theorem 5.6.5 together with the fact that the result holds fibrewise [38, Proposition 3.15]. As in the proof of Corollary 5.6.4, the strict separability of $\mathcal{M}$ ensures that $L^2(\mathcal{M})$ is countably generated.

Let $T \in (J\mathcal{N}J)'$, $x \in X$ and $\epsilon > 0$. Since passing to fibres is a *-homomorphism (Proposition 2.11.16), $T_x \in ((J\mathcal{N}J)_x)'$, and $(J\mathcal{N}J)_x = J_x\mathcal{N}_xJ_x$ by Proposition 3.5.4. Since $(\mathcal{M}_1)_x$ is the basic construction for $\mathcal{N}_x \subset \mathcal{M}_x$, we have $(J_x\mathcal{N}_xJ_x)' = (\mathcal{M}_1)_x$ by [38, Proposition 3.15]. Hence, there is $S^{(x)} \in \mathcal{M}_1$ with $S_x^{(x)} = T_x$. As the map $y \mapsto d_y(T_y, S_y^{(x)})$ is continuous, there is an open neighbourhood $U^{(x)}$ of x such that $\sup_{y\in U^{(x)}} d_y(T_y, S_y^{(x)}) < \epsilon$. By Theorem 5.6.5, there is $R \in \mathcal{M}_1$ with $R_y = T_y$ for all $y \in X$. Therefore, $T = R$ and $T \in \mathcal{M}_1$.

Similarly, if $T \in \mathcal{M}_1'$, $x \in X$ and $\epsilon > 0$, then $T_x \in (\mathcal{M}_1)_x' = (J_x \mathcal{N}_x J_x) = (J\mathcal{N}J)_x$. Hence, there is $S^{(x)} \in \mathcal{N}$ with $S_x^{(x)} = (JTJ)_x$. As the map $y \mapsto d_y((JTJ)_y, S_y^{(x)})$ is continuous, there is an open neighbourhood $U^{(x)}$ of x such that $\sup_{y \in U^{(x)}} d_y((JTJ)_y, S_y^{(x)}) < \epsilon$. By Theorem 5.6.5, there is $R \in \mathcal{N}$ with $R_y = (JTJ)_y$ for all $y \in X$. Therefore, $JTJ = R$ and $T = JRJ \in J\mathcal{N}J$. $\square$

Theorem 3.5.5 and Corollary 5.6.6 give two examples of cases where generalised W*-bundles are equal to their own bicommutant. This is not always the case. We end this section with an example of a generalised W*-bundle which is strictly contained in its bicommutant.

Example 5.6.7. Let $H = C([0,1], \mathbb{C}^2)$ be the trivial Hilbert-$C([0,1])$-module with fibre $\mathbb{C}^2$. Then $\mathcal{L}(H) = C([0,1], \mathbb{M}_2(\mathbb{C}))$ acting by pointwise multiplication.

Let $\mathcal{M} = \{f \in C([0,1], \mathbb{M}_2(\mathbb{C})) : f(1) \in \mathbb{C}1_{\mathbb{M}_2(\mathbb{C})}\}$. Then $\mathcal{M} \subseteq \mathcal{L}(H)$ is a generalised W*-bundle over $[0,1]$. Suppose $g \in \mathcal{M}' \subseteq \mathcal{L}(H)$. Then, for $x \in [0,1)$, $g(x) \in \mathbb{M}_2(\mathbb{C})'$, so $g(x) \in \mathbb{C}1_{\mathbb{M}_2(\mathbb{C})}$. By continuity, $g(1) \in \mathbb{C}1_{\mathbb{M}_2(\mathbb{C})}$. Hence, $\mathcal{M}' = C([0,1], \mathbb{C}1_{\mathbb{M}_2(\mathbb{C})})$ and $\mathcal{M}'' = L(H) \supsetneq \mathcal{M}$.

Note, $\mathcal{M}$ is also a W*-bundle, so can be viewed as a generalised W*-bundle $\mathcal{M} \subseteq L^2(\mathcal{M})$. In this representation, $\mathcal{M} = \mathcal{M}''$ by Theorem 3.5.5.

5.7 The Iterated Basic Construction

If $N \subset M$ is a subfactor of finite index, then the basic construction M_1 is a II_1 factor and $M \subset M_1$ is a subfactor of the same index as $N \subset M$ [38, Proposition 3.1.7]. Hence, the basic construction can be iterated, producing a tower of subfactors

$$N \subset M \subset M_1 \subset M_2 \subset M_3 \subset \cdots . \tag{5.7.1}$$

In this section, we investigate when the basic construction for sub-W*-bundles can be iterated. Attempts at a global argument based on Jones' proof run into difficulties due to the lack of a suitable comparison theory for W*-bundles. Fibrewise arguments, building on the results of Jones, have proven more successful.

The set up for this section is the following: suppose $\mathcal{N} \subset_{\mathcal{E}_\mathcal{N}} \mathcal{M}$ is a expected sub-W*-bundle over the compact Hausdorff space X with $\mathcal{M}$ strictly separable and $\mathcal{N}_x \subset \mathcal{M}_x$ a finite index subfactor for all $x \in X$, and let $\mathcal{M}_1 = \langle \mathcal{M}, e_\mathcal{N} \rangle$ be the basic construction. The assumptions on the sub-W*-bundle ensure that, for each $x \in X$, $(\mathcal{M}_1)_x$ is the basic construction for the subfactor $\mathcal{N}_x \subset \mathcal{M}_x$, that $(\mathcal{M}_1)_x$ is a II_1 factor and that $\mathcal{M}_x \subset (\mathcal{M}_1)_x$ is a subfactor of that same index as $\mathcal{N}_x \subset \mathcal{M}_x$.

We wish to determine when the $C(X)$-valued trace E can be extended from $\mathcal{M}$ to $\mathcal{M}_1$ giving rise to a W*-bundle structure on $\mathcal{M}_1$. In order for the basic construction to be iterated, it is further necessary that the resulting sub-W*-bundle $\mathcal{M} \subset \mathcal{M}_1$ is an expected sub-W*-bundle. Also, we will need to show that $\mathcal{M}_1$ is strictly separable.[4]

The necessary and sufficient condition for $\mathcal{M}_1$ to be a W*-bundle and $\mathcal{M} \subset \mathcal{M}_1$ an expected sub-W*-bundle is that the index function $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$ is continuous. Proving that this condition is necessary is straightforward and given in Proposition 5.7.1; proving the reverse implication is more technical and will be handled in a number of steps.

Proposition 5.7.1. *Suppose $\mathcal{N} \subset_{\mathcal{E}_\mathcal{N}} \mathcal{M}$ is a expected sub-W*-bundle with $\mathcal{M}$ strictly separable and $\mathcal{N}_x \subset \mathcal{M}_x$ a finite index subfactor for all $x \in X$. Let $\mathcal{M}_1 = \langle \mathcal{M}, e_\mathcal{N} \rangle$ be the basic construction. Suppose the $C(X)$-valued trace E can be extended from $\mathcal{M}$ to $\mathcal{M}_1$ giving rise to a W*-bundle structure on $\mathcal{M}_1$. Then the map $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$ is continuous.*

Proof. Fix $x \in X$. By Proposition 5.2.3, $\mathcal{M}_x \subset (\mathcal{M}_1)_x$ is a subfactor and, by Proposition 5.6.4, is the basic construction for the subfactor $\mathcal{N}_x \subset \mathcal{M}_x$. Let τ_x denote the extension of the trace on $\mathcal{M}_x$ to $(\mathcal{M}_1)_x$. Then $[\mathcal{M}_x : \mathcal{N}_x] = \tau_x(e_{\mathcal{N}_x})^{-1}$ by [38, Proposition 3.1.7]. By hypothesis E can be extended from $\mathcal{M}$ to $\mathcal{M}_1$ giving rise to a W*-bundle structure on $\mathcal{M}_1$. Therefore, $E(a)(x) = \tau_x(a(x))$ for all $a \in \mathcal{M}_1$. In particular, $[\mathcal{M}_x, \mathcal{N}_x] = \tau_x(e_{\mathcal{N}_x})^{-1} = E(e_\mathcal{N})(x)^{-1}$. But E is $C(X)$-valued. So $x \mapsto [\mathcal{M}_x, \mathcal{N}_x]$ is continuous. ☐

For the remainder of this section, fix an expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_\mathcal{N}} \mathcal{M}$ with $\mathcal{M}$ strictly separable and $\mathcal{N}_x \subset \mathcal{M}_x$ a finite index subfactor for all $x \in X$, and let $\mathcal{M}_1 = \langle \mathcal{M}, e_\mathcal{N} \rangle$ be the basic construction. Assume further that the index map $x \mapsto [\mathcal{M}_x, N_x]$ is continuous. Denote this continuous map $[\mathcal{M} : \mathcal{N}]$.

In light of Propositions 5.2.3 and 5.4.1, there is only one candidate for the extension of the $C(X)$-valued trace E to $\mathcal{M}_1$: we must have that $E(a)(x) = \tau_x(a(x))$, where τ_x is the trace on $(\mathcal{M}_1)_x$. It is, however, not clear with this definition whether $E(a) \in C(X)$ for all $a \in \mathcal{M}_1$.

[4] It follows from Proposition 5.5.3(iv) that $\mathcal{M}_1$ is a strictly separable subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$, but we need to show that $\mathcal{M}_1$ is a strictly separable subalgebra of $\mathcal{L}(L^2(\mathcal{M}_1))$. Unlike with von Neumann algebras, we can't rely on general results saying that the strict topology on $\|\cdot\|$-norm bounded sets is independent of the representation.

Similarly, there is only one candidate for an E-preserving conditional expectation $\mathcal{E}_{\mathcal{M}} : \mathcal{M}_1 \to \mathcal{M}$: we must have $\mathcal{E}_{\mathcal{M}}(a)(x) = E_{\mathcal{M}_x}(a(x))$, where $E_{\mathcal{M}_x}$ is the trace-preserving conditional expectation $(\mathcal{M}_1)_x \to \mathcal{M}_x$. It is, however, not clear that with this definition that there actually exist $\mathcal{E}_{\mathcal{M}}(a) \in \mathcal{M}$ with $\mathcal{E}_{\mathcal{M}}(a)(x) = E_{\mathcal{M}_x}(a(x))$.

First, let us consider the the problem of extending the $C(X)$-valued trace E to $\mathcal{M}_1$. We begin by showing that E is $C(X)$-valued on a certain subalgebra of $\mathcal{M}_1$.

Lemma 5.7.2. *Let A be the $\|\cdot\|$-norm closure of $\{a + \sum_{\text{finite}} a_i e_{\mathcal{N}} b_i : a, a_i, b_i \in \mathcal{M}\}$ in $\mathcal{M}_1$. Then A is a pre-W*-bundle with respect the conditional expectation E defined above.*

Proof. Let $a, a_i, b_i \in \mathcal{M}$ for $i = 0, \ldots, m$. We have

$$E(a + \sum_{i=0}^{m} a_i e_{\mathcal{N}} b_i)(x) = \tau_x(a(x) + \sum_{i=0}^{m} a_i(x) e_{\mathcal{N}}(x) b_i(x)) \tag{5.7.2}$$

$$= \tau_x(a(x)) + \sum_{i=0}^{m} \tau_x(e_{\mathcal{N}}(x) b_i(x) a_i(x)) \tag{5.7.3}$$

$$= \tau_x((a(x)) + \sum_{i=0}^{m} [\mathcal{M}_x, \mathcal{N}_x]^{-1} \tau_x(b_i(x) a_i(x)) \tag{5.7.4}$$

$$= E(a)(x) + \sum_{i=0}^{m} [\mathcal{M}_x, \mathcal{N}_x]^{-1} E(b_i a_i)(x), \tag{5.7.5}$$

where in the third line we use [38, Proposition 3.1.7].

Since E is $C(X)$-valued on $\mathcal{M}$ and the index map $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$ is continuous by hypothesis, E is $C(X)$-valued on the subalgebra $\{a + \sum_{\text{finite}} a_i e_{\mathcal{N}} b_i : a, a_i, b_i \in \mathcal{M}\}$. Furthermore, we have $|E(a)(x)| = |\tau_x(a(x))| \leq \|a(x)\| \leq \|a\|$, so E is $C(X)$-valued on the $\|\cdot\|$-norm closure of $\{a + \sum_{\text{finite}} a_i e_{\mathcal{N}} b_i : a, a_i, b_i \in \mathcal{M}\}$, which is A.

Since τ_x is a faithful trace on $(\mathcal{M}_1)_x$ for each x, it follows that E is a conditional expectation onto $C(X) \subseteq \mathcal{M} \subseteq A$ and that the axioms (T) and (F) hold. $\square$

The pre-W*-bundle A has two natural representations: the representation on $L^2(\mathcal{M})$, which comes from that fact that $A \subseteq \mathcal{M}_1$ and $\mathcal{M}_1$ is defined as a subalgebra of $\mathcal{L}(L^2(\mathcal{M}))$, and the standard form representation on $L^2(A) = L^2(A, E)$. There are, therefore, two strict topologies on A. We will show that these strict topologies agree on bounded sets.

Lemma 5.7.3. *Let $B = \{\sum_{\text{finite}} a_i e_{\mathcal{N}} b_i : a_i, b_i \in \mathcal{M}\} \subseteq A$. Then B is $\|\cdot\|_{L^2(A,E)}$-dense in $L^2(A, E)$.*

Proof. Since B is closed under right multiplication by elements of $\mathcal{M}$, it's enough to show that $1_{\mathcal{M}}$ lies in the $\|\cdot\|_{L^2(A,E)}$-closure of B. This is shown using the machinery from the proof of Theorem 3.2.10. Let $\epsilon > 0$ and $x \in X$. By [67, Lemma 1.1],

$\{\sum_{\text{finite}} a_i(x)e_{\mathcal{N}}(x)b_i(x) : a_i, b_i \in \mathcal{M}\}$ is a dense *-subalgbera of $(\mathcal{M}_1)_x$. Hence, there is $b^{(x)} \in B$ with $\|b^{(x)}(x) - 1_x\|_{2,\tau_x} < \epsilon$. The map $y \mapsto \|1_y - b^{(x)}(y)\|_{2,\tau_y}$ is continuous by Lemma 5.7.2 as $1 - b^{(x)} \in A$. Hence, there is an open neighbourhood $U^{(x)}$ of x such that $\sup_{y\in U^{(x)}} \|1_y - b^{(x)}(y)\|_{2,\tau_y} < \epsilon$.

Then open sets $\{U^{(x)} : x \in X\}$ form an open cover for X. By compactness, a finite subcover exists. Denote the open sets in this finite subcover by $U_1, \dots, U_m$ and the corresponding elements of $\mathcal{M}$ by $b_1, \dots, b_m$. Let $\phi_1, \dots, \phi_m$ be a partition of unity subordinate to $U_1, \dots, U_m$. Set $b = \sum_{i=1}^m \phi_i b_i \in B$. Using the fact that $\phi_1, \dots, \phi_n$ form a partition of unity, we find that

$$\sup_{y\in X} \|1_y - b(y)\|_{2,\tau_y} < \epsilon. \tag{5.7.6}$$

Therefore, $\|1_{\mathcal{M}} - b\|_{2,u} < \epsilon$. As ϵ was arbitrary, we see that $1_{\mathcal{M}}$ is in the $\|\cdot\|_{L^2(A,E)}$-norm closure of B in $L^2(A, E)$. ☐

Proposition 5.7.4. *Let (a_λ) be a $\|\cdot\|$-bounded net in A and $a \in A$. The following are equivalent:*

(i) $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$,

(ii) $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(A))$,

(iii) $a_\lambda \to a$ in $\|\cdot\|_{2,u}$.

Proof. The equivalence of (ii) and (iii) was shown in Proposition 3.2.15. We prove that (i) and (ii) are equivalent. We write $\widehat{\cdot}$ for the inclusion of $\mathcal{M}$ in $L^2(\mathcal{M})$ and $\widetilde{\cdot}$ for the inclusion of A in $L^2(A)$.

Let $a_i, b_i, b, c, d \in \mathcal{M}$ for $i = 0, \dots, m$. Using Proposition 5.5.3(i), we compute that

$$(b + \sum_{i=0}^m a_i e_{\mathcal{N}} b_i)\widehat{c} = \widehat{bc} + \sum_{i=0}^m (a_i \mathcal{E}_{\mathcal{N}}(b_i c))\widehat{\ }, \tag{5.7.7}$$

$$(b + \sum_{i=0}^m a_i e_{\mathcal{N}} b_i)\widetilde{c e_{\mathcal{N}} d} = (bce_{\mathcal{N}}d)\widetilde{\ } + \sum_{i=0}^m (a_i \mathcal{E}_{\mathcal{N}}(b_i c) e_{\mathcal{N}} d)\widetilde{\ }. \tag{5.7.8}$$

Hence, $\|T\widetilde{c e_{\mathcal{N}} d}\| \le \|T\widehat{c}\|\|e_{\mathcal{N}} d\|$ for all $T \in A$.

Suppose $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$. Then $\|(a_\lambda - a)\widetilde{c e_{\mathcal{N}} d}\| \to 0$ for all $c, d \in \mathcal{M}$. By Lemma 5.7.3 and Proposition 2.11.23, it follows that $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(A))$.

Conversely, suppose $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(A))$. Let $[\mathcal{M}:\mathcal{N}]$ denote the continuous function $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$. The map $\theta : L^2(\mathcal{M}) \to L^2(A)e_{\mathcal{N}}$ given by $\widehat{c} \mapsto [\mathcal{M}:\mathcal{N}]^{1/2}\widetilde{ce_{\mathcal{N}}}$ is a well-defined isomorphism of Hilbert modules since

$$\langle \widetilde{ce_{\mathcal{N}}}, \widetilde{ce_{\mathcal{N}}}\rangle_{L^2(A)} = E(ce_{\mathcal{N}}e_{\mathcal{N}}^*c^*) \tag{5.7.9}$$

$$= E(e_{\mathcal{N}}c^*c) \tag{5.7.10}$$

$$= [\mathcal{M}:\mathcal{N}]^{-1}E(c^*c) \tag{5.7.11}$$

$$= [\mathcal{M}:\mathcal{N}]^{-1}\langle \widehat{c}, \widehat{c}\rangle_{L^2(\mathcal{M})}. \tag{5.7.12}$$

From (5.7.7) and (5.7.8) with $d = 1$, we see that θ intertwines the left A actions. Hence, $a_\lambda \to a$ in the strict topology on $\mathcal{L}(L^2(\mathcal{M}))$. $\square$

We now apply Proposition 5.7.4 to show that $\|\cdot\|$-bounded sequences in A are $\|\cdot\|_{2,u}$-Cauchy if and and only if they are Cauchy with respect to either of the strict topologies on A. This allows us to identify the completion $\overline{A}$ of the pre-W*-bundle A with the strict closure of A in either of the representations, in particular with $\mathcal{M}_1$.

Proposition 5.7.5. *Let $\overline{A}$ be the completion of the pre-W*-bundle A. The inclusions $A \subseteq \mathcal{L}(L^2(\mathcal{M}))$ and $A \subseteq \mathcal{L}(L^2(A))$ extend to embeddings $\iota_1 : \overline{A} \to \mathcal{L}(L^2(\mathcal{M}))$ and $\iota_2 : \overline{A} \to \mathcal{L}(L^2(A))$ with $\iota_1(\overline{A}) = \overline{A}^{\mathrm{st}} \subseteq \mathcal{L}(L^2(\mathcal{M}))$ and $\iota_2(\overline{A}) = \overline{A}^{\mathrm{st}} \subseteq \mathcal{L}(L^2(A))$. On bounded subsets of $\overline{A}$, the $\|\cdot\|_{2,u}$-topology agrees with the strict topologies coming from $\mathcal{L}(L^2(\mathcal{M}))$ and $\mathcal{L}(L^2(A))$.*

Proof. From Proposition 3.4.21,

$$\overline{A} = \frac{\{(a_i)_{i=1}^\infty \in \ell^\infty(A) : (a_i)_{i=1}^\infty \text{ is } \|\cdot\|_{2,u}\text{-Cauchy}\}}{\{(a_i)_{i=1}^\infty \in \ell^\infty(\mathcal{M}) : (a_i)_{i=1}^\infty \text{ is } \|\cdot\|_{2,u}\text{-null}\}}. \tag{5.7.13}$$

The maps ι_1 and ι_2 are the maps induced on the quotient by $(a_n) \mapsto \lim_{n\to\infty} a_n$, where the limit is taken in the respective strict topology. Since $(a_n - a_m)_{(n,m)\in\mathbb{N}^2}$ is a bounded net whenever (a_n) is a bounded sequence, Proposition 5.7.4 together with the strict completeness of the unit balls of $\mathcal{L}(L^2(\mathcal{M}))$ and $\mathcal{L}(L^2(A))$ ensures that these limits exist. A second application of Proposition 5.7.4, shows that ι_1 and ι_2 are well-defined on the quotient and injective.

The proof of Proposition 5.7.4 can now be applied to $\overline{A}$. Since $\|ab\|_{2,u} \leq \|a\|_{2,u}\|b\|$, we can take limits to obtain that $\|T\widetilde{ce_{\mathcal{N}}d}\| \leq \|T\widehat{c}\|\|e_{\mathcal{N}}d\|$ for all $T \in \overline{A}$ and that that the map $\theta : L^2(\mathcal{M}) \to L^2(A)e$ intertwines the left $\overline{A}$-actions. It follows that, on bounded subsets of $\overline{A}$, the $\|\cdot\|_{2,u}$-topology agrees with the strict topologies coming from $\mathcal{L}(L^2(\mathcal{M}))$ and $\mathcal{L}(L^2(A))$. $\square$

Proving that the sub-W*-bundle $\mathcal{M} \subset \mathcal{M}_1$ is an expected sub-W*-bundle is now relatively simple.

Theorem 5.7.6. *Suppose $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ is a expected sub-W*-bundle with $\mathcal{M}$ strictly separable and $\mathcal{N}_x \subset \mathcal{M}_x$ a finite index subfactor for all $x \in X$. Let $\mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle$ be the basic construction. Suppose the map $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$ is continuous. Then the sub-W*-bundle $\mathcal{M} \subset \mathcal{M}_1$ is an expected sub-W*-bundle, $[(\mathcal{M}_1)_x : \mathcal{M}_x] = [\mathcal{M}_x : \mathcal{N}_x]$ for all $x \in X$, and $\mathcal{M}_1$ is strictly separable.*

Proof. By Proposition 5.7.5, $\mathcal{M}_1$ is a W*-bundle. So $\mathcal{M} \subset \mathcal{M}_1$ is a sub-W*-bundle. Let $a, a_i, b_i \in \mathcal{M}$ for $i = 0, \ldots, m$. We have $E_{\mathcal{M}_x}(a(x) + \sum_{i=0}^m a_i(x)e_{\mathcal{N}}(x)b_i(x)) = a(x) + \sum_{i=0}^m [\mathcal{M}_x : \mathcal{N}_x]^{-1} a_i(x)b_i(x)$. Since an element of $\mathcal{M}_1$ is determined by its images in all fibres, we can define $\mathcal{E}_{\mathcal{M}}$ on the subalgebra $\{a + \sum_{\text{finite}} a_i e b_i : a, a_i, b_i \in \mathcal{M}\}$ by $\mathcal{E}_{\mathcal{M}}(a + \sum_{i=0}^m a_i e_N b_i) = a + \sum_{i=0}^m [\mathcal{M} : \mathcal{N}]^{-1} a_i b_i$, where $[\mathcal{M} : \mathcal{N}] \in C(X)$ denotes the map $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$. Note that $\mathcal{E}_{\mathcal{M}}(a) \in \mathcal{M}$ for all a for which it is defined.

Each $E_{\mathcal{M}_x}$ is a trace-preserving conditional expectation, so $\mathcal{E}_M$ is E-preserving. By Corollary 2.5.4, we have $\|\mathcal{E}_{\mathcal{M}}(a)\|_{2,u} \leq \|a\|_{2,u}$. Moreover, $\|\mathcal{E}_{\mathcal{M}}(a)\| \leq \|a\|$ for all a for which it is defined by Proposition 3.2.5 because $\|E_{\mathcal{M}_x}\| \leq 1$ for each $x \in X$. By Proposition 5.5.3(iv) together with Proposition 5.7.5, $\{a + \sum_{\text{finite}} a_i e b_i : a, a_i, b_i \in \mathcal{M}\}$ is $\|\cdot\|_{2,u}$-dense in $\mathcal{M}_1$. It follows that $\mathcal{E}_{\mathcal{M}}$ extends continuously to a bounded linear map $\mathcal{M}_1 \to \mathcal{M}$. Passing to fibres, we see that $\mathcal{E}_{\mathcal{M}}(a)(x) = E_{\mathcal{M}_x}(a(x))$ for all $a \in \mathcal{M}_1$. Since each $E_{\mathcal{M}_x}$ is a trace-preserving conditional expectation, $\mathcal{E}_{\mathcal{M}}$ is an E-preserving conditional expectation.

This proves that $\mathcal{M} \subset \mathcal{M}_1$ is an expected sub-W*-bundle. That $\mathcal{M}_1$ is strictly separable follows from 5.5.3(iv) together with Proposition 5.7.5. That $[(\mathcal{M}_1)_x : \mathcal{M}_x] = [\mathcal{M}_x : \mathcal{N}_x]$ for all $x \in X$ follows from Corollary 5.6.4 and [38, Proposition 3.1.7]. □

Theorem 5.7.6 provides conditions under which the basic construction for sub-W*-bundles can be iterated. This allows us to build a Jones tower of W*-bundles

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \cdots \tag{5.7.14}$$

and define the relative commutants $(\mathcal{M}_i' \cap \mathcal{M}_j)_{i \leq j}$ for such sub-W*-bundles $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$. We will pick up this line of investigation in the final section of this chapter.

5.8 Continuity of Index

In the previous section, we proved that the continuity of the index is the crucial necessary and sufficient condition for the basic construction for an expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ to be iterated. In this section, we investigate to what extent continuity of index is automatic. We show, using a result of Pimsner and Popa, that lower-semicontinuity of the index is automatic, but provide an example of an expected W*-bundle for which the index is not continuous.

First, the automatic lower-semicontinuity.

Theorem 5.8.1. *Let $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ be an expected sub-W*-bundle over X with $\mathcal{N}_x \subset \mathcal{M}_x$ a subfactor for all $x \in X$. Then the map $x \mapsto [\mathcal{M}_x : \mathcal{N}_x]$ is lower-semicontinuous.*

Proof. By [67, Theorem 2.2], we have that

$$[\mathcal{M}_x : \mathcal{N}_x]^{-1} = \inf \left\{ \frac{\|E_{\mathcal{N}_x}(a(x))\|^2_{2,\tau_x}}{\|a(x)\|^2_{2,\tau_x}} : a \in \mathcal{M}, a(x) \neq 0 \right\} \tag{5.8.1}$$

for all $x \in X$.[5]

Fix $x_0 \in X$, $\gamma > 0$ and suppose $[\mathcal{M}_{x_0} : \mathcal{N}_{x_0}] > \gamma$. Then there exists $a \in \mathcal{M}$ with $a(x_0) \neq 0$ such that

$$[\mathcal{M}_{x_0} : \mathcal{N}_{x_0}]^{-1} \leq \frac{\|E_{\mathcal{N}_{x_0}}(a(x_0))\|^2_{2,\tau_{x_0}}}{\|a(x_0)\|^2_{2,\tau_{x_0}}} < \frac{1}{\gamma}. \tag{5.8.2}$$

By Proposition 5.4.1, $E_{\mathcal{N}_x}(a(x)) = \mathcal{E}_{\mathcal{N}}(a)(x)$ for all $x \in X$. Hence, by applying Proposition 3.2.6, the map $x \mapsto \|E_{\mathcal{N}_x}(a(x))\|^2_{2,\tau_x} / \|a(x)\|^2_{2,\tau_x}$ is well-defined and continuous in a neighbourhood of x_0.

Therefore, there exists an open neighbourhood U of x_0 such that $a(x) \neq 0$ and

$$\frac{\|E_{\mathcal{N}_x}(a(x))\|^2_{2,\tau_x}}{\|a(x)\|^2_{2,\tau_x}} < \frac{1}{\gamma} \tag{5.8.3}$$

whenever $x \in U$. Thus, by (5.8.1), $[\mathcal{M}_x : \mathcal{N}_x]^{-1} < \frac{1}{\gamma}$ whenever $x \in U$, so $[\mathcal{M}_x : \mathcal{N}_x] > \gamma$ whenever $x \in U$. □

Now, the counterexample.

Example 5.8.2. Let $X = \mathbb{N} \cup \{\infty\}$ be the one point compactification of the natural numbers. Let $M = \bigotimes_1^\infty \mathcal{R} \cong \mathcal{R}$ and α_k be the automorphism of M which transposes the

[5] This formula is also valid when $[\mathcal{M}_x : \mathcal{N}_x] = \infty$ provided one makes the convention $\infty^{-1} = 0$.

k-th and $(k+1)$-th tensor factors for $k \in \mathbb{N}$. Set $N_k = M^{\alpha_k}$ to be the fix point set of M under the automorphism α_k for $k \in \mathbb{N}$, and set $N_\infty = M$.

We consider the sub-W*-bundle $\mathcal{N}$ of the trivial bundle $\mathcal{M} = C_\sigma(X, M)$ defined by the family of von Neumann subalgebras $\{N_k\}_{k \in X}$. We now check the continuity conditions of Proposition 5.3.1 and Theorem 5.4.2 to show that $\mathcal{N} \subset \mathcal{M}$ is an expected sub-W*-bundle with $\mathcal{N}_k = N_k$ for all $k \in X$.

Let $a \in \mathcal{M}$ and $\epsilon > 0$. There exists $K_1 \in \mathbb{N}$ and $b \in \bigotimes_1^{K_1} \mathcal{R} \subseteq \bigotimes_1^\infty \mathcal{R}$ such that $\|a(\infty) - b\|_{2,\tau_M} < \frac{\epsilon}{2}$. Since $a \in \mathcal{M}$, there is $K_2 \in \mathbb{N}$ such that $\|a(\infty) - a(k)\|_{2,\tau_M} < \frac{\epsilon}{2}$ whenever $k > K_2$. Let $k > \max(K_1, K_2)$. Then $\alpha_k(b) = b$ and

$$\begin{aligned}
\|a(k) - b\|_{2,\tau_M} &\leq \|a(k) - a(\infty)\|_{2,\tau_M} + \|a(\infty) - b\|_{2,\tau_M} && (5.8.4)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (5.8.5)\\
&= \epsilon. && (5.8.6)
\end{aligned}$$

Hence, $b \in N_k$ and $\mathrm{dist}_{\|\cdot\|_{2,\tau_M}}(a(k), N_k) < \epsilon$. This proves that the map $k \mapsto \mathrm{dist}_{\|\cdot\|_{2,\tau_M}}(a(k), N_k)$ is continuous at $k = \infty$, which is the only point that must be checked.

We have now shown that $\mathcal{N} \subset \mathcal{M}$ is an expected sub-W*-bundle and turn to the index function. Since α_k is an outer automorphism of M of order 2, the fixed point subalgebra $N_k \subset M$ has index 2 by [38, Example 2.3.3]. Hence, we have

$$[\mathcal{M}_k : \mathcal{N}_k] = \begin{cases} 2, & k \in \mathbb{N}, \\ 1, & k = \infty, \end{cases} \tag{5.8.7}$$

which is lower-semicontinuous but not upper-semicontinuous.

5.9 The Family of Subfactors $\mathcal{R}_\beta \subset \mathcal{R}$

This sections is devoted to the construction of a particular sub-W*-bundle. This sub-W*-bundle combines the family of subfactors obtained by considering the algebras generated by Jones projections $e_0^{(\beta)}, e_1^{(\beta)}, e_2^{(\beta)}, \ldots$ for a subfactor of finite index β (see [38, Theorem 4.1.1]. These projections satisfy the following relations:

$$\begin{aligned}
e_i^{(\beta)} &= e_i^{(\beta)*} = e_i^{(\beta)2} && (i \in \mathbb{N}_0) && (5.9.1)\\
e_i^{(\beta)} e_j^{(\beta)} &= e_j^{(\beta)} e_i^{(\beta)} && (i, j \in \mathbb{N}_0, |i-j| \geq 2), && (5.9.2)\\
e_i^{(\beta)} e_{i\pm1}^{(\beta)} e_i^{(\beta)} &= \beta^{-1} e_i^{(\beta)} && (i \in \mathbb{N}_0), && (5.9.3)\\
\mathrm{tr}(w e_n^{(\beta)}) &= \beta^{-1} \mathrm{tr}(w) && (n \in \mathbb{N}_0, w \in \mathrm{Alg}\{1, e_0^{(\beta)}, \ldots, e_{n-1}^{(\beta)}\}). && (5.9.4)
\end{aligned}$$

We follow the notation of [30, Section 3.4], writing $\mathcal{R} = \{1, e_0^{(\beta)}, e_1^{(\beta)}, e_2^{(\beta)}, \ldots\}''$ and $\mathcal{R}_\beta = \{1, e_1^{(\beta)}, e_2^{(\beta)}, \ldots\}''$. In [38], this family of subfactors is denoted $P_\tau \subset P$, where $\tau = \beta^{-1}$.

Let $J = \{4\cos^2(\frac{\pi}{n}) : n = 3, 4, \ldots\} \cup [4, \infty)$ be the set of allowed finite indices [38, Theorem 4.3.1], and let X be a compact subset of J. For each $\beta \in X$, let $e_0^{(\beta)}, e_1^{(\beta)}, e_2^{(\beta)}, \ldots \in \mathcal{R}$ be a sequence of Jones projections for a subfactor of index $\beta \in J$.

The sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ will be constructed in the ambient space $\prod_{\beta \in X} \mathcal{R}$. We embed $\ell^\infty(X)$ into $\prod_{\beta\in X} \mathcal{R}$ diagonally via the map $g \mapsto (g(\beta))_{\beta \in X}$ and define a conditional expectation $E : \prod_{\beta \in X} \mathcal{R} \to \ell^\infty(X)$ via $(a_\beta)_{\beta \in X} \mapsto (\tau_{\mathcal{R}}(a_\beta))$.

Set $e_i = (e_i^{(\beta)})_{\beta \in X}$ for all $i \in \mathbb{N}_0$. Let $\mathcal{M}_0$ be the C*-algebra generated by the continuous functions $C(X) \subseteq \ell^\infty(X) \subseteq \prod_{\beta \in X} \mathcal{R}$ together with $\{e_0, e_1, e_2, \ldots\}$ and let $\mathcal{N}_0$ be the C*-algebra generated by $C(X)$ together with $\{e_1, e_2, \ldots\}$.

Proposition 5.9.1. *For all $a \in \mathcal{M}_0$, $E(a) \in C(X)$.*

Proof. Let $A = \{a \in \mathcal{M}_0 : E(a) \in C(X)\}$. We first observe that A is a $\|\cdot\|$-closed subspace of $\mathcal{M}_0$. Furthermore, we have $E(ga)(\beta) = \tau_{\mathcal{R}}(g(\beta)a_\beta) = g(\beta)\tau_{\mathcal{R}}(a_\beta) = g(\beta)E(a)(\beta)$. Hence, A is a $C(X)$-submodule of $\mathcal{M}_0$.

It suffices, therefore, to show that all words in $\{e_i : i \in \mathbb{N}_0\}$ are in A. Let w be a word in $e_0, \ldots, e_n$. Following the technique of [38, Lemmas 4.1.2 and 4.16], we can compute the trace of w in each fibre by converting w into a totally reduced word by a sequence of cyclic permutations together with the rules $e_ie_j \leftrightarrow e_je_i$ for $|i-j| \geq 2$, $e_ie_i \leftrightarrow e_i$ and $e_ie_{i\pm1}e_i \leftrightarrow e_i$. We find that $\tau_{\mathcal{R}}(w(\beta)) = \beta^{-r}\tau_{\mathcal{R}}(e_{i_1}^{(\beta)}e_{i_2}^{(\beta)} \ldots e_{i_k}^{(\beta)})$ for some $r \in \mathbb{N}_0$ and indices $i_1, i_2 \ldots, i_k$ with $|i_s - i_t| \geq 2$ for $s \neq t$. Consequently, $\tau_{\mathcal{R}}(w(\beta)) = \beta^{-r-k}$ by (5.9.4). Hence, $E(w) \in C(X)$. □

It follows that $\mathcal{M}_0$ is a pre-W*-bundle and can be completed to a W*-bundle $\mathcal{M}$ over X.[6] Let $\mathcal{N}$ be the $\|\cdot\|_{2,u}$-norm closure of $\mathcal{N}_0$ in $\mathcal{M}$. Passing to fibres, we see that $\mathcal{N}_\beta \subset \mathcal{M}_\beta$ is isomorphic to the subfactor $\mathcal{R}_\beta \subset \mathcal{R}$.

The sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ is an expected sub-W*-bundle. By [38, Corollary 4.1.12], we have that $E_{\mathcal{N}_x}(e_0^{(\beta)}) = \beta^{-1}1_{\mathcal{R}}$. Together with the bimodule property of conditional expectations, it follows that, if w is a word in $e_0^{(\beta)}, \ldots, e_n^{(\beta)}$, then $E_{\mathcal{N}_x}(w) = \beta^{-r}w'$, where r is the number of occurrences of $e_0^{(\beta)}$ in the word w and w' is the word obtained from w by deleting all occurrences of $e_0^{(\beta)}$. Since $\beta \mapsto \beta^{-r}$ is continuous on X, the fibrewise-defined

[6]In fact, we can just take the $\|\cdot\|_{2,u}$-norm closure of $\mathcal{M}_0$ in $\prod_{\beta \in X} \mathcal{R}$.

conditional expectation $\mathcal{E}_{\mathcal{N}}(a)(x) = E_{\mathcal{N}_x}(a(x))$ does map $\mathcal{M}_0$ into $\mathcal{N}_0$. Hence, $\mathcal{E}_{\mathcal{N}}$ is a well-defined E-preserving conditional expectation $\mathcal{M} \to \mathcal{N}$.

A slight modification of the construction, gives the following result of independent interest.

Theorem 5.9.2. *Let X be a compact Hausdorff space. For any continuous function $f : X \to J$, where J is the Jones set of allowable finite indices, there is an expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ with index function $[\mathcal{M} : \mathcal{N}] = f$.*

Proof. The sub-W*-bundle $\mathcal{N} \subset \mathcal{M}$ will be constructed in the ambient space $\prod_{x\in X} \mathcal{R}$. We embed $\ell^\infty(X)$ into $\prod_{x\in X} \mathcal{R}$ diagonally via the map $g \mapsto (g(x))_{x\in X}$ and define a conditional expectation $E : \prod_{x\in X} \mathcal{R} \to \ell^\infty(X)$ via $(a_x)_{x\in X} \mapsto (\tau_{\mathcal{R}}(a_x))$.

We define $e_i = (e_i^{(f(x))})_{x\in X}$, where $e_0^{(\beta)}, e_1^{(\beta)}, e_2^{(\beta)}, \ldots \in \mathcal{R}$ are Jones projections for a subfactor of index $\beta \in J$. As in the main construction of this section, we let $\mathcal{M}_0$ be the C*-algebra generated by the continuous functions $C(X) \subseteq \ell^\infty(X) \subseteq \prod_{\beta\in X} \mathcal{R}$ together with $\{e_0, e_1, e_2, \ldots\}$ and let $\mathcal{N}_0$ be the C*-algebra generated by $C(X)$ together with $\{e_1, e_2, \ldots\}$. We then complete to give the required expected sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ as in the main construction of this section.

Passing to the fibre at $x \in X$, we see that $\mathcal{N}_x \subset \mathcal{M}_x$ is isomorphic to the subfactor $\mathcal{R}_{f(x)} \subset \mathcal{R}$, which has index $f(x)$. Hence, the index function of the sub-W*-bundle $\mathcal{N} \subset_{\mathcal{E}_{\mathcal{N}}} \mathcal{M}$ is $[\mathcal{M} : \mathcal{N}] = f$. □

5.10 Outlook

The results of this chapter are just the foundations of the theory of sub-W*-bundles. We end this chapter by highlighting future directions for research on sub-W*-bundles.

The next goal for the abstract theory of sub-W*-bundles is to define and investigate an analogue of the standard invariant for sub-W*-bundles. In the setting of finite index subfactors, the standard invariant can be viewed as the collection of higher relative commutants endowed with the additional structure of a planar algebra (see [39]). Under amenability hypotheses on the subfactor, the standard invariant is a complete invariant [68, Theorem 2]. Moreover, there is a reconstruction theorem that, given a suitable planar algebra, builds a subfactor with this planar algebra as its standard invariant [69, Theorem 3.1]. To what extent does all this carry over to the world of sub-W*-bundles?

The further development of the abstract theory of sub-W*-bundles should go hand in hand with the construction of new examples. The example of the previous section encodes the family of subfactors $R_\beta \subset \mathcal{R}$ as a sub-W*-bundle. This family has the property that $\mathcal{R}'_\beta \cap \mathcal{R} = \mathbb{C}1_{\mathcal{R}}$ for $\beta \leq 4$ but $\mathcal{R}'_\beta \cap \mathcal{R} \neq \mathbb{C}1_{\mathcal{R}}$ for $\beta > 4$ [38, Corollary 2.2.4 and Section 5.3]. We say that $R_\beta \subset \mathcal{R}$ is an irreducible subfactor for $\beta \leq 4$ and a reducible subfactor for $\beta > 4$. This leads to the following question: can a sub-W*-bundle be constructed which is irreducible in each fibre and encompasses both the discrete and continuous parts of the Jones set of allowed indices?

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